

# Identities for Poincaré Polynomials via Kostant Cascades

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Communicated by M. Schlichenmaier

**Abstract.** We propose and prove an identity relating the Poincaré polynomials of stabilizer subgroups of the affine Weyl group and of the corresponding stabilizer subgroups of the Weyl group.

*Mathematics Subject Classification:* 20F55, 17B22.

*Key Words:* Affine Weyl Group, Poincaré polynomial.

## 1. Introduction

Let  $\Phi$  be a finite indecomposable root system and  $W$  be its Weyl group. Recall that the affine Weyl group  $W^{\text{aff}}$  is the semidirect product of  $W$  and the root lattice  $\mathbb{Z}\Phi$ . It acts on  $\mathbf{R}\Phi$  as a reflection group with (closed) alcoves as fundamental domains. Denote by  $\mathfrak{A}$  the alcove contained in the dominant Weyl chamber such that  $0 \in \mathfrak{A}$ . For  $\lambda \in \mathfrak{A}$ , we consider the Poincaré polynomials of both its stabiliser  $W_\lambda$  in  $W$

$$P(W_\lambda) = \sum_{w \in W_\lambda} t^{l(w)}, \quad \text{and its stabiliser } W_\lambda^{\text{aff}} \text{ in } W^{\text{aff}}, \quad P(W_\lambda^{\text{aff}}) = \sum_{w \in W_\lambda^{\text{aff}}} t^{l(w)},$$

where  $l$  is the length function. Now  $W_\lambda$  is a parabolic subgroup of  $W_\lambda^{\text{aff}}$ , so  $P_\lambda(t)$  divides  $P_\lambda^{\text{aff}}(t)$ , and we will establish a decomposition of the quotient  $P(W_\lambda^{\text{aff}})/P(W_\lambda)$  in terms of stabilisers. For this purpose, define  $W_{\lambda,\xi} := \text{Stab}_{W_\lambda}(\xi)$  in  $W_\lambda$ , where  $\xi \in \mathbf{Z}\Phi$  is in the root lattice. Then

$$P(W_\lambda/W_{\lambda,\xi}) := P(W_\lambda)/P(W_{\lambda,\xi})$$

is again a polynomial. We then define an integral function on  $\mathbf{Z}\Phi$  given by

$$\ell_\xi := \deg(P(W_\xi^{\text{aff}})/P(W_\xi)).$$

Here  $W_\xi$  is the stabilizer of  $\xi$  in  $W$ , while  $W_\xi^{\text{aff}}$  is obtained by adding to  $W_\xi$  the affine reflection with respect to the hyperplane  $(x, \beta_1) = 1$ , with  $\beta_1$  being the largest short root.

Further, let  $\mathcal{K}$  be the Kostant cascade of orthogonal roots for  $\Phi$  defined in [8], which is a collection of positive roots with a partial ordering “ $\preceq$ ” that we recall in Section 5. Then we have

**Theorem 1.1.** *The Poincaré polynomial of  $W_\lambda^{\text{aff}}$  factorizes as follows,*

$$P\left(W_\lambda^{\text{aff}}/W_\lambda\right) = 1 + \sum_{\xi_\beta} t^{\ell_{\xi_\beta}} P\left(W_\lambda/W_{\lambda, \xi_\beta}\right), \quad (1)$$

where the summation is taken over all dominant  $\xi_\beta = \sum_{\beta' \preceq \beta} \beta'$ , with  $\beta \in \mathcal{K}$  such that  $\langle \xi_\beta, \xi_\beta \rangle = 2\langle \xi_\beta, \lambda \rangle$ .

The formula in Theorem 1.1 has its origin in the study of the quantization of the moduli stack of parabolic Higgs bundles associated with  $\mathbb{P}^1$  with two marked points, which we briefly explain now.

Let  $M_H$  be the moduli stack of  $G$ -Higgs bundles on a Riemann surface  $\Sigma$ , and  $L$  be the determinant line bundle over  $M_H$ . The Verlinde formula for Higgs bundles, motivated by quantum physics in [3, 4] and proved in [1] and [5], gives the graded dimension of  $H^0(M_H, L^k)$

$$\dim_t H^0(M_H, L^k) = \sum_{n=0}^{\infty} \dim H_n^0(M_H, L^k) t^n.$$

Here, we have used the lift of the  $\mathbb{C}^*$ -action on  $M_H$  to the space of holomorphic sections on  $L$  to decompose  $H^0(M_H, L^k)$ , with  $H_n^0(M_H, L^k)$  being a subspace where  $\mathbb{C}^*$  acts by the  $n$ -th tensor power of the defining action of  $\mathbb{C}^*$  on  $\mathbb{C}$ .

The Verlinde formula for Higgs bundles extends to the case of parabolic Higgs bundles and naturally defines a one parameter (denoted  $t$ ) family of 2d TQFT as established in [1]. Such a one parameter family of TQFT is uniquely determined by a one parameter family of commutative Frobenius algebras, whose underlying family of finite dimensional vector space  $V_t = V$  is independent of  $t$  and has a basis given by the integrable weights of  $G$  at level  $k$ . The associated one parameter family of bilinear forms

$$B^{(t)} : V^{\otimes 2} \rightarrow \mathbb{C}$$

is determined by applying the TQFT functor to  $\mathbb{P}^1$  with two marked points. Given two integrable weights  $\lambda_1$  and  $\lambda_2$ , one can consider the matrix element  $B_{\lambda_1 \lambda_2}^{(t)}$ , and Theorem 1.1 in the present paper was used in [1] to prove the following elegant formula

$$B_{\lambda_1 \lambda_2}^{(t)} = \delta_{\lambda_1 \lambda_2^*} P(W_{\lambda_1}^{\text{aff}}).$$

More precisely,  $B_{\lambda_1, \lambda_2}^{(t)}$  is given by an index over the moduli stack of  $G$ -bundles on  $\mathbb{P}^1$  and can be computed by summing over the Shatz strata, labeled by elements in the co-root lattice of  $G$  (see Section 5 of [1] for details). The main theorem of this paper ensures that contributions from different Shatz strata can be nicely summed together.<sup>1</sup>

Furthermore, this interpretation of the main result of this paper also hints at several possible ways to generalize our main theorem. For example, as the Poincaré polynomial  $P(W_\lambda)$  arises as the normalization factor for the Hall-Littlewood polynomials, it is natural to consider replacing  $P(W_\lambda)$  with the normalization factor for other types of orthogonal polynomials associated with root systems.

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<sup>1</sup> Notice that  $\Phi$  in this paper is in fact the coroot lattice of  $G$ , which results in our convention for  $W_\lambda^{\text{aff}}$ , the inner product  $(\cdot, \cdot)$  and  $\mathcal{K}$ .

Our factorization formula also has the following geometric interpretation. Denote the maximal compact subgroup of  $G$  as  $K$ , whose maximal torus is denoted as  $T$  with Cartan subalgebra  $\mathfrak{t}$ . Given an element  $\lambda$  in the Weyl alcove in  $\mathfrak{t}$ , we can define two subgroups of  $K$ . The first,  $K_\lambda$ , is the stabilizer of  $\lambda$ , while the second,  $K'_\lambda$ , is the stabilizer of  $\exp(\lambda) \in K$ . Then their Weyl groups will be  $W_\lambda$  and  $W_\lambda^{\text{aff}}$  respectively. On the other hand, using the following fibration of (generalized) flag varieties,

$$K_\lambda/T \rightarrow K'_\lambda/T \rightarrow K'_\lambda/K_\lambda,$$

we can relate their cohomology groups using the Serre spectral sequence. As all their cohomology groups are in even degrees, all differentials in the spectral sequence vanish, and we have

$$H^*(K'_\lambda/T) = H^*(K_\lambda/T) \otimes H^*(K'_\lambda/K_\lambda).$$

This gives a factorization of Poincaré polynomials of these flag varieties. As

$$P(K_\lambda/T) = P(W_\lambda) \quad \text{and} \quad P(K'_\lambda/T) = P(W_\lambda^{\text{aff}}),$$

we get the following formula for the Poincaré polynomial of the partial flag variety  $K'_\lambda/K_\lambda$ ,

$$P(K'_\lambda/K_\lambda) = P(W_\lambda^{\text{aff}}) / P(W_\lambda).$$

Therefore, the main theorem of the present paper gives an explicit expression for this Poincaré polynomial.

The proof of the main theorem stretches over Section 2 to Section 7. A very preliminary remark about Theorem 1.1 is that if  $\lambda$  is not on the affine wall of  $\mathfrak{A}$ , then (1) is trivially satisfied, since the left-hand side is 1 and so is the right-hand side, where the sum is over the empty set.

In Section 2 we present proofs of elementary identities between Gaussian polynomials, which we use in the following Section 3 to obtain equation (25), which is a preliminary version of formula (1) from Theorem 1.1 above in the special case where  $\lambda$  is the minuscule fundamental weight associated to a simple root, and the range of the sum on the left-hand side of (1) is not yet understood to be given as stated in Theorem 1.1, but is simply given in Table 2 for the different types of root systems and the exponents of  $t$  are not yet determined.

In Section 4 we establish that formula (25) can be rewritten into formula (28), which refers to the affine stabilisers as opposed to the full Weyl group, which is the case in (25), thus bringing it one step closer to (1). A remark following the proof of Lemma 4.4 establishes that the right-hand side of (1) can be identified for any  $\lambda$  on the affine wall with the corresponding right-hand side of (25) for some minuscule fundamental weight associated to a simple root of some other root system. There remains now to prove that the exponents of  $t$  is given as in (1) or equivalently as in (34) and of course also that the summation range and each coefficient is correct. As is stated in Lemma 4.6, by inspecting through Table 3, one sees that the exponents, which are involved in (25), are indeed given by (34) for the case of minuscule fundamental weights. That the coefficient from (1) works is the content of Proposition 4.8. But it still leaves the determination of the summation range.

In order to determine a suitable summation range, Section 5 recalls fundamentals about the Kostant cascade of an indecomposable root system and the following section introduces the set (47) of dominant weights which constitute the summation

range in (1). Proposition 6.2 and the following five lemmas establishes various properties which are needed in the next section.

Section 7 begins with Lemma 7.1, which establishes that for the minuscule weights associated to simple roots, the summation range given in Table 2 is indeed given by the set (47). The following two lemmas provide further properties of the set (47), which allow us to prove the main theorem in its reformulated version of Theorem 7.5 in Section 7. This is now done by inspection, where we use the result of Section 4 to map the problem for general  $\lambda$  to a problem concerning the minuscule fundamental weight associated to a simple root of some other root system and then identify the sum on the right hand of (1) with the corresponding sum for this minuscule fundamental weight. This is done by matching up the summation range, the exponents of  $t$  and their corresponding coefficients.

**Acknowledgment.** Part of JEA and DP's work was supported by the center of excellence grant "Center for Quantum Geometry of Moduli Space" from the Danish National Research Foundation (DNRF95). DP's work was also supported by the Walter Burke Institute for Theoretical Physics, and JEA thanks the Walter Bruke Institute for Theoretical Physics for hospitality during several visits.

### Index of notation

$\Phi$ : an indecomposable finite root system;  
 $W$ : the Weyl group of  $\Phi$ ;  $s_\alpha \in W$ : the reflection with respect to  $\alpha \in W$ ;  
 $\Delta = \{\alpha_i \mid 1 \leq i \leq n\}$ : a basis for  $\Phi$  numbered as in [B];  
 $\Delta_\lambda = \{\alpha \in \Delta \mid (\lambda, \alpha^\vee) = 0\}$ ;  
 $\varpi_\alpha$ : the fundamental weight corresponding to  $\alpha \in \Delta$ ;  
 $s_i = s_{\alpha_i}$  and  $\varpi_i = \varpi_{\alpha_i}$  for  $1 \leq i \leq n$ ;  
 $\text{supp}$ : the support of a linear combination of simple root as in (44);  
 $\beta_1$ : the dominant short root;  $\alpha_0 = -\beta_1$ ;  $s_0$ : the affine reflection  $s_{\beta_1, 1}$ ;  
 $W^{\text{aff}}$ : the affine Weyl group generated by  $W$  and  $s_0$ ;  
 $\mathfrak{A}$ : the fundamental alcove for  $W^{\text{aff}}$  as in (26);  
 $W'_\lambda$ : the stabilizer of  $\lambda$  in a subgroup  $W'$  of  $W^{\text{aff}}$ ;  
 $P(W')$ : the Poincaré polynomial of a subgroup  $W'$  of  $W^{\text{aff}}$ ;  
 $J(\lambda) \cup I(\lambda)$ : the decomposition of  $\Delta_\lambda$  as defined in (29);  
 $(\mathcal{K}, \preceq)$ : the Kostant cascade of  $\Phi$  with its partial order;  
 $\xi_\beta = \sum_{\beta' \preceq \beta} \beta$  for any  $\beta \in \mathcal{K}$ .

## 2. Gaussian polynomials

We work with polynomials over  $\mathbf{Z}$  in one variable  $t$ . For any integers  $n \geq r \geq 0$  consider the Gaussian polynomial

$$\begin{bmatrix} n \\ r \end{bmatrix} := \frac{(t^n - 1)(t^{n-1} - 1) \dots (t^{n-r+1} - 1)}{(t^r - 1)(t^{r-1} - 1) \dots (t - 1)}. \quad (2)$$

One has  $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 = \begin{bmatrix} n \\ n \end{bmatrix}$  and for  $n > 0$

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r \end{bmatrix} + t^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} \quad (3)$$

where the first summand on the right is to be interpreted as 0 if  $r = n$  and the second one as 0 if  $r = 0$ . A classical result (see e.g. Example 3 on page 26 in [9]) says: If  $u$  is another variable, then

$$\prod_{l=0}^{n-1} (1 + t^l u) = \sum_{r=0}^n t^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} u^r. \tag{4}$$

This can be proved by induction on  $n$  by using (3).

If we plug in  $tu$  for  $u$  in (4) and use  $r + \frac{1}{2}r(r-1) = \frac{1}{2}r(r+1)$ , then we get

$$\prod_{l=1}^n (1 + t^l u) = \sum_{r=0}^n t^{r(r+1)/2} \begin{bmatrix} n \\ r \end{bmatrix} u^r. \tag{5}$$

**Claim 2.1.**

If  $0 \leq m \leq n$ , then 
$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \sum_{s=0}^m t^{s^2} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix}. \tag{6}$$

**Proof.** We start with (4) for  $m+n$ :

$$\begin{aligned} & \sum_{p=0}^{m+n} t^{p(p-1)/2} \begin{bmatrix} m+n \\ p \end{bmatrix} u^p = \prod_{l=0}^{m-1} (1 + t^l u) \prod_{l=m}^{m+n-1} (1 + t^l u) \\ &= \prod_{l=0}^{m-1} (1 + t^l u) \prod_{l=0}^{n-1} (1 + t^l t^m u) = \left( \sum_{r=0}^m t^{r(r-1)/2} \begin{bmatrix} m \\ r \end{bmatrix} u^r \right) \left( \sum_{s=0}^n t^{s(s-1)/2} \begin{bmatrix} n \\ s \end{bmatrix} t^{ms} u^s \right). \end{aligned}$$

A comparison of the coefficient of  $u^p$  on both sides yields

$$t^{\binom{p}{2}} \begin{bmatrix} m+n \\ p \end{bmatrix} = \sum_{s=0}^p t^{\binom{p-s}{2} + \binom{s}{2} + ms} \begin{bmatrix} m \\ p-s \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix},$$

in particular for  $p = m$  (using  $m \leq n$ )

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \sum_{s=0}^m t^{\binom{m-s}{2} + \binom{s}{2} + ms - \binom{m}{2}} \begin{bmatrix} m \\ m-s \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix}.$$

A simple calculation shows that  $\binom{m-s}{2} + \binom{s}{2} + ms - \binom{m}{2} = s^2$ . Now the claim follows from  $\begin{bmatrix} m \\ m-s \end{bmatrix} = \begin{bmatrix} m \\ s \end{bmatrix}$ . ■

**Claim 2.2.** Let  $n$  be a positive integer. Set

$$S_n^{\text{ev}} = \sum_{0 \leq r \leq n, r \text{ even}} t^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix}, \quad S_n^{\text{odd}} = \sum_{1 \leq r \leq n, r \text{ odd}} t^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix}.$$

Then 
$$S_n^{\text{ev}} = S_n^{\text{odd}} = \prod_{l=1}^{n-1} (1 + t^l). \tag{7}$$

**Proof.** Note that (4) evaluated at  $u = 1$  yields

$$S_n^{\text{ev}} + S_n^{\text{odd}} = 2 \prod_{l=1}^{n-1} (1 + t^l),$$

where the 2 arises from the factor  $1 + u$  for  $l = 0$  in (4). Similarly we get at  $u = -1$  in (4) that

$$S_n^{\text{ev}} - S_n^{\text{odd}} = \prod_{l=0}^{n-1} (1 - t^l) = 0.$$

The claim follows immediately from this. ■

### 3. Poincaré polynomials

Let  $\Phi$  be a finite root system. We denote by  $\Delta$  a basis for  $\Phi$  and by  $W$  its Weyl group. We shall below use the same numbering as in the tables in [2] for simple roots  $(\alpha_i)$ , simple reflections  $(s_i = s_{\alpha_i})$  and fundamental weights  $(\varpi_i)$  of a finite indecomposable root system.

