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A Grassmann and Graded Approach to Coboundary Lie Bialgebras, their Classification, and Yang-Baxter Equations

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Abstract. We devise geometric, graded algebra, and Grassmann methods to study and to classify finite-dimensional coboundary Lie bialgebras. Mathematical structures on Lie algebras, like Killing forms, root decompositions, and gradations, are extended to their Grassmann algebras. The classification of real three-dimensional coboundary Lie bialgebras and $\mathfrak{gl}_2$ up to Lie algebra automorphisms is retrieved throughout devised methods. The structure of modified classical Yang-Baxter equations on $\mathfrak{so}(2,2)$ and $\mathfrak{so}(3,2)$ are studied and r-matrices are found.

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1. Introduction

Lie bialgebras [9, 10, 11, 23, 24] emerged in the study of integrable systems [13, 14]. A *Lie bialgebra* consists of a Lie algebra $\mathfrak{g}$ and a Lie algebra structure on its dual, $\mathfrak{g}^*$, that amounts to a cocycle in a Chevalley-Eilenberg cohomology on $\mathfrak{g}$. Lie bialgebras also occur in quantum gravity [3, 4, 28, 29] and other research fields [9, 23, 30].

The classification of Lie bialgebras is an unfinished task. Lie bialgebras with $\dim\mathfrak{g}=2$ and $\dim\mathfrak{g}=3$ have been classified [15, 17]. Specific instances of Lie bialgebras with $\dim\mathfrak{g}>3$, e.g. for a semi-simple $\mathfrak{g}$, have also been studied [1, 6, 26, 30, 31]. So far, employed techniques are mostly algebraic and not much effective to analyse higher-dimensional Lie bialgebras [1, 9, 15, 17]. Hence, new approaches to the study and determination of Lie bialgebras are interesting.

Coboundary Lie bialgebras represent a remarkable type of Lie bialgebras. They are characterised by the so-called *r-matrices*: bivectors on $\mathfrak{g}$ that are solutions to the *modified classical Yang-Baxter equations* (mCYBEs) [9, 17]. This work introduces novel geometric, graded algebra, and Grassmann procedures to help in classifying and to look into the structure of coboundary Lie bialgebras. We show that our methods can effectively be applied to Lie bialgebras with a relatively high-dimensional, not necessarily semi-simple, $\mathfrak{g}$. Let us comment in detail the novelties of our work.

First, the introduced $\mathfrak{g}$-*invariant multilinear maps* on $\mathfrak{g}$-modules allow for extending a Killing form on $\mathfrak{g}$ to its Grassmann algebra $\Lambda\mathfrak{g}$, and to depict other structures, like Casimir elements [6] and invariants on $\mathfrak{g}$-modules (cf. [32]). For example, Theorem 3.4 extends $\mathfrak{g}$-invariant multilinear maps on a $\mathfrak{g}$-module V to ΛV.

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Second, we endow each Lie algebra $\mathfrak{g}$ with a G-graded Lie algebra structure $\mathfrak{g} = \bigoplus_{\alpha\in G}\mathfrak{g}^{(\alpha)}$ for a commutative group $(G,\star)$, where $G\subset\mathbb{R}^n$ but the composition law $\star$ need not be the vector addition, such that $[\mathfrak{g}^{(\alpha)},\mathfrak{g}^{(\beta)}]\subset\mathfrak{g}^{(\alpha\star\beta)}$ for all $\alpha,\beta\in G$. We call this structure a *G-gradation* on the Lie algebra $\mathfrak{g}$. We show that the G-gradation of $\mathfrak{g}$ induces a decomposition in the space of k-vectors on $\mathfrak{g}$ given by $\Lambda^k\mathfrak{g} = \bigoplus_{\alpha\in G}(\Lambda^k\mathfrak{g})^{(\alpha)}$ and $[(\Lambda^m\mathfrak{g})^{(\alpha)},(\Lambda^l\mathfrak{g})^{(\beta)}]_S\subset(\Lambda^{m+l-1}\mathfrak{g})^{(\alpha\star\beta)}$ for every $\alpha,\beta\in G$ and $m,l\in\mathbb{Z}$, where $[\cdot,\cdot]_S:\Lambda\mathfrak{g}\times\Lambda\mathfrak{g}\to\Lambda\mathfrak{g}$ is the *algebraic Schouten bracket* (see [23, 36]). Our gradations are useful to analyse relatively high-dimensional Lie algebras, as illustrated by studying $\mathfrak{so}(2,2)$ and $\mathfrak{so}(3,2)$ (see Figures 1 and 2, Example 6.5). Our gradations can be applied to general Lie algebras as in Table 1, which details G-gradations for Lie algebras with $\dim\mathfrak{g}=3$ and the induced decompositions on $\Lambda^2\mathfrak{g}$ and $\Lambda^3\mathfrak{g}$. A generalisation of root decompositions to general Lie algebras, called *root gradations*, is also suggested and studied.

Previous structures are applied to studying and classifying coboundary Lie bialgebras up to Lie algebra automorphisms in an algorithmic way (see [7, 19] for related topics). Let us sketch the procedure. Let $(\Lambda^k\mathfrak{g})^{\mathfrak{g}} := \{w\in\Lambda^k\mathfrak{g} : [v,w]_S = 0, \forall v\in\mathfrak{g}\}$. Then, $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ and $(\Lambda^3\mathfrak{g})^{\mathfrak{g}}$ are analysed through the decomposition in $\Lambda\mathfrak{g}$ induced by a G-gradation in $\mathfrak{g}$ and other new findings detailed in Sections 6 and 7 relating the structures of $\mathfrak{g}$, $\Lambda\mathfrak{g}$, and $(\Lambda\mathfrak{g})^{\mathfrak{g}} = \oplus_{k\in\mathbb{Z}}(\Lambda^k\mathfrak{g})^{\mathfrak{g}}$.

Recall that a coboundary Lie bialgebra on $\mathfrak{g}$ is determined by an r-matrix, which is an element of $\Lambda^2\mathfrak{g}$. Since r-matrices differing in an element of $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ give rise to the same Lie bialgebra (cf. [9, 15]), coboundary Lie bialgebras should be investigated via $\Lambda^2_R\mathfrak{g} := \Lambda^2\mathfrak{g}/(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$, whose elements are hereafter called *reduced bivectors*. We prove how $\mathfrak{g}$-invariant multilinear maps, gradations, algebraic Schouten brackets, and other introduced structures can be defined on each $\Lambda^m_R\mathfrak{g}$. In particular, the equivalence class in $\Lambda^2_R\mathfrak{g}$ of an r-matrix is called a *reduced r-matrix*.

Next, $\mathfrak{g}$-invariant k-linear maps are employed to look into the equivalence up to inner automorphisms of the coboundary Lie bialgebras on $\mathfrak{g}$. This is more general than standard techniques based on Casimir elements [5]. It also gives a more geometrical picture of the classification of Lie bialgebras up to automorphisms thereof. The determination of automorphisms of Lie algebras is a complicated problem (cf. [15]), but it will be partially skipped in our approach. Indeed, we generally restrict ourselves to studying the equivalence under inner Lie algebra automorphisms [20]. Then, the determination of some appropriate not inner Lie algebra automorphisms leads to obtaining the classification.

Although the classification of real three-dimensional coboundary Lie bialgebras up to Lie algebra automorphisms is known [15, 17], we retrieve it to illustrate our techniques, to fill in a minor gap in [15], to give a new more geometrical approach, and to show that determining of all Lie algebra automorphisms is rather unnecessary. Our results are summarised in Table 1 at the end of Section 10.

The structure of the paper goes as follows. Section 2 surveys the main notions on Lie bialgebras and presents the notation to be used. Section 2 also introduces $\mathfrak{g}$-modules, proposes new structures related to them, and gives several examples of posterior interest. Section 3 defines $\mathfrak{g}$-invariant multilinear maps and analyses its applications to Grassmann algebras. This provides methods to generate such maps on subspaces of Grassmann algebras through ad-invariant maps on Lie algebras.

Section 4 studies properties of Killing-type forms, namely symmetric multilinear maps on certain subspaces of a Grassmann algebra $\Lambda\mathfrak{g}$ that generalise and/or extend the standard Killing form on $\mathfrak{g}$. The existence of $\mathfrak{g}$-invariant bilinear maps in $\mathfrak{g}$-modules is investigated in Section 5. Meanwhile, Section 6 shows and studies how a G-gradation on a Lie algebra of a certain type induces a decomposition in its Grassmann algebra and how the algebraic Schouten bracket respects this decomposition. Section 7 investigates the properties of $\mathfrak{g}$-invariant elements in $\Lambda\mathfrak{g}$ and develops methods for their calculation. The classification of coboundary Lie bialgebras on $\mathfrak{g}$ is simplified in Section 8 via a certain quotient of $\Lambda\mathfrak{g}$. Section 9 details simple but useful results on the existence of automorphisms of Lie algebras. Section 10 applies all previous methods to the classification up to Lie algebra automorphisms of three-dimensional coboundary Lie bialgebras. Section 11 details the classification of Lie bialgebras on $\mathfrak{gl}_2$. Section 12 summarises our achievements and sketches future lines of research.

2. On Lie bialgebras, r-matrices, and $\mathfrak{g}$-modules

From now on, $\mathfrak{g}$ is a finite-dimensional Lie algebra. Let us briefly survey the theory of Lie bialgebras and $\mathfrak{g}$-modules (see [9, 23] for details) and establish our notation. We use a more geometric approach than in standard works, e.g. [9, 23]. All structures are over the reals, but complex analogues of our findings can be obtained similarly.

Let $\mathcal{V}^m M$ be the space of m-vector fields on a manifold M.
The *Schouten-Nijenhuis bracket* [27, 36] on $\mathcal{V}M := \oplus_{m\in\mathbb{Z}}\mathcal{V}^m M$ is the unique bilinear map $[\cdot,\cdot] : \mathcal{V}M \times \mathcal{V}M \to \mathcal{V}M$ satisfying: (a) $[f, g] = 0$ for arbitrary $f, g \in C^\infty(M)$, (b) if X is a vector field on M, then $[X, f] = Xf = -[f, X]$, c) we have

$$\begin{aligned}&[X_1 \wedge \ldots \wedge X_s, Y_1 \wedge \ldots \wedge Y_l] \\ &:= \sum_{\substack{i=1,\ldots,s\\ j=1,\ldots,l}}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge X_s \wedge Y_1 \wedge \ldots \wedge \widehat{Y}_j \wedge \ldots \wedge Y_l,\end{aligned} \tag{1}$$

where $X_1, \ldots, X_s, Y_1, \ldots, Y_l$ are vector fields on M, the $\widehat{X}_i, \widehat{Y}_j$ are omitted in the exterior products in (1), and $[X_i, Y_j]$ is the Lie bracket[1] of X_i and Y_j (see [27] for more details). Remarkably, $[\mathcal{X}, \mathcal{Y}] \in \mathcal{V}^{k+l-1}M$ for $\mathcal{X} \in \mathcal{V}^k M$ and $\mathcal{Y} \in \mathcal{V}^l M$. The space $\mathcal{V}^L G$ of left-invariant elements of $\mathcal{V}G$ for a Lie group G is closed relative to $[\cdot,\cdot]$ and $\mathcal{V}^L G$ can be identified with the Grassmann algebra $\Lambda\mathfrak{g}$ of the Lie algebra, $\mathfrak{g}$, of G. Moreover, $[\cdot,\cdot]$ can be restricted to $\mathcal{V}^L G$ leading to the *algebraic Schouten bracket* on $\Lambda\mathfrak{g}$ [36].

A *Lie bialgebra* is a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ has Lie bracket $[\cdot,\cdot]_\mathfrak{g}$, while $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$, the *cocommutator*, is linear, its transpose $\delta^* : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$, and

$$\delta([v_1, v_2]_\mathfrak{g}) = [v_1, \delta(v_2)] + [\delta(v_1), v_2], \qquad \forall v_1, v_2 \in \mathfrak{g}. \tag{2}$$

A *Lie bialgebra homomorphism* is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $(\phi \otimes \phi) \circ \delta_\mathfrak{g} = \delta_\mathfrak{h} \circ \phi$ for the cocommutators $\delta_\mathfrak{g}$ for $\mathfrak{g}$ and $\delta_\mathfrak{h}$ of $\mathfrak{h}$. A *coboundary Lie bialgebra* is a Lie bialgebra $(\mathfrak{g}, \delta_r)$ such that $\delta_r(v) := [v, r]$ for an $r \in \Lambda^2\mathfrak{g}$, a so-called *$r$-matrix*, and every $v \in \mathfrak{g}$. To characterise r-matrices, we use the following notions. The identification of $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields on

[1] We denote similar structures with the same symbol, but their meaning is clear from context.

G allows us to understand the tensor algebra $\mathfrak{T}(\mathfrak{g})$ on $\mathfrak{g}$ as the tensor algebra, $\mathfrak{T}^L(G)$, of left-invariant tensor fields on G. This gives rise to a Lie algebra representation $\mathrm{ad} : v \in \mathfrak{g} \mapsto \mathrm{ad}_v \in \mathfrak{gl}(\mathfrak{T}(\mathfrak{g}))$, where $\mathfrak{gl}(\mathfrak{T}(\mathfrak{g}))$ is the space of endomorphisms on $\mathfrak{T}(\mathfrak{g})$, we define $\mathrm{ad}_v(w) := \mathcal{L}_v w$ for every $w \in \mathfrak{T}(\mathfrak{g})$, where $\mathcal{L}_v w$ is the Lie derivative of $w \in \mathfrak{T}^L(G)$ relative to the vector field $v \in \mathfrak{T}^L(G)$. The geometric notation $\mathcal{L}_v w$ is conciser than algebraic ones (cf. [9]). A $q \in \mathfrak{T}(\mathfrak{g})$ is called $\mathfrak{g}$*-invariant* if $\mathcal{L}_v q = 0$ for all $v \in \mathfrak{g}$. We denote the set of $\mathfrak{g}$-invariant elements of $\mathfrak{T}(\mathfrak{g})$ by $\mathfrak{T}(\mathfrak{g})^{\mathfrak{g}}$. The map ad admits a restriction $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\Lambda^m \mathfrak{g})$. We recall that $(\Lambda^m \mathfrak{g})^{\mathfrak{g}}$ stands for the space of $\mathfrak{g}$-invariant m-vectors.

Theorem 2.1. *The map $\delta_r : v \in \mathfrak{g} \mapsto [v, r] \in \Lambda^2 \mathfrak{g}$, for $r \in \Lambda^2 \mathfrak{g}$, is a cocommutator if and only if $[r, r] \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$.*

We call $[r, r] \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$ the *modified classical Yang-Baxter equation* (mCYBE), while $[r, r] = 0$ is referred to as the *classical Yang-Baxter equation* (CYBE) and its solutions amount to left-invariant Poisson bivectors on Lie groups [36].

Note that two r-matrices $r_1, r_2 \in \Lambda^2 \mathfrak{g}$ satisfy $\delta_{r_1} = \delta_{r_2}$ if and only if $r_1 - r_2 \in (\Lambda^2 \mathfrak{g})^{\mathfrak{g}}$. Then, what really matters to the determination of coboundary Lie bialgebras is not r-matrices, but their equivalence classes in $\Lambda^2_R \mathfrak{g} = \Lambda^2 \mathfrak{g} / (\Lambda^2 \mathfrak{g})^{\mathfrak{g}}$.

Let V be a linear space, $GL(V)$ and $\mathfrak{gl}(V)$ stand for the Lie group of automorphisms and the Lie algebra of endomorphisms on V, respectively. A $\mathfrak{g}$*-module* is a pair (V, ρ), where $\rho : v \in \mathfrak{g} \mapsto \rho_v \in \mathfrak{gl}(V)$ is a Lie algebra morphism. A $\mathfrak{g}$-module (V, ρ) will be represented just by V, while $\rho_v(x)$, for any $v \in \mathfrak{g}$ and $x \in V$, will be written simply as vx if ρ is understood from context.

Example 2.2. Let $\mathrm{ad} : v \in \mathfrak{g} \mapsto [v, \cdot]_{\mathfrak{g}} \in \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$. Then, $(\mathfrak{g}, \mathrm{ad})$ is a $\mathfrak{g}$-module [16]. Since each $[v, \cdot]_{\mathfrak{g}}$, with $v \in \mathfrak{g}$, is a derivation of the Lie algebra $\mathfrak{g}$ [16], the map ad can be considered as a mapping $\mathrm{ad} : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{g})$, where $\mathfrak{Der}(\mathfrak{g})$ is the Lie algebra of derivations on $\mathfrak{g}$. ■

Example 2.3. The group $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra automorphisms of $\mathfrak{g}$ admits a Lie group structure [33] and its Lie algebra is denoted by $\mathfrak{aut}(\mathfrak{g})$. The tangent map at $\mathrm{id}_{\mathfrak{g}} \in \mathrm{Aut}(\mathfrak{g})$ to the injection $\iota : \mathrm{Aut}(\mathfrak{g}) \hookrightarrow GL(\mathfrak{g})$ induces a Lie algebra morphism

$$\widehat{\mathrm{ad}} : \mathfrak{aut}(\mathfrak{g}) \simeq \mathrm{T}_{\mathrm{id}_{\mathfrak{g}}} \mathrm{Aut}(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{g}) \simeq \mathrm{T}_{\mathrm{id}_{\mathfrak{g}}} GL(\mathfrak{g})$$

and $(\mathfrak{g}, \widehat{\mathrm{ad}})$ becomes an $\mathfrak{aut}(\mathfrak{g})$-module. ■

In view of the properties of the algebraic Schouten bracket, each $\mathfrak{g}$ gives rise to a $\mathfrak{g}$-module $(\Lambda \mathfrak{g}, \mathrm{ad})$, where $\mathrm{ad} : v \in \mathfrak{g} \mapsto [v, \cdot] \in \mathfrak{gl}(\Lambda \mathfrak{g})$ (cf. [36]). This can be understood as a consequence of Proposition 2.4 to be proved next. To grasp this fact and related ones, recall that every $T \in \mathfrak{gl}(V)$ gives rise to the maps $\Lambda^m T \colon \lambda \in \Lambda^m V \mapsto 0 \in \Lambda^m V$ for $m \leq 0$, and the maps $\Lambda^m T \in \mathfrak{gl}(\Lambda^m V)$, for $m > 0$, of the form

$$\Lambda^m T := T \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}(m - \text{operators}) + \ldots + \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes T(m - \text{operators}),$$

where id is the identity on V and $\Lambda^m T$ is assumed to be restricted to $\Lambda^m V$. Moreover, $\Lambda T := \bigoplus_{m \in \mathbb{Z}} \Lambda^m T \in \mathfrak{gl}(\Lambda V)$. If T is considered as an element of $GL(V)$, we define $\Lambda^m T := T \otimes \ldots \otimes T(m - \text{operators})$ for $m \geq 1$ and $\Lambda^m T$ is the identity on $\Lambda^m V$ for $m \leq 0$. Finally, $\Lambda T := \bigoplus_{m \in \mathbb{Z}} \Lambda^m T$.

Proposition 2.4 follows easily from the fact that if

$$T_i(v_1,\ldots,v_m) := v_1 \otimes \ldots \otimes T(v_i) \otimes \ldots \otimes v_m\,,$$

for $T \in \mathfrak{gl}(V)$, then $T_i \circ S_j = S_j \circ T_i$ for $i \neq j$ and every $T, S \in \mathfrak{gl}(V)$.

Proposition 2.4. *If (V,ρ) is a $\mathfrak{g}$-module and $m \in \mathbb{N}$, then we obtain $\mathfrak{g}$-modules $(\Lambda^m V, \Lambda^m\rho)$ and $(\Lambda V, \Lambda\rho)$ by $\Lambda^m\rho\,(v) := \Lambda^m(\rho_v)$ and $\Lambda(\rho)_v := \Lambda\rho_v$ for $v \in \mathfrak{g}$.*

The lemma below stems from the fact that every element of a connected Lie group is a product of elements in the image of its exponential map (see [34, p. 228]).

Lemma 2.5. *Let (V,ρ) be a $\mathfrak{g}$-module and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If $\Phi : G \to GL(V)$ is a Lie group morphism making commutative the diagram (3), where $\exp_G$ and $\exp$ are exponential maps on $\mathfrak{g}$ and $\mathfrak{gl}(V)$ respectively, then $\Phi(G)$ is an immersed Lie subgroup of $GL(V)$ generated by the elements $\exp(\rho(\mathfrak{g}))$.*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \rho\ } & \mathfrak{gl}(V) \\ \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\ \Phi\ } & GL(V) \end{array} \qquad (3)$$

The Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ in diagram (3) gives rise to a Lie group morphism $\Phi : \widetilde{G} \to GL(V)$, where $\widetilde{G}$ is connected and simply connected, so that the associated diagram of the form (3) is commutative [12]. Since $\Phi(\widetilde{G})$ is generated by the elements of $\exp(\rho(\mathfrak{g}))$, then $\Phi(\widetilde{G})$ is the smallest group containing $\exp(\rho(\mathfrak{g}))$. Hence, $\Phi(G) = \Phi(\widetilde{G})$. Although $\Phi(\widetilde{G})$ may not be an embedded submanifold in $GL(V)$, it is always a Lie group [25]. Previous facts justify the following definition and Proposition 2.7.

Definition 2.6. The *Lie group* of a $\mathfrak{g}$-module (V,ρ) is the (immersed) Lie subgroup $GL(\rho)$ of $GL(V)$ generated by $\exp(\rho(\mathfrak{g}))$.

Proposition 2.7. *Let $\mathrm{Ad} : g \in G \mapsto \mathrm{Ad}_g \in GL(\mathfrak{g})$ be the adjoint action of a connected Lie group G on its Lie algebra $\mathfrak{g}$. The Lie group of the $\mathfrak{g}$-module $(\mathfrak{g}, \mathrm{ad})$ is equal to $\mathrm{Ad}(G)$.*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \mathrm{ad}\ } & \mathfrak{gl}(\mathfrak{g}) \\ \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\ \mathrm{Ad}\ } & GL(\mathfrak{g}) \end{array}$$

Proposition 2.8. *The Lie group of the $\mathfrak{aut}(\mathfrak{g})$-module $(\mathfrak{g}, \widehat{\mathrm{ad}})$ is given by the connected component, $\mathrm{Aut}_c(\mathfrak{g})$, of the neutral element of $\mathrm{Aut}(\mathfrak{g})$.*

Proof. The inclusion $\iota : \mathrm{Aut}_c(\mathfrak{g}) \hookrightarrow GL(\mathfrak{g})$ has a tangent map $\widehat{\mathrm{ad}} : \mathfrak{aut}(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{g})$ at $\mathrm{id}_\mathfrak{g} \in \mathrm{Aut}_c(\mathfrak{g})$, which is a Lie algebra morphism. This leads to the commutativity between the right and central columns of the diagram below. Let $\widetilde{\mathrm{Aut}}(\mathfrak{g})$ be the connected and simply connected Lie group associated with $\mathfrak{aut}(\mathfrak{g})$ and let $\widetilde{\iota} : \widetilde{\mathrm{Aut}}(\mathfrak{g}) \to GL(\mathfrak{g})$ be the induced Lie group morphism.

The commutativity of the left and central columns of the diagram aside comes from the properties of $\widetilde{\iota}$ for $\widetilde{\mathrm{Aut}}(\mathfrak{g})$ and $\widehat{\mathrm{ad}}$. From the latter and Lemma 2.5, it follows that $GL(\widehat{\mathrm{ad}}) = \widetilde{\iota}(\widetilde{\mathrm{Aut}}(\mathfrak{g})) = \mathrm{Aut}_c(\mathfrak{g})$. ■

$$\begin{array}{ccccc} \mathfrak{aut}(\mathfrak{g}) & \xrightarrow{\ \widehat{\mathrm{ad}}\ } & \mathfrak{gl}(\mathfrak{g}) & \xleftarrow{\ \widehat{\mathrm{ad}}\ } & \mathfrak{aut}(\mathfrak{g}) \\ \downarrow{\scriptstyle \exp_{\widetilde{\mathrm{Aut}}(\mathfrak{g})}} & & \downarrow{\scriptstyle \exp} & & {\scriptstyle \exp_{\mathrm{Aut}_c(\mathfrak{g})}}\downarrow \\ \widetilde{\mathrm{Aut}}(\mathfrak{g}) & \xrightarrow{\ \widetilde{\iota}\ } & GL(\mathfrak{g}) & \xleftarrow{\ \iota\ } & \mathrm{Aut}_c(\mathfrak{g}) \end{array}$$

Propositions 2.7 and 2.8 show that the Lie groups of the $\mathfrak{g}$-module $(\mathfrak{g}, \mathrm{ad})$ and the $\mathfrak{aut}(\mathfrak{g})$-module $(\mathfrak{g}, \widehat{\mathrm{ad}})$ are $\mathrm{Inn}(\mathfrak{g})$ and $\mathrm{Aut}_c(\mathfrak{g})$, respectively. These Lie groups, and related structures, play a role in our classification of coboundary Lie bialgebras up to Lie algebra automorphisms. As a $\mathfrak{g}$-module, (V,ρ) induces new ones $(\Lambda^m V, \Lambda^m\rho)$, while $GL(\rho)$ is related to $GL(\Lambda^m\rho)$ as explained next.

Proposition 2.9. *If (V,ρ) is a $\mathfrak{g}$-module, the $\mathfrak{g}$-module $(\Lambda^m V, \Lambda^m\rho)$, for $m\in\mathbb{Z}$, has $GL(\Lambda^m\rho)=\{\Lambda^m T | T\in GL(\rho)\}$.*

Proof. Assume first $m>0$. As $[T_i,T_j]=0$ for $i,j=1,...,m$, and $T\in\mathfrak{gl}(\mathfrak{g})$, then

$$\exp(\Lambda^m\rho_v)=\exp(\rho_v\otimes \mathrm{id}\otimes\ldots\otimes\mathrm{id}+\ldots+\mathrm{id}\otimes\ldots\otimes\mathrm{id}\otimes\rho_v)=\exp(\rho_v)\otimes\ldots\otimes\exp(\rho_v)$$

for all $v\in\mathfrak{g}$. Hence, $GL(\Lambda^m\rho)$ is generated by the composition of the operators $T\otimes\ldots\otimes T$ (m-times), where T is a composition of operators $\exp(\rho_v)$ with $v\in\mathfrak{g}$. As the maps $\exp(\rho_v)$, for every $v\in\mathfrak{g}$, generate $GL(\rho)$, then T is any element of $GL(\rho)$, which finishes the proof for $m>0$. The case $m\leq 0$ is immediate. ■

A group action $\Phi\colon g\in G\mapsto\Phi_g\in GL(V)$ leads to a new one

$$\Lambda^m\Phi : g\in G\mapsto \Lambda^m\Phi_g\in GL(\Lambda^m V) \text{ for } m\in\mathbb{Z}.$$

Let $\mathfrak{inn}(\mathfrak{g})$ be the Lie algebra of $\mathrm{Inn}(\mathfrak{g})$. Note that $\mathfrak{inn}(\mathfrak{g})$ and $\mathrm{Inn}(\mathfrak{g})$ can be naturally embedded in $\mathfrak{gl}(\mathfrak{g})$ and $GL(\mathfrak{g})$, respectively. Let us denote both embeddings by ι_1 and ι_2. Previous comments and the fact that $\mathrm{Ad}(G)=\mathrm{Inn}(\mathfrak{g})$ allow us to extend the diagram of Proposition 2.7 as follows.

$$\begin{array}{ccccc}
\mathfrak{g} & \xrightarrow{\Lambda^m\mathrm{ad}} & \mathfrak{gl}(\Lambda^m\mathfrak{g}) & \xleftarrow{\Lambda^m\iota_1} & \mathfrak{inn}(\mathfrak{g})\\
\downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp} & & {\scriptstyle \exp_{\mathrm{Inn}(\mathfrak{g})}}\downarrow\\
G & \xrightarrow{\Lambda^m\mathrm{Ad}} & GL(\Lambda^m\mathfrak{g}) & \xleftarrow{\Lambda^m\iota_2} & \mathrm{Inn}(\mathfrak{g})
\end{array}$$

Proposition 2.10. *The dimension of the orbit $\mathcal{O}_w$ of the action of $\mathrm{Inn}(\mathfrak{g})$ on $\Lambda^m\mathfrak{g}$ through $w\in\Lambda^m\mathfrak{g}$ is $\dim\mathrm{Im}\,\Theta^m_w$, where $\Theta^m_w : v\in\mathfrak{inn}(\mathfrak{g})\mapsto[v,w]\in\Lambda^m\mathfrak{g}$.*

Proof. The orbit of $w\in\Lambda^m\mathfrak{g}$ relative to $\mathrm{Inn}(\mathfrak{g})$ is given by the points $g\cdot w := \Lambda^m\mathrm{Ad}_g w$ for every $g\in G$. Define $\exp(tv)=: g_t$, $g_1 := g$ for $v\in\mathfrak{g}$. Then, $\dim G\cdot w=\dim(\mathfrak{g})-\dim(\mathfrak{g})_w$, where G_w is the isotropy group of $w\in\Lambda^m\mathfrak{g}$. The Lie algebra $\mathfrak{g}_w$ of G_w is given by those $v\in\mathfrak{g}$ such that $\frac{d}{dt}\big|_{t=0}\Lambda^m\mathrm{Ad}_{g_t}(w)=[v,w]=0$. This amounts to $v\in\ker\Theta^m_w$. Hence, $\dim\mathcal{O}_w=\dim\mathfrak{g}-\dim\mathfrak{g}_w=\dim\mathrm{Im}\,\Theta^m_w$. ■

3. The $\mathfrak{g}$-invariant maps on Grassmann algebras $\Lambda\mathfrak{g}$

Let us extend and analyse notions on Lie algebras, like the ad-invariance, to $\mathfrak{g}$-modules and Grassmann algebras. This is aimed at studying Lie bialgebras hereafter.

Definition 3.1. A k-linear map $b: V^{\otimes k}\to\mathbb{R}$ is $GL(\rho)$*-invariant* relative to a $\mathfrak{g}$-module (V,ρ) if $T^*b=b$ for every $T\in GL(\rho)$, i.e. $b(Tx_1,\ldots,Tx_k)=b(x_1,\ldots,x_k)$ for all $x_1,\ldots,x_k\in V$. Moreover, b is $\mathfrak{g}$*-invariant* relative to (V,ρ) if

$$b(\rho_v(x_1),\ldots,x_k)+\ldots+b(x_1,\ldots,\rho_v(x_k))=0,\quad \forall v\in\mathfrak{g},\quad\forall x_1,\ldots,x_k\in V. \tag{4}$$

Example 3.2. The Killing form on $\mathfrak{g}$ induced by a $\mathfrak{g}$-module (V,ρ) takes the form $\kappa_\rho(v_1,v_2) := \mathrm{tr}(\rho_{v_1}\circ\rho_{v_2})$ for every $v_1,v_2\in\mathfrak{g}$ and it satisfies the equality $\kappa_\rho(\mathrm{ad}_v(v_1),v_2)+\kappa_\rho(v_1,\mathrm{ad}_v(v_2))=0$ for all $v,v_1,v_2\in\mathfrak{g}$ (cf. [33]). Killing forms are recovered by considering the $\mathfrak{g}$-module $(\mathfrak{g},\mathrm{ad})$ and they are called, due to its invariance, ad-*invariant.* The Killing form $\kappa_\mathfrak{g}$ is $\mathfrak{g}$-invariant relative to $(\mathfrak{g},\mathrm{ad})$.

Note that $\mathfrak{g}$-invariance can be interpreted as an extension of ad-invariance to $\mathfrak{g}$-modules. As shown in Proposition 3.3, the invariance of a k-linear map on a $\mathfrak{g}$-module (V,ρ) relative to $GL(\rho)$ can be characterised by the $\mathfrak{g}$-invariance of the k-linear map. The proof is not detailed as it is quite immediate.

Proposition 3.3. *A k-linear map $b : V^{\otimes k} \to \mathbb{R}$ is $GL(\rho)$-invariant relative to a $\mathfrak{g}$-module (V,ρ) if and only if b is $\mathfrak{g}$-invariant relative to (V,ρ).*

Subsequently, if $\{v_1,\ldots,v_r\}$ is a basis of V, then we define $v_J := v_{J(1)} \wedge \ldots \wedge v_{J(m)}$, where $J := (J(1),\ldots,J(m))$ with $J(1),\ldots,J(m) \in \{1,\ldots,r\}$ represents a multi-index of length $|J| = m$, the S_m is the permutation group of m elements, and $\mathrm{sg}(\sigma)$ stands for the sign of a permutation $\sigma \in S_m$.

Theorem 3.4. *Every $\mathfrak{g}$-invariant k-linear map $b : V^{\otimes k} \to \mathbb{R}$ relative to a $\mathfrak{g}$-module V induces a $\mathfrak{g}$-invariant k-linear map, $b_{\Lambda V}$, on ΛV relative to the induced $\mathfrak{g}$-module on ΛV by imposing that:*

(1) *the spaces $\Lambda^m V$, with $m \in \mathbb{Z}$, are orthogonal between themselves relative to $b_{\Lambda V}$;*

(2) *$b_{\Lambda V}(1,\ldots,1) = 1$;*

(3) *the restriction, $b_{\Lambda^m V}$, of $b_{\Lambda V}$ to $\Lambda^m V$, with $m \in \mathbb{N}$, satisfies*

$$b_{\Lambda^m V}(v_{J_1},\ldots,v_{J_k}) := \sum_{\sigma_1,\ldots,\sigma_k \in S_m} \mathrm{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))},\ldots,v_{J_k(\sigma_k^{-1}(r))}\right). \tag{5}$$

Proof. Since $1 \in \Lambda^0 V$, the $v_J = v_{J(1)} \wedge \ldots \wedge v_{J(m)}$ span ΛV and $b_{\Lambda V}$ is k-linear, then conditions 1, 2, and 3 establish $b_{\Lambda V}$. Condition 3 establishes a well-defined value of $b_{\Lambda^m V}$ independently of the representative for each v_{J_s}, with $s \in \{1,\ldots,k\}$. Indeed, defining $\sigma v_J := v_{J(\sigma^{-1}(1))} \wedge \ldots \wedge v_{J(\sigma^{-1}(m))}$ and $\tilde{\tilde{\sigma}}_j := \tilde{\sigma}_j \cdot \sigma_j$, we obtain

$$\begin{aligned} b_{\Lambda^m V}(\tilde{\sigma}_1 v_{J_1},\ldots,\tilde{\sigma}_k v_{J_k}) &= \sum_{\sigma_1,\ldots,\sigma_k \in S_m} \mathrm{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}\tilde{\sigma}_1^{-1}(r))},\ldots,v_{J_k(\sigma_k^{-1}\tilde{\sigma}_k^{-1}(r))}\right) \\ &= \sum_{\tilde{\tilde{\sigma}}_1,\ldots,\tilde{\tilde{\sigma}}_k \in S_m} \mathrm{sg}(\tilde{\tilde{\sigma}}_1 \ldots \tilde{\tilde{\sigma}}_k)\mathrm{sg}(\tilde{\sigma}_1 \ldots \tilde{\sigma}_k) \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\tilde{\tilde{\sigma}}_1^{-1}(r))},\ldots,v_{J_k(\tilde{\tilde{\sigma}}_k^{-1}(r))}\right). \end{aligned}$$

Hence, $b_{\Lambda^m V}(\tilde{\sigma}_1 v_{J_1},...,\tilde{\sigma}_k v_{J_k}) = \mathrm{sg}(\tilde{\sigma}_1...\tilde{\sigma}_k) b_{\Lambda^m V}(v_{J_1},...,v_{J_k})$ and $b_{\Lambda V}$ is well-defined. Let us prove that $b_{\Lambda V}$ is $\mathfrak{g}$-invariant relative to the $\mathfrak{g}$-module on ΛV induced by the $\mathfrak{g}$-module on V. By Proposition 3.3, the $\mathfrak{g}$-invariance of $b_{\Lambda V}$ is inferred from its $GL(\Lambda\rho)$-invariance. This also reduces to the $GL(\Lambda^m \rho)$-invariance of the restrictions $b_{\Lambda^m V}$ for $m \in \overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$. Since b is $GL(\rho)$-invariant and setting $e^{\rho_v} := \exp(\rho_v)$ for every $v \in \mathfrak{g}$ and $m \in \mathbb{N}$, $A = b_{\Lambda^m V}(\Lambda^m e^{\rho_v}(v_{J_1}),\ldots,\Lambda^m e^{\rho_v}(v_{J_k}))$ reads

$$\begin{aligned} A &= \sum_{\sigma_1,\ldots,\sigma_k \in S_m} \mathrm{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b\left(e^{\rho_v} v_{J_1(\sigma_1^{-1}(r))},\ldots,e^{\rho_v} v_{J_k(\sigma_k^{-1}(r))}\right) \\ &= \sum_{\sigma_1,\ldots,\sigma_k \in S_m} \mathrm{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))},\ldots,v_{J_k(\sigma_k^{-1}(r))}\right) = b_{\Lambda^m V}(v_{J_1},\ldots,v_{J_k}). \end{aligned}$$

Since the invariance of $b_{\Lambda^0 V}$ is obvious, $b_{\Lambda V}$ is $GL(\Lambda\rho)$-invariant and Proposition 3.3 ensures that is $\mathfrak{g}$-invariant. ∎

Since each Killing form $\kappa_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant relative to $(\mathfrak{g}, \mathrm{ad})$, it can be extended to any $\Lambda^m\mathfrak{g}$ via Theorem 3.4. Extensions to $\Lambda^2\mathfrak{g}$ and $\Lambda^3\mathfrak{g}$ are called the *double* and *triple Killing forms* of $\mathfrak{g}$, respectively.

The next corollary gives an immediate consequence of Proposition 3.3 and the proof of Theorem 3.4.

Corollary 3.5. *If b is a $\mathfrak{g}$-invariant k-linear map on V, then $b_{\Lambda^m V}$ is $GL(\Lambda^m\rho)$-invariant, i.e. $b_{\Lambda^m V}(\Lambda^m T\cdot, \ldots, \Lambda^m T\cdot) = b_{\Lambda^m V}(\cdot, \ldots, \cdot)$ for every $T \in GL(\rho)$.*

As shown next, certain extensions of a $\mathfrak{g}$-invariant k-linear symmetric map are zero and, therefore, useless.

Proposition 3.6. *If b is a $\mathfrak{g}$-invariant k-linear map on V, then $b_{\Lambda^m V} = 0$ for $m > 1$ and odd $k > 1$.*

Proof. Let us first prove that we can gather the summands appearing in (5) into families that sum up to zero. We introduce the equivalence relation in $S_m^k := S_m \times \overset{k \text{ times}}{\cdots} \times S_m$ given by assuming that $(\sigma_1, \ldots, \sigma_k) \equiv (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k)$ if and only if there exists $\sigma \in S_m$ such that $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k) = (\sigma\sigma_1, \ldots, \sigma\sigma_k)$. Let $[(\sigma_1, \ldots, \sigma_k)]$ be the equivalence class of $(\sigma_1, \ldots, \sigma_k) \in S_m^k$ and let $\mathcal{R}$ be the space of equivalence classes. If we denote by $\mathbf{w}$ a generic element of S_m^k, the map $b_{\Lambda^m V}$ satisfies

$$b_{\Lambda^m V}(v_{J_1}, \ldots, v_{J_k}) := \sum_{\substack{[\mathbf{w}] \in \mathcal{R} \\ (\sigma_1, \ldots, \sigma_k) \in [\mathbf{w}]}} \mathrm{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right).$$

Since every equivalence class reads $[(\sigma_1, \ldots, \sigma_k)] = \{(\sigma\sigma_1, \ldots, \sigma\sigma_k) : \sigma \in S_m\}$,

$$b_{\Lambda^m V}(v_{J_1}, \ldots, v_{J_k}) := \sum_{\substack{[(\sigma_1, \ldots, \sigma_k)] \in \mathcal{R} \\ \sigma \in S_m}} \mathrm{sg}(\sigma\sigma_1 \ldots \sigma\sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}\sigma^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}\sigma^{-1}(r))}\right).$$

Let us show that the above sum vanishes for every equivalence class of $\mathcal{R}$. First,

$$\prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}\sigma^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}\sigma^{-1}(r))}\right) = \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right).$$

Define $\mathrm{sg}(\mathbf{w}) := \mathrm{sg}(\widehat{\sigma}_1 \ldots \widehat{\sigma}_k)$ and $\sigma\mathbf{w} := (\sigma\widehat{\sigma}_1, \ldots, \sigma\widehat{\sigma}_k)$ for $\mathbf{w} = (\widehat{\sigma}_1, \ldots, \widehat{\sigma}_k)$ and $\sigma \in S_m$. As k is odd, one has $\mathrm{sg}(\sigma\mathbf{w}) = \mathrm{sg}(\sigma)^k \mathrm{sg}(\mathbf{w}) = \mathrm{sg}(\sigma)\mathrm{sg}(\mathbf{w})$.
Every equivalence class of $\mathcal{R}$ has $m!$ elements having the same absolute value. Half of them are odd for $m > 1$ and the elements of the other half are even. Hence,

$$\sum_{\sigma \in S_m} \mathrm{sg}(\sigma\mathbf{w}) \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right) = 0 \implies b_{\Lambda^m V} = 0. \qquad \blacksquare$$

Example 3.7. Consider the Lie algebra $\mathfrak{su}_2$ and its Killing form $\kappa_{\mathfrak{su}_2}$, which is a $\mathfrak{su}_2$-invariant, bilinear, symmetric map on $\mathfrak{su}_2$. Take the basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}_2$ given in Table 1. Theorem 3.4 extends $\kappa_{\mathfrak{su}_2}$ to the double- and triple-Killing forms $\kappa_{\Lambda^2\mathfrak{su}_2}$, $\kappa_{\Lambda^3\mathfrak{su}_2}$ on $\Lambda^2\mathfrak{su}_2$ and $\Lambda^3\mathfrak{su}_2$, respectively. In the bases $\{e_{12}, e_{13}, e_{23}\}$, $\{e_{123}\}$ for the spaces $\Lambda^2\mathfrak{su}_2, \Lambda^3\mathfrak{su}_2$ (see Table 1), we then obtain $[\kappa_{\mathfrak{su}_2}] = -2\mathrm{Id}_{3\times 3}$, $[\kappa_{\Lambda^2\mathfrak{su}_2}] = 4\mathrm{Id}_{3\times 3}$, $[\kappa_{\Lambda^3\mathfrak{su}_2}] = (-8)$.

Example 3.7 shows that $\kappa_{\mathfrak{su}_2}$ and its extensions to $\Lambda^2\mathfrak{su}_2$ and $\Lambda^3\mathfrak{su}_2$ are simultaneously diagonal and non-degenerate. The corollary below explains this fact.

Corollary 3.8. *If b is a symmetric $\mathfrak{g}$-invariant k-linear mapping on an r-dimensional $\mathfrak{g}$-module V, then $b_{\Lambda V}$ is symmetric. If b is bilinear, then: a) it diagonalises in a basis $\{e_1,\dots,e_r\}$ and $b_{\Lambda^m V}$ diagonalises in the basis $\{e_J\}_{|J|=m}$; b) b is non-degenerate if and only if $b_{\Lambda V}$ is so.*

Proof. If b is a symmetric $\mathfrak{g}$-invariant k-linear mapping on V, then Theorem 3.4 ensures that $b_{\Lambda V}$ is symmetric. Indeed, condition 3 of Theorem 3.4 guarantees the symmetry of $b_{\Lambda^m V}$ on decomposable elements of $\Lambda^m V$, $m \in \mathbb{N}$, whereas condition 2 ensures the same for $m=0$. Since $b_{\Lambda V}$ is multilinear, it is symmetric on ΛV.

If b is bilinear and symmetric, it can always be put into diagonal form in a certain basis $\{e_1,\dots,e_r\}$ for V. This gives rise to a basis $\{e_J\}_{r\geq|J|\geq 0}$ of ΛV. Using the expression for $b_{\Lambda V}$, we see that it is also diagonal. The element in the diagonal $b_{\Lambda V}(e_J,e_J)$, for $|J|\geq 1$, reads $\prod_{j=1}^{|J|} b(e_{J(j)},e_{J(j)})$. Thus, b is non-degenerate if and only if the induced symmetric $b_{\Lambda^m V}$ on each $\Lambda^m V$ is so as well. ■

Example 3.9. Consider the Lie algebra $\mathfrak{sl}_2$ and a basis $\{e_1,e_2,e_3\}$ satisfying the commutation relations in Table 1. In the induced bases $\{e_{12},e_{13},e_{23}\}$ and $\{e_{123}\}$ in $\Lambda^2\mathfrak{sl}_2$ and $\Lambda^3\mathfrak{sl}_2$, respectively, one has

$$[\kappa_{\mathfrak{sl}_2}]=\begin{pmatrix} 2 & 0 & 0\\ 0 & 0 & 2\\ 0 & 2 & 0\end{pmatrix},\qquad [\kappa_{\Lambda^2\mathfrak{sl}_2}]=\begin{pmatrix} 0 & 4 & 0\\ 4 & 0 & 0\\ 0 & 0 & -4\end{pmatrix},\qquad [\kappa_{\Lambda^3\mathfrak{sl}_2}]=(-8). \tag{6}$$

Since $\mathfrak{sl}_2$ is simple, the Cartan criterion states that $\kappa_{\mathfrak{sl}_2}$ is non-degenerate. Then, Corollary 3.8 ensures that $\kappa_{\Lambda^2\mathfrak{sl}_2}$ and $\kappa_{\Lambda^3\mathfrak{sl}_2}$ must be non-degenerate. This agrees with their expressions showed in (6). ■

4. Killing-type forms

This section describes the invariance properties of certain multilinear symmetric maps on the spaces $\Lambda^m\mathfrak{g}$ induced by Killing forms. Our methods give rise to symmetric bilinear maps that are invariant under the action of $\mathrm{Aut}(\mathfrak{g})$, which will be of interest in the description of coboundary cocommutators in Section 10.

Proposition 4.1. *The Killing form $\kappa_{\Lambda\mathfrak{g}}$ is invariant relative to the action of $\mathrm{Aut}(\mathfrak{g})$ on $\Lambda\mathfrak{g}$. In particular, $\kappa_{\Lambda\mathfrak{g}}$ is $\mathfrak{aut}(\mathfrak{g})$-invariant relative to the $\mathfrak{aut}(\mathfrak{g})$-module $(\Lambda\mathfrak{g},\widehat{\Lambda\mathrm{ad}})$.*

Proof. The Killing form $\kappa_{\mathfrak{g}}$ is invariant relative to the action of $\mathrm{Aut}(\mathfrak{g})$ [18]. If v_{J_1},v_{J_2} are decomposable elements of $\Lambda^m\mathfrak{g}$, for $m\geq 1$, and $T\in\mathrm{Aut}(\mathfrak{g})$, then

$$\begin{aligned}\kappa_{\Lambda\mathfrak{g}}(\Lambda^m T v_{J_1},\Lambda^m T v_{J_2}) &:= \sum_{\sigma_1,\sigma_2\in S_m}\mathrm{sg}(\sigma_1\sigma_2)\frac{1}{m!}\prod_{r=1}^m \kappa_{\mathfrak{g}}\left(Tv_{J_1(\sigma_1^{-1}(r))},Tv_{J_2(\sigma_2^{-1}(r))}\right)\\ &= \sum_{\sigma_1,\sigma_2\in S_m}\mathrm{sg}(\sigma_1\sigma_2)\frac{1}{m!}\prod_{r=1}^m \kappa_{\mathfrak{g}}\left(v_{J_1(\sigma_1^{-1}(r))},v_{J_2(\sigma_2^{-1}(r))}\right)=\kappa_{\Lambda\mathfrak{g}}(v_{J_1},v_{J_2}).\end{aligned}\tag{7}$$

Since $\kappa_{\Lambda\mathfrak{g}}$ is bilinear and (7) is satisfied for decomposable elements of $\Lambda^m\mathfrak{g}$, which span $\Lambda^m\mathfrak{g}$, the $\kappa_{\Lambda^m\mathfrak{g}}$ is invariant relative to the action of $\mathrm{Aut}(\mathfrak{g})$ on $\Lambda^m\mathfrak{g}$. As the latter is true for every m, and the spaces $\Lambda^m\mathfrak{g}$ for different m are orthogonal relative to $\kappa_{\Lambda\mathfrak{g}}$, this map is invariant relative to the action of $\mathrm{Aut}(\mathfrak{g})$. In view of Proposition 3.3, the map $\kappa_{\Lambda\mathfrak{g}}$ is $\mathfrak{aut}(\mathfrak{g})$-invariant if and only if $\kappa_{\Lambda\mathfrak{g}}$ is $GL(\widehat{\Lambda\mathrm{ad}})$-invariant, which in turn means that $\kappa_{\Lambda\mathfrak{g}}(\Lambda T\cdot,\Lambda T\cdot)=\kappa_{\Lambda\mathfrak{g}}(\cdot,\cdot)$ for every $T\in\mathrm{Aut}_c(\mathfrak{g})$, where $\mathrm{Aut}_c(\mathfrak{g})$ stands for the connected part of the neutral element of $\mathrm{Aut}(\mathfrak{g})$. This is an immediate consequence of (7) and it proves the second part of the proposition. ■

Since $\kappa_{\Lambda\mathfrak{g}}$ is invariant under the maps ΛT, with $T\in\mathrm{Aut}(\mathfrak{g})$, it is therefore invariant under ΛT with $T\in\mathrm{Inn}(\mathfrak{g})$. In view of Proposition 3.3, the $\kappa_{\Lambda\mathfrak{g}}$ is also $\mathfrak{g}$-invariant relative to $(\Lambda\mathfrak{g},\mathrm{ad})$.