The Poincaré polynomial of a finite Coxeter group such as  $W$  will be denoted by  $P(W) = \sum_{w \in W} t^{l(w)}$ , and we write  $P(X_n) = P(W)$  if the root system has type  $X_n$ . The degree of  $P(W)$  is the maximal length in  $W$ , equal to the number of positive roots in  $\Phi$ . One has in general

$$P(W) = \prod_{i=1}^n \frac{t^{m_i+1} - 1}{t - 1} \tag{8}$$

where  $m_1, m_2, \dots, m_n$  are the exponents of the root system. One gets explicitly for the classical types

$$P(A_n) = \prod_{i=1}^n \frac{t^{i+1} - 1}{t - 1}, \quad n \geq 1 \tag{9}$$

$$P(B_n) = P(C_n) = \prod_{i=1}^n \frac{t^{2i} - 1}{t - 1}, \quad n \geq 1 \tag{10}$$

and 
$$(D_n) = \frac{t^n - 1}{t - 1} \prod_{i=1}^{n-1} \frac{t^{2i} - 1}{t - 1}, \quad n \geq 2 \tag{11}$$

where we use the convention that  $B_1 = C_1 = A_1$  and  $D_3 = A_3$  and  $D_2 = A_1 \times A_1$ . It will be convenient to set  $P(A_0) = P(C_0) = 1$  and even  $P(A_{-1}) = 1$ . We need also

$$P(E_6) = \frac{(t^2 - 1)(t^5 - 1)(t^6 - 1)(t^8 - 1)(t^9 - 1)(t^{12} - 1)}{(t - 1)^6} \quad \text{and} \tag{12}$$

$$P(E_7) = \frac{(t^2 - 1)(t^6 - 1)(t^8 - 1)(t^{10} - 1)(t^{12} - 1)(t^{14} - 1)(t^{18} - 1)}{(t - 1)^7}. \tag{13}$$

Note: If the root system is a direct sum of several root systems, then its Poincaré polynomial is the product of the Poincaré polynomials of its summands.

If  $I$  is a subset of  $\Delta$ , then we denote by  $W_I$  the subgroup of  $W$  generated by all  $s_i$  with  $\alpha_i \in I$ . The set  $\Phi_I = \Phi \cap \mathbf{Z}I$  is a root subsystem of  $\Phi$  with basis  $I$  and

Weyl group  $W_I$ . We set  $P(W/W_I) = P(W)/P(W_I)$ . This quotient is in fact a polynomial, equal to  $\sum_w t^{\ell(w)}$  where we sum over minimal length representatives for the cosets of  $W_I$  in  $W$ . The degree of  $P(W/W_I)$  is equal to the number of positive roots in  $\Phi$  not in  $\Phi_I$ .

Assume now that  $\Phi$  is indecomposable. We are now going to exhibit some explicit expressions for  $P(W/W_I)$  in case  $I = \Delta \setminus \{\alpha\}$  where the simple root  $\alpha$  has the property that the corresponding fundamental weight is minuscule.

We start in type  $A_n$ . Here any fundamental weight is minuscule. If  $\alpha = \alpha_i$  with  $1 \leq i \leq n$ , then the remaining simple roots span a root system of type  $A_{i-1} \times A_{n-i}$ . We get then (as is well-known)

$$\frac{P(A_n)}{P(A_{i-1}) \cdot P(A_{n-i})} = \frac{\prod_{s=2}^{n+1} (t^s - 1)}{(t - 1) \prod_{p=2}^i (t^p - 1) \prod_{q=2}^{n-i+1} (t^q - 1)} \tag{14}$$

$$= \frac{\prod_{s=n+2-i}^{n+1} (t^s - 1)}{\prod_{p=1}^i (t^p - 1)} = \begin{bmatrix} n + 1 \\ i \end{bmatrix}. \tag{15}$$

Note that this also works for  $i = 1$  and  $i = 0$  thanks to our convention. For  $i = 1$  we get more explicitly

$$\frac{P(A_n)}{P(A_{n-1})} = \frac{t^{n+1} - 1}{t - 1}.$$

Since 
$$\frac{t^{n+1} - 1}{t - 1} = 1 + t \frac{t^n - 1}{t - 1}$$

we get thus 
$$\frac{P(A_n)}{P(A_{n-1})} = 1 + t \frac{P(A_{n-1})}{P(A_{n-2})}, \quad n \geq 1. \tag{16}$$

On the other hand, we get from Claim 2.1 for  $0 \leq m \leq n$

$$\frac{P(A_{m+n-1})}{P(A_{m-1}) \cdot P(A_{n-1})} = \sum_{s=0}^m t^{s^2} \frac{P(A_{m-1}) \cdot P(A_{n-1})}{P(A_{s-1}) P(A_{m-s-1}) P(A_{s-1}) P(A_{n-s-1})}. \tag{17}$$

(For  $m = 1$  we get (16) back.) In type  $B_n$  our only choice is  $\alpha = \alpha_n$ . Then the remaining simple roots span a root system of type  $A_{n-1}$ . We get then

$$\frac{P(B_n)}{P(A_{n-1})} = \frac{\prod_{i=1}^n (t^{2i} - 1)}{(t - 1) \prod_{i=2}^n (t^i - 1)} = \prod_{i=1}^n (t^i + 1).$$

Now (15) and (5) yield

$$\frac{P(B_n)}{P(A_{n-1})} = \sum_{s=0}^n t^{s(s+1)/2} \frac{P(A_{n-1})}{P(A_{s-1}) \cdot P(A_{n-s-1})}, \quad n \geq 2. \tag{18}$$

In type  $C_n$  our only choice is  $\alpha = \alpha_1$ . Then the remaining simple roots span a root system of type  $C_{n-1}$ .

We get then 
$$\frac{P(C_n)}{P(C_{n-1})} = \frac{\prod_{i=1}^n (t^{2i} - 1)}{(t-1) \prod_{i=1}^{n-1} (t^{2i} - 1)} = \frac{t^{2n} - 1}{t - 1}, \quad n \geq 1.$$

Since 
$$\frac{t^{2n} - 1}{t - 1} = 1 + t \frac{t^{2(n-1)} - 1}{t - 1} + t^{2n-1}$$

we get thus 
$$\frac{P(C_n)}{P(C_{n-1})} = 1 + t \frac{P(C_{n-1})}{P(C_{n-2})} + t^{2n-1}, \quad n \geq 2. \tag{19}$$

In type  $D_n$  there are three choices for  $\alpha$ . We take first  $\alpha = \alpha_1$ ; here the remaining simple roots span a root system of type  $D_{n-1}$ . We get then for  $n \geq 3$ :

$$\frac{P(D_n)}{P(D_{n-1})} = \frac{(t^n - 1) \prod_{i=1}^{n-1} (t^{2i} - 1)}{(t-1) (t^{n-1} - 1) \prod_{i=1}^{n-2} (t^{2i} - 1)} = \frac{(t^n - 1) (t^{2(n-1)} - 1)}{(t-1) (t^{n-1} - 1)} = \frac{(t^n - 1) (t^{n-1} + 1)}{t - 1}.$$

Since 
$$\frac{(t^n - 1) (t^{n-1} + 1)}{t - 1} = 1 + t \frac{(t^{n-1} - 1) (t^{n-2} + 1)}{t - 1} + t^{2(n-1)}$$

we get thus 
$$\frac{P(D_n)}{P(D_{n-1})} = 1 + t \frac{P(D_{n-1})}{P(D_{n-2})} + t^{2(n-1)}, \quad n \geq 4. \tag{20}$$

If we take  $\alpha = \alpha_{n-1}$  or  $\alpha = \alpha_n$  in type  $D_n$ , then the remaining simple roots span a root system of type  $A_{n-1}$ . We get then

$$\frac{P(D_n)}{P(A_{n-1})} = \frac{(t^n - 1) \prod_{i=1}^{n-1} (t^{2i} - 1)}{(t-1) \prod_{i=2}^n (t^i - 1)} = \prod_{i=1}^{n-1} \frac{t^{2i} - 1}{t^i - 1} = \prod_{i=1}^{n-1} (t^i + 1).$$

Therefore Claim 2.2 says

$$\begin{aligned} \frac{P(D_n)}{P(A_{n-1})} &= \sum_{0 \leq r \leq n, r \text{ even}} t^{r(r-1)/2} \frac{P(A_{n-1})}{P(A_{r-1}) \cdot P(A_{n-r-1})} \\ &= \sum_{0 \leq 2s \leq n} t^{s(2s-1)} \frac{P(A_{n-1})}{P(A_{2s-1}) \cdot P(A_{n-2s-1})}, \quad n \geq 4. \end{aligned} \tag{21}$$

In type  $E_6$  we can take  $\alpha = \alpha_1$  or  $\alpha = \alpha_6$ . In both cases the remaining simple roots span a root system of type  $D_5$ . In type  $E_7$  only  $\alpha = \alpha_7$  is possible; in this case the remaining simple roots span a root system of type  $E_6$ . The corresponding quotients are

$$\frac{P(E_6)}{P(D_5)} = \frac{(t^9 - 1) (t^{12} - 1)}{(t-1) (t^4 - 1)} = \frac{(t^9 - 1) (t^8 + t^4 + 1)}{(t-1)}$$

and 
$$\frac{P(E_7)}{P(E_6)} = \frac{(t^{10} - 1) (t^{14} - 1) (t^{18} - 1)}{(t-1) (t^5 - 1) (t^9 - 1)} = \frac{(t^5 + 1) (t^{14} - 1) (t^9 + 1)}{(t-1)}.$$

A little calculation shows now that

$$\frac{P(E_6)}{P(D_5)} = 1 + t \frac{P(D_5)}{P(A_4)} + t^8 \frac{P(D_5)}{P(D_4)} \tag{22}$$

and 
$$\frac{P(E_7)}{P(E_6)} = 1 + (t + t^{10}) \frac{P(E_6)}{P(D_5)} + t^{27}. \tag{23}$$

Here is a table listing the possible pairs  $(\Phi, \alpha)$  together with the type of  $\Delta \setminus \{\alpha\}$  and the degree of  $P(W/W_I)$ .

$\Phi$	$\alpha$	type $\Delta \setminus \{\alpha\}$	degree
$A_n$	$\alpha_j$	$A_{j-1} \times A_{n-j}$	$j(n+1-j)$
$B_n$	$\alpha_n$	$A_{n-1}$	$n(n+1)/2$
$C_n$	$\alpha_1$	$C_{n-1}$	$2n-1$
$D_n$	$\alpha_1$	$D_{n-1}$	$2(n-1)$
$D_n$	$\alpha_{n-1}, \alpha_n$	$A_{n-1}$	$n(n-1)/2$
$E_6$	$\alpha_1, \alpha_6$	$D_5$	16
$E_7$	$\alpha_7$	$E_6$	27

Table 1

We can give the equations (16) – (23) a uniform appearance. For any dominant (not necessarily integral) element  $\lambda$  in the real span  $\mathbf{R}\Phi$  of the roots set

$$\Delta_\lambda = \{ \alpha \in \Delta \mid s_\alpha \lambda = \lambda \} = \{ \alpha \in \Delta \mid (\lambda, \alpha^\vee) = 0 \}. \tag{24}$$

Then the stabiliser  $W_\lambda$  of  $\lambda$  in  $W$  is equal to  $W_{\Delta_\lambda}$ . For example,  $\Delta_{\varpi_\alpha} = \Delta \setminus \{\alpha\}$  for any  $\alpha \in \Delta$ . So the denominators on the left hand sides of our equations are just  $P(W_{\varpi_\alpha})$ .

Note that the summands for  $s = 0$  in (17), (18), and (21) are equal to 1. The denominators on the right hand side of our equations can be interpreted as Poincaré polynomials of intersections  $W_{\varpi_\alpha, \xi} = W_{\varpi_\alpha} \cap W_\xi$  for suitable  $\xi$ . We get thus with the above notations

$$P(W/W_{\varpi_\alpha}) = 1 + \sum_{\xi \in \mathcal{M}} t^{\ell_\xi} P(W_{\varpi_\alpha}/W_{\varpi_\alpha, \xi}) \tag{25}$$

where the  $\ell_\xi$  are suitable positive integers and the set  $\mathcal{M}$  is given by Table 2 below (where we skip two cases that are symmetric to some we include).

$\Phi$	$\alpha$	$\mathcal{M}$
$A_n$	$\alpha_j$	$\varpi_s + \varpi_{n+1-s}, 1 \leq s \leq \min(j, n-j+1)$
$B_n$	$\alpha_n$	$\varpi_s, 1 \leq s < n, 2\varpi_n$
$C_n$	$\alpha_1$	$\varpi_2, 2\varpi_1$
$D_n$	$\alpha_1$	$\varpi_2, 2\varpi_1$
$D_n$	$\alpha_n$	if $n$ even: $\varpi_{2s}, 1 \leq s < n/2, 2\varpi_n$ if $n$ odd: $\varpi_{2s}, 1 \leq s < (n-1)/2, \varpi_{n-1} + \varpi_n$
$E_6$	$\alpha_6$	$\varpi_2, \varpi_1 + \varpi_6$
$E_7$	$\alpha_7$	$\varpi_1, \varpi_6, 2\varpi_7$

Table 2

Note that these elements are not uniquely determined by (25). We shall see that there is a general recipe that produces exactly the set  $\mathcal{M}$  above and also find a general expression for the exponents  $\ell_\xi$ .

#### 4. Stabilisers in the affine Weyl group

Assume from now on that  $\Phi$  is indecomposable. We consider here the affine Weyl group  $W^{\text{aff}}$  of the root system as the group generated by  $W$  and all translations by

roots (not by coroots). Denote the largest short root in  $\Phi$  by  $\beta_1$ . (Our convention is that all roots are short if all roots have the same length.) Set  $s_0 = s_{\beta_1,1} \in W^{\text{aff}}$  equal to the affine reflection with respect to the hyperplane  $(x, \beta_1^\vee) = 1$ . Now  $W^{\text{aff}}$  is a Coxeter group with Coxeter generators

$$S^{\text{aff}} = \{s_\alpha \mid \alpha \in \Delta\} \cup \{s_0\}.$$

Set  $\alpha_0 = -\beta_1$ . The extended Dynkin diagram of  $\Phi$  has vertices corresponding to  $\Delta \cup \{\alpha_0\}$  and is constructed after the usual rules. In particular, the vertex corresponding to  $\alpha_0$  is linked to some  $\alpha \in \Delta$  if and only if  $(\alpha_0, \alpha^\vee) < 0$ . Equivalently: if and only if  $(\beta_1, \alpha^\vee) > 0$ , and the type of the link is determined by the fact that  $\alpha_0$  is short. The vertices in  $\Delta \cup \{\alpha_0\}$  are in bijection with the elements in  $S^{\text{aff}}$ , and the Coxeter graph of the Coxeter system  $(W^{\text{aff}}, S^{\text{aff}})$  is the Coxeter graph associated to the extended Dynkin diagram. The fundamental alcove

$$\mathfrak{A} = \{ \mu \in \mathbf{R}\Phi \mid 0 \leq (\mu, \alpha_i^\vee) \text{ for all } i, 1 \leq i \leq n, \text{ and } (\mu, \beta_1^\vee) \leq 1 \} \quad (26)$$

is a fundamental domain for  $W^{\text{aff}}$ . For any  $\mu \in \mathfrak{A}$  the stabiliser  $W_\mu^{\text{aff}}$  of  $\mu$  in  $W^{\text{aff}}$  is generated by all  $s \in S^{\text{aff}}$  with  $s\mu = \mu$ .