Proposition 3.3 shows that b is $\mathfrak{aut}(\mathfrak{g})$-invariant if and only if $T^*b=b$ for every $T\in\mathrm{Aut}_c(\mathfrak{g})$. Proposition 4.2 follows from the previous idea and the well-known proof of the fact that every Killing metric on $\mathfrak{g}$ is invariant relative to $\mathrm{Aut}(\mathfrak{g})$.

Proposition 4.2. *The map $b(v_1,\dots,v_k):=\sum_{\sigma\in S_k}\mathrm{Tr}\Big(\mathrm{ad}_{v_{\sigma(1)}}\circ\dots\circ\mathrm{ad}_{v_{\sigma(k)}}\Big)$, where $v_1,\dots,v_k\in\mathfrak{g}$, is invariant under the action of $\mathrm{Aut}(\mathfrak{g})$ and $\mathfrak{aut}(\mathfrak{g})$-invariant relative to $(\mathfrak{g},\widehat{\mathrm{ad}})$. The k-linear anti-symmetric map given by $b(v_1,\dots,v_k):=\sum_{\sigma\in S_k}\mathrm{sg}(\sigma)\mathrm{Tr}(\mathrm{ad}_{v_{\sigma(1)}}\circ\dots\circ\mathrm{ad}_{v_{\sigma(k)}})$, for all $v_1,\dots,v_k\in\mathfrak{g}$, is $\mathfrak{aut}(\mathfrak{g})$-invariant with respect to the $\mathfrak{aut}(\mathfrak{g})$-module $(\mathfrak{g},\widehat{\mathrm{ad}})$.*

Let us prove that a polynomial Casimir element C of order k, i.e. a symmetric element of $\mathfrak{g}^{\otimes k}$ satisfying that $\mathcal{L}_vC=0$ for every $v\in\mathfrak{g}$, leads to a $\mathfrak{g}$-invariant k-linear symmetric map on $\mathfrak{g}$. Recall that $\kappa_{\mathfrak{g}}$ induces a map $\widetilde{\kappa}_{\mathfrak{g}}:v\in\mathfrak{g}\mapsto\kappa_{\mathfrak{g}}(v,\cdot)\in\mathfrak{g}^*$ and there exists a natural isomorphism $\mathfrak{g}^{\otimes k}\simeq\left[(\mathfrak{g}^*)^{\otimes k}\right]^*$.

Theorem 4.3. *Every polynomial Casimir element C of order k on a Lie algebra $\mathfrak{g}$ induces a $\mathfrak{g}$-invariant k-linear symmetric map on $\mathfrak{g}$ given by $b(v_1,\dots,v_k):=C(\widetilde{\kappa}_{\mathfrak{g}}(v_1),\dots,\widetilde{\kappa}_{\mathfrak{g}}(v_k))$ for every $v_1,\dots,v_k\in\mathfrak{g}$.*

Proof. Since $\kappa_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant, we have

$$I:=\sum_{j=1}^k b(\mathrm{ad}_vv_j,v_1,...,\widehat{v_j},\dots,v_k)=\sum_{j=1}^k C(\widetilde{\kappa}_{\mathfrak{g}}(\mathrm{ad}_vv_j),\widetilde{\kappa}_{\mathfrak{g}}(v_1),...,\widehat{\widetilde{\kappa}_{\mathfrak{g}}(v_j)},...,\widetilde{\kappa}_{\mathfrak{g}}(v_k))$$

and, for all $v,v_1,v_2\in\mathfrak{g}$,

$$[\mathrm{ad}_v^*\circ\widetilde{\kappa}_{\mathfrak{g}}(v_1)](v_2)=\widetilde{\kappa}_{\mathfrak{g}}(v_1)(\mathrm{ad}_vv_2)=\kappa_{\mathfrak{g}}(v_1,\mathrm{ad}_vv_2)=-\kappa_{\mathfrak{g}}(\mathrm{ad}_vv_1,v_2)=-[\widetilde{\kappa}_{\mathfrak{g}}\circ\mathrm{ad}_v(v_1)](v_2),$$

Hence, $\widetilde{\kappa}_{\mathfrak{g}}\circ\mathrm{ad}_v=-\mathrm{ad}_v^*\circ\widetilde{\kappa}_{\mathfrak{g}}$ for every $v\in\mathfrak{g}$. As C is a Casimir element, $\mathcal{L}_vC=0$, which along with the above expression and the fact that $\mathcal{L}_v\theta=-\mathrm{ad}_v^*\theta$ for every $\theta\in\mathfrak{g}^*$ (where θ can be understood also as a left-invariant one-form on a Lie group G with Lie algebra $\mathfrak{g}$) gives, for all $v_1,\dots,v_k,v\in\mathfrak{g}$, that

$$I=-\sum_{j=1}^k C(\mathrm{ad}_v^*\widetilde{\kappa}_{\mathfrak{g}}(v_j),\widetilde{\kappa}_{\mathfrak{g}}(v_1),...,\widehat{\widetilde{\kappa}_{\mathfrak{g}}(v_j)},...,\widetilde{\kappa}_{\mathfrak{g}}(v_k))=(\mathcal{L}_vC)(\widetilde{\kappa}_{\mathfrak{g}}(v_1),...,\widetilde{\kappa}_{\mathfrak{g}}(v_k))=0.$$ ■

If $\mathfrak{g}$ is semi-simple, then the proof of Theorem 4.3 can be reversed and a $\mathfrak{g}$-invariant k-linear symmetric amounts to a Casimir element. If $\mathfrak{g}$ is not semi-simple, $\widetilde{\kappa}_{\mathfrak{g}}$ is not invertible and $\mathfrak{g}$-invariant multilinear symmetric maps may be more versatile, as they not need to come from Casimir elements.

5. On the existence of $\mathfrak{g}$-invariant bilinear maps

Next, simple remarks simplify the calculation of $\mathfrak{g}$-invariant maps, which may be difficult when $\dim \mathfrak{g}$ is large. Our methods will be illustrated in Section 10 and here.

Proposition 5.1. *Let b be a $\mathfrak{g}$-invariant bilinear map on a $\mathfrak{g}$-module V. Then, $b(\operatorname{Im}\rho_v, \ker\rho_v) = 0$ for every $v \in \mathfrak{g}$ and $x \in V$. If b is also symmetric, then $b(\rho_v(x), x) = 0$. If b is anti-symmetric relative to the $\mathfrak{g}$-module $(\mathfrak{g}, \mathrm{ad})$, then $b(\mathrm{ad}_v(w), w) = 0$ for every $v, w \in \mathfrak{g}$.*

Proof. Every $x_1 \in \mathrm{Im}\rho_v$ can be written as $x_1 := \rho_v(x_3)$ for some $x_3 \in V$. Assume that $x_2 \in \ker\rho_v$. As b is $\mathfrak{g}$-invariant, $b(x_1, x_2) = b(\rho_v(x_3), x_2) = -b(x_3, \rho_v(x_2)) = 0$ and $b(\operatorname{Im}\rho_v, \ker\rho_v) = 0$. Using the symmetricity of b, we get, for all $v \in \mathfrak{g}$ and $x \in V$, $b(\rho_v(x), x) = -b(x, \rho_v(x)) = -b(\rho_v(x), x)$. Therefore, $b(\rho_v(x), x) = 0$ for every $x \in V$ and $v \in \mathfrak{g}$. If b is anti-symmetric, then

$$b(\mathrm{ad}_{v_1}(v_2), v_2) = -b(\mathrm{ad}_{v_2}(v_1), v_2) = b(v_1, \mathrm{ad}_{v_2}(v_2)) = 0$$

for every $v_1, v_2 \in \mathfrak{g}$. ■

Proposition 5.2. *Let $b : V \otimes V \to \mathbb{R}$ be a $\mathfrak{g}$-invariant bilinear map relative to the $\mathfrak{g}$-module V. If W is a two-dimensional linear subspace of V satisfying that $vW \subset W$ for any $v \in \mathfrak{g}$, then for any linearly independent $f_s, f_t \in V$ one has*

$$\mathrm{Tr}(\rho_v|_W) b(f_s, f_t) f_t \wedge f_s = b(f_s, f_s)(vf_t) \wedge f_t + b(f_t, f_t) f_s \wedge (vf_s). \tag{8}$$

Proof. Since $vW \subset W$, there exists constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $vf_s = \alpha_1 f_s + \alpha_2 f_t$ and $vf_t = \beta_1 f_s + \beta_2 f_t$. The $\mathfrak{g}$-invariance of b ensures that

$$\alpha_1 b(f_s, f_t) + \alpha_2 b(f_t, f_t) = b(vf_s, f_t) = -b(f_s, vf_t) = -\beta_1 b(f_s, f_s) - \beta_2 b(f_s, f_t).$$

After rearranging, the above expression gives (8). ■

Example 5.3. Consider the 3D Heisenberg Lie algebra $\mathfrak{h}$. Take a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{h}$ as in Table 1. Since $\mathfrak{h}$ is nilpotent, its Killing form $\kappa_{\mathfrak{h}}$ vanishes [18, pg. 480]. In the bases $\{e_{12}, e_{13}, e_{23}\}$ and $\{e_{123}\}$ of $\Lambda^2\mathfrak{h}$ and $\Lambda^3\mathfrak{h}$, respectively, a $\mathfrak{h}$-invariant b and its extensions to $\Lambda^2\mathfrak{h}$ and $\Lambda^3\mathfrak{h}$ given by Propositions 3.4 and 5.1 read

$$[b] := \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [b_{\Lambda^2\mathfrak{h}}] := \begin{pmatrix} \alpha_1\alpha_4 - \alpha_2\alpha_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [b_{\Lambda^3\mathfrak{g}}] := (0),$$

for $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$. Using again Proposition 5.1, we can compute the general $\mathfrak{h}$-invariant bilinear map $\widetilde{b}$ on $\Lambda^2\mathfrak{h}$, which reads

$$[\widetilde{b}] := \begin{pmatrix} \beta_3 & \beta_2 & \beta_1 \\ \beta_4 & 0 & 0 \\ \beta_5 & 0 & 0 \end{pmatrix}.$$

If $\widetilde{b}$ is symmetric, Proposition 5.1 yields $\beta_1 = \beta_5 = 0$ and $\beta_2 = \beta_4 = 0$. If $\widetilde{b}$ is anti-symmetric, then $\beta_5 = -\beta_1$, $\beta_4 = -\beta_2$ and $\beta_3 = 0$. All given $\widetilde{b}$ are $\mathfrak{h}$-invariant. ■

Example 5.4. Let us consider the Lie algebra $\mathfrak{r}_{3,1} := \langle e_1, e_2, e_3\rangle$ with commutation relations given in Table 1. Proposition 5.1 yields necessary conditions for a bilinear form $\omega_{\Lambda^2\mathfrak{r}_{3,1}}$ on $\Lambda^2\mathfrak{r}_{3,1}$ to be $\mathfrak{r}_{3,1}$-invariant. In particular,

$$[\omega_{\Lambda^2\mathfrak{r}_{3,1}}] := \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \forall a, b \in \mathbb{R},$$

in the basis $\{e_{12}, e_{13}, e_{23}\}$ of $\Lambda^2\mathfrak{r}_{3,1}$. From Proposition 5.2 and assuming $v := e_1$, one gets that $a = b = 0$ and there are no $\mathfrak{r}_{3,1}$-invariant forms on $\Lambda^2\mathfrak{r}_{3,1}$. ■

6. *G*-Gradations, Grassmann algebras, and CYBEs

Let us show that a type of graded Lie algebra $\mathfrak{g}$ gives a decomposition in $\Lambda\mathfrak{g}$ compatible, in a way to be stated, with its algebraic Schouten bracket. This will be used to study the structure and solutions to mCYBEs for quite general Lie algebras.

Definition 6.1. We say that $\mathfrak{g}$ admits a *G-gradation* if $\mathfrak{g} = \bigoplus_{\alpha\in G\subset\mathbb{R}^n} \mathfrak{g}^{(\alpha)}$, where $(G \subset \mathbb{R}^n, \star)$ is a commutative group, the $\mathfrak{g}^{(\alpha)}$ are subspaces of $\mathfrak{g}$, and $[\mathfrak{g}^{(\alpha)}, \mathfrak{g}^{(\beta)}] \subset \mathfrak{g}^{(\alpha\star\beta)}$ for all $\alpha, \beta \in G$. The spaces $\mathfrak{g}^{(\alpha)}$, for $\alpha \in G$, are called the *homogeneous spaces* of the gradation and α is the degree of $\mathfrak{g}^{(\alpha)}$. ■

Although every $\mathfrak{g}$ admits a trivial $\mathbb{Z}$-gradation with $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(\alpha)} = \langle 0\rangle$ with $\alpha \neq 0$, this will not be useful to our purposes, as follows from posterior considerations. If $\mathfrak{g}$ admits a G-gradation and G is known from context or of minor importance, we will simply say that $\mathfrak{g}$ admits a gradation.

Example 6.2. It stems from the commutation relations in Table 1 that $\mathfrak{su}_2$ is isomorphic to the Lie algebra on $\mathbb{R}^3$ with the Lie bracket given by the vector product $\times$. Consider a basis $\{v, v_1^\perp, v_2^\perp\}$ of $\mathbb{R}^3$, where $v \in \mathbb{R}^3\backslash\{0\}$, the vector $v_1^\perp$ is perpendicular to v, and $v_2^\perp := v \times v_1^\perp$. Then, $\mathfrak{su}_2$ admits a $\mathbb{Z}_2$-gradation $\mathfrak{g}^{(0)} = \langle v\rangle$ and $\mathfrak{g}^{(1)} = \langle v_1^\perp, v_2^\perp\rangle$. Since v is not unique, $\mathfrak{su}_2$ admits several $\mathbb{Z}_2$-gradations.

A G-gradation is a type of graded Lie algebra [36]. Although some of our results can be extended for G being a semigroup, this will not be necessary to us.

If $\mathfrak{g}$ admits a root decomposition, it has a $\mathbb{Z}^k$-gradation relative to the group $(\mathbb{Z}^k, +)$. Table 1 shows that nontrivial gradations can be found for all three-dimensional Lie algebras. Figures 1 and 2 give $\mathbb{Z}^2$-gradations for the special pseudo-orthogonal Lie algebras $\mathfrak{so}(3,2)$ and $\mathfrak{so}(2,2)$ [8, 21]. A type of G-gradation whose properties are close to standard root decompositions, but applicable to more general Lie algebras, is given in Definition 6.3. This will be used to derive $\mathfrak{g}$-invariant subspaces in $\Lambda\mathfrak{g}$.

Definition 6.3. A Lie algebra $\mathfrak{g}$ has a *root $\mathbb{Z}^k$-gradation* if it admits a $\mathbb{Z}^k$-gradation such that $\dim \mathfrak{g}^{(0)} = k$ and $\Xi : \alpha \in \mathbb{Z}^k \mapsto \boldsymbol{\alpha} \in \mathfrak{g}^{(0)*}$ such that $[e, e^{(\alpha)}] = \boldsymbol{\alpha}(e)e^{(\alpha)}$ for every $e \in \mathfrak{g}^{(0)}$ and $e^{(\alpha)} \in \mathfrak{g}^{(\alpha)}$ is an injective group morphism. ■

Since $\mathfrak{g}^{(0)}$ is abelian, every root decomposition gives rise to a root $\mathbb{Z}^k$-gradation. For instance, Figure 6.5 shows a root decomposition for $\mathfrak{so}(2,2)$ leading to a root $\mathbb{Z}^2$-gradation for $\Xi : (i,j) \in \mathbb{Z}^2 \mapsto ie^0 + jh^0 \in (\mathfrak{so}(2,2)^{(0)})^*$, where $\{e^0, f^0\}$ is the dual basis to the basis $\{e_0, f_0\}$ of $\mathfrak{so}(2,2)^{(0)}$.

The theorem below shows that a G-gradation on $\mathfrak{g}$ leads to a decomposition of each $\Lambda^m\mathfrak{g}$ into the so-called *homogeneous spaces* so that the algebraic Schouten bracket maps homogeneous spaces of $\Lambda\mathfrak{g}$ onto homogeneous spaces in a manner determined by the group structure in G.

Theorem 6.4. *If $\mathfrak{g}$ admits a G-gradation $\mathfrak{g} = \bigoplus_{\alpha\in G\subset\mathbb{R}^n}\mathfrak{g}^{(\alpha)}$, then each $\Lambda^m\mathfrak{g}$ admits a decomposition into so-called homogeneous spaces of the form*

$$\Lambda^m\mathfrak{g} = \bigoplus_{\alpha\in G}(\Lambda^m\mathfrak{g})^{(\alpha)}, \quad (\Lambda^m\mathfrak{g})^{(\alpha)} := \bigoplus_{\substack{\{\alpha_1,\ldots,\alpha_m\}\subset G\\ \alpha_1\star\ldots\star\alpha_m=\alpha}}\mathfrak{g}^{(\alpha_1)}\wedge\ldots\wedge\mathfrak{g}^{(\alpha_m)}, m>0, \quad (\Lambda^0\mathfrak{g})^{(0)} = \mathbb{R}, \tag{9}$$

so that $[(\Lambda^p\mathfrak{g})^{(\alpha)}, (\Lambda^q\mathfrak{g})^{(\beta)}]_S \subset \Lambda^{p+q-1}\mathfrak{g}^{(\alpha\star\beta)}$, for $p,q\in\mathbb{Z}$, $\alpha,\beta\in G$.

Proof. The m exterior products among the elements of a basis $\mathfrak{g}$ adapted to its G-gradation, with $m\geq 1$, give rise to a basis of $\Lambda^m\mathfrak{g}$. Since G is commutative, the exterior product $e^{(\alpha_1)}\wedge\ldots\wedge e^{(\alpha_m)}$, where $e^{(\alpha_i)}\in\mathfrak{g}^{(\alpha_i)}$ for every $i=1,\ldots,m$, belongs to $(\Lambda^m\mathfrak{g})^{(\alpha)}$ for $\alpha=\alpha_1\star\ldots\star\alpha_m$. Elements $e^{(\alpha_1)}\wedge\ldots\wedge e^{(\alpha_m)}$ with $\alpha=\alpha_1\star\ldots\star\alpha_m$ span a basis of $(\Lambda^m\mathfrak{g})^{(\alpha)}$. Repeating the previous process for every $\alpha\in G$, we obtain a basis of $\Lambda^m\mathfrak{g}$ and the decomposition in (9).

Finally, the algebraic Schouten bracket of elements of $(\Lambda^p\mathfrak{g})^{(\alpha)}$ and $(\Lambda^q\mathfrak{g})^{(\beta)}$ follows from (1). Due to the gradation in $\mathfrak{g}$, each term of the sum (1) belongs to $(\Lambda^{p+q-1}\mathfrak{g})^{(\alpha\star\beta)}$. ∎

Example 6.5. Consider the basis $\{e_-,e_0,e_+,f_-,f_0,f_+\}$ of $\mathfrak{so}(2,2)\simeq\mathfrak{sl}_2\oplus\mathfrak{sl}_2$, where $\{e_-,e_0,e_+\}$, $\{f_-,f_0,f_+\}$ are bases of each copy of $\mathfrak{sl}_2$ within $\mathfrak{so}(2,2)$. Then, $\mathfrak{so}(2,2)$ admits a root $\mathbb{Z}^2$-gradation, given in the first diagram of Figure 1. This gives rise to induced decompositions on $\Lambda^2\mathfrak{so}(2,2)$ and $\Lambda^3(\mathfrak{so}(2,2))$, whose non-zero homogeneous spaces are indicated by bold points in Figure 1. Hence, $\mathfrak{so}(2,2)^{(0)} = \langle f_0, e_0\rangle$. We write $\{e^0,f^0\}$ for the dual basis of $\{e_0,f_0\}$.

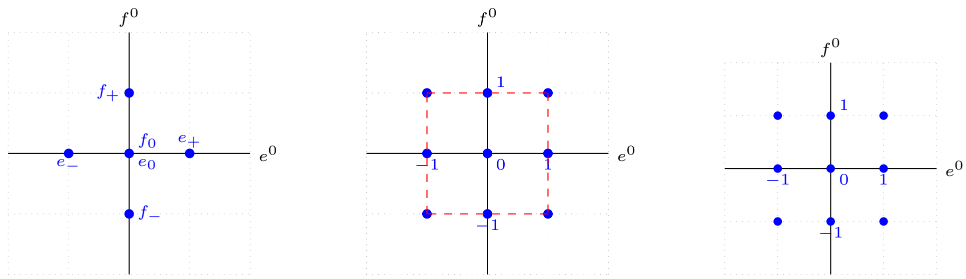


Figure 1: Root decomposition of $\mathfrak{so}(2,2)$ (left), and the induced decompositions on $\Lambda^2\mathfrak{so}(2,2)$ (center) and $\Lambda^3\mathfrak{so}(2,2)$ (right) given by Theorem 6.4. Non-zero homogeneous spaces are indicated by bold points. A basis for each homogeneous subspace of $\mathfrak{so}(2,2)$ is detailed in the first diagram. Limit homogeneous spaces are represented by points over a dashed line.

Let us prove that a G-graded Lie algebra $\mathfrak{g}$ induces a decomposition in $\Lambda_R\mathfrak{g}$.

Proposition 6.6. *If $\mathfrak{g}$ admits a G-gradation, then*

$$(\Lambda^m\mathfrak{g})^{\mathfrak{g}} = \bigoplus\nolimits_{\alpha\in G}(\Lambda^m\mathfrak{g})^{(\alpha)}\cap(\Lambda^m\mathfrak{g})^{\mathfrak{g}} \quad \textit{and} \quad \Lambda^m_R\mathfrak{g} = \bigoplus\nolimits_{\alpha\in G}(\Lambda^m_R\mathfrak{g})^{(\alpha)},$$

where $(\Lambda^m_R\mathfrak{g})^{(\alpha)} = (\Lambda^m\mathfrak{g})^{(\alpha)}/((\Lambda^m\mathfrak{g})^{\mathfrak{g}}\cap(\Lambda^m\mathfrak{g})^{(\alpha)})$.

Proof. Due to the G-gradation of $\mathfrak{g}$, every $w \in \Lambda^m\mathfrak{g}$ can be written in a unique way as $w = \sum_{\alpha\in G} w^{(\alpha)}$ for some $w^{(\alpha)} \in (\Lambda^m\mathfrak{g})^{(\alpha)}$. Since w is $\mathfrak{g}$-invariant, $0 = \sum_{\alpha\in G}[v^{(\beta)}, w^{(\alpha)}]$ for every $v^{(\beta)} \in \mathfrak{g}^{(\beta)}$. Since G is a group, the elements $\beta + \alpha$, for a fixed β and different values of α, are different and the m-vectors $[v^{(\beta)}, w^{(\alpha)}]$ belong to different homogeneous subspaces of $\Lambda^m\mathfrak{g}$. Hence, they vanish separately and $w^{(\alpha)} \in ((\Lambda^m\mathfrak{g})^{(\alpha)})^{\mathfrak{g}}$. Hence, $(\Lambda^m\mathfrak{g})^{\mathfrak{g}} \subset \bigoplus_{\alpha\in G}(\Lambda^m\mathfrak{g})^{(\alpha)} \cap (\Lambda^m\mathfrak{g})^{\mathfrak{g}}$. The converse and the induced decomposition in the spaces $\Lambda^m_R\mathfrak{g}$, for $m \in \mathbb{Z}$, are immediate. ■

Let us now describe a few hints on how gradations and their induced decompositions on Grassmann algebras allow us to obtain certain solutions to CYBEs.

Definition 6.7. A *limit homogeneous space* of a graded decomposition of $\Lambda^2\mathfrak{g}$ is a homogeneous subspace $(\Lambda^2\mathfrak{g})^{(\alpha)} \subset \Lambda^2\mathfrak{g}$ such that $(\Lambda^3\mathfrak{g})^{(\alpha\star\alpha)} = 0$.

Elements of limit homogeneous spaces are solutions of the CYBE. If $r_1 \in (\Lambda^2\mathfrak{g})^{(\alpha)}$ and $r_2 \in (\Lambda^2\mathfrak{g})^{(\beta)}$ are r-matrices belonging to limit homogeneous spaces, and furthermore $(\Lambda^3\mathfrak{g})^{(\alpha+\beta)} = \{0\}$, then any linear combination of elements r_1, r_2 is an r-matrix.

Consider the diagram of $\Lambda^2\mathfrak{so}(2,2)$ in Figure 1 showing its limit homogeneous subspaces. By direct computation, one can verify that all the elements of these subspaces are solutions of the CYBE. Moreover, Table 1 details many solutions of the CYBEs for three-dimensional Lie algebras that can be obtained by using limit homogeneous spaces.

A short calculation shows that $f_0 \wedge e_0$ is an r-matrix for $\mathfrak{so}(2,2)$. This is a particular case of the following immediate and more general result.

Proposition 6.8. *If $\mathfrak{g}$ admits a root gradation, then $\Lambda^2(\mathfrak{g}^{(0)})$ is a subspace of solutions of the CYBE.*

When $\dim\mathfrak{g} = 5$ or larger, the spaces $\Lambda^2\mathfrak{g}, \Lambda^3\mathfrak{g}$ are so large that it is difficult to determine their $\mathfrak{g}$-invariant elements, homogeneous subspaces, and other of their properties related to mCYBEs. To help in analysing these topics, the use of $\mathrm{Aut}(\mathfrak{g})$ can be useful. If $\mathrm{Aut}(\mathfrak{g})$ maps homogeneous spaces into homogeneous spaces of a gradation of $\mathfrak{g}$, e.g. when $\mathrm{Aut}(\mathfrak{g})$ preserves the Cartan subalgebra of a root decomposition of $\mathfrak{g}$, then $\mathrm{Aut}(\mathfrak{g})$ is useful to obtain decompositions in $\Lambda\mathfrak{g}$ induced by gradations in $\mathfrak{g}$. To illustrate our claims, we will now study $\mathfrak{so}(3,2)$.

Let $\{j_\pm, j_3, k_\pm, k_3, s_\pm, r_\pm\}$ be a basis of $\mathfrak{so}(3,2)$ satisfying the non-vanishing commutation relations [8]

$$
\begin{array}{llll}
[j_\pm, k_\pm] = \pm r_\pm, & [j_\mp, k_\pm] = \pm s_\pm, & [j_\mp, r_\pm] = \pm 2k_\pm, & [j_3, r_\pm] = \pm r_\pm, \\
[j_\pm, s_\pm] = \pm 2k_\pm, & [j_3, s_\pm] = \mp s_\pm, & [k_\mp, r_\pm] = \pm 2j_\pm, & [k_3, r_\pm] = \pm r_\pm, \\
[s_+, s_-] = -4(k_3 - j_3), & [k_\mp, s_\pm] = \pm 2j_\mp, & [k_3, s_\pm] = \pm s_\pm, & [r_+, r_-] = -4(k_3 + j_3), \\
 & [k_-, k_+] = 2k_3, & [j_-, j_+] = -2j_3. &
\end{array}
$$

The diagrams for the homogeneous spaces of $\Lambda\mathfrak{so}(3,2)$ appear in Figure 2. Consider $T_1, T_2 \in \mathrm{Aut}(\mathfrak{so}(3,2))$ that act on the diagram for $\mathfrak{so}(3)$ as reflections on the OY and OX axis, that is $T_1(j_3) = -j_3$, $T_1(k_3) = k_3$, $T_2(j_3) = j_3$, $T_2(k_3) = -k_3$, and they act as a permutation in the rest of elements of the chosen basis of $\mathfrak{so}(3,2)$. Evidently, T_1, T_2 do not preserve the subspaces of the root $\mathbb{Z}^2$-gradation, but they map homogeneous subspaces of $\mathfrak{so}(3,2)$ onto homogeneous subspaces.

Then, they map homogeneous spaces of the induced decomposition in $\Lambda\mathfrak{so}(3,2)$ into homogeneous spaces of the decomposition. For instance, this allows one to obtain a basis adapted to the decomposition of any $\Lambda^m\mathfrak{so}(3,2)$ from the bases of homogeneous spaces with $(i \geq 0, j \geq 0)$, which can be used to simplify mCYBEs, for example, by searching for r-matrices belonging to certain subfamilies of homogeneous subspaces.

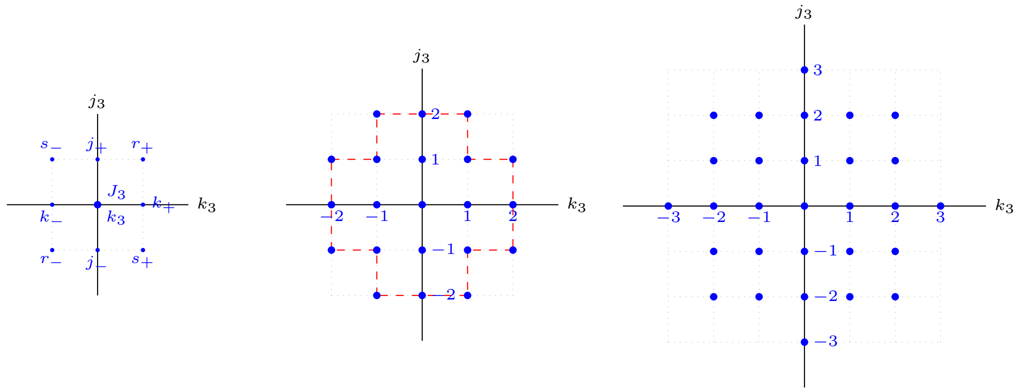


Figure 2: Graded and induced decompositions of $\mathfrak{so}(3,2)$ (left), $\Lambda^2\mathfrak{so}(3,2)$ (center), $\Lambda^3\mathfrak{so}(3,2)$ (right) and limit homogeneous spaces (subspaces over the dashed line).

7. Geometry of $\mathfrak{g}$-invariant elements

This section addresses the study and characterisation of the spaces $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant m-vectors for Lie algebras. The interest in the spaces $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ is due to their occurrence in the analysis of Lie bialgebras, mCYBEs, and $\mathrm{Aut}(\mathfrak{g})$ [9]. The first part of this section studies $(\Lambda\mathfrak{g})^{\mathfrak{g}}$ for general Lie algebras and, then, Lie algebras with a root gradation. The second part is focused on $(\Lambda\mathfrak{g})^{\mathfrak{g}}$ for nilpotent Lie algebras $\mathfrak{g}$. Although our results do not characterise completely $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$, they are general enough to obtain many of its elements and, sometimes, to determine the whole $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$. Let us begin with a simple practical fact.

Proposition 7.1. *The space $(\Lambda\mathfrak{g})^{\mathfrak{g}}$ is an $\mathbb{R}$-algebra relative to the exterior product and each space $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ is $\mathfrak{aut}(\mathfrak{g})$-invariant.*

Proof. It is immediate that $(\Lambda\mathfrak{g})^{\mathfrak{g}}$ is a subalgebra of $\Lambda\mathfrak{g}$. By Proposition 3.3, the second part of the proposition amounts to the fact that $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ is invariant relative to $\Lambda^m T$ for every $T \in \mathrm{Aut}_c(\mathfrak{g})$. But if $w \in (\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ and $T \in \mathrm{Aut}_c(\mathfrak{g})$, then

$$[v, \Lambda^m T w] = [TT^{-1}v, \Lambda^m T w] = \Lambda^m T[T^{-1}v, w] = 0, \ \forall v \in \mathfrak{g} \Rightarrow \Lambda^m T w \in (\Lambda^m\mathfrak{g})^{\mathfrak{g}}.$$

Hence, $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ is invariant relative to the action of $\mathrm{Aut}_c(\mathfrak{g})$ on it. ∎

To prove our following results, it is appropriate to introduce the next notion.

Definition 7.2. A *traceless ideal* of $\mathfrak{g}$ is an ideal $0 \neq \mathfrak{h} \subset \mathfrak{g}$ such that the restriction of each ad_v, with $v \in \mathfrak{g}$, to $\mathfrak{h}$, say $\mathrm{ad}_v|_{\mathfrak{h}}$, is traceless. If the elements of $\mathrm{ad}(\mathfrak{g})$ are traceless, then $\mathfrak{g}$ is called *unimodular*.

The Lie algebras of abelian, compact, semi-simple, or nilpotent groups are unimodular [2, 5, 18]. One of our reasons to study unimodular Lie algebras and their traceless ideals is given by the following proposition.

Proposition 7.3. *Every traceless ideal $\mathfrak{h} \subset \mathfrak{g}$ is such that $\Lambda^{\dim\mathfrak{h}}\mathfrak{h} \subset (\Lambda^{\dim\mathfrak{h}}\mathfrak{g})^{\mathfrak{g}}$.*

Proof. The ideal $\mathfrak{h}$ gives a one-dimensional space $\Lambda^{\dim \mathfrak{h}}\mathfrak{h}$. Let $r \in \Lambda^{\dim \mathfrak{h}}\mathfrak{h}$. As $\mathfrak{h}$ is a traceless ideal, $[v, r] = \mathrm{Tr}(\mathrm{ad}_v|_{\mathfrak{h}})r = 0$ for every $v \in \mathfrak{g}$ and $r \in (\Lambda^{\dim \mathfrak{h}}\mathfrak{g})^{\mathfrak{g}}$. ■

Proposition 7.3 says how to determine traceless ideals in $\mathfrak{g}$ via $(\Lambda\mathfrak{g})^{\mathfrak{g}}$. The following theorem gives a method to determine the decomposable elements of $(\Lambda\mathfrak{g})^{\mathfrak{g}}$.

Theorem 7.4. *The space spanned by a decomposable $\Omega \in (\Lambda\mathfrak{g})^{\mathfrak{g}}$ amounts to a unique traceless $\mathfrak{h} \subset \mathfrak{g}$ such that $\langle\Omega\rangle = \Lambda^{\dim \mathfrak{h}}\mathfrak{h}$.*

Proof. A decomposable $\Omega \in (\Lambda\mathfrak{g})^{\mathfrak{g}}$ takes the form $\Omega = v_1 \wedge \ldots \wedge v_m$ for some linearly independent $v_1, \ldots, v_m \in \mathfrak{g}$. This defines a unique $\mathfrak{h} := \langle v_1, \ldots, v_m\rangle \subset \mathfrak{g}$ that is independent of the chosen $v_1, \ldots, v_m$, and $\langle\Omega\rangle = \Lambda^{\dim \mathfrak{h}}\mathfrak{h}$. All elements of $\langle\Omega\rangle$ give rise to the same $\mathfrak{h}$ also. Let us prove that $\mathfrak{h}$ is a traceless ideal. If $\widehat{\Omega} : \beta \in \mathfrak{g}^* \mapsto \iota_\beta\Omega \in \Lambda\mathfrak{g}$, then $\ker \widehat{\Omega} = \mathfrak{h}^\circ$. Assuming $v \in \mathfrak{g}$, $\theta \in \mathfrak{h}^\circ$, we obtain

$$\begin{aligned}\iota_{\mathrm{ad}_v^*\theta}\Omega &= \mathrm{ad}_v^*\theta(v_1) \wedge \ldots \wedge v_m + \ldots + (-1)^m v_1 \wedge \ldots \wedge \mathrm{ad}_v^*\theta(v_m)\\ &= \iota_\theta([v, v_1] \wedge \ldots \wedge v_m + \ldots + (-1)^m v_1 \wedge \ldots \wedge [v, v_m]) = \iota_\theta[v, \Omega] = 0.\end{aligned}$$

Then, $\mathrm{ad}_v^*\theta \in \mathfrak{h}^\circ$ for every $\theta \in \mathfrak{h}^\circ$. Consequently, $\mathrm{ad}_v\mathfrak{h} \subset \mathfrak{h}$ for every $v \in \mathfrak{g}$ and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Since $[v, \Omega] = (\mathrm{Tr}\,\mathrm{ad}_v|_{\mathfrak{h}})\Omega = 0$, one gets that $\mathrm{ad}_v|_{\mathfrak{h}}$ is traceless.

Conversely, if $\mathfrak{h}$ is a non-zero ideal, $\Lambda^{\dim \mathfrak{h}}\mathfrak{h}$ is one-dimensional and it admits a basis, Ω, given by the exterior product of the elements of a basis of $\mathfrak{h}$. Since every ad_v, with $v \in \mathfrak{g}$, acts on $\mathfrak{h}$ tracelessly by assumption, $[v, \Omega] = (\mathrm{Tr}\ \mathrm{ad}_v|_{\mathfrak{h}})\Omega = 0$ and $\Omega \in (\Lambda^{\dim \mathfrak{h}}\mathfrak{g})^{\mathfrak{g}}$. All elements in $\langle\Omega\rangle$ belong similarly to $(\Lambda^{\dim \mathfrak{h}}\mathfrak{g})^{\mathfrak{g}}$. ■

Lie algebras and root gradations. Let us study the relation between induced decompositions on $\Lambda^m\mathfrak{g}$ by root gradations on $\mathfrak{g}$ and $\mathfrak{g}$-invariant metrics on $\Lambda^m\mathfrak{g}$.

Proposition 7.5. *If $\mathfrak{g}$ admits a root gradation, then $(\Lambda^m\mathfrak{g})^{\mathfrak{g}} \subset (\Lambda^m\mathfrak{g})^{(0)}$.*

Proof. If $w \in (\Lambda^m\mathfrak{g})^{\mathfrak{g}}$, then $w = \sum_{\alpha\in G} w^{(\alpha)}$ for a family of elements $w^{(\alpha)} \in (\Lambda^m\mathfrak{g})^{(\alpha)}$ for every $\alpha \in G$. Proposition 6.6 yields that each $w^{(\alpha)}$ belongs to $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$. For every $e \in \mathfrak{g}^{(0)}$, one has that $[e, w^{(\alpha)}] = \boldsymbol{\alpha}(e)w^{(\alpha)} = 0$ and therefore $\boldsymbol{\alpha} = 0$. Since the mapping Ξ of the root gradation is injective, $\alpha = 0$ and $w \subset (\Lambda^m\mathfrak{g})^{(0)}$. ■

Proposition 7.5 allows us to restrict the search for elements of $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ to $(\Lambda^m\mathfrak{g})^{(0)}$. In turn, $(\Lambda^m\mathfrak{g})^{(0)}$ can also be obtained via the root gradation of $\mathfrak{g}$ as shown in Theorem 6.4 and parts of Section 6.

Let us now analyse $(\Lambda^m\mathfrak{g})^{(\alpha)}$ in relation to $\mathfrak{g}$-invariant maps $b_{\Lambda^m\mathfrak{g}}$. This illustrates the structure of each $\Lambda^m\mathfrak{g}$ and facilitates finding the elements of $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$.

Theorem 7.6. *If b is a $\mathfrak{g}$-invariant bilinear symmetric map on $\mathfrak{g}$, then its extension $b_{\Lambda^m\mathfrak{g}}(v_{(\alpha)}, v_{(\beta)}) = 0$ for every $v_{(\alpha)} \in \Lambda^m\mathfrak{g}^{(\alpha)}$ and $v_{(\beta)} \in \Lambda^m\mathfrak{g}^{(\beta)}$ with $\alpha + \beta \neq 0$.*

Proof. Since $b_{\Lambda\mathfrak{g}}$ is $\mathfrak{g}$-invariant, $b_{\Lambda^m\mathfrak{g}}([h, v_{(\alpha)}]_S, v_{(\beta)}) = -b_{\Lambda^m\mathfrak{g}}(v_{(\alpha)}, [h, v_{(\beta)}]_S)$, for every $h \in \mathfrak{g}^{(0)}$, $v_{(\alpha)} \in \Lambda^p\mathfrak{g}^{(\alpha)}$, and $v_{(\beta)} \in \Lambda^p\mathfrak{g}^{(\beta)}$. Hence, $(\boldsymbol{\alpha}+\boldsymbol{\beta})(h)b_{\Lambda^m\mathfrak{g}}(v_{(\alpha)}, v_{(\beta)}) = 0$. Since $\alpha + \beta \neq 0$ by assumption, the injectivity of Ξ of the root gradation gives $\boldsymbol{\alpha}+\boldsymbol{\beta} \neq 0$. Thus, there is $h \in \mathfrak{g}^{(0)}$ so that $(\boldsymbol{\alpha}+\boldsymbol{\beta})(h) \neq 0$ and $b_{\Lambda^m\mathfrak{g}}(v_{(\alpha)}, v_{(\beta)}) = 0$. ■

Example 7.7. Let us illustrate Theorem 7.6 for $\Lambda^2\mathfrak{so}(2,2)$. Using the basis $\{e_-, e_0, e_+, f_-, f_0, f_+\}$ of $\mathfrak{so}(2,2)$ used in Example 6.5, we obtain that non-zero products between elements of the chosen basis read

$$\kappa_{\mathfrak{so}(2,2)}(e_0, e_0) = \kappa_{\mathfrak{so}(2,2)}(f_0, f_0) = 2, \quad \kappa_{\mathfrak{so}(2,2)}(e_-, e_+) = \kappa_{\mathfrak{so}(2,2)}(f_-, f_+) = 2.$$

This calculation enables us to determine an orthogonal basis of $\Lambda^2\mathfrak{so}(2,2)$ relative to $\kappa_{\Lambda^2\mathfrak{so}(2,2)}$ given by $\{e_-\wedge f_- \pm e_+\wedge f_+,\ e_+\wedge f_- \pm e_-\wedge f_+,\ e_0\wedge f_- \pm e_0\wedge f_+,\ f_-\wedge f_0 \pm f_0\wedge f_+,\ e_-\wedge e_0 \pm e_0\wedge e_+,\ e_-\wedge f_0 \pm e_+\wedge f_0,\ e_0\wedge f_0,\ e_-\wedge e_+,\ f_-\wedge f_+\}$, which satisfies the orthogonality relations determined by Theorem 7.6. ■

Corollary 7.8. *If b is a non-degenerate bilinear symmetric form on a semi-simple $\mathfrak{g}$, the restrictions of $b_{\Lambda^m\mathfrak{g}}$ to $(\Lambda^m\mathfrak{g})^{(0)}$ and $\Lambda^m\mathfrak{g}^{(\alpha)} \oplus \Lambda^m\mathfrak{g}^{(-\alpha)}$ are non-degenerate.*

Proof. We prove both results by reduction to absurdity. If the restriction of $b_{\Lambda^m\mathfrak{g}}$ to $(\Lambda^m\mathfrak{g})^{(0)}$ is degenerate, there exists a $w \in (\Lambda^m\mathfrak{g})^{(0)}$ perpendicular (with respect to $b_{\Lambda^m\mathfrak{g}}$) to every element of $(\Lambda^m\mathfrak{g})^{(0)}$. Theorem 7.6 yields that w is perpendicular to $\Lambda^m\mathfrak{g}$ and $b_{\Lambda^m\mathfrak{g}}$ is degenerate, which goes against our initial assumption in view of Theorem 3.8. Hence, $b_{\Lambda^m\mathfrak{g}}$ is non-degenerate on $(\Lambda^m\mathfrak{g})^{(0)}$.

Similarly, if $w \in \Lambda^m\mathfrak{g}^{(\alpha)} \oplus \Lambda^m\mathfrak{g}^{(-\alpha)}$ is orthogonal to $\Lambda^m\mathfrak{g}^{(\alpha)} \oplus \Lambda^m\mathfrak{g}^{(-\alpha)}$, then it stems from Theorem 7.6 that w is orthogonal to $\Lambda^m\mathfrak{g}$, which contradicts our assumed non-degeneracy of $b_{\Lambda^m\mathfrak{g}}$. ■

Example 7.9. A short calculation shows that the basis of $(\Lambda^3\mathfrak{so}(2,2))^{(0)}$, namely $\{e_-\wedge e_0\wedge e_+, e_-\wedge e_+\wedge f_0, e_0\wedge f_-\wedge f_+, f_-\wedge f_0\wedge f_+\}$, is orthogonal relative to $\kappa_{\Lambda^3\mathfrak{so}(2,2)}$ as claimed in Corollary 7.8. ■

Nilpotent Lie algebras. Every nilpotent Lie algebra $\mathfrak{g}$ possesses a flag of ideals, called the *lower central series* of $\mathfrak{g}$, defined recurrently as $\mathfrak{g}_{s)} := [\mathfrak{g}, \mathfrak{g}_{s-1)}]$ for $s \in \mathbb{N}$ with $\mathfrak{g}_{0)} := \mathfrak{g}$. Then, $\mathfrak{g} \supset \mathfrak{g}_{1)} \supset \ldots \supset \mathfrak{g}_{p-1)} \supset \mathfrak{g}_{p)} = \{0\}$ for a certain p. Let us use this fact to study $(\Lambda\mathfrak{g})^{\mathfrak{g}}$. First, the nilpotency of $\mathfrak{g}$ allows for the characterisation of certain decomposable elements of $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ via Lie subalgebras of $\mathfrak{g}$. This is done in the next proposition, which is an immediate consequence of Theorem 7.4.

Proposition 7.10. *If $\mathfrak{g}$ is nilpotent, then every decomposable element of $(\Lambda\mathfrak{g})^{\mathfrak{g}}$ expands the space $\Lambda^{\dim\mathfrak{h}}\mathfrak{h}$ of a non-zero ideal $\mathfrak{h}$ of $\mathfrak{g}$ and vice versa.*

The proof of the following proposition is also quite immediate.

Proposition 7.11. *Let $\mathfrak{g}_{p)} = 0$ and $\mathfrak{g}_{p-1)} \neq 0$.*
If $\dim\mathfrak{z}(\mathfrak{g}) = 1$, then $\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}_{p-2)} \subset (\Lambda^2\mathfrak{g})^{\mathfrak{g}}$. If $\mathfrak{g}$ is nilpotent, one has:

(1) *If $\dim\mathfrak{z}(\mathfrak{g}) = 2$, then $\Lambda^2\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}_{p-2)} \subset (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$;*

(2) *If $\dim\mathfrak{z}(\mathfrak{g}) = 1$ and $\dim\mathfrak{g}_{p-2)} > 1$, then $\mathfrak{z}(\mathfrak{g}) \wedge \Lambda^2\mathfrak{g}_{p-2)} \subset (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$;*

(3) *If $\dim\mathfrak{z}(\mathfrak{g}) = 1$ and $\dim\mathfrak{g}_{p-2)} = 1$, then $\mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}_{p-2)} \wedge \mathfrak{g}_{p-3)} \subset (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$.*

8. Reduced mCYBEs

If $\dim\mathfrak{g} \geq 4$, mCYBEs are frequently very complicated to solve. This section shows a simplification of the mCYBE concerning, mainly, a not semi-simple $\mathfrak{g}$ obtained by mapping structures of $\Lambda^m\mathfrak{g}$ onto $\Lambda^m_R\mathfrak{g} := \Lambda^m\mathfrak{g}/(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$.