Any proper subset  $K$  of  $\Delta \cup \{\alpha_0\}$  is the Dynkin diagram of a finite root system. We denote its Weyl group by  $W_K$  and identify it with the subgroup of  $W^{\text{aff}}$  generated by all  $s_\alpha$  with  $\alpha \in K \cap \Delta$  together with  $s_0$  in case  $\alpha_0 \in K$ . (For  $K \subset \Delta$  this is compatible with our earlier definition.)

The statement above on stabilisers in  $W^{\text{aff}}$  for elements in  $\mathfrak{A}$  implies with the notation from (24)

$$\lambda \in \mathfrak{A} \text{ and } (\lambda, \beta_1^\vee) = 1 \implies W_\lambda^{\text{aff}} = W_{\Delta_\lambda \cup \{\alpha_0\}}. \quad (27)$$

**Lemma 4.1.** *Let  $\alpha \in \Delta$  be a simple root such that  $\varpi_\alpha$  is a minuscule fundamental weight. Then there exists an automorphism of the extended Dynkin diagram that interchanges  $\alpha$  and  $\alpha_0$ . If we remove  $\alpha$  from the extended diagram, then the remainder  $(\Delta \setminus \{\alpha\}) \cup \{\alpha_0\}$  is isomorphic to  $\Delta$ , and we get the extended diagram of the remainder by adding  $\{\alpha\}$ .*

**Proof.** By inspection and well-known. ■

Recall that a fundamental weight  $\varpi_\alpha$  is minuscule if and only if  $1 = (\varpi_\alpha, \beta_1^\vee)$ . So we have  $\varpi_\alpha \in \mathfrak{A}$  and can apply (27) to it.

**Corollary 4.2.** *Let  $\alpha \in \Delta$  be a simple root such that  $\varpi_\alpha$  is a minuscule fundamental weight. Then  $W_{\varpi_\alpha}^{\text{aff}}$  is isomorphic to  $W$  as a Coxeter group.*

**Proof.** Since  $\Delta_{\varpi_\alpha} = \Delta \setminus \{\alpha\}$  we get from (27) that  $W_{\varpi_\alpha}^{\text{aff}} = W_{(\Delta \setminus \{\alpha\}) \cup \{\alpha_0\}}$ . An automorphism as in Lemma 4.1 maps  $(\Delta \setminus \{\alpha\}) \cup \{\alpha_0\}$  onto  $\Delta$ . Therefore  $W_{\varpi_\alpha}^{\text{aff}}$  is isomorphic to  $W_\Delta = W$  as a Coxeter group. ■

**Remark 4.3.** We get here in particular that  $P(W_{\varpi_\alpha}^{\text{aff}}) = P(W)$ . So we can rewrite (25) as

$$P(W_{\varpi_\alpha}^{\text{aff}}/W_{\varpi_\alpha}) = 1 + \sum_{\xi \in \mathcal{M}} t^{\ell_\xi} P(W_{\varpi_\alpha}/W_{\varpi_\alpha, \xi}). \quad (28)$$

Our goal is to generalise this formula so that we can replace  $\varpi_\alpha$  by any  $\lambda \in \mathfrak{A}$  satisfying  $(\lambda, \beta_1^\vee) = 1$ . ■

For any dominant  $\lambda$  denote by  $J(\lambda) \subset \Delta_\lambda$  the subset such that

$$J(\lambda) \cup \{\alpha_0\} = \text{the connected component of } \Delta_\lambda \cup \{\alpha_0\} \text{ containing } \alpha_0. \quad (29)$$

Set  $I(\lambda) = \Delta_\lambda \setminus J(\lambda)$ . For example, we have  $J(\varpi_\alpha) = \Delta \setminus \{\alpha\}$  and  $I(\varpi_\alpha) = \emptyset$  for any  $\alpha \in \Delta$  with  $\varpi_\alpha$  minuscule as  $(\Delta \setminus \{\alpha\}) \cup \{\alpha_0\}$  is connected (isomorphic to  $\Delta$ ). On the other hand, we get  $J(\lambda) = \emptyset$  if and only if  $\alpha \notin \Delta_\lambda$  for all  $\alpha \in \Delta$  with  $(\beta_1, \alpha^\vee) > 0$ . This means in particular  $J(\beta_1) = \emptyset$  and  $I(\beta_1) = \Delta_{\beta_1}$ .

For any dominant  $\lambda$  as above we get a direct product decomposition

$$W_\lambda = W_{J(\lambda)} \times W_{I(\lambda)} \quad (30)$$

since  $\Delta_\lambda$  is the disjoint union of  $J(\lambda)$  and  $I(\lambda)$ , and since no root in  $J(\lambda)$  is linked to a root in  $I(\lambda)$  by definition of  $J(\lambda)$ .

If  $\lambda \neq 0$ , then  $\Delta_\lambda \cup \{\alpha_0\}$  and  $J(\lambda) \cup \{\alpha_0\}$  are proper subsets of the extended Dynkin diagram and define finite parabolic subgroups  $W_{\Delta_\lambda \cup \{\alpha_0\}}$  and  $W_{J(\lambda) \cup \{\alpha_0\}}$  of  $W^{\text{aff}}$ . We get similarly to (30) that

$$W_{\Delta_\lambda \cup \{\alpha_0\}} = W_{J(\lambda) \cup \{\alpha_0\}} \times W_{I(\lambda)}. \quad (31)$$

A comparison of (30) and (31) implies for the Poincaré polynomials

$$P(W_{\Delta_\lambda \cup \{\alpha_0\}}/W_\lambda) = P(W_{J(\lambda) \cup \{\alpha_0\}}/W_{J(\lambda)}). \quad (32)$$

The following lemma applies to any set of the form  $J(\lambda) \cup \{\alpha_0\}$  where  $\lambda \neq 0$ .

**Lemma 4.4.** *Let  $K$  be a connected proper subset of the extended Dynkin diagram  $\Delta \cup \{\alpha_0\}$  such that  $\alpha_0 \in K$ . Then  $\alpha_0$  corresponds to a minuscule fundamental weight of the root system with Dynkin diagram  $K$ .*

**Proof.** The claim is obvious if  $K$  has type  $A_r$  for some  $r$  since here all fundamental weights are minuscule. This takes care of  $\Phi$  of type  $A_n$  where any possible  $K$  has type  $A_r$  for some  $r \leq n$ .

So assume that neither  $\Phi$  nor  $K$  is of type  $A_n$ . Then  $\alpha_0$  is an end-vertex of the extended Dynkin diagram  $\Delta \cup \{\alpha_0\}$ , hence also of  $K$ . Furthermore  $\alpha_0$  corresponds to a short root. Now in the classical cases (BCD-types) any short simple root located at an end of the Dynkin diagram corresponds to a minuscule fundamental weight.

So we are left with the possibility that  $K$  has exceptional type. If  $K$  has type  $E_8$ ,  $F_4$ , or  $G_2$ , then  $K$  can occur as a subdiagram in  $\Delta \cup \{\alpha_0\}$  only if  $\Delta$  has the same type as  $K$ , and it has to be the subdiagram  $\Delta$  in  $\Delta \cup \{\alpha_0\}$ . But we assume that  $\alpha_0 \in K$ . So these cases cannot occur.

Let us now suppose that  $K$  has type  $E_6$ . Then  $\Phi$  has to have type  $E_n$  with  $n \geq 6$ . Then  $\alpha_0$  has distance (in an obvious sense) at least 2 from the branching point in  $\Delta \cup \{\alpha_0\}$ , hence also at least distance 2 from the branching point in  $K$ . Therefore  $\alpha_0$  corresponds to one of the two minuscule fundamental weights.

The argument is similar for  $K$  of type  $E_7$ . Now  $\Phi$  has to have type  $E_n$  with  $n \geq 7$ . Then  $\alpha_0$  has distance at least 3 from the branching point in  $\Delta \cup \{\alpha_0\}$ , hence also at least distance 3 from the branching point in  $K$ . Therefore the end-vertex  $\alpha_0$  corresponds to the minuscule fundamental weight. ■

**Remark 4.5.** Consider  $\lambda \neq 0$  as above. The lemma says that there exists an indecomposable finite root system  $\Phi'$  with basis  $\Delta'$  and a root  $\alpha' \in \Delta'$  corresponding to a minuscule fundamental weight such that the pair of Dynkin diagrams  $(J(\lambda) \cup \{\alpha_0\}, J(\lambda))$  is isomorphic to the pair of Dynkin diagrams  $(\Delta', \Delta' \setminus \{\alpha'\})$ . It then follows that  $P(W_{J(\lambda) \cup \{\alpha_0\}}/W_{J(\lambda)}) = P(W_{\Delta'}/W_{\Delta' \setminus \{\alpha'\}})$  is one of the left hand sides considered in (25). If we assume in addition that  $\lambda \in \mathfrak{A}$  with  $(\lambda, \beta_1^\vee) = 1$ , then we have by (27), (29) and (32) that

$$P(W_\lambda^{\text{aff}}/W_\lambda) = P(W_{\Delta'}/W_{\Delta' \setminus \{\alpha'\}}). \tag{33}$$

The question is then whether we can also relate the right hand side in (25) to  $\lambda$ . ■

With a view to this goal we first want to give a general formula for the exponents  $\ell_\xi$  in (25). We set for any dominant  $\xi \neq 0$

$$\ell_\xi = \deg P(W_{J(\xi) \cup \{\alpha_0\}}/W_{J(\xi)}), \tag{34}$$

which obviously agrees with our definition of  $\ell_\xi$  in the introduction. Note that we can apply the considerations above also to  $\xi$ , hence have an isomorphism of pairs of Dynkin diagrams of the form  $(J(\xi) \cup \{\alpha_0\}, J(\xi)) \simeq (\Delta', \Delta' \setminus \{\alpha'\})$  and get then  $\ell_\xi = \deg P(W_{\Delta'}/W_{\Delta' \setminus \{\alpha'\}})$ ; the latter degree can then be read off from Table 1. The following table gives a list of these types and the corresponding  $\ell_\xi$  for certain  $\xi$  that will turn out to play a role later on.

type $\Phi$	$\xi$	type $J(\xi)$	type $J(\xi) \cup \{\alpha_0\}$	$\ell_\xi$
any	$\beta_1$	$\emptyset$	$A_1$	1
$A_n$	$\varpi_i + \varpi_{n+1-i}, 1 < i \leq n/2$	$A_{i-1} \times A_{i-1}$	$A_{2i-1}$	$i^2$
$B_n$	$\varpi_i, 1 < i < n$	$A_{i-1}$	$B_i$	$i(i+1)/2$
$B_n$	$2\varpi_n$	$A_{n-1}$	$B_n$	$n(n+1)/2$
$C_n$	$\varpi_{2i}, 1 < i \leq n/2$	$A_{2i-1}$	$D_{2i}$	$i(2i-1)$
$C_n$	$2\varpi_1$	$C_{n-1}$	$C_n$	$2n-1$
$D_n$	$\varpi_{2i}, 1 < i < (n-1)/2$	$A_{2i-1}$	$D_{2i}$	$i(2i-1)$
$D_n$	$2\varpi_{n-1}$ or $2\varpi_n$	$A_{n-1}$	$D_n$	$n(n-1)/2$
$D_{2m+1}$	$\varpi_{2m} + \varpi_{2m+1}$	$A_{2m-1}$	$D_{2m}$	$m(2m-1)$
$D_n$	$2\varpi_1$	$D_{n-1}$	$D_n$	$2(n-1)$
$E_6$	$\varpi_1 + \varpi_6$	$D_4$	$D_5$	8
$E_7$	$\varpi_6$	$D_5$	$D_6$	10
$E_7$	$2\varpi_7$	$E_6$	$E_7$	27
$E_8$	$\varpi_1$	$D_7$	$D_8$	14
$F_4$	$\varpi_1$	$C_3$	$C_4$	7

Table 3

**Lemma 4.6.** Let  $\alpha \in \Delta$  be a simple root such that  $\varpi_\alpha$  is a minuscule fundamental weight. Then the exponents  $\ell_\xi$  in (25) coincide with the exponents defined by (34). We have  $\alpha \notin J(\xi)$  for all  $\xi \in \mathcal{M}$ .

**Proof.** Each  $\xi$  from Table 2 occurs also in Table 3. One can then easily compare the data in Table 3 with the exponents in (16)–(23). (Note that  $\beta_1$  occurs in different guises in Table 2.) One checks that  $\alpha \notin J(\xi)$  by inspection. ■

Let  $\lambda \in \mathfrak{A}$  with  $(\lambda, \beta_1^\vee) = 1$ . In our intended generalisation of (28) to  $\lambda$ , the right hand side is supposed to involve terms of the form  $P(W_\lambda/W_{\lambda,\xi})$ . We shall have to relate  $W_{\lambda,\xi}$  to  $J(\lambda)$ .

**Lemma 4.7.** *Let  $\lambda \in \mathfrak{A}$  with  $(\lambda, \beta_1^\vee) = 1$ . Let  $\xi \in \mathbf{R}\Phi$ ,  $\xi \neq 0$ , be dominant. If  $I(\lambda) \subset \Delta_\xi$ , then  $P(W_\lambda/W_{\lambda,\xi}) = P(W_{J(\lambda)}/W_{J(\lambda)\cap\Delta_\xi})$ .*

**Proof.** The assumption  $I(\lambda) \subset \Delta_\xi$  implies

$$\Delta_\lambda \cap \Delta_\xi = (J(\lambda) \cup I(\lambda)) \cap \Delta_\xi = (J(\lambda) \cap \Delta_\xi) \cup I(\lambda).$$

Since  $W_{\lambda,\xi} = W_\lambda \cap W_\xi$  is the parabolic subgroup of  $W$  generated by the  $s_\alpha$  with  $\alpha \in \Delta_\lambda \cap \Delta_\xi$  we get as for (30) that  $W_{\lambda,\xi} = W_{J(\lambda)\cap\Delta_\xi} \times W_{I(\lambda)}$ . A comparison with (30) then yields the claim. ■

**Proposition 4.8.** *Let  $\lambda \in \mathfrak{A}$  with  $(\lambda, \beta_1^\vee) = 1$ . There exists a finite set  $\mathcal{M}$  of non-zero dominant weights such that*

$$P(W_\lambda^{\text{aff}}/W_\lambda) = 1 + \sum_{\xi \in \mathcal{M}} t^{\ell_\xi} P(W_\lambda/W_{\lambda,\xi}). \tag{35}$$

**Proof.** Because of (27), (32), and Lemma 4.7 we actually want to find  $\mathcal{M}$  with

$$I(\lambda) \subset \Delta_\xi \quad \text{for all } \xi \in \mathcal{M} \tag{36}$$

such that 
$$P(W_{J(\lambda)\cup\{\alpha_0\}}/W_{J(\lambda)}) = 1 + \sum_{\xi \in \mathcal{M}} t^{\ell_\xi} P(W_{J(\lambda)}/W_{J(\lambda)\cap\Delta_\xi}). \tag{37}$$

Consider  $J(\lambda) \cup \{\alpha_0\}$  as the Dynkin diagram of a finite root system and take its extended Dynkin diagram. It contains an additional vertex  $\alpha'$ . Set  $\Delta' = J(\lambda) \cup \{\alpha'\}$  and regard  $\Delta'$  as the basis of a finite root system  $\Phi'$ . By Lemma 4.4,  $\alpha'$  corresponds to a minuscule fundamental weight of  $\Phi'$ , so we can apply (25) to  $\Phi'$  and  $\alpha'$ . Lemma 4.1 implies that we have an isomorphism of pairs of Dynkin diagrams

$$(J(\lambda) \cup \{\alpha_0\}, J(\lambda)) \xrightarrow{\sim} (\Delta', \Delta' \setminus \{\alpha'\}).$$

Therefore we can rewrite (25) as

$$P(W_{J(\lambda)\cup\{\alpha_0\}}/W_{J(\lambda)}) = 1 + \sum_{\xi' \in \mathcal{M}'} t^{\ell_{\xi'}} P(W_{J(\lambda)}/W_{J(\lambda)\cap\Delta'_{\xi'}}) \tag{38}$$

where  $\mathcal{M}'$  is now a set of dominant weights for  $\Phi'$  and where  $\Delta'_{\xi'}$  is defined as in (24), just working in  $\Phi'$  instead of  $\Phi$ . We now need the following lemma.

**Lemma 4.9.** *There exists a finite set  $\mathcal{M}$  of non-zero dominant weights together with a bijection  $\mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ ,  $\xi \mapsto \xi'$ , such that, for all  $\xi \in \mathcal{M}$ ,*

$$I(\lambda) \subset \Delta_\xi, \quad J(\xi) \subset J(\lambda), \quad \text{and} \quad J(\lambda) \cap \Delta_\xi = J(\lambda) \cap \Delta'_{\xi'}. \tag{39}$$

Let us postpone the proof of the lemma and show that (37) holds for any  $\mathcal{M}$  satisfying this lemma.

To begin with, the third condition in (39) says that the Poincaré polynomials on the right hand side of (38) are the same as those on the right hand side of (37). Thanks to the first condition in (39) they are also the same as in (35).

So it is left to check that  $\ell_\xi = \ell_{\xi'}$  for all  $\xi \in \mathcal{M}$  which means

$$\deg P(W_{J(\xi) \cup \{\alpha_0\}}/W_{J(\xi)}) = \deg P(W_{J'(\xi') \cup \{\alpha_0\}}/W_{J'(\xi')}). \tag{40}$$

Here  $J'(\xi') \subset \Delta' = J(\lambda) \cup \{\alpha'\}$  is defined analogously so that  $J'(\xi') \cup \{\alpha_0\}$  is the connected component of  $\Delta'_{\xi'} \cup \{\alpha_0\}$  containing  $\alpha_0$ . (Note that  $\alpha_0$  is by Lemma 4.1 also the extra vertex in the extended Dynkin diagram of  $\Phi'$ .)

Lemma 4.6 says that  $\alpha' \notin J'(\xi')$ , hence  $J'(\xi') \subset J(\lambda)$ . So what we really want is

$$J(\xi) = J'(\xi') \tag{41}$$

which then implies (40). Since  $J'(\xi') \subset \Delta'_{\xi'} \cap J(\lambda) = \Delta_\xi \cap J(\lambda) \subset \Delta_\xi$  thanks to (39), the connected subset  $J'(\xi') \cup \{\alpha_0\}$  is contained in  $\Delta_\xi \cup \{\alpha_0\}$ , hence in its connected component  $J(\xi) \cup \{\alpha_0\}$ . It follows that  $J'(\xi') \subset J(\xi)$ . Similarly

$$J(\xi) \subset \Delta_\xi \cap J(\lambda) = \Delta'_{\xi'} \cap J(\lambda) \subset \Delta'_{\xi'}$$

implies by the same arguments  $J(\xi) \subset J'(\xi')$ .

So the proposition follows as soon as we have proved the lemma. ■

**Proof of Lemma 4.9.** Any  $\xi \in \mathbf{R}\Phi$  can be written as  $\xi = \sum_{\gamma \in \Delta} (\xi, \gamma^\vee) \varpi_\gamma$ , and we can choose the coordinates  $(\xi, \gamma^\vee)$  arbitrarily. Now given  $\xi' \in \mathcal{M}'$  we want the corresponding  $\xi \in \mathcal{M}$  to satisfy

$$(\xi, \gamma^\vee) = \begin{cases} 0 & \text{if } \gamma \in I(\lambda), \\ (\xi', \gamma^\vee) & \text{if } \gamma \in J(\lambda). \end{cases} \tag{42}$$

This guarantees that the first and the third condition in (39) are satisfied.

It remains to define all  $(\xi, \gamma^\vee)$  with  $\gamma \notin \Delta_\lambda$ , i.e., with  $(\lambda, \gamma^\vee) \neq 0$ , such that also the second condition in (39) holds. Here we require:

$$\text{If } \gamma \in \Delta \setminus \Delta_\lambda \text{ is linked to an element in } J'(\xi') \cup \{\alpha_0\}, \text{ then } (\xi, \gamma^\vee) > 0. \tag{43}$$

Let us show that (43) implies  $J(\xi) \subset J(\lambda)$ . Consider  $\gamma \in J(\xi)$ . The connectedness of  $J(\xi) \cup \{\alpha_0\}$  implies that there exists a sequence  $\gamma = \gamma_0, \gamma_1, \dots, \gamma_r = \alpha_0$  with  $\gamma_i \in J(\xi)$  and  $(\gamma_i, \gamma_{i+1}^\vee) < 0$  for all  $i < r$ .

If  $r = 1$ , then  $(\xi, \gamma^\vee) = 0$  – since  $\gamma \in J(\xi)$  – and (43) imply  $\gamma \in \Delta_\lambda$ , hence  $\gamma \in J(\lambda)$  by the definition of  $J(\lambda)$ .

We now use induction on  $r$ . So we may assume that  $\gamma_i \in J(\lambda)$  for all  $i$  with  $0 < i < r$ . We get then from (42) that  $(\xi', \gamma_i^\vee) = (\xi, \gamma_i^\vee) = 0$  whenever  $0 < i < r$ , hence  $\gamma_i \in \Delta'_{\xi'}$ . And since  $\{\gamma_1, \gamma_2, \dots, \alpha_0\}$  is connected, we even get that  $\gamma_i \in J'(\xi')$ , in particular  $\gamma_1 \in J'(\xi')$ . Now  $(\xi, \gamma^\vee) = 0$  and (43) imply  $\gamma \in \Delta_\lambda$ , hence  $\gamma \in J(\lambda)$  by the definition of  $J(\lambda)$ . ■

**Remark 4.10.** It is clear that  $\mathcal{M}$  is not uniquely determined by the conditions in (39) since they only involve  $\Delta_\xi$  – which then determines  $J(\xi)$  – and not the precise values of the  $(\xi, \gamma^\vee)$  with  $\gamma \notin \Delta_\xi$ . We shall exhibit later on a natural choice for  $\mathcal{M}$  with nice properties. ■

**Example 4.11.** (1) In each case  $\mathcal{M}'$  contains the largest short root  $\beta'_1$  of  $\Phi'$ . Let us show that we can take the largest short root  $\beta_1$  of  $\Phi$  as the corresponding element in  $\mathcal{M}$ . We have observed above that  $J(\beta_1) = \emptyset$ . Any root  $\gamma \in I(\lambda)$  satisfies  $(\alpha_0, \gamma^\vee) = 0$ , hence  $\gamma \in \Delta_{\beta_1}$ . So there is no problem with the first two conditions in (39).

Next,  $\Delta_{\beta_1}$  consists exactly of the simple roots not linked to  $\alpha_0$  in the extended Dynkin diagram, hence  $J(\lambda) \cap \Delta_{\beta_1}$  of those simple roots not linked to  $\alpha_0$  in the extended Dynkin diagram and belonging to  $J(\lambda)$ .

The same applies to  $J(\lambda) \cap \Delta'_{\beta'_1}$  since  $\Delta' \cup \{\alpha_0\} = J(\lambda) \cup \{\alpha_0, \alpha'\}$  is the extended Dynkin diagram for  $\Phi'$ . So also the third condition in (39) holds.

(2) Consider  $\Phi$  of type  $F_4$  with  $J(\lambda) = \{\alpha_2, \alpha_3, \alpha_4\}$ . This actually implies  $\lambda = \frac{1}{2}\varpi_1$  and  $I(\lambda) = \emptyset$ . In the extended Dynkin diagram,  $\alpha_0$ , the negative of the dominant short root, is linked to  $\alpha_4$ . This shows that  $J(\lambda) \cup \{\alpha_0\}$  has type  $C_4$ . In the extended Dynkin diagram of  $J(\lambda) \cup \{\alpha_0\}$  the extra vertex  $\alpha'$  is linked to  $\alpha_4$ . The pair  $(\Delta', \Delta' \setminus \{\alpha'\})$  has type  $(C_4, C_3)$ . If we denote by  $\alpha'_i$  and  $\varpi'_i$  the simple roots and the fundamental weights for  $\Phi'$ , then we have in standard numbering  $\Delta' = \{\alpha'_1 = \alpha', \alpha'_2 = \alpha_4, \alpha'_3 = \alpha_3, \alpha'_4 = \alpha_2\}$  and by Table 2  $\mathcal{M}' = \{\varpi'_2, 2\varpi'_1\}$ . Here  $\varpi'_2 = \beta'_1$  in the notation from above; so we can take  $\beta_1 = \varpi_4$  as the corresponding element in  $\mathcal{M}$ . Consider next  $\xi' = 2\varpi'_1$ . Here (42) says that we shall take  $(\xi, \alpha'_i) = 0$  for all  $i \geq 2$ . Since  $\alpha_1 \notin \Delta_\lambda$  and  $J'(\xi') = \{\alpha_2, \alpha_3, \alpha_4\}$ , we should require  $(\xi, \alpha'_1) > 0$  according to (43). The most natural choice is therefore  $\xi = \varpi_1$ . So we end with  $\mathcal{M} = \{\varpi_4, \varpi_1\}$ .

## 5. The Kostant cascade

We keep the assumptions on  $\Phi$ ,  $\Delta$ , and  $W$  with indecomposable  $\Phi$ . In case all roots have the same length, we call all roots short. We normalise the  $W$ -invariant scalar product on the real span  $\mathbf{R}\Phi$  of  $\Phi$  such that  $(\alpha, \alpha) = 2$  for all *short* roots  $\alpha$ . Then any  $\lambda$  in the integral span  $\mathbf{Z}\Phi$  (the root lattice) of  $\Phi$  satisfies  $(\lambda, \lambda) \in 2\mathbf{Z}$ , and  $(\lambda, \lambda) = 2$  if and only if  $\lambda$  is a short root.

Our normalisation implies that  $(\beta^\vee, \lambda) = (\beta, \lambda)$  for all short roots  $\beta$ , but  $(\beta^\vee, \lambda) = \frac{1}{2}(\beta, \lambda)$  or  $(\beta^\vee, \lambda) = \frac{1}{3}(\beta, \lambda)$  if  $\beta$  is a long root and  $\Phi$  of BCF-type or of type  $G_2$ .

We now introduce the Kostant cascade  $\mathcal{K}$  of  $\Phi$ , actually a variation of the usual one. We start with the largest short root which we denote by  $\beta_1$ . Remove all simple roots  $\alpha$  with  $(\beta_1, \alpha) \neq 0$  from  $\Delta$ . Decompose the set of remaining simple roots (regarded as subsets of the Dynkin diagram) into connected components  $\Delta_1, \Delta'_1, \dots$

Denote by  $\beta_2, \beta'_2, \dots$  the largest short root in the root subsystem consisting of  $\Phi \cap \mathbf{Z}\Delta_1, \Phi \cap \mathbf{Z}\Delta'_1, \dots$ . Remove from  $\Delta_1$  all  $\alpha$  with  $(\beta_2, \alpha) \neq 0$ , remove from  $\Delta'_1$  all  $\alpha$  with  $(\beta'_2, \alpha) \neq 0$ , and so on. Decompose the set of remaining simple roots into connected components  $\Delta_2, \Delta'_2, \dots$ . Set  $\beta_3, \beta'_3, \dots$  equal to the largest short root in  $\Phi \cap \mathbf{Z}\Delta_2, \Phi \cap \mathbf{Z}\Delta'_2, \dots$  respectively. Continue like this until there are no short simple roots left. Now the Kostant cascade  $\mathcal{K}$  is defined as the set of all the largest short roots encountered in this process:

$$\mathcal{K} = \{ \beta_1, \beta_2, \beta'_2, \dots, \beta_3, \beta'_3, \dots \}.$$

The construction implies that the elements in  $\mathcal{K}$  are pairwise orthogonal.

Our definition here differs in the case of two root lengths from the usual one where one always takes the largest root. It follows that  $\{\beta^\vee \mid \beta \in \mathcal{K}\}$  is basically the usual Kostant cascade for the dual root system  $\Phi^\vee$ . (There are differences in types  $G_2$  and  $C_n$  with  $n$  odd, where  $\alpha_n^\vee$  occurs in the usual Kostant cascade for  $\Phi^\vee$ , while we here ignore  $\alpha_n$  because it is long.) For examples see Appendix A.

Recall that the support of a linear combination of simple roots is defined as

$$\text{supp} \left( \sum_{i=1}^n m_i \alpha_i \right) = \{ \alpha_i \mid m_i \neq 0 \}. \tag{44}$$

Recall also the usual order relation  $\leq$  on the real span of the roots where  $\lambda \leq \mu$  if and only if  $\mu - \lambda = \sum_{i=1}^n m_i \alpha_i$  with all  $m_i \in \mathbf{Z}$ ,  $m_i \geq 0$ .

For any  $\beta \in \mathcal{K}$  set  $\Delta(\beta) = \text{supp } \beta$ .

Note that  $\Delta(\beta)$  is then also the connected subset of  $\Delta$  from the construction of the Kostant cascade such that  $\beta$  is the largest short root in  $\Phi \cap \mathbf{Z}\Delta(\beta)$ . Note that  $(\beta, \alpha) \geq 0$  for all  $\alpha \in \Delta(\beta)$  since the largest short root is dominant. In the construction of the Kostant cascade one can therefore replace the condition  $(\beta, \alpha) \neq 0$  by  $(\beta, \alpha) > 0$ .

We call an element  $\beta \in \mathcal{K}$  a predecessor of an element  $\beta' \in \mathcal{K}$  if  $\Delta(\beta')$  is a connected component of

$$\Delta(\beta) \setminus \{ \alpha \in \Delta(\beta) \mid (\beta, \alpha) > 0 \}.$$

One has then  $\Delta(\beta') \subsetneq \Delta(\beta)$  and  $\beta' < \beta$ . (The largest short root in  $\Phi \cap \mathbf{Z}\Delta(\beta)$  is larger than the largest short root in the proper subsystem  $\Phi \cap \mathbf{Z}\Delta(\beta')$ .) Each element in  $\mathcal{K}$  not equal to  $\beta_1$  has a unique predecessor.

We define a partial ordering  $\preceq$  on  $\mathcal{K}$  such that  $\beta \preceq \beta'$  if and only if  $\beta = \beta'$  or there exists a chain  $\beta = \beta^{(1)}, \beta^{(2)}, \dots, \beta^{(s)} = \beta'$  in  $\mathcal{K}$  such that each  $\beta^{(i)}$ ,  $1 \leq i < s$ , is the predecessor of  $\beta^{(i+1)}$ . The discussion above and the construction of the Kostant cascade show for all  $\beta, \beta' \in \mathcal{K}$  that

$$\beta \preceq \beta' \iff \beta' \leq \beta \iff \Delta(\beta') \subset \Delta(\beta). \tag{45}$$

For any  $\beta \in \mathcal{K}$  the set of all  $\beta' \in \mathcal{K}$  with  $\beta' \preceq \beta$  is totally ordered and has the form

$$\beta_1 = \beta^{(1)} \preceq \beta^{(2)} \preceq \dots \preceq \beta^{(r-1)} \preceq \beta^{(r)} = \beta$$

such that each  $\beta^{(i)}$ ,  $1 \leq i < r$ , is the predecessor of  $\beta^{(i+1)}$ .

For our main goal we shall be interested in the supports of the difference  $\beta_1 - \beta$  for  $\beta \in \mathcal{K}$ . One can check case-by-case for all  $\beta \neq \beta_1$  that  $\text{supp}(\beta_1 - \beta)$  has two connected components in type  $A$  and is connected in the other types. We prefer to give a general argument and a version that works in all cases. For this we look at (our version of) the extended Dynkin diagram of  $\Phi$ . Denote by  $\Phi^{\text{aff}}$  the affine root system with Dynkin diagram  $\Delta \cup \{\alpha_0\}$ . In the simply laced case this is the root system of the usual untwisted affine algebra. In the other cases we look at the types that Kac denotes by  $D_{l+1}^{(2)}$ ,  $A_{2l-1}^{(2)}$ ,  $E_6^{(2)}$ , or  $D_4^{(3)}$ , see [7], §4.8, Tables Aff2 and Aff3.

When working with  $\Phi^{\text{aff}}$  then  $\alpha_0$  is no longer the negative of the largest short root  $\beta_1$  in  $\Phi$ . It is now part of a basis for  $\mathbf{R}\Phi^{\text{aff}} = \mathbf{R}\alpha_0 \oplus \mathbf{R}\Phi$ . The basic imaginary root  $\delta$  in  $\Phi^{\text{aff}}$  is given by  $\delta = \alpha_0 + \beta_1$ .