Definition 8.1. The elements of $\Lambda^m_R\mathfrak{g}$, for $m \in \mathbb{Z}$, are called *reduced m-vectors.* If r is an r-matrix, then the class of r in $\Lambda^2_R\mathfrak{g}$ is called a *reduced r-matrix.*

Proposition 8.2. *Let $\pi_p : w_p \in \Lambda^p\mathfrak{g} \mapsto [w_p] \in \Lambda^p_R\mathfrak{g}$, with $p \in \mathbb{Z}$. The algebraic Schouten bracket induces a new bracket, called the reduced Schouten bracket, on $\Lambda_R\mathfrak{g} := \bigoplus_{p\in\mathbb{Z}} \Lambda^p_R\mathfrak{g}$, of the form*

$$[[w_p],[w_q]]_R := [[w_p, w_q]], \qquad \forall w_p \in \Lambda^p\mathfrak{g}, \quad \forall w_q \in \Lambda^q\mathfrak{g}. \tag{10}$$

This new bracket induces a graded Lie superalgebra structure in $\Lambda_R\mathfrak{g}$ and a decomposition on $\Lambda_R\mathfrak{g}$ compatible with $[\cdot,\cdot]_R$ in such a way that $\pi = \bigoplus_{p\in\mathbb{Z}} \pi_p$ satisfies that $[\pi(a),\pi(b)]_R = \pi([a,b])$ for arbitrary $a,b \in \Lambda\mathfrak{g}$. Then, $r \in \Lambda^2\mathfrak{g}$ is an r-matrix if and only if $[\pi(r),\pi(r)]_R = 0$.

Proof. Let us show that (10) is well defined. If $[w_p] = [\bar{w}_p]$ and $[w_q] = [\bar{w}_q]$ for $w_p, \bar{w}_p \in \Lambda^p\mathfrak{g}$ and $w_q, \bar{w}_q \in \Lambda^q\mathfrak{g}$, then $w_p - \bar{w}_p, w_q - \bar{w}_q \in (\Lambda\mathfrak{g})^{\mathfrak{g}}$ and $[(\Lambda\mathfrak{g})^{\mathfrak{g}}, \Lambda\mathfrak{g}] = 0$. Hence, $[[w_p],[w_q]]_R := [[w_p, w_q]] = [[w_p - \bar{w}_p + \bar{w}_p, w_q - \bar{w}_q + \bar{w}_q]] = [[\bar{w}_p],[\bar{w}_q]]_R$.

To prove that $\Lambda_R\mathfrak{g}$ is a graded Lie superalgebra relative to the reduced Schouten bracket, it is enough and immediate to see that (10) satisfies $[\Lambda^p_R\mathfrak{g}, \Lambda^q_R\mathfrak{g}]_R \subset \Lambda^{p+q-1}_R\mathfrak{g}$ and, for all $w_p \in \Lambda^p\mathfrak{g}, w_q \in \Lambda^q\mathfrak{g}, w_s \in \Lambda^s\mathfrak{g}$, we have

(1) $[[w_p],[w_q]]_R = -(-1)^{(p-1)(q-1)}[[w_q],[w_p]]_R$,

(2) $(-1)^{(p-1)(s-1)}[[w_p],[[w_q],[w_s]]_R]_R + (-1)^{(q-1)(p-1)}[[w_q],[[w_s],[w_p]]_R]_R$
$+ (-1)^{(s-1)(q-1)}[[w_s],[[w_p],[w_q]]_R]_R = 0$.

The relation $[\pi(a),\pi(b)]_R = \pi([a,b])$ is immediate from (10). If r is an r-matrix, $[\pi(r),\pi(r)]_R = \pi([r,r]) \in \pi((\Lambda^3\mathfrak{g})^{\mathfrak{g}}) = 0$. The converse is trivial. ∎

Let us hereafter prove that there exists a cohomology on $\Lambda\mathfrak{g}^* \otimes \Lambda^2_R\mathfrak{g}$, which characterises cocommutators on $\mathfrak{g}$. Moreover, $\Lambda^2_R\mathfrak{g}$ is a $\mathfrak{g}$-module as shown next.

Lemma 8.3. *The pair $(\Lambda^m_R\mathfrak{g}, \sigma : v \in \mathfrak{g} \mapsto \sigma_v := [[v],\cdot]_R \in \mathfrak{gl}(\Lambda^m_R\mathfrak{g}))$ is a $\mathfrak{g}$-module and $\Psi : T \in \mathrm{Aut}(\mathfrak{g}) \mapsto [\Lambda^m T] \in GL(\Lambda^m_R\mathfrak{g})$, with $[\Lambda^m T]([w]) := [\Lambda^m T(w)]$ for every $w \in \Lambda^m\mathfrak{g}$, is a Lie group action.*

Proof. Let us show that $(\Lambda^m_R\mathfrak{g}, \sigma)$ is a $\mathfrak{g}$-module. Since the reduced bracket is well defined, σ is well defined. Property 2) in the proof of Proposition 8.2 for $p, q = 1$ and arbitrary s, reduces to

$$[[w_1],[[w_1'],[w_s]]_R]_R + [[w_1'],[[w_s],[w_1]]_R]_R + [[w_s],[[w_1],[w_1']]_R]_R = 0,$$

and σ is a Lie algebra homomorphism as $\sigma_{w_1}\sigma_{w_1'}([w_s]) - \sigma_{w_1'}\sigma_{w_1}([w_s]) = \sigma_{[w_1,w_1']}([w_s])$. Let us prove that $T \in \mathrm{Aut}(\mathfrak{g})$ acts on $\Lambda^m_R\mathfrak{g}$ via Ψ. It is only necessary to verify that $\Psi(T)$ is unambiguous. This amounts to proving that $[\Lambda^m T]([w]) = [\Lambda^m T]([w'])$ for all $w, w' \in [w]$. We have $\Lambda^m T(w) = \Lambda^m T(w - w' + w') = \Lambda^m T(w - w') + \Lambda^m T(w')$. Since $\Lambda^m T(\Lambda^m\mathfrak{g})^{\mathfrak{g}} \subset (\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ and due to Proposition 7.1, $[\Lambda^m T(w)] = [\Lambda^m T(w')]$. Thus, $[\Lambda^m T]([w]) = [\Lambda^m T(w)] = [\Lambda^m T(w')] = [\Lambda^m T]([w'])$. ∎

Let us obtain $\mathfrak{g}$-invariant maps on $\Lambda^m_R\mathfrak{g}$ out of $\mathfrak{g}$-invariant maps on $\Lambda^m\mathfrak{g}$.

Proposition 8.4. *If $b\colon (\Lambda^m\mathfrak{g})^k \to \mathbb{R}$ is a symmetric/anti-symmetric $\mathfrak{g}$-invariant k-linear map and its kernel contains $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$, there is a $\mathfrak{g}$-invariant k-linear map b_R on $\Lambda^m_R\mathfrak{g}$ given by $b_R([w_1],\ldots,[w_k]) := b(w_1,\ldots,w_k)$, for all $w_1,\ldots,w_k \in \Lambda^m\mathfrak{g}$.*

Proof. The map b_R is well defined because if $w_i' \in [w_i]$ for $i = 1, \dots, k$, then $w_i - w_i' \in (\Lambda^m\mathfrak{g})^\mathfrak{g}$. Since the kernel of b contains $(\Lambda^m\mathfrak{g})^\mathfrak{g}$, one has

$$\begin{aligned} b_R([w_1], \dots, [w_k]) &= b(w_1, \dots, w_k) = b(w_1 - w_1' + w_1', \dots, w_k - w_k' + w_k') \\ &= b(w_1', \dots, w_k') = b_R([w_1'], \dots, [w_k']) \end{aligned}$$

and b_R does not depend on the representative of each particular equivalence class of $\Lambda^m_R\mathfrak{g}$. The $\mathfrak{g}$-invariance of b_R stems immediately from the $\mathfrak{g}$-invariance of b. ■

The following result is immediate.

Proposition 8.5. *There is a natural cohomology complex on the spaces* $\Lambda^q\mathfrak{g}^*\otimes\Lambda^2_R\mathfrak{g}$ *making the following diagram commutative*

$$\begin{array}{ccccccccc}
\mathbb{R}\otimes\Lambda^2\mathfrak{g} & \xrightarrow{\mathrm{d}} & \mathfrak{g}^*\otimes\Lambda^2\mathfrak{g} & \xrightarrow{\mathrm{d}} & \Lambda^2\mathfrak{g}^*\otimes\Lambda^2\mathfrak{g} & \xrightarrow{\mathrm{d}} & \Lambda^3\mathfrak{g}^*\otimes\Lambda^2\mathfrak{g} & \xrightarrow{\mathrm{d}} & \dots \\
\downarrow{\scriptstyle \mathrm{id}\otimes\pi_2} & & \downarrow{\scriptstyle \mathrm{id}\otimes\pi_2} & & \downarrow{\scriptstyle \mathrm{id}\otimes\pi_2} & & \downarrow{\scriptstyle \mathrm{id}\otimes\pi_2} & & \\
\mathbb{R}\otimes\Lambda^2_R\mathfrak{g} & \xrightarrow{\mathrm{d}_R} & \mathfrak{g}^*\otimes\Lambda^2_R\mathfrak{g} & \xrightarrow{\mathrm{d}_R} & \Lambda^2\mathfrak{g}^*\otimes\Lambda^2_R\mathfrak{g} & \xrightarrow{\mathrm{d}_R} & \Lambda^3\mathfrak{g}^*\otimes\Lambda^2_R\mathfrak{g} & \xrightarrow{\mathrm{d}_R} & \dots
\end{array}$$

9. On the automorphisms of a Lie algebra

Let us investigate $\mathrm{Aut}(\mathfrak{g})$ and its relations to $\mathrm{Aut}(\mathfrak{g})$-invariant metrics on $\Lambda\mathfrak{g}$. This is to be used to classify solvable Lie bialgebras in Section 10. Let us recall a relevant property of Lie algebra automorphisms [33].

Proposition 9.1. *If* $\mathfrak{Der}(\mathfrak{g})$ *is the Lie algebra of derivations of* $\mathfrak{g}$*,* $\mathfrak{Der}(\mathfrak{g}) \simeq \mathfrak{aut}(\mathfrak{g})$*.*

Derivations of $\mathfrak{g}$ can be derived by determining those $D \in \mathfrak{gl}(\mathfrak{g})$ satisfying that $D([v_1, v_2]) = [D(v_1), v_2] + [v_1, D(v_2)]$, $\forall v_1, v_2 \in \mathfrak{g}$. This can be solved via computer programs even for relatively high-dimensional Lie algebras.

Proposition 9.1 also provides information about the connected part of the neutral element of $\mathrm{Aut}(\mathfrak{g})$. Recall that the determination of the different connected parts of $\mathrm{Aut}(\mathfrak{g})$ can be tricky when $\mathfrak{g}$ is not complex and semi-simple [18, 22].

Let us illustrate our above claim. The Killing form $\kappa_{\mathfrak{sl}_2}$, given by (6) in the basis $\{e_1, e_2, e_3\}$ indicated in Table 1, is indefinite with signature $(2, 1)$. The $\kappa_{\mathfrak{sl}_2}$ induces a quadratic function $f(xe_1 + ye_2 + ze_3) = 2x^2 + 4yz$ on $\mathfrak{sl}_2$. Its level set S_k consists of points (x, y, z), where $2x^2 + 4yz = k$. If $k < 0$, then S_k is a two-sheeted hyperboloid contained in the region of $\mathfrak{sl}_2$ with $z > 0$ or in the region of $\mathfrak{sl}_2$ with $z < 0$. The elements of $\mathrm{Aut}(\mathfrak{sl}_2)$ are isometries of $\kappa_{\mathfrak{sl}_2}$. The elements of $\mathrm{Aut}_c(\mathfrak{sl}_2)$, the component part of the identity in $\mathrm{Aut}(\mathfrak{sl}_2)$, leave invariant each component of each two-sheeted hyperboloid. But $T \in \mathrm{Aut}(\mathfrak{sl}_2)$ such that $T(e_1) = -e_1$, $T(e_2) = -e_3$, and $T(e_3) = -e_2$ does not preserve the sign of z. Consequently, it swaps connected parts of each two-sheeted hyperbolid and $T \notin \mathrm{Aut}_c(\mathfrak{sl}_2)$. Hence, $\mathrm{Aut}(\mathfrak{sl}_2)$ is not connected. Since $\mathrm{Inn}(\mathfrak{g})$ is connected, $\mathrm{Inn}(\mathfrak{sl}_2) \neq \mathrm{Aut}(\mathfrak{sl}_2)$ and the assumption $\mathrm{Inn}(\mathfrak{sl}_2) = \mathrm{Aut}(\mathfrak{sl}_2)$, made in [15], is incorrect.

The $\mathrm{Aut}(\mathfrak{g})$-invariant bilinear symmetric maps are easier to obtain than $\mathrm{Aut}(\mathfrak{g})$, e.g. by Proposition 3.3 they must be $\mathfrak{Der}(\mathfrak{g})$-invariant bilinear symmetric maps. It will be shown in Section 10 that this will frequently be enough to characterise coboundary real three-dimensional Lie bialgebras. To use the above fact in practical applications is convenient to enunciate the following result, which is a straightforward generalisation of Proposition 4.1.

Theorem 9.2. *If b is a k-linear map on $\mathfrak{g}$ invariant under* $\mathrm{Aut}(\mathfrak{g})$*, $b_{\Lambda^m\mathfrak{g}}$ is invariant under* $\mathrm{Aut}(\mathfrak{g})$.

The crux now is that if b is a k-linear symmetric map on $\mathfrak{g}$ invariant relative to $\mathrm{Aut}(\mathfrak{g})$, then the level sets, S_k, where $p(v) := b_{\Lambda^m\mathfrak{g}}(v,\dots,v)$, for all $v \in \Lambda^m\mathfrak{g}$, takes the value k are invariant under the action of $\mathrm{Aut}(\mathfrak{g})$ on $\Lambda^m\mathfrak{g}$. The orbits of $\mathrm{Aut}(\mathfrak{g})$ on $\Lambda^m\mathfrak{g}$ are contained in some S_k, but they do not need to be connected. Since $\mathrm{Inn}(\mathfrak{g})$ is relatively easily to be obtained and it informs about the connected components of $\mathrm{Aut}(\mathfrak{g})$, we can investigate the action of $\mathrm{Aut}(\mathfrak{g})$ by searching elements connecting the different orbits of $\mathrm{Inn}(\mathfrak{g})$ within the same S_k (see Section 10).

Let us now provide hints to characterise automorphisms for Lie algebras. More specifically, let us analyse the properties of Λ^2T for every $T \in \mathrm{Aut}(\mathfrak{g})$. If $\mathfrak{g}$ is a complex simple or semi-simple Lie algebra, $\mathrm{Aut}(\mathfrak{g})$ can be determined by $\mathrm{Inn}(\mathfrak{g})$, which already had a characterisation in this work, and Dynkin diagrams [18, 22]. Meanwhile, automorphisms of general Lie algebras cannot be determined so easily. We here focus on automorphisms of solvable and nilpotent Lie algebras.

The *derived series* of a Lie algebra $\mathfrak{g}$ are the sequence of ideals defined recurrently by $\mathfrak{g}^{p)} := [\mathfrak{g}^{p-1)}, \mathfrak{g}^{p-1)}]$ for all $p \in \mathbb{N}$ and $\mathfrak{g}^{0)} = \mathfrak{g}$ (see [18]). It follows by induction that $T\mathfrak{g}^{p)} = \mathfrak{g}^{p)}$ for every $p \in \mathbb{N}\cup\{0\}$ and $T \in \mathrm{Aut}(\mathfrak{g})$. A similar result applies to lower central series. If $\mathfrak{g}$ is solvable, the elements of $\mathrm{Aut}(\mathfrak{g})$ leave invariant the elementary sequence $\mathfrak{s}_{pq} := \mathfrak{g}_{p)} \wedge \mathfrak{g}_{q)}$, $p \leq q, p, q \in \mathbb{N}\cup\{0\}$. If $\mathfrak{s}_{pq} \neq 0$, then $\mathfrak{s}_{pq} \supset \mathfrak{s}_{lm}$ if and only if $p \leq l$ and $q \leq m$. Similar results apply to $\mathfrak{s}^{pq} := \mathfrak{g}^{p)} \wedge \mathfrak{g}^{q)}, p \leq q$, for $p, q \in \mathbb{N}\cup\{0\}$. Above relations estimate the form of Λ^2T.

10. Study of real three-dimensional coboundary Lie bialgebras

This section uses ours methods to analyse and to classify, up to Lie algebra automorphisms, coboundary real three-dimensional Lie algebras. Gradations allow us to obtain $\mathfrak{g}$-invariant elements of Lie bialgebras and to derive, relatively easily, solutions to CYBEs. We avoid using all automorphisms in the classification of Lie bialgebras (cf. [15]), by focusing on the classification up to inner Lie algebra automorphisms with the help of $\mathfrak{g}$-invariant multilinear maps, which is easier. Next, a few not inner automorphisms lead to the final classification. Our results retrieve geometrically findings in [15, 17], solve a minor gap in [15], and provide a new approach.

General properties. Let us prove a few results on the characterisation of $(\Lambda^m\mathfrak{g})^{\mathfrak{g}}$ and $\mathrm{Aut}(\mathfrak{g})$. From now on, we assume that $\dim\mathfrak{g} = 3$.

Proposition 10.1. *Let $\mathfrak{g}$ be such that $\kappa_{\mathfrak{g}} \neq 0$ and $\mathfrak{g}_{1)} \subset \ker\kappa_{\mathfrak{g}}$ is a two-dimensional abelian Lie subalgebra. If $v \notin \mathfrak{g}_{1)}$, then every $T \in \mathrm{Aut}(\mathfrak{g})$ leaves invariant the set of eigenvectors of* $\mathrm{ad}_v|_{\mathfrak{g}_{1)}}$.

Proof. Let us prove that if $T \in \mathrm{Aut}(\mathfrak{g})$, then $Tv \in v + \mathfrak{g}_{1)}$ or $Tv \in -v + \mathfrak{g}_{1)}$ for every $v \notin \mathfrak{g}_{1)}$. Since $T \in \mathrm{Aut}(\mathfrak{g})$, one has that $\kappa_{\mathfrak{g}}(Tv,Tv) = \kappa_{\mathfrak{g}}(v,v)$. Since v and $\mathfrak{g}_{1)}$ generate $\mathfrak{g}$ and T is injective, $Tv = \lambda v + h$ for an $h \in \mathfrak{g}_{1)}$ and $\lambda \in \mathbb{R}\backslash\{0\}$. As $\mathfrak{g}_{1)} \subset \ker\kappa_{\mathfrak{g}}$, then $\kappa_{\mathfrak{g}}(Tv,Tv) = \lambda^2\kappa_{\mathfrak{g}}(v,v)$ for every $v \in \mathfrak{g}$. Since $\kappa_{\mathfrak{g}} \neq 0$, one has that $\lambda \in \{\pm1\}$. Hence, $Tv \in v + \mathfrak{g}_{1)}$ or $Tv \in -v + \mathfrak{g}_{1)}$. Since $\dim\mathfrak{g}_{1)} = 2$ and $\mathfrak{g}_{1)}$ is an abelian ideal of $\mathfrak{g}$ invariant under automorphisms of $\mathfrak{g}$, one obtains that $\mathrm{ad}_v|_{\mathfrak{g}_{1)}} = \pm\mathrm{ad}_{Tv}|_{\mathfrak{g}_{1)}}$ and then $\mathrm{ad}_v|_{\mathfrak{g}_{1)}} = \pm T|_{\mathfrak{g}_{1)}} \circ \mathrm{ad}_v|_{\mathfrak{g}_{1)}} \circ T^{-1}|_{\mathfrak{g}_{1)}}$. Consequently, if e is an eigenvector of $\mathrm{ad}_v|_{\mathfrak{g}_{1)}}$, then Te is a new eigenvector of $\mathrm{ad}_v|_{\mathfrak{g}_{1)}}$. ■

Proposition 10.1 can be particularised/modified to study $T|_{\mathfrak{g}^{(1)}}$. For instance, if $\mathrm{ad}_v|_{\mathfrak{g}^{1)}}$ has two eigenvectors e_1, e_2 with different eigenvalues λ_1, λ_2 satisfying that $\lambda_1 + \lambda_2 = 0$ and $Tv \in -v + \mathfrak{g}_{1)}$, then $T|_{\mathfrak{g}^{1)}}$ is an anti-diagonal matrix in the basis $\{e_1, e_2\}$. If $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$, then $Tv \in v + \mathfrak{g}_{1)}$ and $T|_{\mathfrak{g}^{1)}}$ is diagonal. Several variations of this reasoning can be applied, e.g. when $\mathrm{ad}_v|_{\mathfrak{g}^{1)}}$ is triangular.

Proposition 10.2. *Let* $\Omega \in (\Lambda^3\mathfrak{g}^*)\backslash\{0\}$ *and assume that* $\Upsilon : \Lambda^2\mathfrak{g} \ni r \mapsto \Omega([r,r]) \in \mathbb{R}$ *is a semi-definite function different from zero. Then, every automorphism of* $\mathfrak{g}$ *has positive determinant.*

Proof. Since $\dim \mathfrak{g} = 3$, the Ω is a basis of $\Lambda^3\mathfrak{g}^*$ and there exists a dual basis $\theta \in \Lambda^3\mathfrak{g}$. As $\Omega([r,r]) = \Upsilon(r)$, then $[r,r] = \Upsilon(r)\theta$. Since Υ is not identically zero, there exists an $r \in \Lambda^2\mathfrak{g}$ such that $[r,r] = \Upsilon(r)\theta \neq 0$. If $T \in \mathrm{Aut}(\mathfrak{g})$, then $\Upsilon(r)\det(T)\,\theta = \det(T)\,[r,r] = \Lambda^3T[r,r]$. Due to the properties of the Schouten bracket, $\Lambda^3T[r,r] = [\Lambda^2Tr, \Lambda^2Tr] = \Upsilon(\Lambda^2Tr)\theta$. Hence, $\Rightarrow \Upsilon(r)\det(T) = \Upsilon(\Lambda^2Tr)$. As Υ is semi-definite and $\Upsilon(v) \neq 0$, then $\det(T) = \Upsilon(\Lambda^2Tr)/\Upsilon(r) > 0$. ■

We hereafter assume that each $\mathfrak{g}$ has a basis $\{e_1, e_2, e_3\}$ satisfying some commutation relations given in Table 1. We also choose the induced bases $\{e_{12}, e_{13}, e_{23}\}$ and $\{e_{123}\}$ in $\Lambda^2\mathfrak{sl}_2$ and $\Lambda^3\mathfrak{sl}_2$, respectively. For each $\mathfrak{g}$, we first analyse $\mathfrak{g}$-invariant elements through gradations, which helps us to determine the form of mCYBEs, solutions to CYBEs, and reduced r-matrices. Recall that if $\mathfrak{g}$ admits a G-gradation, then Proposition 6.6 reduces obtaining $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ to determining the spaces $(\Lambda^2\mathfrak{g})^{(\alpha)} \cap (\Lambda^2\mathfrak{g})^{\mathfrak{g}}$. Meanwhile, Proposition 10.3 simplifies the derivation of $(\Lambda^3\mathfrak{g})^{\mathfrak{g}}$.

Proposition 10.3. *Each* G*-gradation on* $\mathfrak{g}$ *has a unique homogeneous subspace* $(\Lambda^3\mathfrak{g})^{(\alpha)} \neq 0$ *and* $[\mathfrak{g}^{(\beta)}, (\Lambda^3\mathfrak{g})^{(\alpha)}] = 0$ *for* $\beta \neq 0$*. If* $\mathfrak{g}$ *has a root gradation, then* $\Lambda^3\mathfrak{g} = (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$ *if and only if* $\alpha = 0$*.*

Lie algebra $\mathfrak{sl}_2$. Since $\mathfrak{sl}_2$ admits a root decomposition giving rise to a root $\mathbb{Z}$-gradation whose $\Lambda^3\mathfrak{sl}_2$ has zero degree, Proposition 10.3 yields $\Lambda^3\mathfrak{sl}_2 = (\Lambda^3\mathfrak{sl}_2)^{\mathfrak{sl}_2}$ and every element of $\Lambda^2\mathfrak{sl}_2$ satisfies the mCYBE. Since $\mathfrak{sl}_2$ has a root gradation, Proposition 7.5 gives that $(\Lambda^2\mathfrak{sl}_2)^{\mathfrak{sl}_2} \subset (\Lambda^2\mathfrak{sl}_2)^{(0)}$. It is then immediate that $(\Lambda^2\mathfrak{sl}_2)^{\mathfrak{sl}_2} = 0$. Thus, every $r \in \Lambda^2\mathfrak{sl}_2$ induces a different cocommutator $\delta_r(\cdot) := [\cdot, r]$.