The bilinear form  $(, )$  extends to a positive semidefinite bilinear form on  $\mathbf{R}\Phi^{\text{aff}}$  such that  $(\alpha_0, \alpha_0) = 2$  and  $(\delta, \mu) = 0$  for all  $\mu \in \mathbf{R}\Phi^{\text{aff}}$ .

The short real roots in  $\Phi^{\text{aff}}$  are characterised as the elements  $\gamma$  in the root lattice  $\mathbf{Z}(\Delta \cup \{\alpha_0\})$  with  $(\gamma, \gamma) = 2$ . For any such root  $\gamma$  also  $\delta - \gamma$  is a short real root in  $\Phi^{\text{aff}}$  as  $(\delta - \gamma, \delta - \gamma) = (\gamma, \gamma) = 2$  and since  $\delta - \gamma$  again belongs to the root lattice. This applies in particular to all short roots in  $\Phi$ .

**Lemma 5.1.** *Let  $\beta \in \Phi$  be a short root. Then*

- (a)  $\text{supp}(\beta_1 - \beta) \cup \{\alpha_0\}$  is a connected subset of the extended Dynkin diagram, and
- (b) for each  $\alpha \in \Delta$  with  $(\beta, \alpha) > 0$  also  $\text{supp}(\beta_1 - \beta) \cup \{\alpha_0\} \cup \{\alpha\}$  is connected.

**Proof.** (a) Since  $\delta - \beta$  is a short real root in  $\Phi^{\text{aff}}$ , as observed above, its support in  $\Delta \cup \{\alpha_0\}$  is connected. The claim follows since  $\delta - \beta = \alpha_0 + (\beta_1 - \beta)$  has support  $\text{supp}(\beta_1 - \beta) \cup \{\alpha_0\}$ .

(b) The reflection  $s_\alpha$  associated to  $\alpha$  maps  $\beta$  to a short root of the form  $s_\alpha \beta = \beta - m\alpha$  with  $m$  a positive integer. Then  $\beta_1 - s_\alpha \beta = (\beta_1 - \beta) + m\alpha$  satisfies  $\text{supp}(\beta_1 - s_\alpha \beta) = \text{supp}(\beta_1 - \beta) \cup \{\alpha\}$ . Now apply (a). ■

### 6. Dominant weights associated to the Kostant cascade

We are going to construct certain dominant weights that will play a crucial role in our main theorem. For any  $\beta \in \mathcal{K}$  set

$$\xi_\beta = \sum_{\beta' \preceq \beta} \beta'. \tag{46}$$

We are interested in those  $\xi_\beta$  that are dominant and shall determine

$$\Xi = \{ \xi_\beta \mid \beta \in \mathcal{K}, \xi_\beta \text{ dominant} \} \tag{47}$$

case-by-case. Note that  $\xi_{\beta_1} = \beta_1 \in \Xi$  since the largest short root is dominant.

**Remark 6.1.** Joseph constructs in [6] for each  $\beta \in \mathcal{K}$  a dominant weight in  $\beta + \sum_{\beta' \preceq \beta} \mathbf{N}\beta'$ . One can check that this element coincides with  $\xi_\beta$  if and only if  $\xi_\beta$  is dominant. ■

We shall at the same time determine for each  $\xi_\beta \in \Xi$

$$\text{supp}^* \xi_\beta := \text{supp}(\beta_1 - \beta) = \bigcup_{\beta' \preceq \beta} \text{supp}(\beta_1 - \beta'). \tag{48}$$

In order to check the second equal sign, note that  $\beta' \preceq \beta$  implies  $\beta \leq \beta'$ , hence  $\beta_1 - \beta' \leq \beta_1 - \beta$  and thus  $\text{supp}(\beta_1 - \beta') \subset \text{supp}(\beta_1 - \beta)$ . Note also: If  $m$  is the number of  $\beta' \in \mathcal{K}$  with  $\beta' \preceq \beta$ , then

$$(\xi_\beta, \xi_\beta) = 2m \quad \text{and} \quad \text{supp}^* \xi_\beta = \text{supp}(m\beta_1 - \xi_\beta). \tag{49}$$

We have clearly  $\text{supp}^* \xi_\beta = \emptyset$  for  $\beta = \beta_1$  in all cases.

A detailed case-by-case description of  $\mathcal{K}$  and  $\Xi$  can be found in Appendix A.

**Proposition 6.2.** *Let  $\beta \in \mathcal{K}$ . Then  $\xi_\beta$  is dominant if and only if  $\text{supp}(\beta_1 - \beta) \neq \Delta$ . If so, then  $J(\xi_\beta) = \text{supp}(\beta_1 - \beta)$ .*

This could be proved by inspecting all cases in Appendix A. We prefer to give a general proof that minimises case-by-case arguments. We shall proceed via a series of lemmas. Only the first one requires case-by-case consideration.

**Lemma 6.3.** *Let  $\beta \in \mathcal{K}$  such that  $\beta_1$  is the predecessor of  $\beta$ . If  $\alpha \in \Delta$  with  $(\beta, \alpha) > 0$ , then  $\alpha \notin \text{supp}(\beta_1 - \beta)$ .*

**Proof.** Since  $(\beta, \alpha) > 0$  implies that  $\alpha$  occurs in the support of  $\beta$ , the claim is obvious if  $\alpha$  occurs with coefficient 1 in  $\beta_1$ . This is always the case in types *A* and *B*. In types *C* and *D* this works for  $\alpha$  an endpoint of the Dynkin diagram. The only other  $\alpha$  in types *C* and *D* is  $\alpha = \alpha_4$  where one gets  $\text{supp}(\beta_1 - \beta) = \{\alpha_1, \alpha_2, \alpha_3\}$ . For the exceptional types please consult the lists in Appendix A. ■

**Lemma 6.4.** *Let  $\beta, \beta' \in \mathcal{K}$  such that  $\beta'$  is the predecessor of  $\beta$ . Then we have  $(\beta + \beta', \alpha^\vee) = 0$  for all  $\alpha \in \Delta(\beta') \setminus \Delta(\beta)$ .*

**Proof.** (cf. Lemma 2.7 in [6]) Recall that  $\Delta(\beta)$  is a connected component of

$$\Delta^0(\beta') = \{ \gamma \in \Delta(\beta') \mid (\beta', \gamma) = 0 \}.$$

So if  $\alpha \in \Delta(\beta') \setminus \Delta(\beta)$  satisfies  $(\beta', \alpha^\vee) = 0$ , then  $\alpha$  belongs to another connected component, which then implies that  $\alpha$  is orthogonal to all roots in  $\Delta(\beta)$ , hence  $(\beta, \alpha^\vee) = 0$  and  $(\beta + \beta', \alpha^\vee) = 0$ .

So suppose that  $(\beta', \alpha^\vee) \neq 0$ . Since  $\beta'$  is dominant in  $\Phi \cap \mathbf{Z}\Delta(\beta')$ , we get then  $(\beta', \alpha^\vee) > 0$ , and since  $\beta'$  is short,  $(\beta', \alpha^\vee) = 1$ . (We could a priori have  $(\beta', \alpha^\vee) = 2$  and  $\beta' = \alpha$ . But then  $\Delta(\beta') = \{\alpha\}$ , and  $\beta'$  cannot have a successor.) We have to prove that  $(\beta, \alpha^\vee) = -(\beta', \alpha^\vee) = -1$ .

Suppose first that  $\Delta(\beta')$  is not of type  $A_n$  with  $n \geq 2$ . Then  $\alpha$  is uniquely determined and  $\Delta(\beta)$  is a connected component of  $\Delta(\beta') \setminus \{\alpha\}$ . Since  $\beta$  belongs to  $\Phi \cap \mathbf{Z}\Delta(\beta')$  and since  $\beta'$  is the only dominant short root in this subsystem, there exists  $\gamma \in \Delta(\beta')$  with  $(\beta, \gamma^\vee) < 0$ . We have clearly  $\gamma \notin \Delta(\beta)$ , and  $\gamma$  cannot belong to one of the other components of  $\Delta(\beta') \setminus \{\alpha\}$ , since the other components are orthogonal to  $\Delta(\beta)$ . It follows that  $\gamma = \alpha$ . And since  $\beta$  is short, we get  $(\beta, \alpha^\vee) = -1$  as desired.

For  $\Delta(\beta')$  of type  $A_n$  we need  $n \geq 3$  in order for a successor to exist. Here there are two simple roots  $\alpha, \alpha' \in \Delta(\beta')$  with  $(\beta', \alpha^\vee) = 1 = (\beta', \alpha'^\vee)$  and we have  $\beta = \beta' - \alpha - \alpha'$  which easily yields  $(\beta, \alpha^\vee) = -1 = (\beta, \alpha'^\vee)$ . ■

**Lemma 6.5.** *Let  $\beta \in \mathcal{K}$ .*

- (a) *If  $\xi_\beta$  is dominant, then  $\xi_\beta = \sum_{\alpha \in \Delta(\beta)} (\beta, \alpha^\vee) \varpi_\alpha$ .*
- (b) *If  $\xi_\beta$  is dominant, then  $\xi_{\beta'}$  is dominant for all  $\beta' \in \mathcal{K}$  with  $\beta' \preceq \beta$ .*
- (c) *Let  $\beta' \in \mathcal{K}$  be the predecessor of  $\beta$ . If  $\xi_{\beta'}$  is dominant, then  $\xi_\beta$  is dominant if and only if  $(\beta, \alpha^\vee) = 0$  for all  $\alpha \in \Delta \setminus \Delta(\beta')$ .*

**Proof.** We use induction on  $\beta_1 - \beta$ . If  $\beta = \beta_1$ , then (a) is clear since  $\Delta(\beta_1) = \Delta$  while (b) and (c) are empty.

So suppose that  $\beta \neq \beta_1$ . Let  $\beta' \in \mathcal{K}$  be the predecessor of  $\beta$ . Any  $\alpha \in \Delta(\beta)$  is not only orthogonal to  $\beta'$ , but also to all elements in  $\mathcal{K}$  constructed earlier. This implies

$$(\xi_{\beta'}, \alpha^\vee) = 0 \quad \text{and} \quad (\xi_\beta, \alpha^\vee) = (\xi_{\beta'} + \beta, \alpha^\vee) = (\beta, \alpha^\vee) \geq 0.$$

If  $\alpha \in \Delta \setminus \Delta(\beta)$ , then  $\alpha \notin \text{supp } \beta$ , hence  $(\beta, \alpha^\vee) \leq 0$  and

$$(\xi_{\beta'}, \alpha^\vee) = (\xi_\beta - \beta, \alpha^\vee) \geq (\xi_\beta, \alpha^\vee).$$

This shows that  $\xi_{\beta'}$  is dominant if  $\xi_\beta$  is so. The more general claim in (b) follows by induction.

Suppose now conversely that  $\xi_{\beta'}$  is dominant. Since  $(\xi_\beta, \alpha^\vee) \geq 0$  for all  $\alpha \in \Delta(\beta)$ , we see that  $\xi_\beta$  is dominant if and only if  $(\xi_\beta, \alpha^\vee) \geq 0$  for all  $\alpha \in \Delta \setminus \Delta(\beta)$ .

If  $\alpha \in \Delta(\beta') \setminus \Delta(\beta)$ , then we have  $(\xi_{\beta'}, \alpha^\vee) = (\beta', \alpha^\vee)$ , hence, thanks to Lemma 6.4,

$$(\xi_\beta, \alpha^\vee) = (\xi_{\beta'} + \beta, \alpha^\vee) = (\beta' + \beta, \alpha^\vee) = 0$$

Recall that  $\lambda = \sum_{\alpha \in \Delta} (\lambda, \alpha^\vee) \varpi_\alpha$  for any  $\lambda$ . If  $\alpha \in \Delta \setminus \Delta(\beta')$ , then (a) applied inductively to  $\xi_{\beta'}$  yields  $(\xi_{\beta'}, \alpha^\vee) = 0$ , hence

$$(\xi_\beta, \alpha^\vee) = (\xi_{\beta'} + \beta, \alpha^\vee) = (\beta, \alpha^\vee) \leq 0,$$

where the final inequality follows from the fact that  $\alpha \notin \text{supp } \beta$ . It follows that  $\xi_\beta$  is dominant if and only if this inequality is an equality for each  $\alpha$ . Therefore (c) holds. Now (a) follows from the results above on all  $(\xi_\beta, \alpha)$ . ■

**Remark 6.6.** Note that the condition in (c) is equivalent to

$$\{\alpha \in \Delta \mid (\beta, \alpha) < 0\} = \{\alpha \in \Delta(\beta') \mid (\beta', \alpha) > 0\}. \quad \blacksquare$$

**Lemma 6.7.** Let  $\beta, \beta' \in \mathcal{K}$  such that  $\beta'$  is the predecessor of  $\beta$ . Suppose that  $\xi' := \xi_{\beta'}$  is dominant and that  $J(\xi') = \text{supp } (\beta_1 - \beta)$ .

- (a) We have either  $\Delta(\beta) \subset J(\xi')$  or  $\Delta(\beta) \subset I(\xi')$ .
- (b) If  $\Delta(\beta) \subset I(\xi')$ , then  $\xi_\beta$  is dominant.
- (c) If  $\Delta(\beta) \subset J(\xi')$ , then  $\xi_\beta$  is not dominant.

**Proof.** (a) We have  $\Delta_{\xi'} = J(\xi') \cup I(\xi')$ , hence

$$\Delta(\beta) = (\Delta(\beta) \cap (J(\xi') \cup \{\alpha_0\})) \cup (\Delta(\beta) \cap I(\xi'))$$

Since  $\Delta(\beta)$  is connected, one of the two intersections on the right hand side has to be empty.

(b) Suppose that  $\Delta(\beta) \subset I(\xi')$ . Let  $\gamma \in \Delta \setminus \Delta(\beta')$ . We have  $\gamma \in \text{supp } (\beta_1 - \beta')$  by the construction of the Kostant cascade, hence  $\gamma \in J(\xi')$  by our assumption.

We want to show  $(\alpha, \gamma) = 0$  for all  $\alpha \in \Delta(\beta)$ , since then also  $(\beta, \gamma) = 0$ , and Lemma 6.5(c) implies that  $\xi_\beta$  is dominant.

Suppose by contradiction that there exists  $\alpha \in \Delta(\beta)$  with  $(\alpha, \gamma) \neq 0$ . Then  $\{\alpha, \gamma\}$  is a connected subset of  $\Delta_{\xi'} \cup \{\alpha_0\}$  and has a non-trivial intersection with the connected component  $J(\xi') \cup \{\alpha_0\}$ , hence is contained in  $J(\xi') \cup \{\alpha_0\}$ . This yields  $\alpha \in J(\xi')$  in contradiction with  $\alpha \in \Delta(\beta) \subset I(\xi')$ .

(c) Suppose that  $\Delta(\beta) \subset J(\xi')$ . We want to show that there exist  $\gamma \in \Delta \setminus \Delta(\beta')$  and  $\alpha \in \Delta(\beta)$  with  $(\alpha, \gamma) < 0$ . If so, then also  $(\beta, \gamma) < 0$ , hence  $\xi_\beta$  is not dominant by Lemma 6.5(c).

Start with an arbitrary  $\alpha \in \Delta(\beta)$ . Since  $\Delta(\beta) \subset J(\xi')$  and since  $J(\xi') \cup \{\alpha_0\}$  is connected, there exists a sequence  $\alpha^{(0)} = \alpha_0, \alpha^{(1)}, \dots, \alpha^{(r)} = \alpha$  with  $(\alpha^{(i)}, \alpha^{(i+1)}) < 0$  for all  $i < r$  and with  $\alpha^{(i)} \in J(\xi')$  for all  $i > 0$ . By changing  $\alpha$ , we may assume that  $\alpha^{(r-1)} \notin \Delta(\beta)$ . Note that  $(\alpha_0, \alpha^{(1)}) < 0$  implies  $(\beta_1, \alpha^{(1)}) > 0$ , hence  $\alpha^{(1)} \notin \Delta(\beta)$ . So we have  $r > 1$ , hence  $\alpha^{(r-1)} \neq \alpha_0$  and  $\alpha^{(r-1)} \in J(\xi') \setminus \Delta(\beta)$ . Now  $(\xi', \alpha^{(r-1)}) = 0$  and  $(\alpha^{(r-1)}, \alpha^{(r)}) < 0$  imply: If  $\alpha^{(r-1)} \in \Delta(\beta')$ , then  $\alpha^{(r-1)}$  belongs to the connected component  $\Delta(\beta)$  of  $\Delta^0(\beta')$  contradicting  $\alpha^{(r-1)} \notin \Delta(\beta)$ . It follows that  $\gamma \in \Delta \setminus \Delta(\beta')$ , so we can take  $\gamma = \alpha^{(r-1)}$ . ■

**Lemma 6.8.** *Let  $\beta \in \mathcal{K}$ .*

- (a) *If  $\xi_\beta$  is dominant, then  $\alpha \notin \text{supp}(\beta_1 - \beta)$  for all  $\alpha \in \Delta(\beta)$  with  $(\beta, \alpha) > 0$ ; we have then  $J(\xi_\beta) = \text{supp}(\beta_1 - \beta)$ .*
- (b) *If  $\xi_\beta$  is not dominant, then  $\text{supp}(\beta_1 - \beta) = \Delta$ .*

**Proof.** (a) We use induction on  $\beta_1 - \beta$ . The case  $\beta = \beta_1$  is trivial. So assume that  $\beta \neq \beta_1$  and that  $\xi_\beta$  is dominant. Denote the predecessor of  $\beta$  by  $\beta'$ . Then  $\xi_{\beta'}$  is dominant by Lemma 6.5(b); we may assume  $J(\xi_{\beta'}) = \text{supp}(\beta_1 - \beta')$  by induction. Now Lemma 6.7 implies  $\Delta(\beta) \subset I(\xi_{\beta'})$ , hence

$$\Delta(\beta) \cap \text{supp}(\beta_1 - \beta') = \Delta(\beta) \cap J(\xi_{\beta'}) = \emptyset.$$

It then follows that  $\Delta(\beta) \cap \text{supp}(\beta_1 - \beta) = \Delta(\beta) \cap \text{supp}(\beta' - \beta)$  and finally that  $\text{supp}(\beta_1 - \beta') \subset \Delta_{\xi_\beta}$  by Lemma 6.5(a).

We now apply Lemma 6.4 to the root system  $\Phi \cap \mathbf{Z} \Delta(\beta')$ . It follows that all  $\alpha \in \Delta(\beta)$  with  $(\beta, \alpha) > 0$  do not belong to  $\text{supp}(\beta' - \beta)$ , hence  $\text{supp}(\beta' - \beta) \subset \Delta_{\xi_\beta}$ , hence  $\text{supp}(\beta_1 - \beta) \subset \Delta_{\xi_\beta}$ . Since  $\text{supp}(\beta_1 - \beta) \cup \{\alpha_0\}$  is connected by Lemma 5.1(a), we get  $\text{supp}(\beta_1 - \beta) \cup \{\alpha_0\} \subset J(\xi_\beta) \cup \{\alpha_0\}$ , i.e.,  $\text{supp}(\beta_1 - \beta) \subset J(\xi_\beta)$ .

We want to prove equality. Suppose that we have a root  $\alpha \in J(\xi_\beta) \setminus \text{supp}(\beta_1 - \beta)$ . Since  $\Delta \setminus \Delta(\beta) \subset \text{supp}(\beta_1 - \beta)$ , we get  $\alpha \in \Delta(\beta)$ . We have then  $(\beta_1, \alpha) = 0$  as  $\alpha \in \Delta(\beta)$  and  $(\beta, \alpha) = 0$  as  $\alpha \in J(\xi_\beta)$ , hence  $(\beta_1 - \beta, \alpha) = 0$ . Since  $\alpha \notin \text{supp}(\beta_1 - \beta)$  we get that  $(\alpha', \alpha) = 0$  for all  $\alpha' \in \text{supp}(\beta_1 - \beta)$ .

As  $J(\xi_\beta) \cup \{\alpha_0\}$  is connected and contains  $\alpha$ , there is a sequence  $\gamma_1 = \alpha, \gamma_2, \dots, \gamma_{r-1}, \gamma_r = \alpha_0$  with  $\gamma_i \in J(\xi_\beta)$  for all  $i < r$  and  $(\gamma_i, \gamma_{i+1}) < 0$  for all  $i < r$ . As we have  $(\beta_1, \gamma_{r-1}) = -(\alpha_0, \gamma_{r-1}) > 0$ , we cannot have  $\gamma_{r-1} \in \Delta(\beta)$ , hence  $\gamma_{r-1} \in \text{supp}(\beta_1 - \beta)$ . Let  $i > 1$  be minimal for  $\gamma_i \in \text{supp}(\beta_1 - \beta)$ . Then  $\gamma_{i-1} \in J(\xi_\beta) \setminus \text{supp}(\beta_1 - \beta)$  implies as above  $(\gamma_{i-1}, \gamma_i) = 0$  contradicting our choice of the sequence. So there is no  $\alpha$  as above.

(b) Let  $\beta \in \mathcal{K}$  be the smallest element with  $\beta' \preceq \beta$  such that  $\xi_{\beta'}$  is not dominant. Since  $\beta_1 = \xi_{\beta_1}$  is dominant, we have  $\beta \neq \beta_1$ , so  $\beta'$  has a predecessor  $\beta''$ . Then  $\xi_{\beta''}$  is dominant by the minimality of  $\beta'$ . Thanks to (a) we can now apply Lemma 6.7 and get  $\Delta(\beta') \subset J(\xi_{\beta''}) = \text{supp}(\beta_1 - \beta'')$ , hence

$$\Delta(\beta) \subset \Delta(\beta') \subset \text{supp}(\beta_1 - \beta'') \subset \text{supp}(\beta_1 - \beta).$$

Since any  $\alpha \in \Delta \setminus \Delta(\beta)$  clearly belongs to  $\text{supp}(\beta_1 - \beta)$ . ■

**Remark 6.9.** Comparing both parts of the lemma we see

$$\xi_\beta \text{ dominant} \iff \text{supp}(\beta_1 - \beta) \neq \Delta.$$

Together with the first part of the lemma this shows that we have proved Proposition 6.2. ■

### 7. Proof of the main theorem

We start with another look at the minuscule case.

**Lemma 7.1.** *Let  $\alpha \in \Delta$  such that  $\varpi_\alpha$  is minuscule. We then have for any  $\beta \in \mathcal{K}$ :*

$$\alpha \in \Delta(\beta) \iff \alpha \notin \text{supp}(\beta_1 - \beta). \tag{50}$$

The set  $\mathcal{M}$  from Table 2 for  $\alpha$  is equal to the set of all  $\xi_\beta$  with  $\alpha \in \Delta(\beta)$ .

**Proof.** Recall that a fundamental weight  $\varpi_\alpha$  is minuscule if and only if we have  $1 = (\varpi_\alpha, \beta_1^\vee) = (\varpi_\alpha, \beta_1)$ . The condition  $(\varpi_\alpha, \beta_1^\vee) = 1$  implies that  $\alpha$  occurs with coefficient 1 when we write  $\beta_1$  as a linear combination of the simple roots. This fact yields the equivalence in (50). The claim on the set from Table 2 follows by inspection. ■

**Remark 7.2.** The lemma implies that all  $\xi_\beta$  with  $\alpha \in \Delta(\beta)$  are dominant. ■

Fix now  $\lambda \in \mathfrak{A}$  with  $(\lambda, \beta_1^\vee) = 1$ . Let us write  $\lambda = \sum_{i=1}^n r_i \varpi_i$ . So we have  $r_i \geq 0$  for all  $i$  and  $\Delta_\lambda = \{\alpha_i \mid 1 \leq i \leq n, r_i = 0\}$ . Since  $\beta \leq \beta_1$  for all  $\beta \in \mathcal{K}$ , we get  $0 \leq (\beta, \lambda) \leq (\beta_1, \lambda) = 1$  and

$$(\beta, \lambda) = 1 \iff r_i = 0 \text{ for all } \alpha_i \in \text{supp}(\beta_1 - \beta) \iff \text{supp}(\beta_1 - \beta) \subset \Delta_\lambda.$$

Set now  $\Xi(\lambda) := \{\xi \in \Xi \mid \text{supp}^* \xi \subset \Delta_\lambda\}$ .

If  $\lambda = \varpi_\alpha$  is a minuscule fundamental weight, then  $\Delta_{\varpi_\alpha} = \Delta \setminus \{\alpha\}$ , so  $\text{supp}^* \xi \subset \Delta_\lambda$  is equivalent to  $\alpha \notin \text{supp}^* \xi$ . Therefore Lemma 7.1 shows that  $\Xi(\varpi_\alpha)$  is the set  $\mathcal{M}$  from Table 2.

**Lemma 7.3.** *We have  $\Xi(\lambda) = \{\xi \in \Xi \mid (\xi, \xi) = 2(\xi, \lambda)\}$ .*

**Proof.** Suppose  $\xi = \xi_\beta$  with  $\beta \in \mathcal{K}$ . Set  $m$  equal to the number of  $\beta' \in \mathcal{K}$  with  $\beta' \preceq \beta$ . We have then  $(\xi, \xi) = 2m$  and

$$(\xi, \lambda) = \sum_{\beta' \preceq \beta} (\beta', \lambda) \leq m$$

with equality if and only if  $\text{supp}(\beta_1 - \beta') \subset \Delta_\lambda$  for all  $\beta' \preceq \beta$ , hence by (48) if and only if  $\text{supp}^* \xi \subset \Delta_\lambda$ . ■

**Lemma 7.4.** *If  $\xi \in \Xi(\lambda)$ , then  $J(\xi) \subset J(\lambda)$  and  $I(\lambda) \subset \Delta_\xi$ .*

**Proof.** We have  $J(\xi) \cup \{\alpha_0\} = \text{supp}^* \xi \cup \{\alpha_0\} \subset \Delta_\lambda \cup \{\alpha_0\}$ . Since the left-hand side is connected, it is contained in the connected component  $J(\lambda) \cup \{\alpha_0\}$  of the right hand side containing  $\alpha_0$ . This implies  $J(\xi) \subset J(\lambda)$ .

Consider now  $\alpha \in \Delta_\lambda$  with  $\alpha \notin \Delta_\xi$ , hence  $(\xi, \alpha) > 0$ . We have to show that  $\alpha \in J(\lambda)$ . Let  $\beta \in \mathcal{K}$  with  $\xi = \xi_\beta$ , hence with  $J(\xi) = \text{supp}(\beta_1 - \beta)$  by Lemma 6.8.

We have then  $(\beta, \alpha) = (\xi_\beta, \alpha) > 0$  by Lemma 6.5(a). We get in particular  $\alpha \in \Delta(\beta)$ , so Lemma 5.1(b) shows that

$$\text{supp}(\beta_1 - \beta) \cup \{\alpha_0\} \cup \{\alpha\} = J(\xi) \cup \{\alpha_0\} \cup \{\alpha\}$$

is connected. As it is a subset of  $\Delta_\lambda \cup \{\alpha_0\}$ , it is already contained in  $J(\lambda) \cup \{\alpha_0\}$ . This yields  $\alpha \in J(\lambda)$ . ■

We are now ready to prove our main theorem, slightly reformulated in a more compact form as below:

**Theorem 7.5** (reformulated). *We have*

$$P(W_\lambda^{\text{aff}}/W_\lambda) = 1 + \sum_{\xi \in \Xi(\lambda)} t^{\ell_\beta} P(W_\lambda/W_{\lambda,\xi}). \tag{51}$$

**Proof.** If  $\lambda$  is a minuscule fundamental weight, then the claim follows from Lemma 7.1. In general, we have to show that we can take  $\mathcal{M} = \Xi(\lambda)$  in Proposition 4.8. Recall the construction there. We consider the extended Dynkin diagram associated to  $J(\lambda) \cup \{\alpha_0\}$ , constructed by adding a vertex  $\alpha'$ . We consider the root system  $\Phi'$  with Dynkin diagram  $\Delta' = J(\lambda) \cup \{\alpha'\}$ . Set  $\mathcal{M}'$  equal to the set of weights for  $\Phi'$  associated to  $\Phi'$  and  $\alpha'$  in Table 2. By Lemma 4.9 it now suffices to find a bijection  $\Xi(\lambda) \xrightarrow{\sim} \mathcal{M}'$ ,  $\xi \mapsto \xi'$ , satisfying (39). Note that the first two conditions in (39) hold thanks to Lemma 7.4. So we only have to check that  $J(\lambda) \cap \Delta_\xi = J(\lambda) \cap \Delta'_{\xi'}$  or, equivalently, that  $J(\lambda) \setminus \Delta_\xi = J(\lambda) \setminus \Delta'_{\xi'}$ .

We proceed case-by-case looking at the distinct possibilities for the type of the pair  $(J(\lambda) \cup \{\alpha_0\}, J(\lambda))$ , equal to the type of one of the pairs  $(\Phi', \Delta' \setminus \{\alpha'\})$ . We shall denote the simple roots in  $\Delta'$  and the corresponding fundamental weights by  $\alpha'_i$  and  $\varpi'_i$  respectively, using standard numbering.

**Type  $(\mathbf{A}_{k+1}, \mathbf{A}_k)$ :** This possibility corresponds to the subcase  $j = 1$  of the type-A case in Table 1. (It includes the case  $J(\lambda) = \emptyset$  for  $k = 0$ .) In this case the largest short root  $\beta'$  of  $\Phi'$  is the only element of  $\mathcal{M}'$ . The largest short root  $\beta_1 = \xi_1$  belongs to (any!)  $\Xi(\lambda)$  and the pair  $(\beta_1, \beta'_1)$  satisfies (39) according to the first example following Proposition 4.8.

So we only have to show that  $\beta_1$  is the only element in  $\Xi(\lambda)$ . If  $\xi \in \Xi(\lambda)$ , then the connected subset  $J(\xi) \cup \{\alpha_0\}$  is contained in  $J(\lambda) \cup \{\alpha_0\}$ . Since  $\alpha_0$  is an end-vertex of  $J(\lambda) \cup \{\alpha_0\}$ , the pair  $(J(\xi) \cup \{\alpha_0\}, J(\xi))$  has to have type  $(A_{s+1}, A_s)$  for some  $s \leq k$ . However Table 3 over all elements in  $\Xi$  shows that  $\xi = \beta_1$  is the only element in  $\Xi$  where this type occurs.

**Type  $(\mathbf{A}_{j+k+1}, \mathbf{A}_j \times \mathbf{A}_k)$  with  $k \geq j > 0$ :** Since  $J(\lambda)$  is not connected, this possibility can occur only for  $\Phi$  of type  $A_n$  for some  $n$ . We may assume up to symmetry that  $J(\lambda) = \{\alpha_1, \alpha_2, \dots, \alpha_j, \alpha_{n+1-k}, \dots, \alpha_{n-1}, \alpha_n\}$  and get

$$\Delta' = \{\alpha'_1 = \alpha_1, \alpha'_2 = \alpha_2, \dots, \alpha'_j = \alpha_j, \alpha'_{j+1} = \alpha', \alpha'_{j+2} = \alpha_{n+1-k}, \dots, \alpha'_{j+k+1} = \alpha_n\}.$$

We get now from Table 2 and from a look at all  $\text{supp}^* \xi$  that

$$\mathcal{M}' = \{\varpi'_i + \varpi'_{j+k+2-i} \mid 1 \leq i \leq j+1\} \quad \text{and} \quad \Xi(\lambda) = \{\varpi_i + \varpi_{n+1-i} \mid 1 \leq i \leq j+1\}.$$

Then we have a bijection given by  $\xi_i := \varpi_i + \varpi_{n+1-i} \mapsto \xi'_i := \varpi'_i + \varpi'_{j+k+2-i}$  and it has the right property since

$$J(\lambda) \setminus \Delta_{\xi_i} = \{\alpha_i, \alpha_{n+1-i}\} = \{\alpha'_i, \alpha'_{j+k+2-i}\} = J(\lambda) \setminus \Delta'_{\xi'_i}$$

if  $i < j + 1$ . For  $i = j + 1 < k + 1$  both sets are equal to  $\{\alpha_{n+1-j}\}$ , and for  $i = j + 1 = k + 1$  both sets are empty.