A simple calculation and Proposition 2.10 ensure that the orbit of w relative to the action of $\mathrm{Inn}(\mathfrak{sl}_2)$ on $\Lambda^2\mathfrak{sl}_2$, let us say $\mathcal{O}_w$, satisfies $\dim \Theta^2_w = 2$ for $w \in \Lambda^2\mathfrak{sl}_2\backslash\{0\}$ and $\dim \Theta^2_0 = 0$. Since $\mathrm{Inn}(\mathfrak{sl}_2)$ is connected, the $\mathcal{O}_w$ are two- or zero-dimensional connected immersed submanifolds. Each $\mathcal{O}_w$ must be contained in a connected submanifold of a level set, S_k, of $f_{\Lambda^2\mathfrak{sl}_2} : r \in \Lambda^2\mathfrak{sl}_2 \mapsto \kappa_{\Lambda^2\mathfrak{sl}_2}(r,r) \in \mathbb{R}$.

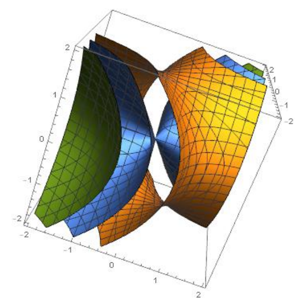


Figure 3: Orbits of $\mathrm{Aut}(\mathfrak{sl}_2)$ acting on $\Lambda^2\mathfrak{sl}_2$.

If $r = xe_{12} + ye_{13} + ze_{23}$, then $f_{\Lambda^2\mathfrak{sl}_2}(r) := 8xy - 4z^2$ and $f_{\Lambda^2\mathfrak{sl}_2}$ admits three types of S_k according to the sign of k. If $k < 0$, then S_k is a one-sheeted hyperboloid; S_0 consists of two cones (without their vertex), one opposite to the other, and the origin of $\Lambda^2\mathfrak{sl}_2$; meanwhile S_k for $k > 0$ is a two-sheeted hyperboloid with two parts contained within the region $x > 0, y > 0$ and $x < 0, y < 0$, respectively (see Figure 3).

Each S_k is the union of different orbits $\mathcal{O}_w$. Then, each S_k, for $k < 0$, is an orbit $\mathcal{O}_w$; each S_k with $k > 0$ consists of two connected orbits; meanwhile, S_0 has three orbits given by two cones for points with $z > 0$ or $z < 0$, and $(0, 0, 0)$.

Consequently, there are six families of inequivalent classes of r-matrices on $\mathfrak{sl}_2$ relative to the action of $\mathrm{Inn}(\mathfrak{sl}_2)$ (cf. [15]). The representatives of each class are $r_0 = 0, r = ae_{23}$, with $a > 0$ (one-sheeted hyperboloids), $r = a(e_{12} + e_{13})$, with $a \neq 0$, (two-sheeted hyperboloids), and $r = \pm e_{12}$ (cones).

Now the orbits of the action of $\mathrm{Aut}(\mathfrak{sl}_2)$ on $\Lambda^2\mathfrak{sl}_2$ can easily be derived. Derivations of $\mathfrak{sl}_2$ are of the form ad_v for a certain $v \in \mathfrak{sl}_2$ [22]. In view of Proposition 9.1, $\mathfrak{inn}(\mathfrak{sl}_2) = \mathfrak{der}(\mathfrak{sl}_2) = \mathfrak{aut}(\mathfrak{sl}_2)$. Hence, $\mathrm{Inn}(\mathfrak{sl}_2) = \mathrm{Aut}_c(\mathfrak{sl}_2)$ and each orbit of the action of $\mathrm{Aut}(\mathfrak{sl}_2)$ on $\Lambda^2\mathfrak{sl}_2$ is the sum of some $\mathcal{O}_w$. As $\kappa_{\mathfrak{sl}_2}$ is invariant under the action of $\mathrm{Aut}(\mathfrak{sl}_2)$, and Theorem 9.2 yields that $\kappa_{\Lambda^2\mathfrak{sl}_2}$ is invariant under the action of $\mathrm{Aut}(\mathfrak{sl}_2)$ on $\Lambda^2\mathfrak{sl}_2$ and each of its orbits must be contained in a S_k.

The $T \in \mathrm{Aut}(\mathfrak{sl}_2)$ such that $T(e_1) := e_1$, $T(e_2) := -e_2$, $T(e_3) := -e_3$ can be extended to $\Lambda^2 T$ giving rise to a map satisfying

$$\Lambda^2 T(e_{12}) = -e_{12}, \ \Lambda^2 T(e_{13}) = -e_{13}, \ \text{ and } \ \Lambda^2 T(e_{23}) = e_{23},$$

which connects the two connected parts of the two-sheeted hyperboloids in S_k for each fixed $k > 0$. It also maps the two cones contained in S_0. This leads to types of non-zero r-matrices up to the action of $\mathrm{Aut}(\mathfrak{sl}_2)$. One is related to a solution of the CYBE that can be derived via gradations (see Table 1).

Our result agrees with [17], but they do not match [15, p. 56]), where authors assumed $\mathrm{Inn}(\mathfrak{sl}_2) = \mathrm{Aut}(\mathfrak{sl}_2)$. The latter was refuted in Section 9.

Lie algebra $\mathfrak{su}_2$. The $\mathbb{Z}_2$-gradations of $\mathfrak{su}_2$ and their associated decompositions for $\Lambda^3\mathfrak{su}_2$ (see Table 1 and Example 6.2) show that, in view of Proposition 10.3, the space $\Lambda^3\mathfrak{su}_2$ is invariant under e_b, e_c. Since this holds for all given $\mathbb{Z}_2$-gradations, namely e_b, e_c are arbitrary non-zero orthogonal vectors, $\Lambda^3\mathfrak{su}_2 = (\Lambda^3\mathfrak{su}_2)^{\mathfrak{su}_2}$ and every r-matrix is a solution to the mCYBE.

By Proposition 6.6, the space $(\Lambda^2\mathfrak{su}_2)^{\mathfrak{su}_2}$ is the linear combination of $\mathfrak{su}_2$-invariant elements within homogeneous spaces of $\Lambda^2\mathfrak{su}_2$. Then, $(\Lambda^2\mathfrak{su}_2)^{\mathfrak{su}_2} = \{0\}$ and every $r \in \Lambda^2\mathfrak{su}_2$ induces a different cocomutator. Then, the classification of coboundary cocomutators of $\mathfrak{su}_2$ up to $\mathrm{Aut}(\mathfrak{su}_2)$ amounts to classifying r-matrices.

Let us study the equivalence of r-matrices under inner automorphisms by using $\mathfrak{su}_2$-invariant metrics on $\mathfrak{su}_2, \Lambda^2\mathfrak{su}_2$, and $\Lambda^3\mathfrak{su}_2$. In the bases given in Table 1, $[\kappa_{\mathfrak{su}_2}] = -2\ \mathbb{I}_{3\times 3}$, $[\kappa_{\Lambda^2\mathfrak{su}_2}] = 4\ \mathbb{I}_{3\times 3}$, $[\kappa_{\Lambda^3\mathfrak{su}_2}] = -8\ \mathbb{I}_{1\times 1}$. Due to Proposition 2.10 and since $\mathrm{Inn}(\mathfrak{su}_2)$ is connected, the orbits of the action of $\mathrm{Inn}(\mathfrak{su}_2)$ on $\Lambda^2\mathfrak{su}_2$ have a dimension given by $\mathrm{Im}\,\Theta^2_w$: two for $w \in \Lambda^2\mathfrak{su}_2 \backslash \{0\}$ and zero otherwise.

The orbits of the action of $\mathrm{Inn}(\mathfrak{su}_2)$ on $\Lambda^2\mathfrak{su}_2$ are connected immersed submanifolds contained in the level sets, S_k, where the quadratic function

$$f_{\Lambda^2\mathfrak{su}_2}(r) := \kappa_{\Lambda^2\mathfrak{su}_2}(r, r) = 4(x^2 + y^2 + z^2)$$

takes the value k. Since the orbits of $\mathrm{Inn}(\mathfrak{su}_2)$ must be open relative to the topology of each S_k (with $k \geq 0$), which are connected, each orbit of $\mathrm{Inn}(\mathfrak{su}_2)$ must be the whole S_k for each $k \geq 0$.

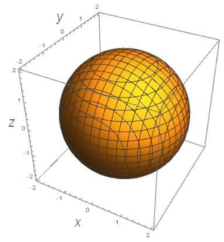


Figure 4: Orbit of $\mathrm{Aut}(\mathfrak{su}_2)$ acting on $\Lambda^2\mathfrak{su}_2$.

Hence, non-equivalent $r \in \Lambda^2\mathfrak{su}_2$, with respect to the action of $\mathrm{Inn}(\mathfrak{su}_2)$, are given by elements r with different modulus, e.g. $r_a = ae_{12}$, with $a \geq 0$. Since the orbits of the action of $\mathrm{Aut}(\mathfrak{su}_2)$ on $\Lambda^2\mathfrak{su}_2$ are given by the sum of orbits of $\mathrm{Inn}(\mathfrak{su}_2)$ and they are contained in the surfaces S_k, the orbits of the action of $\mathrm{Aut}(\mathfrak{su}_2)$ in $\Lambda^2\mathfrak{su}_2$ are indeed the spheres S_k with $k > 0$ and the point $k = 0$ (see Figure 4).

The above retrieves the results given in [15, 17] in a new and geometric fashion.

Lie algebra $\mathfrak{h}$. Let us analyse the three-dimensional (3D) *Heisenberg algebra* $\mathfrak{h}$ [15] depicted in Table 1. This is the only three-dimensional nilpotent Lie algebra [35].

In view of Proposition 10.3 and the fact that non-zero homogeneous spaces in $\Lambda^k\mathfrak{h}$ for $k = 2, 3$ are related to non-zero degrees, $(\Lambda^3\mathfrak{h})^{\mathfrak{h}} = \Lambda^3\mathfrak{h}$ and every element of $\Lambda^2\mathfrak{h}$ is a solution to the mCYBE. Since $\mathfrak{g}$ admits a $\mathbb{Z}$-gradation, Proposition 6.6 yields that $(\Lambda^2\mathfrak{h})^{\mathfrak{h}}$ is the sum of $\mathfrak{h}$-invariant elements on each homogeneous space of $\Lambda^2\mathfrak{h}$, which is easily derivable. This gives that $(\Lambda^2\mathfrak{h})^{\mathfrak{h}} = \langle e_{13}, e_{12}\rangle$. Figure 6 depicts the equivalence classes of $\Lambda^2_R\mathfrak{h}$ in $\Lambda^2\mathfrak{h}$.

Every class of $\Lambda^2_R\mathfrak{h}$ gives rise to a unique Lie bialgebra. Then, to classify coboundary Lie bialgebras, one can restrict oneself to studying reduced r-matrices in $\Lambda^2_R\mathfrak{h}$. Define the maps $T_\alpha \in \mathrm{Aut}(\mathfrak{g})$, with $\alpha \in \mathbb{R}\backslash\{0\}$, given by $T_\alpha(e_1) := \alpha e_1, T(e_2) := e_2, T(e_3) := \alpha e_3$. Therefore, $\Lambda^2 T_\alpha(e_{12}) = \alpha e_{12}$ for any $\alpha \neq 0$. Since $(\Lambda^2\mathfrak{h})^{\mathfrak{h}}$ is invariant under the action of $\mathrm{Aut}(\mathfrak{h})$, there exists an induced action of $\Lambda^2 T$ on $\Lambda^2_R\mathfrak{h}$, which has two orbits given by $[0]$ and $[e_{12}]$.

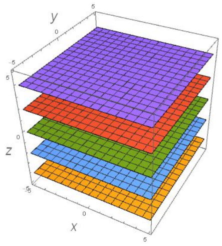


Figure 5: Orbits of $\mathrm{Inn}(\mathfrak{h})$ acting on $\Lambda^2\mathfrak{h}$.

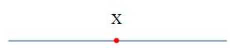


Figure 6: Orbits of $\mathrm{Aut}(\mathfrak{h})$ on $\Lambda^2_R\mathfrak{h}$.

So, there is only one class of non-zero coboundary cocommutators, related to $r := e_{12}$ (see Figure 6). The gradation of $\mathfrak{h}$ and the induced decompositions in $\Lambda^2\mathfrak{h}$ and $\Lambda^3\mathfrak{h}$ give that e_{12} is a solution to the mCYBE on $\mathfrak{h}$.

Lie algebra $\mathfrak{r}'_{3,0}$. Let us analyse $(\Lambda^2\mathfrak{r}'_{3,0})^{\mathfrak{r}'_{3,0}}$ and $(\Lambda^3\mathfrak{r}'_{3,0})^{\mathfrak{r}'_{3,0}}$. In view of Proposition 10.3 and Table 1, the only non-zero homogeneous subspace of $(\Lambda^3\mathfrak{r}'_{3,0})^{\mathfrak{r}'_{3,0}}$ is invariant relative to e_2, e_3, which have non-zero degree. The invariance of $\Lambda^3\mathfrak{r}'_{3,0}$ relative to e_1 is immediate. Then $(\Lambda^3\mathfrak{r}'_{3,0})^{\mathfrak{r}'_{3,0}} = \Lambda^3\mathfrak{r}'_{3,0}$, all the elements of $\Lambda^2\mathfrak{r}'_{3,0}$ are r-matrices, and the study of coboundary commutators reduces to analysing $\Lambda^2_R\mathfrak{r}'_{3,0}$.

Recall that $(\Lambda^2\mathfrak{r}'_{3,0})^{\mathfrak{r}'_{3,0}}$ is the sum of homogeneous $\mathfrak{r}'_{3,0}$-invariant elements in $\Lambda^2\mathfrak{r}'_{3,0}$. It easily follows by using the gradations in $\mathfrak{r}'_{3,0}$ that $\langle e_{23}\rangle \subset (\Lambda^2\mathfrak{r}'_{3,0})^{(2)}$ is $\mathfrak{r}'_{3,0}$-invariant. To obtain the $\mathfrak{r}'_{3,0}$-invariant elements within $\Lambda^2(\mathfrak{r}'_{3,0})^{(1)}$, we consider an arbitrary element $e_1 \wedge \lambda(e_2, e_3) \in \Lambda^2(\mathfrak{r}'_{3,0})^{(1)}$, where $\lambda(e_2, e_3)$ stands for a linear combination of e_2 and e_3. Then we obtain $[e_2, e_1 \wedge \lambda(e_2, e_3)] = -e_3 \wedge \lambda(e_2, e_3) = 0$ and $[e_3, e_1 \wedge \lambda(e_2, e_3)] = e_2 \wedge \lambda(e_2, e_3) = 0$. Hence, $\lambda(e_2, e_3) = 0$ and $(\Lambda^2\mathfrak{r}'_{3,0})^{\mathfrak{r}'_{3,0}} = \langle e_{23}\rangle$.

Let us classify the non-equivalent (up to inner Lie algebra automorphisms of $\mathfrak{r}'_{3,0}$) coboundary cocommutators on $\mathfrak{r}'_{3,0}$ by using $\mathfrak{r}'_{3,0}$-invariant metrics on $\Lambda^2_R\mathfrak{r}'_{3,0}$. Let us discuss the existence of $\mathfrak{r}'_{3,0}$-invariant metrics on $\Lambda^2_R\mathfrak{r}'_{3,0}$. Consider the basis $\{[e_{12}],[e_{13}]\}$ of $\Lambda^2_R\mathfrak{r}'_{3,0}$. If $\Lambda^2_R\mathrm{ad}: v\in\mathfrak{r}'_{3,0}\mapsto[[v],\cdot]_R\in\mathfrak{gl}(\Lambda^2_R\mathfrak{r}'_{3,0})$, where $[\cdot,\cdot]_R$ is the bracket on $\Lambda^2_R\mathfrak{r}'_{3,0}$ induced by the algebraic bracket on $\Lambda^2\mathfrak{r}'_{3,0}$, then

$$\mathrm{Im}\,\Lambda^2_R\mathrm{ad}_{e_1}=\langle[e_{13}],[e_{12}]\rangle,\quad \ker\,\Lambda^2_R\mathrm{ad}_{e_1}=\langle[0]\rangle,\quad \mathrm{Im}\,\Lambda^2_R\mathrm{ad}_{e_2}=\langle[0]\rangle,$$
$$\ker\,\Lambda^2_R\mathrm{ad}_{e_2}=\langle[e_{13}]\rangle,\quad \mathrm{Im}\,\Lambda^2_R\mathrm{ad}_{e_3}=\langle[e_{23}]\rangle=\langle[0]\rangle,\quad \ker\,\Lambda^2_R\mathrm{ad}_{e_3}=\langle[e_{12}]\rangle,$$

$$b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([e_{12}],[e_{12}])=b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([[e_1],[e_{13}]]_R,[e_{12}])=-b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([e_{13}],[[e_1],[e_{12}]]_R)=b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([e_{13}],[e_{13}]),$$
$$b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([e_{13}],[e_{12}])=-b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([[e_1],[e_{12}]]_R,[e_{12}])=b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([e_{12}],[[e_1],[e_{12}]]_R)=-b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([e_{12}],[e_{13}]).$$

Therefore, Propositions 5.1–5.2 yield that $[b^R_{\Lambda^2\mathfrak{r}'_{3,0}}]=a_1\mathrm{Id}$ for $a_1\in\mathbb{R}$, which is an $\mathfrak{r}'_{3,0}$-invariant metric. For simplicity, we hereafter assume that $a_1=1$. The equivalence classes in $\Lambda^2_R\mathfrak{r}'_{3,0}$ can be written as $x[e_{12}]+y[e_{13}]$ in the basis $\{[e_{12}],[e_{13}]\}$.

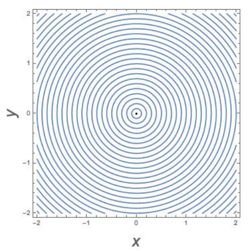


Figure 7: Some orbits of the action of $\mathrm{Inn}(\mathfrak{r}'_{3,0})$ on $\Lambda^2_R\cdot\mathfrak{r}'_{3,0}$

Let us study the equivalence of reduced r-matrices up to $\mathrm{Inn}(\mathfrak{r}'_{3,0})$. It follows that $b^R_{\Lambda^2\mathfrak{r}'_{3,0}}([r],[r])=x^2+y^2$. The image of Θ^2_r has dimensional one for $x^2+y^2\neq0$ and zero otherwise. Thus, the orbits in $\Lambda^2_R\mathfrak{r}'_{3,0}$ relative to the action of $\mathrm{Inn}(\mathfrak{r}'_{3,0})$ are circles and the centre. Therefore, there exists a nontrivial family of r-matrices $r=\mu e_{12}$, $\mu\in\mathbb{R}^+$, giving rise to different non-zero cocommutators, which are not equivalent up to elements of $\mathrm{Inn}(\mathfrak{r}'_{3,0})$.

Let us classify coboundary Lie bialgebras on $\mathfrak{r}'_{3,0}$. Consider the $T_\alpha\in\mathrm{Aut}(\mathfrak{r}'_{3,0})$, with $\alpha\in\mathbb{R}\backslash\{0\}$, satisfying $T_\alpha(e_1):=e_1, T_\alpha(e_2):=\alpha e_2, T_\alpha(e_3):=\alpha e_3$. Then, each $\Lambda^2T_\alpha\in GL(\Lambda^2\mathfrak{r}'_{3,0})$ satisfies that $\Lambda^2T_\alpha(e_{12})=\alpha e_{12}$ and gives rise to an automorphism on $\Lambda^2_R\mathfrak{r}'_{3,0}$ of the form $\Lambda^2_RT_\alpha=\alpha\mathrm{Id}_{\Lambda^2_R\mathfrak{r}'_{3,0}}$. The functions $\Lambda^2_RT_\alpha$ map the circles $x^2+y^2=k$, with $k>0$ among themselves. Hence, their union forms the only orbit of $\mathrm{Aut}(\mathfrak{r}'_{3,0})$ on $\Lambda^2_R\mathfrak{r}'_{3,0}$ related to a non-zero coboundary cocommutator. There is then only one class of non-zero coboundary cocommutators, up to the action of $\mathrm{Aut}(\mathfrak{r}'_{3,0})$, represented by $r=e_{12}$ (see Figure 8). This matches the results in [15]. The gradation of $\mathfrak{r}'_{3,0}$ easily shows that e_{12} is a solution to the mCYBE.

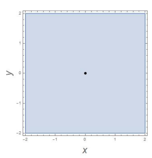


Figure 8: Orbits of $\mathrm{Aut}(\mathfrak{r}'_{3,0})$ acting on $\Lambda^2_R\mathfrak{r}'_{3,0}$.

Lie algebra $\mathfrak{r}_{3,-1}$. Since $\mathfrak{r}_{3,-1}$ admits a root gradation (see Table 1) and $\Lambda^3\mathfrak{r}_{3,-1}$ has degree 0, Proposition 10.3 shows that $\Lambda^3\mathfrak{r}_{3,-1}=(\Lambda^3\mathfrak{r}_{3,-1})^{\mathfrak{r}_{3,-1}}$ and every $r\in\Lambda^2\mathfrak{r}_{3,-1}$ is an r-matrix, while Proposition 7.5 yields that $(\Lambda^2\mathfrak{r}_{3,-1})^{\mathfrak{r}_{3,-1}}\subset(\Lambda^2\mathfrak{r}_{3,-1})^{(0)}$. It is then immediate that $(\Lambda^2\mathfrak{r}_{3,-1})^{\mathfrak{r}_{3,-1}}=\langle e_{23}\rangle$ and $\Lambda^2_R\mathfrak{r}_{3,-1}=\langle[e_{13}],[e_{12}]\rangle$.

Let us classify cocommutators on $\mathfrak{r}_{3,-1}$ via $\mathfrak{r}_{3,-1}$-invariant metrics, $b^R_{\Lambda^2\mathfrak{r}_{3,-1}}$, on $\Lambda^2_R\mathfrak{r}_{3,-1}$. Define $\Lambda^2_R\mathrm{ad}: v\in\mathfrak{r}_{3,-1}\mapsto[[v],\cdot]_R\in\mathfrak{gl}(\Lambda^2_R\mathfrak{r}_{3,-1})$. In the basis $\{[e_{12}],[e_{13}]\}$ of $\Lambda^2_R\mathfrak{r}_{3,-1}$, one gets the expressions $b^R_{\Lambda^2\mathfrak{r}_{3,-1}}([[e_1],[e_{12}]]_R,[e_{12}])=b^R_{\Lambda^2\mathfrak{r}_{3,-1}}([e_{12}],[e_{12}])$, and $b^R_{\Lambda^2\mathfrak{r}_{3,-1}}([[e_1],[e_{13}]]_R,[e_{13}])=-b^R_{\Lambda^2\mathfrak{r}_{3,-1}}([e_{13}],[e_{13}])$.

A short calculation shows that the $\mathfrak{r}_{3,-1}$-invariant metrics on $\Lambda^2_R\mathfrak{r}_{3,-1}$ are given by

$$[b^R_{\Lambda^2\mathfrak{r}_{3,-1}}]=\begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}, \quad \beta\in\mathbb{R}.$$

Let $\{x,y\}$ be the dual basis to the basis $\{[e_{12}],[e_{13}]\}$ of $\Lambda^2_R\mathfrak{r}_{3,-1}$. Then, $r_R = x[e_{12}]+y[e_{13}]$ and the quadratic function related to $b^R_{\Lambda^2\mathfrak{r}_{3,-1}}$ reads $f^R_{\Lambda^2\mathfrak{r}_{3,-1}}(r_R) = 2xy$. The image of Θ^2_r has dimensional one for $x^2+y^2\neq 0$ and zero otherwise.