**Type  $(\mathbf{B}_{k+1}, \mathbf{A}_k)$**  with  $k > 0$ : This possibility can occur only for  $\Phi$  of type  $B_n$  with  $n \geq k + 1$ , and we get  $J(\lambda) = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ . We may assume that  $k < n - 1$  since otherwise  $\lambda$  is the minuscule fundamental weight  $\varpi_n$ . A look at all  $\text{supp}^* \xi$  shows that  $\Xi(\lambda) = \{\varpi_i \mid 1 \leq i \leq k + 1\}$ . We get

$$\Delta' = \{\alpha'_1 = \alpha_1, \alpha'_2 = \alpha_2, \dots, \alpha'_k = \alpha_k, \alpha'_{k+1} = \alpha'\}$$

and now Table 2 yields  $\mathcal{M}' = \{\varpi'_i \mid 1 \leq i \leq k\} \cup \{2\varpi'_{k+1}\}$ . Now the obvious bijection with  $\varpi_i \mapsto \varpi'_i$  for  $i \leq k$  and  $\varpi_{k+1} \mapsto 2\varpi'_{k+1}$  works.

**Type  $(\mathbf{D}_{k+1}, \mathbf{A}_k)$**  with  $k \geq 3$ : This possibility can occur only for  $\Phi$  of type  $C_n$  or  $D_n$  with  $n \geq k + 1$ , and we get  $J(\lambda) = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$  (up to symmetry in type  $D_n$ ). Let us assume  $k \leq n - 2$  in case  $\Phi$  has type  $D_n$  so to exclude the case that  $\lambda$  is the minuscule fundamental weight  $\varpi_n$ . A look at all  $\text{supp}^* \xi$  shows that  $\Xi(\lambda) = \{\varpi_{2i} \mid 1 \leq i \leq (k + 1)/2\}$  where we have to replace  $\varpi_{n-1}$  in type  $D_n$  by  $\varpi_{n-1} + \varpi_n$ .

We have  $\Delta' = \{\alpha'_1 = \alpha_1, \alpha'_2 = \alpha_2, \dots, \alpha'_k = \alpha_k, \alpha'_{k+1} = \alpha'\}$ . (Note that  $\alpha'$  is linked to  $\alpha_{k-1}$  in the diagram.) We get from Table 2 that  $\mathcal{M}' = \{\varpi'_{2i} \mid 1 \leq i \leq (k + 1)/2\}$  where we have to replace  $\varpi'_k$  by  $\varpi'_k + \varpi'_{k+1}$  for  $k$  even, and  $\varpi'_{k+1}$  by  $2\varpi'_{k+1}$  for  $k$  odd. Now the map  $\varpi_{2i} \mapsto \varpi'_{2i}$  with the obvious modification works.

**Types  $(\mathbf{C}_{k+1}, \mathbf{C}_k)$  and  $(\mathbf{D}_{k+1}, \mathbf{D}_k)$** : Here  $\mathcal{M}'$  has order 2 and contains besides  $\beta'_1 = \varpi'_2$  the element  $2\varpi'_1$ . We have here  $\alpha'_1 = \alpha'$ , hence  $\Delta'_{2\varpi'_1} = J(\lambda)$ . So we have to show that also  $\Xi(\lambda)$  consists of two elements:  $\beta_1 = \xi_1$  corresponding to  $\beta'_1$ , and another one, say  $\xi$ , that has to satisfy  $J(\lambda) \subset \Delta_\xi$ .

Except for  $\Phi$  of type  $C_{k+1}$  or  $D_{k+1}$  with  $\lambda$  a minuscule fundamental weight, there are four possibilities for  $\Phi$ :

- $\Phi$  has type  $F_4$  with  $J(\lambda) = \{\alpha_2, \alpha_3, \alpha_4\}$ ,
- $\Phi$  has type  $E_6$  with  $J(\lambda) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ ,
- $\Phi$  has type  $E_7$  with  $J(\lambda) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ ,
- $\Phi$  has type  $E_8$  with  $J(\lambda) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ .

In each case a look at all  $\text{supp}^* \xi$  shows that  $\Xi(\lambda) = \{\xi, \xi_2\}$  with  $\xi_2 = \varpi_1/\varpi_1 + \varpi_6/\varpi_6/\varpi_1$ , hence with  $J(\lambda) \subset \Delta_{\xi_2}$ .

**Types  $(\mathbf{E}_6, \mathbf{D}_5)$  and  $(\mathbf{E}_7, \mathbf{E}_6)$** : One checks that these possibilities occur only for  $\Phi$  of type  $E_6$  or  $E_7$  respectively with  $\lambda$  a minuscule fundamental weight. So we do not have to do any work in these cases. ■

### A. Tables for $\mathcal{K}$ and $\Xi$

Below we usually write  $\xi_i = \xi_{\beta_i}$ ,  $\xi'_i = \xi_{\beta'_i}$ , and so on.

**Type  $\mathbf{A}_n$** : Here we have

$$\mathcal{K} = \{\beta_i = \varepsilon_i - \varepsilon_{n+2-i} = \alpha_i + \alpha_{i+1} + \dots + \alpha_{n+1-i} \mid 1 \leq i < (n + 2)/2\}.$$

The set is totally ordered: We have  $\beta_1 \preceq \beta_2 \preceq \dots \preceq \beta_m$  where  $m = \lfloor n/2 \rfloor$ , and get

$$\Xi = \{ \xi_i = \beta_1 + \beta_2 + \dots + \beta_i = \varpi_i + \varpi_{n+1-i} \}.$$

(For  $n$  odd this means  $\xi_{(n+1)/2} = 2\varpi_{(n+1)/2}$ .) The support is given by

$$\text{supp}^* \xi_i = \{ \alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{n+2-i}, \dots, \alpha_{n-1}, \alpha_n \}.$$

**Type  $B_n$ ,  $n \geq 2$ :** Here we have

$$\mathcal{K} = \{ \beta_i = \varepsilon_i = \alpha_i + \alpha_{i+1} + \dots + \alpha_n \mid 1 \leq i \leq n \}.$$

The set is totally ordered: We have  $\beta_1 \preceq \beta_2 \preceq \dots \preceq \beta_n$  and get

$$\Xi = \{ \xi_i = \beta_1 + \beta_2 + \dots + \beta_i \mid 1 \leq i \leq n \}, \quad \xi_i = \begin{cases} \varpi_i & \text{if } 1 \leq i < n, \\ 2\varpi_n & \text{if } i = n. \end{cases}$$

The support is given by  $\text{supp}^* \xi_i = \{ \alpha_1, \alpha_2, \dots, \alpha_{i-1} \}$ .

**Type  $C_n$ ,  $n \geq 3$ :** Here  $\mathcal{K}$  is the union of

$$\{ \beta_i = \varepsilon_{2i-1} + \varepsilon_{2i} = \alpha_{2i-1} + 2\alpha_{2i} + \dots + 2\alpha_{n-1} + \alpha_n \mid 1 \leq i \leq n/2 \}$$

and  $\{ \beta'_i = \varepsilon_{2i-3} - \varepsilon_{2i-2} = \alpha_{2i-3} \mid 2 \leq i \leq (n+2)/2 \}$ .

We have  $\beta_1 \preceq \beta_2 \preceq \dots \preceq \beta_m$  with  $m = \lfloor n/2 \rfloor$  and  $\beta_i \preceq \beta'_{i+1}$  for all  $i$ ,  $1 \leq i \leq n/2$ . We get

$$\Xi = \{ \xi_i = \beta_1 + \beta_2 + \dots + \beta_i = \varpi_{2i} \mid 1 \leq i \leq n/2 \} \cup \{ \xi'_2 = \beta_1 + \beta'_2 = 2\varpi_1 \}.$$

The support is given by (for  $i > 1$ )

$$\text{supp}^* \xi_i = \{ \alpha_1, \alpha_2, \dots, \alpha_{2i-1} \} \quad \text{and} \quad \text{supp}^* \xi'_2 = \{ \alpha_2, \alpha_3, \dots, \alpha_n \}.$$

**Type  $D_n$ ,  $n \geq 4$ :** Here  $\mathcal{K}$  is the union of  $\{ \beta_i = \varepsilon_{2i-1} + \varepsilon_{2i} \mid 1 \leq i \leq n/2 \}$ , where

$$\beta_i = \begin{cases} \alpha_{2i-1} + 2\sum_{j=2i}^{n-2} \alpha_j + \alpha_{n-1} + \alpha_n & \text{if } 2i \leq n-1 \\ \alpha_n & \text{if } 2i = n, \end{cases}$$

and  $\{ \beta'_i = \varepsilon_{2i-3} - \varepsilon_{2i-2} = \alpha_{2i-3} \mid 2 \leq i \leq (n+1)/2 \}$

and, if  $n = 2m$  is even,  $\beta''_m = \varepsilon_{n-1} - \varepsilon_n = \alpha_{n-1}$ .

We have  $\beta_1 \preceq \beta_2 \preceq \dots \preceq \beta_m$  with  $m = \lfloor n/2 \rfloor$  and  $\beta_i \preceq \beta'_{i+1}$  for all  $i$ ,  $1 \leq i \leq \lfloor (n-1)/2 \rfloor$  and for even  $n = 2m$  also  $\beta_{m-1} \preceq \beta''_m$ . Now  $\Xi$  is the union of

$$\{ \xi_i = \beta_1 + \beta_2 + \dots + \beta_i \mid 1 \leq i \leq n/2 \}, \quad \xi_i = \begin{cases} \varpi_{2i} & \text{if } 2i < n-1, \\ \varpi_{n-1} + \varpi_n & \text{if } 2i = n-1, \\ 2\varpi_n & \text{if } 2i = n, \end{cases}$$

together with  $\xi'_2 = \beta_1 + \beta'_2 = 2\varpi_1$

and for even  $n = 2m$  also  $\xi''_m = \beta_1 + \beta_2 + \dots + \beta_{m-1} + \beta''_m = 2\varpi_{n-1}$ .

The support is given by (for  $i > 1$ )

$$\text{supp}^* \xi_i = \{ \alpha_1, \alpha_2, \dots, \alpha_{2i-1} \} \quad \text{and} \quad \text{supp}^* \xi'_2 = \{ \alpha_2, \alpha_3, \dots, \alpha_n \}$$

and (in case  $n = 2m$  is even)  $\text{supp}^* \xi''_m = \{ \alpha_1, \alpha_2, \dots, \alpha_{n-2}, \alpha_n \}$ .

In the exceptional cases we are going to list only those  $\beta \in \mathcal{K}$  where  $\text{supp}(\beta_1 - \beta)$  is not the full set of all simple roots. The other ones are not of interest for our applications; one can also check that the corresponding  $\xi_\beta$  are not dominant, cf. also Lemma 6.8(b). The remaining  $\beta \in \mathcal{K}$  are linearly ordered under  $\preceq$ .

**Type E<sub>6</sub>:** Besides  $\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \varpi_2 = \xi_1$  we have  $\beta_2 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = \varpi_1 + \varpi_6 - \varpi_2$  with corresponding element in  $\Xi$  given by  $\xi_2 = \beta_1 + \beta_2 = \varpi_1 + \varpi_6$ . We get

$$\text{supp}^* \xi_2 = \{ \alpha_2, \alpha_3, \alpha_4, \alpha_5 \}.$$

**Type E<sub>7</sub>:** Besides  $\beta_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \varpi_1 = \xi_1$  we have  $\beta_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 = \varpi_6 - \varpi_1$  and  $\beta_3 = \alpha_7 = 2\varpi_7 - \varpi_6$ . The corresponding elements in  $\Xi$  are  $\xi_2 = \beta_1 + \beta_2 = \varpi_6$  and  $\xi_3 = \beta_1 + \beta_2 + \beta_3 = 2\varpi_7$ . We get

$$\text{supp}^* \xi_2 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \}, \quad \text{supp}^* \xi_3 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \}.$$

**Type E<sub>8</sub>:** Besides  $\beta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \varpi_8 = \xi_1$  we have  $\beta_2 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \varpi_1 - \varpi_8$ . The corresponding element in  $\Xi$  is  $\xi_2 = \beta_1 + \beta_2 = \varpi_1$ . We get

$$\text{supp}^* \xi_2 = \{ \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \}.$$

**Type F<sub>4</sub>:** Besides  $\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \varpi_4 = \xi_1$  we have  $\beta_2 = \alpha_1 + \alpha_2 + \alpha_3 = \varpi_1 - \varpi_4$ . The corresponding element in  $\Xi$  is  $\xi_2 = \beta_1 + \beta_2 = \varpi_1$ . We get

$$\text{supp}^* \xi_2 = \{ \alpha_2, \alpha_3, \alpha_4 \}.$$

**Type G<sub>2</sub>:** Here only  $\beta_1 = 2\alpha_1 + \alpha_2 = \varpi_1 = \xi_1$  occurs.

If we compare these lists with Table 3 above, then we will see that the weights in that table are just the elements in  $\Xi$  for a given type. Thus a comparison with the results on  $\text{supp}^* \xi$  yields:

**Lemma A.1.** *We have that  $J(\xi) = \text{supp}^* \xi$  for each  $\xi \in \Xi$ .*

### B. Alternative proof for the main theorem

First, we are going to look in more detail at the set of all  $\beta \in \mathcal{K}$  with  $\text{supp}(\beta_1 - \beta) \subset I$  for some proper subset  $I$  of  $\Delta$ . The results will be applied to  $I = J(\lambda)$ .

**Lemma B.1.** *Let  $I \subset \Delta$  be a proper subset. Let  $\beta \in \mathcal{K}$  with  $\text{supp}(\beta_1 - \beta) \subset I$ . Then each  $\beta' \in \mathcal{K}$  with  $\beta' \preceq \beta$  also satisfies  $\text{supp}(\beta_1 - \beta') \subset I$ . There exists at most one root  $\beta'' \in \mathcal{K}$  with  $\beta$  as predecessor and  $\text{supp}(\beta_1 - \beta'') \subset I$ .*

**Proof.** For the first claim, recall that  $\text{supp}(\beta_1 - \beta') \subset \text{supp}(\beta_1 - \beta)$  for  $\beta' \preceq \beta$ , as argued right after (48).

If  $\beta'' \in \mathcal{K}$  has  $\beta$  as predecessor, then there exists a connected component  $J$  of  $\Delta(\beta) \cap \Delta_\beta$  with  $\Delta(\beta'') = \text{supp} \beta'' = J$ . If  $\text{supp}(\beta_1 - \beta'') \subset I$ , then we get  $\Delta \setminus J \subset \text{supp}(\beta_1 - \beta'') \subset I$ .

If also  $\gamma \in \mathcal{K}$  has  $\beta$  as predecessor, then  $\Delta(\gamma) \cap J = \emptyset$  by the construction of  $\mathcal{K}$ . It follows that  $J \subset \text{supp}(\beta_1 - \gamma)$ . If now also  $\text{supp}(\beta_1 - \gamma) \subset I$ , then we would get  $\Delta = J \cup (\Delta \setminus J) \subset I$  — a contradiction since we assume  $I$  to be proper. So  $\beta''$  is unique (if it exists). ■

**Remark B.2.** This shows that the roots  $\beta \in \mathcal{K}$  with  $\text{supp}(\beta_1 - \beta) \subset I$  form a complete interval  $\beta_1 \preceq \beta_2 \preceq \dots \preceq \beta_r$  in  $\mathcal{K}$  with respect to  $\preceq$ . ■

**Lemma B.3.** *Let  $I \subset \Delta$  be a proper subset, let  $\alpha \in \Delta \setminus I$ .*

- (a) *If  $\beta \in \Phi$  is a root with  $\beta \leq \beta_1$  and  $\text{supp}(\beta_1 - \beta) \subset I$ , then  $(\beta, \alpha) \geq (\beta_1, \alpha) \geq 0$ .*
- (b) *Suppose that  $\beta \in \mathcal{K}$  with  $\text{supp}(\beta_1 - \beta) \subset I$ . If  $\alpha$  is not linked to any element of  $I \cup \{\alpha_0\}$ , then  $(\xi_\beta, \alpha) = 0$ . If  $\alpha$  is linked to some element of  $\text{supp}(\beta_1 - \beta) \cup \{\alpha_0\}$ , then  $(\xi_\beta, \alpha) > 0$ .*

**Proof.** (a) Since  $\alpha \notin \text{supp}(\beta_1 - \beta)$  and since  $\beta_1 - \beta$  is a linear combination of the simple roots with non-negative coefficients, we have  $(\beta_1 - \beta, \alpha) \leq 0$ . This implies the claim since  $\beta_1$  is dominant.

(b) Note that  $\alpha \notin I$ , hence  $\alpha \notin \text{supp}(\beta_1 - \beta)$ , implies  $\alpha \in \text{supp} \beta$ . Since  $\xi_\beta$  is dominant by Lemma 6.8, we get thus  $(\xi_\beta, \alpha) = (\beta, \alpha)$ .

If  $\alpha$  is not linked to any  $\gamma \in I \cup \{\alpha_0\}$ , then we have on the one hand  $(\alpha_0, \alpha) = 0$ , hence  $(\beta_1, \alpha) = 0$ . On the other hand we get  $(\gamma, \alpha) = 0$  for all  $\gamma \in \text{supp}(\beta_1 - \beta) \subset I$ . It follows that  $(\beta_1 - \beta, \alpha) = 0$ , hence also  $0 = (\beta, \alpha) = (\xi_\beta, \alpha)$ .

Suppose now that  $\alpha$  is linked to some  $\gamma \in \text{supp}(\beta_1 - \beta) \cup \{\alpha_0\}$ . In case  $\gamma = \alpha_0$  we get  $(\alpha_0, \gamma) < 0$ , hence  $(\beta_1, \alpha) > 0$ . Now (a) yields  $0 < (\beta, \alpha) = (\xi_\beta, \alpha)$ .

If  $\gamma \in \text{supp}(\beta_1 - \beta)$ , then  $(\gamma, \alpha) < 0$  implies  $(\beta_1 - \beta, \alpha) < 0$ , hence  $(\beta, \alpha) > 0$  by (a). ■

**Set-up.** Consider the following situation: We have two indecomposable finite root systems  $\Phi_1$  and  $\Phi_2$  with bases  $\Delta_1$  and  $\Delta_2$ . Denote by  $\beta_1$  resp.  $\beta'_1$  the largest short root and use it to construct the affine root system  $\Phi_1^{\text{aff}}$  resp.  $\Phi_2^{\text{aff}}$ . Write  $\alpha_0$  resp.  $\alpha'_0$  for the extra basis element for  $\Phi_1^{\text{aff}}$  resp.  $\Phi_2^{\text{aff}}$ , and  $\delta = \alpha_0 + \beta_1$  and  $\delta' = \alpha'_0 + \beta'_1$  for the basic imaginary roots.

Let now  $I_i \subset \Delta_i$  be proper subsets for  $i = 1, 2$  such that  $I_1 \cup \{\alpha_0\}$  and  $I_2 \cup \{\alpha'_0\}$  are connected. Suppose that we have a bijection

$$f: I_1 \cup \{\alpha_0\} \xrightarrow{\sim} I_2 \cup \{\alpha'_0\} \quad \text{with } f(\alpha_0) = \alpha'_0$$

that induces an isomorphism of Dynkin diagrams. Extend  $f$  to a linear isomorphism  $\mathbf{R}(I_1 \cup \{\alpha_0\}) \xrightarrow{\sim} \mathbf{R}(I_2 \cup \{\alpha'_0\})$ . The assumption involving the Dynkin diagram implies for the corresponding simple reflections

$$s_{f(\alpha)} f(\beta) = f(s_\alpha \beta) \quad \text{for all } \alpha, \beta \in I_1 \cup \{\alpha_0\}.$$

Since  $\Phi_I = W_I I$  for any proper subset  $I$  of the basis of one of the  $\Phi_i^{\text{aff}}$ , this shows that  $f$  maps  $\Phi_{I_1 \cup \{\alpha_0\}}$  bijectively onto  $\Phi_{I_2 \cup \{\alpha'_0\}}$ . That assumption implies also that  $(f(\lambda), f(\mu)) = (\lambda, \mu)$  first for all  $\lambda, \mu \in I_1 \cup \{\alpha_0\}$  and then for all  $\lambda, \mu \in \mathbf{R}(I_1 \cup \{\alpha_0\})$ . (Start with the fact that  $(\alpha_0, \alpha_0) = 2 = (\alpha'_0, \alpha'_0)$  and use the connectedness of  $I_1 \cup \{\alpha_0\}$ .)

In order to simplify notation, let us now identify  $I_1 \cup \{\alpha_0\}$  and  $I_2 \cup \{\alpha'_0\}$  via  $f$ . So we write  $I = I_1 = I_2$  and  $\alpha'_0 = \alpha_0$ . We regard  $\mathbf{R}(I \cup \{\alpha_0\})$  as a subspace both of  $\mathbf{R}\Phi_1^{\text{aff}}$  and of  $\mathbf{R}\Phi_2^{\text{aff}}$ . The root subsystem  $\Phi_{I \cup \{\alpha_0\}}$  generated by  $I \cup \{\alpha_0\}$  is the same in  $\Phi_1^{\text{aff}}$  and in  $\Phi_2^{\text{aff}}$ .

Set  $\Psi_1$  equal to the set of all short roots  $\beta \in \Phi_1$  with  $\text{supp}(\beta_1 - \beta) \subset I$ , set  $\Psi_2$  equal to the set of all short roots  $\beta \in \Phi_2$  with  $\text{supp}(\beta'_1 - \beta) \subset I$ . Note that  $\Psi_1$  and  $\Psi_2$  consist of positive roots since (e.g.)  $\text{supp}(\beta_1 - \beta) = \Delta_1$  for any negative root  $\beta \in \Phi_1$ .

**Lemma B.4.** *There exists a bijection  $\tau: \Psi_1 \xrightarrow{\sim} \Psi_2$  such that, for all  $\beta \in \Psi_1$ ,*

$$\beta_1 - \beta = \beta'_1 - \tau(\beta). \tag{B1}$$

*We have  $(\tau(\beta), \alpha) = (\beta, \alpha)$  for all  $\alpha \in I \cup \{\alpha_0\}$  and  $(\tau(\beta), \tau(\beta')) = (\beta, \beta')$  for all  $\beta, \beta' \in \Psi_1$ .*

**Proof.** Let  $\beta \in \Psi_1$ . Write  $\beta_1 - \beta = \eta \in \mathbf{Z}I$ . We observed before Lemma 5.1 that  $\gamma := \delta - \beta$  is a short real root in  $\Phi_1^{\text{aff}}$ . We have  $\gamma = \alpha_0 + \beta'_1 - \beta = \alpha_0 + \eta$ , hence  $\gamma \in \Phi_{I \cup \{\alpha_0\}}$ . So  $\gamma$  is also a short real root in  $\Phi_2^{\text{aff}}$ . Therefore  $\delta' - \gamma$  is another short real root in  $\Phi_2^{\text{aff}}$ . Define then  $\tau$  by  $\tau(\beta) = \delta' - \gamma = \beta' - \eta$ . We see that  $\tau(\beta) \in \Phi_2$  and that (B1) holds. Further (B1) implies  $\text{supp}(\beta' - \tau(\beta)) = \text{supp}(\beta_1 - \beta) \subset I$ , hence that  $\tau(\beta) \in \Psi_2$ .

We get for all  $\beta \in \Psi_1$  and  $\alpha \in I \cup \{\alpha_0\}$

$$(\beta, \alpha) = -(\delta - \beta, \alpha) = -(\delta' - \tau(\beta), \alpha) = (\tau(\beta), \alpha)$$

and for all  $\beta, \beta' \in \Psi_1$

$$(\beta, \beta') = (\delta - \beta, \delta - \beta') = (\delta' - \tau(\beta), \delta' - \tau(\beta')) = (\tau(\beta), \tau(\beta')).$$

Since the set-up is symmetric in  $\Delta_1$  and  $\Delta_2$  we get similarly a map  $\tau': \Psi_2 \rightarrow \Psi_1$  with  $\beta'_1 - \gamma = \beta - \tau'(\gamma)$  for all  $\gamma \in \Psi_2$ . Then  $\tau$  and  $\tau'$  are clearly inverse to each other, hence bijections. ■

Let us denote the Kostant cascade in  $\Phi_i$  by  $\mathcal{K}_i$ ,  $i = 1, 2$ .

**Proposition B.5.** *The bijection  $\tau$  maps the set of all  $\beta \in \mathcal{K}_1$  with  $\text{supp}(\beta_1 - \beta) \subset I$  onto the set of all  $\gamma \in \mathcal{K}_2$  with  $\text{supp}(\beta'_1 - \gamma) \subset I$ .*

**Proof.** Let  $\beta \in \mathcal{K}_1$  with  $\text{supp}(\beta_1 - \beta) \subset I$ . We want to show that  $\tau(\beta) \in \mathcal{K}_2$ . We use induction on the number of elements  $\beta'$  in the Kostant cascade with  $\beta' \preceq \beta$ . The induction starts since  $\tau(\beta_1) = \beta'_1$  belongs to  $\mathcal{K}_2$ .

Suppose now that  $\beta \neq \beta_1$ . Then  $\beta$  has a predecessor  $\beta'$  that by Lemma B.1 satisfies the same assumptions. Write  $\gamma = \tau(\beta)$  and  $\gamma' = \tau(\beta')$ . We may assume by induction that  $\gamma'$  belongs to  $\mathcal{K}_2$ .

Set  $\eta = \beta_1 - \beta$  and  $\eta' = \beta_1 - \beta'$ . Now  $\beta' \geq \beta$  implies  $\eta' \leq \eta$ , hence we have  $\gamma' = \beta'_1 - \eta' \geq \beta'_1 - \eta = \gamma$ . This implies in particular  $S := \text{supp} \gamma \subset S' := \text{supp} \gamma'$ .

Since  $\gamma'$  belongs to the Kostant cascade, it is the largest short root with support  $S'$  and thus satisfies  $(\gamma', \alpha) \geq 0$  for all  $\alpha \in S'$ , hence also for all  $\alpha \in S$ . On the other

hand, we have  $(\beta', \beta) = 0$  since these roots are two distinct members of a Kostant cascade. Now Lemma B.4 implies  $(\gamma', \gamma) = 0$ . Write  $\gamma = \sum_{\alpha \in S} m_\alpha \alpha$  with integers  $m_\alpha > 0$ . Then  $0 = (\gamma', \gamma) = \sum_{\alpha \in S} m_\alpha (\gamma', \alpha)$  implies by the observation above that  $(\gamma', \alpha) = 0$  for all  $\alpha \in S$ . We get thus

$$S \subset S' \cap (\Delta_2)_{\gamma'}. \tag{B2}$$

Since  $\text{supp}(\beta'_1 - \gamma) = \text{supp} \eta \subset I$ , Lemma B.3(a) implies

$$(\gamma, \alpha) \geq (\beta'_1, \alpha) \geq 0 \quad \text{for all } \alpha \in \Delta_2 \setminus I. \tag{B3}$$

We claim next that

$$\alpha \in \Delta_2 \text{ with } (\gamma, \alpha) < 0 \implies (\gamma', \alpha) > 0. \tag{B4}$$

To start with  $(\gamma, \alpha) < 0$  implies  $\alpha \in I$  by (B3), hence  $(\beta, \alpha) < 0$  by Lemma B.4. Since  $\xi_\beta$  is dominant, Lemma 6.5 (applied to  $\Delta_1$ ) shows that  $\alpha \in \text{supp}(\beta')$ . We get now from Lemma 6.4 and Lemma B.4

$$(\gamma', \alpha) = (\beta', \alpha) = -(\beta, \alpha) > 0$$

as claimed.

Let us show next that  $S$  is a connected component of  $S' \cap (\Delta_2)_{\gamma'}$ . If not, then there exist  $\alpha \in (S' \cap (\Delta_2)_{\gamma'}) \setminus S$  and  $\alpha' \in S$  with  $(\alpha, \alpha') < 0$ . This implies  $(\gamma, \alpha) < 0$ , hence  $(\gamma', \alpha) > 0$  by (B4), contradicting  $\alpha \in (\Delta_2)_{\gamma'}$ .

We claim now that  $\gamma$  is the largest short root in  $\Phi_S$ , or, equivalently, that  $(\gamma, \alpha) \geq 0$  for all  $\alpha \in S$ . But this follows from (B4) since  $S \subset (\Delta_2)_{\gamma'}$ .

Now  $\gamma' \in \mathcal{K}_2$  implies  $\gamma \in \mathcal{K}_2$  by the construction of the Kostant cascade; furthermore  $\gamma'$  is the predecessor of  $\gamma$ . ■

We have thus shown that  $\tau(\mathcal{K}_1 \cap \Psi_1) \subset \mathcal{K}_2 \cap \Psi_2$ . Since our set-up is symmetric in  $\Phi_1$  and  $\Phi_2$  and since  $\tau^{-1}$  is defined symmetrically, we get now equality:  $\tau(\mathcal{K}_1 \cap \Psi_1) = \mathcal{K}_2 \cap \Psi_2$ .

**Remark B.6.** Recall that the roots  $\beta \in \mathcal{K}_1$  with  $\text{supp}(\beta_1 - \beta) \subset I$  form a complete interval  $\beta_1 \preceq \beta_2 \preceq \dots \preceq \beta_r$  in  $\mathcal{K}_1$  with respect to  $\preceq$ . The proof shows more precisely that  $\tau(\beta_1) \preceq \tau(\beta_2) \preceq \dots \preceq \tau(\beta_r)$  is the complete interval of all roots  $\gamma \in \mathcal{K}_2$  with  $\text{supp}(\beta'_1 - \gamma) \subset I$ . ■

Now we return to the set-up of Proposition 4.8. There we had fixed  $\lambda \in \mathfrak{A}$  with  $(\lambda, \beta_1^\vee) = 1$ . We consider there a root system  $\Phi'$  with basis  $\Delta' = J(\lambda) \cup \{\alpha'\}$ . The construction there shows that the assumptions of the set-up above before Lemma B.4 are satisfied with  $\Phi_1 = \Phi$  and  $\Phi_2 = \Phi'$  and  $I = J(\lambda)$ .

Since  $\alpha'$  corresponds to a minuscule fundamental weight for the root system  $\Phi'$ , it occurs with coefficient 1 in  $\beta'_1$ . So we have by Lemma 7.1 for any  $\beta'$  in the Kostant cascade for  $\Phi'$  that

$$\alpha' \in \text{supp} \beta' \iff \text{supp}(\beta'_1 - \beta') \subset J(\lambda).$$

Denote by  $\beta'_1 \preccurlyeq \beta'_2 \preccurlyeq \cdots \preccurlyeq \beta'_r$  the elements in the Kostant cascade for  $\Phi'$  containing  $\alpha'$  in its support. In particular,  $\beta'_1$  is the largest short root in  $\Phi'$ . Set  $\xi'_j = \sum_{i=1}^j \beta'_i$  for each  $j$ . So the elements  $\xi'_j$  are dominant and form the set  $\mathcal{M}'$  in the proof of Proposition 4.8.

Set  $\beta_i = \tau^{-1}(\beta'_i)$  for all  $i$ . Proposition B.5 tells us that  $\beta_1 \preccurlyeq \beta_2 \preccurlyeq \cdots \preccurlyeq \beta_r$  are the elements  $\beta$  in the Kostant cascade for  $\Phi$  with  $\text{supp}(\beta_1 - \beta) \subset J(\lambda)$ . Let us write  $\xi_i = \xi_{\beta_i}$  for  $1 \leq i \leq r$ . We have then

$$\Xi(\lambda) = \{ \xi_i \mid 1 \leq i \leq r \}.$$

**Lemma B.7.** *We can take  $\mathcal{M} = \Xi(\lambda)$  in Lemma 4.9 with the bijection  $\mathcal{M} \rightarrow \mathcal{M}'$  given by  $\xi_i \mapsto \xi'_i$ .*

**Proof.** We have  $J(\xi_i) \subset J(\lambda)$  and  $I(\lambda) \subset \Delta_{\xi_i}$  for all  $i$  thanks to Lemma 7.4. So it remains to check for all  $i$  that

$$J(\lambda) \cap \Delta_{\xi_i} = J(\lambda) \cap \Delta_{\xi'_i},$$

i.e., that for all  $\alpha \in J(\lambda)$ :  $(\xi_i, \alpha) = 0 \iff (\xi'_i, \alpha) = 0$ .

But we have even stronger

$$(\xi_i, \alpha) = (\xi'_i, \alpha) \quad \text{for all } \alpha \in J(\lambda)$$

as Lemma B.4 yields  $(\beta_j, \alpha) = (\beta'_j, \alpha)$  for all  $j$ . ■

Now we are ready to give another proof of the main theorem:

**Proof of Theorem 1.1.** If  $\lambda$  is a minuscule fundamental weight, then the claim follows from Lemma 7.1. In general, combine Lemma B.7 and Proposition 4.8. ■

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Received December 22, 2019  
and in final form May 14, 2020