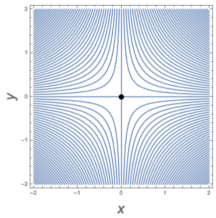


Figure 9: Some orbits of $\mathrm{Inn}(\mathfrak{r}_{3,-1})$ acting on $\Lambda^2_R\mathfrak{r}_{3,-1}$.

The representatives of inequivalent reduced r-matrices (see Figure 9), up to the action of $\mathrm{Inn}(\mathfrak{r}_{3,-1})$, are parametrised by $a>0$ in the form $r^{(\pm,\pm)} = a(\pm[e_{12}]\pm[e_{13}])$, $r_2^{(\pm)} = \pm[e_{12}]$, $r_3^{(\pm)} = \pm[e_{13}]$, $r_0 = [e_{23}]$. The Lie algebra $\mathfrak{r}_{3,-1}$ satisfies the conditions given in Proposition 10.1. Hence, automorphisms of $\mathfrak{r}_{3,-1}$ must match one of the following automorphisms

$$T_{\alpha,\beta}(e_1) := e_1+v, \qquad T_{\alpha,\beta}(e_2) := \alpha e_2, \qquad T_{\alpha,\beta}(e_3) := \beta e_3, \qquad \forall\alpha,\beta\in\mathbb{R}\backslash\{0\},$$
$$T'_{\alpha,\beta}(e_1) := -e_1+v, \qquad T'_{\alpha,\beta}(e_2) := \alpha e_3, \qquad T'_{\alpha,\beta}(e_3) := \beta e_2, \qquad \forall\alpha,\beta\in\mathbb{R}\backslash\{0\},$$

for certain $v\in\langle e_2,e_3\rangle$. Then,

$$\Lambda^2_R T_{\alpha,\beta}([e_{12}]) = \alpha[e_{12}], \qquad \Lambda^2_R T_{\alpha,\beta}([e_{13}]) = \beta[e_{13}], \qquad \forall\alpha,\beta\in\mathbb{R}\backslash\{0\},$$
$$\Lambda^2_R T'_{\alpha,\beta}([e_{12}]) = -\alpha[e_{13}], \qquad \Lambda^2_R T'_{\alpha,\beta}([e_{13}]) = -\beta[e_{12}], \qquad \forall\alpha,\beta\in\mathbb{R}\backslash\{0\}.$$

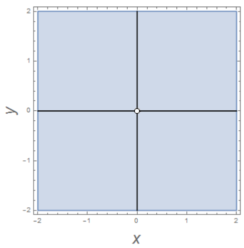


Figure 10: Orbits of $\mathrm{Aut}(\mathfrak{r}_{3,-1})$ acting on $\Lambda^2_R\mathfrak{r}_{3,-1}$.

Since the maps $\Lambda^2_R T_{\alpha,\beta}, \Lambda^2_R T'_{\alpha,\beta}$, do not preserve the sets $S_k = [f^R_{\Lambda^2\mathfrak{r}_{3,-1}}]^{-1}(k)$, one has that, in general, $T_{\alpha,\beta}, T'_{\alpha,\beta}\notin \mathrm{Inn}(\mathfrak{g})$ and non-zero coboundary cocommutators on $\mathfrak{r}_{3,-1}$ are related to only two non-equivalent reduced r-matrices: $r=[e_{12}]$ or $r'=[e_{12}-e_{13}]$. Recall also that $r_0 = e_{23}$ gives rise to a zero cocommutator. Figure 10 depicts the orbits of the action of $\mathrm{Aut}(\mathfrak{r}_{3,-1})$ on $\Lambda^2_R\mathfrak{r}_{3,-1}$.

Lie algebra $\mathfrak{r}_{3,1}$. Since $\mathfrak{r}_{3,1}$ admits a root decomposition and the unique homogeneous space in $\Lambda^3\mathfrak{r}_{3,1}$ is not related to the zero element of the group (see Table 1), Proposition 10.3 shows that $(\Lambda^3\mathfrak{r}_{3,1})^{\mathfrak{r}_{3,1}} = \{0\}$. Moreover, the root decomposition of $\mathfrak{r}_{3,1}$ tells us that $(\Lambda^2\mathfrak{r}_{3,1})^{\mathfrak{r}_{3,1}}\subset\Lambda^2(\mathfrak{r}_{3,1})^{(0)}$. It is then immediate that $(\Lambda^2\mathfrak{r}_{3,1})^{\mathfrak{r}_{3,1}} = 0$. Therefore, the determination of r-matrices demands solving the corresponding mCYBE and every r-matrix gives rise to a different cocommutator.

In the coordinates $\{x,y,z\}$ corresponding to the basis $\{e_{12},e_{13},e_{23}\}$ of $\Lambda^2\mathfrak{r}_{3,1}$, one has $r = xe_{12}+ye_{13}+ze_{23}$ and $[r,r]=0$ for every $r\in\Lambda^2\mathfrak{r}_{3,1}$. Hence, every element of $\Lambda^2\mathfrak{r}_{3,1}$ is an r-matrix.

The fundamental vector fields of the action of $\mathrm{Inn}(\mathfrak{r}_{3,1})$ on $\Lambda^2\mathfrak{r}_{3,1}$ are spanned by

$$X_1 := x\partial_x + y\partial_y + 2z\partial_z, \qquad X_2 := -y\partial_z, \qquad X_3 := x\partial_z.$$

They generate an integrable two-dimensional distribution off the line $x = y = 0$ with integrals given by semi-planes of the form given in Figure 11.

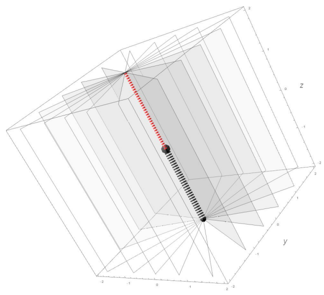


Figure 11: Representative orbits of $\mathrm{Inn}(\mathfrak{r}_{3,1})$ acting on $\Lambda^2\mathfrak{r}_{3,1}$.

The line $x = y = 0$ can also be divided into three orbits of the action of $\mathrm{Inn}(\mathfrak{r}_{3,1})$ consisting of the points with the same sign of z.

Let us study now the equivalence of r-matrices up to the action of $\mathrm{Aut}(\mathfrak{r}_{3,1})$. Elements of $\mathrm{Aut}(\mathfrak{r}_{3,1})$ leave the first derived ideal $[\mathfrak{r}_{3,1}, \mathfrak{r}_{3,1}] = \langle e_2, e_3\rangle$ invariant. Then, the induced action of $\mathrm{Aut}(\mathfrak{r}_{3,1})$ on $\Lambda^2\mathfrak{r}_{3,1}$ must leave the subspace $\langle e_{23}\rangle$ invariant and every point within it must be contained in an orbit within $\langle e_{23}\rangle$.

Obviously, the $r = 0$ is an orbit of the action of $\mathrm{Aut}(\mathfrak{r}_{3,1})$ on $\Lambda^2\mathfrak{r}_{3,1}$. Moreover, the Lie algebra automorphisms $T_{\alpha,\beta,\gamma,\delta}$ given by

$$T_{\alpha,\beta,\gamma,\delta}(e_1) := e_1, \quad T_{\alpha,\beta,\gamma,\delta}(e_2) := \alpha e_2 + \beta e_3, \quad T_{\alpha,\beta,\gamma,\delta}(e_3) = \gamma e_2 + \delta e_3, \quad \alpha\delta - \beta\gamma \neq 0,$$

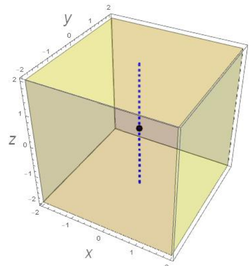


Figure 12: Representative orbits of the action of $\mathrm{Aut}(\mathfrak{r}_{3,1})$ on $\Lambda^2\mathfrak{r}_{3,1}$.

are such that the $\Lambda^2 T_{\alpha,\beta,\gamma,\delta}$ connect different semi-planes in $\Lambda^2\mathfrak{r}_{3,1}$ and, in particular, they connect the parts $z > 0$ and $z < 0$ of the line $x = y = 0$. Hence, there exist two non-zero non-equivalent coboundaries induced by the r-matrices $r_1 = e_{13}$ and $r_2 = e_{23}$. Note that r_2 can be shown to be a solution of the CYBE through the gradations in $\mathfrak{r}_{3,1}$ and its induced decompositions in $\Lambda\mathfrak{r}_{3,1}$.

Lie algebra $\mathfrak{r}_3$. In view of Table 1 and Proposition 10.3, the unique non-zero homogeneous space of $\mathfrak{r}_3$ in $\Lambda^3\mathfrak{r}_3$ is invariant under e_1, e_2. Nevertheless, $[e_3, (\Lambda^3\mathfrak{r}_3)^{(2)}] \neq 0$ and the determination of r-matrices requires solving the mCYBE. As the space of r-matrices, YB, is invariant under the action of $\mathrm{Aut}(\mathfrak{r}_3)$, classifying inequivalent coboundary Lie bialgebras reduces to studying r-matrices in YB.

Let us determine $(\Lambda^2\mathfrak{r}_3)^{\mathfrak{r}_3}$ to know whether different r-matrices induce different coboundary cocommutators. Since $\mathfrak{r}_3$ admits a $\mathbb{Z}$-gradation, $(\Lambda^2\mathfrak{r}_3)^{\mathfrak{r}_3}$ is the sum of the $\mathfrak{r}_3$-invariant elements on each homogeneous subspace of $\Lambda^2\mathfrak{r}_3$. Using the gradation of $\mathfrak{r}_3$, one sees that $[v^{(\alpha)}, w^{(\beta)}] = 0$, for $v^{(\alpha)} \subset \mathfrak{r}_3^{(\alpha)}, w^{(\beta)} \in (\Lambda^2\mathfrak{r}_3)^{(\beta)}$ when $\alpha + \beta \neq 2$. Inspecting remaining commutators, one obtains $(\Lambda^2\mathfrak{r}_3)^{\mathfrak{r}_3} = \{0\}$ and every r-matrix induces a different coboundary cocommutator.

Let $\{x, y, z\}$ be the coordinates on $\Lambda^2\mathfrak{r}_3$ induced by the basis $\{e_{12}, e_{13}, e_{23}\}$. The mCYBE, where $r = xe_{12} + ye_{13} + ze_{23}$, reads $[r, r] = -2z^2 e_{123}$. Hence, $YB = \langle e_{12}, e_{13}\rangle$ is the space of solutions to the mCYBE, presented in Figure 13.

A long but simple calculation shows that $\Lambda^2\mathfrak{r}_3$ does not admit a non-zero $\mathfrak{r}_3$-invariant metrics. Nevertheless, one can still classify r-matrices up to the action of $\mathrm{Inn}(\mathfrak{r}_3)$ on $\Lambda^2\mathfrak{r}_3$, whose fundamental vector fields are spanned by

$$X_1 := z\partial_x, \quad X_2 := (-y + z)\partial_x, \quad X_3 := 2x\partial_x + (y + z)\partial_y + z\partial_z.$$

Since $\mathrm{Inn}(\mathfrak{r}_3)$ maps solutions of mCYBE onto new solutions, the above vector fields are tangent to YB and they take on YB the form $X_1|_{YB} = 0$, $X_2|_{YB} = -y\partial_x$, $X_3|_{YB} = 2x\partial_x + y\partial_y$, which span the tangent space to YB when $y \neq 0$; they span $\langle \partial/\partial x \rangle$ for $y = 0$ and $x \neq 0$; and they span a distribution of rank zero for $z = y = x = 0$. In consequence, there exist five orbits of $\mathrm{Inn}(\mathfrak{r}_3)$ depicted in Figure 13. Let us classify coboundary cocommutators up to the action of elements of $\mathrm{Aut}(\mathfrak{r}_3)$ on YB. Since $\mathfrak{r}_3$ obeys the assumptions of Proposition 10.1 and $[r, r]$ satisfies the condition in Proposition 10.2, one has that all automorphisms must take the form $T_{\alpha,\beta}(e_3) = e_3 + v$, $T_{\alpha,\beta}(e_1) = \alpha e_1$, $T_{\alpha,\beta}(e_2) = \alpha e_2 + \beta e_1$ for all $\alpha \in \mathbb{R}\backslash\{0\}, \beta \in \mathbb{R}, v \in \langle e_1, e_2 \rangle$. Thus, for all $\alpha \in \mathbb{R}\backslash\{0\}, \beta \in \mathbb{R}, v \in \langle e_2, e_3 \rangle$,

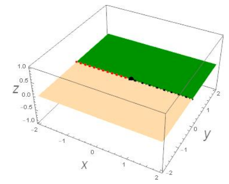


Figure 13: Orbits of the action of $\mathrm{Inn}(\mathfrak{r}_3)$ on $YB \subset \Lambda^2\mathfrak{r}_3$.

$$\Lambda^2 T_{\alpha,\beta}(e_{12}) = \alpha^2 e_{12}, \quad \Lambda^2 T_{\alpha,\beta}(e_{13}) = \alpha e_{13}, \quad \Lambda^2 T_{\alpha,\beta}(e_{23}) = \alpha e_{23} + \beta e_{13} + T_{\alpha,\beta}(e_2) \wedge v.$$

It was proven in Section 9 that $[\mathfrak{r}_3, \mathfrak{r}_3] = \langle e_1, e_2 \rangle$ is invariant under the action of $\mathrm{Aut}(\mathfrak{r}_3)$. Thus, $\langle e_{12} \rangle$ is invariant under the action of $\mathrm{Aut}(\mathfrak{r}_3)$ on $\Lambda^2\mathfrak{r}_3$. It follows from $\Lambda^2 T_{\alpha\beta}$ that $\langle e_{12} \rangle$ has three orbits: the $0 \in \Lambda^2\mathfrak{r}_3$ and the orbits of $\pm e_{12}$. Since $\langle e_{12} \rangle \subset$YB, it is clear that $\pm e_{12}$ are r-matrices giving rise to non-zero cocommutators. Since there exist automorphisms on $\mathfrak{g}$ inverting the coordinate y and leaving x invariant, there exists only one equivalence class of non-zero solutions in YB without $y = z = 0$ given by $r_1 = e_{13}$.

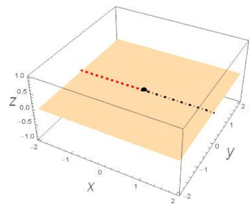


Figure 14: Orbits of $\mathrm{Aut}(\mathfrak{r}_3)$ acting on $YB \subset \Lambda^2\mathfrak{r}_3$.

Hence, we have the equivalence classes related to the r-matrices $r_0 = 0$, $r_\pm = \pm e_{12}$, $r = e_{13}$, as depicted in Figure 14. Note that $\pm e_{12}$ can be seen to be solutions to the CYBE via the gradation of $\mathfrak{r}_3$ and the induced decompositions in $\Lambda\mathfrak{r}_3$.

Lie algebra $\mathfrak{r}_{3,\lambda}$, $\lambda \in (-1, 1)$. In view of Proposition 10.3 and the fact that $\mathfrak{r}_{3,\lambda}$ admits a root decomposition and the unique non-zero homogeneous space in $\Lambda^3\mathfrak{r}_{3,\lambda}$ has degree three, one has that $(\Lambda^3\mathfrak{r}_3)^{\mathfrak{r}_{3,\lambda}} = \{0\}$. If we write $r = xe_{12} + ye_{13} + ze_{23}$, then the mCYBE reads $[r, r] = 2(\lambda - 1)yz\, e_{123}$. Hence, the space of r-matrices, YB, consists of the sum of the plane of points with $y = 0$ and the plane of points with $z = 0$. Since $\mathfrak{r}_{3,\lambda}$ admits a root decomposition, $(\Lambda^2\mathfrak{r}_{3,\lambda})^{\mathfrak{r}_{3,\lambda}} \subset (\Lambda^2\mathfrak{r}_{3,\lambda})^{(0)} = \{0\}$.

Let us classify coboundary Lie bialgebras up to inner automorphisms of $\mathfrak{r}_{3,\lambda}$. Since $(\Lambda^2\mathfrak{r}_{3,\lambda})^{\mathfrak{r}_{3,\lambda}} = 0$, we have to obtain classes of solutions of the mCYBE (equivalent up to inner automorphisms of $\mathfrak{r}_{3,\lambda}$). Let us study the solutions to the mCYBE in three subsets: a) $y = 0$ with $z \neq 0$, denoted by YB_1; b) $z = 0$ with $y \neq 0$, denoted by YB_2, and c) the line $y = z = 0$ denoted by YB_3.

The desired classification follows from analysing the fundamental vector fields of the action of $\mathrm{Inn}(\mathfrak{r}_{3,\lambda})$ on $\Lambda^2\mathfrak{r}_{3,\lambda}$. These are spanned by

$$Z_1 := z\partial_x, \quad Z_2 := -\lambda y\partial_x, \quad Z_3 := (1 + \lambda)x\partial_x + y\partial_y + \lambda z\partial_z.$$

Assume $\lambda \neq 0$. Let us consider YB_3. The distribution $\mathcal{D}$ spanned by Z_1, Z_2, Z_3 on YB_3 and $x \neq 0$ has rank one. Meanwhile, $\mathcal{D}$ has rank zero at $(x, y, z) = 0$. Hence, YB_3 is divided into three orbits of $\mathrm{Inn}(\mathfrak{g})$ for points $(x, 0, 0)$ with $x > 0$, $x < 0$, and $x=0$. Then, $Z_1|_{YB_1}=z\partial_x, Z_2|_{YB_1}=0, Z_3|_{YB_1}=(1+\lambda)x\partial_x+\lambda z\partial_z$ span the tangent space to YB_1.

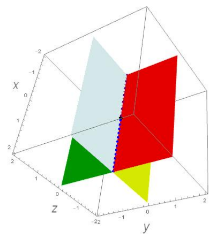


Figure 15: Orbits of the action $\mathrm{Inn}(\mathfrak{r}_{3,\lambda})$ on $YB\subset\Lambda^2\mathfrak{r}_{3,\lambda}$.

Hence, this gives rise to two orbits in YB_1 for its points with $z<0$ and $z>0$, respectively. Meanwhile, $Z_1|_{YB_2}=0$, $Z_2|_{YB_2}=-\lambda y\partial_x$, $Z_3|_{YB_2}=(1+\lambda)x\partial_x+y\partial_y$ and span the tangent space to YB_2. Then, we have two orbits of points in YB_2 with $y>0$ and $y<0$, respectively (see Figure 15).

Let us now classify coboundary cocommutators up to the action of $\mathrm{Aut}(\mathfrak{r}_{3,\lambda})$. Since $[\mathfrak{r}_{3,\lambda},\mathfrak{r}_{3,\lambda}]=\langle e_1,e_2\rangle$ is invariant under $\mathrm{Aut}(\mathfrak{r}_{3,\lambda})$, the space $\langle e_{12}\rangle$ is also invariant relative to the action of $\mathrm{Aut}(\mathfrak{r}_{3,\lambda})$. Moreover, the automorphisms of the form $T_{\alpha,\beta}(e_1)=\beta e_1, T_{\alpha,\beta}(e_2)=\alpha e_2, T_{\alpha,\beta}(e_3)=e_3$, for all $\alpha\in\mathbb{R}\backslash\{0\}$, are such that the induced $\Lambda^2T_{\alpha,\beta}$ show that $\langle e_{12}\rangle$ has only two equivalence classes: 0 and e_{12}. This finishes the study of solutions with $y=z=0$.

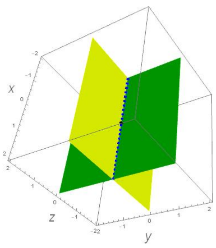


Figure 16: Orbits of the action of $\mathrm{Aut}(\mathfrak{r}_{3,\lambda})$ on $YB\subset\Lambda^2\mathfrak{r}_{3,\lambda}$.

The $\Lambda^2T_{\alpha,\beta}$ change the sign of y and z. This maps the two semiplane orbits relative of $\mathrm{Inn}(\mathfrak{r}_{3,\lambda})$ for the r-matrices in YB_1 and YB_2. Therefore, we get three classes of inequivalent non-zero coboundary Lie bialgebras (up to the action of $\mathrm{Aut}(\mathfrak{r}_{3,\lambda})$) induced by the r-matrices: $r_0=e_{12}$, $r_y=e_{23}$, and $r_z=e_{13}$. This is depicted in Figure 16.

Let us now tackle the case $\lambda=0$, namely $\mathfrak{r}_{3,0}$. The analysis of solutions to the mCYBEs goes as in the previous case. The fundamental vector fields of the action of $\mathrm{Inn}(\mathfrak{r}_{3,0})$ read $Z_1:=z\partial_x, Z_2:=0, Z_3:=x\partial_x+y\partial_y$. On YB_3, the distribution spanned by Z_1,Z_2,Z_3 has rank one for $x\neq 0$ and zero for $x=0$. Therefore, we obtain three orbits gathering those points with $z=y=0$ and equal sign of x.

Restricting to YB_1, we get $Z_1|_{YB_1}=z\partial_x, Z_2|_{YB_1}=0, Z_3|_{YB_1}=x\partial_x$, which span $\langle\partial_x\rangle$. Thus, the orbits of the action of $\mathrm{Inn}(\mathfrak{r}_{3,0})$ on this space are lines $(x,0,z_0)$ with a constant $z_0\neq 0$. Restricting to YB_2, i.e. $z=0$ and $y\neq 0$, we get a unique non-zero restriction of Z_1,Z_2,Z_3 given by $Z_3|_{YB_1}=x\partial_x+y\partial_y$. Thus, the orbits of the action of $\mathrm{Inn}(\mathfrak{r}_{3,0})$ on this space are lines $(\mu x,\mu y,0)$ with $\mu>0$ and $y\neq 0$.

The maps $T_{\alpha,\beta,\gamma}\in\mathrm{Aut}(\mathfrak{r}_{3,0})$ such that $T_{\alpha,\beta,\gamma}(e_1):=\alpha e_1+\gamma e_2$, $T_{\alpha,\beta,\gamma}(e_2):=\beta e_2$, $T_{\alpha,\beta,\gamma}(e_3):=e_3$, with $\alpha,\beta\in\mathbb{R}\backslash\{0\}$ and $\gamma\in\mathbb{R}$, are such that the $\Lambda^2T_{\alpha,\beta,\gamma}$ identify the lines $(x,0,z_0)$ and $(\mu x_0,\mu y_0,0)$ with different $z_0\neq 0$ and $x_0,y_0\neq 0$ among themselves, respectively. Then, we get two r-matrices $r_y:=e_{13}$ and $r_z:=e_{23}$.

If $z=y=0$, the $\Lambda^2T_{\alpha,\beta,\gamma}$ map points with positive and negative values of x.

We get three classes of nonequivalent non-zero coboundary Lie bialgebras up to Aut $(\mathfrak{r}_{3,0})$ induced by the r-matrices given by the non-zero r-matrices $r_0=e_{12}$, $r_y=e_{23}$ and $r_z=e_{13}$, as shown in Figure 16. All of them are trivial solutions of the CYBE in view of the gradation in $\mathfrak{r}_{3,0}$ and the induced decompositions in $\Lambda\mathfrak{r}_{3,0}$.

	$[e_1,e_2]$	$[e_1,e_3]$	$[e_3,e_2]$	$\mathcal{S}_k$	G	$\mathfrak{g}$	$\Lambda^2\mathfrak{g}$	$\Lambda^3\mathfrak{g}$	Root
$\mathfrak{sl}_2$	e_2	$-e_3,$	$-e_1$	$2xy-z^2=k,\ k\in\mathbb{R}_+$ $2xy-z^2=0, z\neq 0,\ k=0$ $2xy-z^2=k,\ k\in\mathbb{R}_-$	$\mathbb{Z}$	$e_3 \quad e_1 \quad e_2$ $(-1)\ (0)\ (1)$	$\mathbf{e_{13}} \quad e_{23} \quad \mathbf{e_{12}}$ $(-1)\ (0)\ (1)$	e_{123} / (0)	Yes
$\mathfrak{su}_2$	e_3	$-e_2$	$-e_1$	$x^2+y^2+z^2=k,\ k\in\mathbb{R}_+$	$\mathbb{Z}_2$	$e_a \quad e_b, e_c$ $(0)\ (1)$	$e_{bc} \quad e_{ba}, e_{ac}$ $(0)\ (1)$	e_{abc} / (0)	No
$\mathfrak{h}$	e_3	0	0	$z\neq 0,\ k=1$	$\mathbb{Z}$	$e_1 \quad e_2 \quad e_3$ $(1)\ (2)\ (3)$	$e_{12} \quad \mathbf{e_{13}} \quad \mathbf{e_{23}}$ $(3)\ (4)\ (5)$	e_{123} / (6)	No
$\mathfrak{r}'_{3,0}$	$-e_3$	e_2	0	$x^2+y^2>0,\ k=1$	$\mathbb{Z}$	$e_1 \quad e_2, e_3$ $(0)\ (1)$	$e_{12}, e_{13} \quad \mathbf{e_{23}}$ $(1)\ (2)$	e_{123} / (2)	No
$\mathfrak{r}_{3,-1}$	e_2	$-e_3$	0	$xy=0,\ x^2+y^2\neq 0,\ k=1,$ $xy\neq 0\neq 0,\ k=2$	$\mathbb{Z}$	$e_3 \quad e_1 \quad e_2$ $(-1)\ (0)\ (1)$	$\mathbf{e_{13}} \quad e_{23} \quad \mathbf{e_{12}}$ $(-1)\ (0)\ (1)$	e_{123} / (0)	Yes
$\mathfrak{r}_{3,1}$	e_2	e_3	0	$x^2+y^2\neq 0, z\in\mathbb{R},\ k=1,$ $x=y=0, z\neq 0,\ k=2$	$\mathbb{Z}$	$e_1 \quad e_2, e_3$ $(0)\ (1)$	$e_{12}, e_{13} \quad \mathbf{e_{23}}$ $(1)\ (2)$	e_{123} / (2)	Yes
					$\mathbb{Z}^2$	$(0,1)\ e_3$ $e_1 \quad e_2$ $(0,0)\ (1,0)$	$\mathbf{e_{13}} \quad \mathbf{e_{23}}\ (1,1)$ $(0,1)\ (1,0)\ \mathbf{e_{12}}$	e_{123} / $(1,1)$	No
$\mathfrak{r}_3$	0	$-e_1$	e_1+e_2	$x>0, y=0, z=0\ k=1,$ $x<0, y=0, z=0,\ k=2,$ $y\neq 0, z=0,\ k=3$	$\mathbb{Z}$	$e_3 \quad e_1, e_2$ $(0)\ (1)$	$e_{13}, e_{23} \quad \mathbf{e_{12}}$ $(1)\ (2)$	e_{123} / (2)	No
$\mathfrak{r}_{3,\lambda\in(-1,1)}$	0	$-e_1$	λe_2	$y=0, z\neq 0, x\in\mathbb{R},\ k=1,$ $z=0, y\neq 0, x\in\mathbb{R},\ k=2,$ $y=0, z=0, x\in\mathbb{R},\ k=3$	$\mathbb{R}$	$e_3 \quad e_1 \quad e_2$ $(0)\ (1)\ (\lambda)$	$\mathbf{e_{13}} \quad \mathbf{e_{23}} \quad \mathbf{e_{12}}$ $(1)\ (\lambda)\ (1+\lambda)$	e_{123} / $(1+\lambda)$	Yes
$\mathfrak{r}'_{3,\lambda>0}$	0	$e_2-\lambda e_1$	λe_2+e_1	$x>0, y=0, z=0,\ k=1,$ $x<0, y=0, z=0,\ k=2$	$\mathbb{Z}$	$e_3 \quad e_1, e_2$ $(0)\ (1)$	$e_{13}, e_{23} \quad \mathbf{e_{12}}$ $(1)\ (2)$	e_{123} / (2)	No

Table 1: For an explanation of the Table see the following page.

Explanation of Table 1: Commutation relations, non-zero orbits of equivalent r-matrices, and G-gradations of $\mathfrak{g}$ with $\dim\mathfrak{g}=3$ along with decompositions on their Grassmann algebras. The letters a,b,c stand for arbitrary different values within $\{1,2,3\}$. As throughout the work, $e_{i_1\ldots i_r}:=e_{i_1}\wedge\ldots\wedge e_{i_r}$, with $i_1,\ldots,i_r\in\overline{1,r}$, and $\{x,y,z\}$ is the dual basis to $\{e_{12},e_{13},e_{23}\}$. Solutions of CYBEs obtained via gradations are written in bold. The spaces $\mathcal{S}_k$ are the equivalence classes of reduced r-matrices.

Lie algebra $\mathfrak{r}'_{3,\lambda}$, $\lambda>0$. It stems from Table 1 that the unique non-zero homogeneous space in $\Lambda^3\mathfrak{r}'_{3,\lambda}$ is not invariant relative to the action of e_3 and hence $(\Lambda^3\mathfrak{r}'_{3,\lambda})^{\mathfrak{r}'_{3,\lambda}}=0$.

The corresponding mCYBE read $[r,r]=-2(y^2+z^2)e_{123}$. Its space of solutions read $YG:=\{(x,y,z)\in\Lambda^2\mathfrak{r}'_{3,\lambda}:y=z=0\}$. The space $(\Lambda^2\mathfrak{r}'_{3,\lambda})^{\mathfrak{r}'_{3,\lambda}}$ can be easily determined as it is spanned by $\mathfrak{r}'_{3,\lambda}$-invariant elements within each homogeneous subspace in $\Lambda^2\mathfrak{r}'_{3,\lambda}$. By using Table 1 and, some easy calculations, one obtains, since $\lambda\neq 0$, that $(\Lambda^2\mathfrak{r}'_{3,\lambda})^{\mathfrak{r}'_{3,\lambda}}=0$.

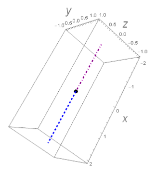


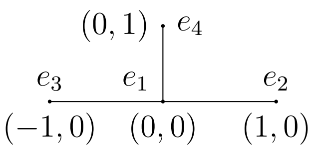
Figure 17: Orbits of the action of $\mathrm{Aut}(\mathfrak{r}'_{3,\lambda})$ on $YB\subset\Lambda^2\mathfrak{r}'_{3,\lambda}$.

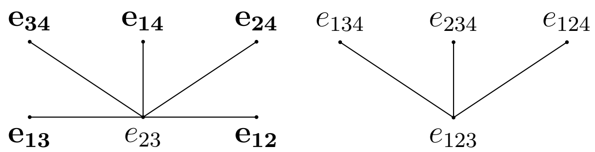
The invariance of $\kappa_{\mathfrak{r}'_{3,\lambda}}$ under $\mathrm{Aut}(\mathfrak{r}'_{3,\lambda})$ and the Lie bracket of $\mathfrak{r}'_{3,\lambda}$ show that Proposition 10.1 applies and each $T\in\mathrm{Aut}(\mathfrak{r}'_{3,\lambda})$ reads $T(e_1)=\alpha e_1, T(e_2)=\alpha e_2, T(e_3)=e_3$, for an $\alpha\in\mathbb{R}\backslash\{0\}$. Then, $\Lambda^2T(e_{12})=\alpha^2e_{12}$ and we obtain three nonequivalent coboundary cocommutators invariant under $\mathrm{Aut}(\mathfrak{r}'_{3,\lambda})$ given by the r-matrices $\pm e_{12}$ and 0. These are solutions to the CYBE in view of the gradation of $\mathfrak{r}'_{3,\lambda}$ and the induced decompositions in $\Lambda\mathfrak{r}'_{3,\lambda}$ (see Figure 17).

11. A four-dimensional case

Let us apply our formalism to classifying coboundary Lie algebras on $\mathfrak{gl}_2$. This illustrates that our methods can be applied to four-dimensional Lie bialgebras, which have been little studied [1]. We also investigate the geometric structures involved in determining r-matrices, e.g. the orbits of the adjoint action of GL_2 on $\Lambda^2\mathfrak{gl}_2$.

Let $\{e_1,\ldots,e_4\}$ be a basis of $\mathfrak{gl}_2=\mathfrak{sl}_2\oplus\mathbb{R}$, where e_1,e_2,e_3 satisfy the conditions for $\mathfrak{sl}_2$ in Table 1 and $\langle e_4\rangle$ is the centre of $\mathfrak{gl}_2$. Let us use the decompositions





It easily follows that $(\Lambda^2\mathfrak{gl}_2)^{\mathfrak{gl}_2}=\{0\}$ and $(\Lambda^3\mathfrak{gl}_2)^{\mathfrak{gl}_2}=\{e_{123}\}$. Consider the basis $\{e_{12},e_{13},e_{14},e_{23},e_{24},e_{34}\}$ of $\Lambda^2\mathfrak{gl}_2$, its dual one $\{\lambda_{12},\lambda_{13},\lambda_{14},\lambda_{23},\lambda_{24},\lambda_{34}\}$, and

$$x=\frac{\lambda_{24}+\lambda_{34}}{\sqrt{2}},\ y=\lambda_{14},\ z=\frac{\lambda_{34}-\lambda_{24}}{\sqrt{2}},\ \dot{x}=\frac{\lambda_{12}+\lambda_{13}}{\sqrt{2}},\ \dot{y}=\lambda_{23},\ \dot{z}=\frac{\lambda_{12}-\lambda_{13}}{\sqrt{2}},\quad (11)$$

which can be understood as a global adapted coordinate system to $T\mathbb{R}^3$. Hence $\Lambda^2\mathfrak{gl}_2$ and $T\mathbb{R}^3$ are diffeomorphic. The invariant metrics on $\Lambda^2\mathfrak{gl}_2$ are linear combinations of $f_1=z^2-x^2-y^2, f_2=\dot{z}^2-\dot{x}^2-\dot{y}^2, f_3=x\dot{x}+y\dot{y}+z\dot{z}$. Moreover,

$$X_1 = \dot{x}\partial_{\dot{z}} + \dot{z}\partial_{\dot{x}} - x\partial_z - z\partial_x,\ X_3 = -\dot{y}\partial_{\dot{x}} - (\dot{x}+\dot{z})\partial_{\dot{y}} + \dot{y}\partial_{\dot{z}} + y\partial_x - (x-z)\partial_y + y\partial_z,$$

$$X_2 = -\dot{y}\partial_{\dot{x}} - (\dot{x}-\dot{z})\partial_{\dot{y}} - \dot{y}\partial_{\dot{z}} - y\partial_x + (x+z)\partial_y + y\partial_z,$$

span the fundamental vector fields of the action of $\mathrm{Inn}(\mathfrak{gl}_2)$ on $\Lambda^2\mathfrak{gl}_2$. Writing $p = (x, y, z, \dot{x}, \dot{y}, \dot{z})$, an easy but tedious calculation gives that $\mathcal{D}_p = \langle (X_1)_p, (X_2)_p, (X_3)_p \rangle$ has dimension three if and only if $\dim\langle(-x, -y, z), (\dot{x}, \dot{y}, \dot{z})\rangle = 2$. Another long but easy computation shows that this amounts to $df_1 \wedge df_2 \wedge df_3 \neq 0$. Moreover, $\dim \mathcal{D}_0 = 0$ and $\dim \mathcal{D}_p = 2$ elsewhere. The connected components of the orbits of the action of $\mathrm{Inn}(\mathfrak{gl}_2)$ on $\Lambda^2\mathfrak{gl}_2$ are, analogously to the previous section, the maximal connected submanifolds of each strata $S_{(k_1,k_2,k_3)} = f_1^{-1}(k_1) \cap f_2^{-1}(k_2) \cap f_3^{-1}(k_3)$.

The isomorphism $(x, y, z) \in \mathbb{R}^3 \mapsto (x, y, z) \in T_\xi\mathbb{R}^3$, for any $\xi \in \mathbb{R}^3$ endows $T_\xi\mathbb{R}^3$ with an inner product induced by the one in $\mathbb{R}^3$. Consider f_1 as a function on $\mathbb{R}^3$. Each $S_{(k_1,k_2,k_3)}$ is given by points $(\xi, \dot{\xi}) \in T\mathbb{R}^3$, where $\xi = (x, y, z), \dot{\xi} = (\dot{x}, \dot{y}, \dot{z})$, such that $\xi \in f_1^{-1}(k_1)$, while $\dot{\xi}$ belongs to the intersection of $f_2^{-1}(k_2) \cap T_\xi\mathbb{R}^3$ and the plane $\pi_{\xi,k_3} = \{\dot{\xi} \in T_\xi\mathbb{R}^3 : \xi \cdot \dot{\xi} = k_3\}$, whose normal vector is $(x, y, z) \in T_\xi\mathbb{R}^3$. Proposition 11.1 studies the orbits of $\mathrm{Aut}(SL_2)$ acting on $T\mathbb{R}^3$. This will be relevant.

Proposition 11.1. *Let $\mathcal{O}_k := f_1^{-1}(k) \subset \mathbb{R}^3$ for any $k \in \mathbb{R}$. The distribution $\mathcal{D}$ has rank two at*[2]

$$\mathcal{A}_2 = T_0\mathbb{R}^3\backslash\{0_0\} \bigcup \{(\xi, \dot{\xi}) \in T\mathbb{R}^3 : \xi \in \mathbb{R}^3\backslash\{0\}, \dot{\xi} \in (T_\xi\mathcal{O}_{f_1(\xi)})^\perp\},$$

and three at $\mathcal{A}_3 = \{(\xi, \dot{\xi}) \in T\mathbb{R}^3 : \xi \in \mathbb{R}^3\backslash\{0\}, \dot{\xi} \notin (T_\xi\mathcal{O}_{f_1(\xi)})^\perp\}$.

Then, $\mathcal{A}_2/\mathrm{Aut}(SL_2)$ is the sum of the level sets of f_2 in $T_0\mathbb{R}^3\backslash\{0_0\}$, the

$$A_{k_1,k_3} = \{(\xi, \dot{\xi}) \in T\mathbb{R}^3 : \xi \in \mathcal{O}_{k_1}\backslash\{0\}, \dot{\xi} \in (T_\xi\mathcal{O}_{k_1})^\perp, \xi \cdot \dot{\xi} = k_3\} \text{ for } k_1 \neq 0,$$

and $A_{0,\lambda} = \mathrm{Aut}(SL_2)((1, 0, 1), \lambda\dot{\xi})$ for $\dot{\xi} \in (T_{(1,0,1)}\mathcal{O}_0)^\perp\backslash\{0_{(1,0,1)}\}$ and $\lambda \in \mathbb{R}$. The equivalence classes in $\mathcal{A}_3/\mathrm{Aut}(SL_2)$ can be parametrised by (k_1, k_2, k_3) for $k_1, k_2, k_3 \in \mathbb{R}$ and an additional parameter $\kappa \in \{\pm 1\}$ for some values of k_1, k_2, k_3 with $k_1 < 0$.

Proof. If $\xi = (x, y, z) \in \mathbb{R}^3\backslash\{0\}$, then $T_\xi\mathcal{O}_{f_1(\xi)} = \{\xi' \in T_\xi\mathbb{R}^3 : \xi' \cdot (-x, -y, z) = 0\}$. Hence, $\dim \mathcal{D}_{(\xi,\dot{\xi})} < 3$ if and only if $\dot{\xi} \in (T_\xi\mathcal{O}_{f_1(\xi)})^\perp$. If $\xi = 0$, $\dim \mathcal{D}_{(\xi,\dot{\xi})} < 3$ for each $(0, \dot{\xi}) \in T_0\mathbb{R}^3$. Since $\dim \mathcal{D}_{(0,0)} = 0$, the rank of $\mathcal{D}$ is two exactly on $\mathcal{A}_2$.

Let us work out the orbits of $\mathrm{Aut}(SL_2)$ acting on $\Lambda^2\mathfrak{gl}_2$. Since $\mathrm{Aut}(SL_2)$ respects the decomposition $\mathfrak{gl}_2 \simeq \langle e_1, e_2, e_3\rangle \oplus \langle e_4\rangle$ due to (11), then $g(\xi, \dot{\xi}) = (g\xi, g\dot{\xi})$, where $g \in \mathrm{Aut}(SL_2)$ acts linearly on each component. Hence, $gT_0\mathbb{R}^3 = T_0\mathbb{R}^3$ and the orbits of $\mathrm{Aut}(SL_2)$ in $T_0\mathbb{R}^3$ are 0 and $\{f_2^{-1}(k)\backslash\{0_0\} \cap T_0\mathbb{R}^3\}_{k\in\mathbb{R}}$ (note that $f_3(0, \dot{\xi}) = 0$). Meanwhile, if $(\xi, \dot{\xi}) \in \mathcal{A}_2$ and $\xi \in \mathbb{R}^3\backslash\{0\}$, then $\mathrm{Aut}(SL_2)\xi = \mathcal{O}_{f_1(\xi)}\backslash\{0\}$. For every $g\xi$, we have $g\dot{\xi} \in (T_{g\xi}\mathcal{O}_{f_1(g\xi)})^\perp$ and $k_3 = f_3(\xi, \dot{\xi}) = f_3(g\xi, g\dot{\xi})$. If $k_1 \neq 0$, the plane $\pi_{g\xi,k_3}$ cuts $(T_{g\xi}\mathcal{O}_{f_1(g\xi)})^\perp$ at one point determining $g\dot{\xi}$. The case $k_1 = 0$ is immediate.

The action of $\mathrm{Aut}(SL_2)$ on $\Lambda^2\mathfrak{gl}_2$ leaves $0_0 \in T_0\mathbb{R}^3$ invariant. Let us analyse the orbits of $\mathrm{Aut}(SL_2)$ on $\mathcal{A}_3$. Take a point $(\xi, \dot{\xi}) \in \mathcal{A}_3$. The action of $\mathrm{Aut}(SL_2)$ leaves invariant S_{k_1,k_2,k_3}. Hence, the orbit of $v_\xi = (\xi, \dot{\xi})$ is included in $S_{(k_1,k_2,k_3)}$ for $k_i = f_i(v_\xi)$. Hence, $\xi \in \mathcal{O}_{k_1}$, $f_2(\dot{\xi}) = k_2$, and $f_3(\xi, \dot{\xi}) = k_3$. Then, the $g\xi$ generate $\mathcal{O}_{k_1}\backslash\{0\}$. Let G_ξ be the isotropy group of ξ. At the points of $G_\xi\dot{\xi}$, the surface $S_{\xi,k_2} = f_2^{-1}(k_2) \cap T_\xi\mathbb{R}^3$ cannot be tangent to π_{ξ,k_3} as this amounts to $(\xi, \dot{\xi}) \in \mathcal{A}_2$

[2] An $((x, y, z), (a, b, c)) \in T\mathbb{R}^3$ is also written as $(a, b, c)_{(x,y,z)}$ and $(0, 0, 0)$ is denoted by 0

also. Then, $G_\xi\dot{\xi}$ is a one-dimensional submanifold of $T_\xi\mathbb{R}^3$ within the intersection of S_{ξ,k_2} with π_{ξ,k_3}. We have some possibilities of non-empty intersection.

First, if $k_1 = f_1(\xi) > 0$, every $v_\xi \in \mathcal{A}_3$ is such that the plane π_{ξ,k_3} intersects S_{ξ,k_2} in a circle or ellipse I_{ξ,k_2,k_3}, which is connected. Hence, the orbit of $(\xi,\dot{\xi})$ is $\bigcup_{\xi\in\mathcal{O}_{k_1}}\{\xi\}\times I_{\xi,k_2,k_3}$. Such orbits are parametrised by k_1,k_2,k_3. Second, if $k_1 = f_1(\xi) < 0$, each $v_\xi \in \mathcal{A}_3$ is such that the plane π_{ξ,k_3} intersects S_{ξ,k_2} in two hyperbolas, which have two non-connected parts, or two crossing lines, whose intersection is a point of $\mathcal{A}_2$. Hence, $G_\xi\dot{\xi}$ connects among themselves the parts of the crossing lines invariant under inversion $z \leftrightarrow -z$ (it follows from considering a particular case $\xi = (0,1,0)$, $\dot{\xi} = (0,1,0)$) or the two hyperbolas. For cases with hyperbolas, k_1,k_2,k_3 parametrise orbits. Otherwise, we add parameter $\kappa \in \{-1,1\}$. If $0 = k_1 = f_1(\xi)$ and $v_\xi \in \mathcal{A}_3$, the plane π_{ξ,k_3} intersects S_{ξ,k_2} giving rise to a parabola, which is connected. All cases are parametrised by k_2,k_3. ■

As $\mathfrak{gl}_2 \simeq \mathfrak{sl}_2 \oplus \mathbb{R}$, then $\mathrm{Aut}(\mathfrak{gl}_2) = \mathrm{Aut}(\mathfrak{sl}_2) \otimes \mathrm{Aut}(\mathbb{R})$ and the linear maps $T_\alpha(e_4) = \alpha e_4$, $T_\alpha(e_i) = e_i$, $i = 1,2,3$, $\alpha \neq 0$, belong to $\mathrm{Aut}(\mathfrak{gl}_2)$. The $\{\Lambda^2 T_\alpha\}_{\alpha\neq 0}$ connect the $\{S_{(\alpha^2 k_1,k_2,\alpha k_3)}\}_{\alpha\neq 0}$. Then, $(\xi,\dot{\xi})$ can be mapped by $g \in \mathrm{Aut}(\mathfrak{gl}_2)$ to $g(\xi,\dot{\xi}) = (g\xi, g\dot{\xi})$, where $g\xi$, with $\xi \neq 0$, is $(1,0,0),(1,0,1),(0,0,1)$ - if $f_1(\xi) < 0, f_1(\xi) = 0, f_1(\xi) > 0$, correspondingly – or 0 for $\xi = 0$. We can fix $f_3(g(\xi,\dot{\xi})) \geq 0$.

Let us solve the mCYBEs on $\mathfrak{gl}_2$, which read

$$x\dot{y} + y\dot{x} = 0, \qquad z\dot{y} + y\dot{z} = 0, \qquad 2y\dot{y} - x\dot{x} - z\dot{z} = 0. \tag{12}$$

Case (a). $(y,\dot{y}) = 0$ with ξ or $\dot{\xi}$ being zero. All are solutions.

Case (b). $(y,\dot{y}) = 0$, $\xi = (x,y,z) \neq 0$, and $\dot{\xi} = (\dot{x},\dot{y},\dot{z}) \neq 0$. Then (12) amounts to $\xi,\dot{\xi}$ being perpendicular.

Case (c). $(y,\dot{y}) \neq 0$. The first two equations in (12) yield that $(x,\dot{x})$ and $(z,\dot{z})$ are proportional to $(y,-\dot{y})$. Thus, $(x,\dot{x}) = \mu_1(y,-\dot{y}),(z,\dot{z}) = \mu_2(y,-\dot{y})$ for some μ_1,μ_2. The later and the last equation in (12) yield $y\dot{y} = 0$. If $y = 0$, solutions to (12) read $(0,\dot{\xi})$ for $\dot{\xi}$ with $\dot{y} \neq 0$. If $\dot{y} = 0$, solutions to (12) read $(\xi,0)$ for any $\xi \in \mathbb{R}^3\backslash\{0\}$ with $y \neq 0$. Thus, solutions to (12) read 0_ξ,ξ_0 for all $\xi \in \mathbb{R}^3\backslash\{0\}$ (Types I and II, respectively), and $(\xi,\dot{\xi})$, with perpendicular $\xi,\dot{\xi}$ and $y = \dot{y} = 0$ (Type III). 0_0 is Type 0 solution.

As mentioned, the action of $\mathrm{Aut}(\mathfrak{gl}_2)$ connects each point in $\Lambda^2\mathfrak{gl}_2$ to one $(\xi,\dot{\xi})$ where ξ is $(1,0,0),(1,0,1),(0,0,1)$ or $(0,0,0)$ and $k_3 \geq 0$. Let us restrict us to them. Type I solutions read $0_{(1,0,0)},0_{(1,0,1)},0_{(0,0,1)}$. Type II solutions are $(\lambda,0,0)_0$, $(0,0,\lambda)_0$, for $\lambda > 0$, and $(1,0,1)_0$. Since $gT_0\mathbb{R}^3 = T_0\mathbb{R}^3$ for every $g \in \mathrm{Aut}(\mathfrak{gl}_2)$ and the $\{T_\alpha\}_{\alpha\neq 0}$ do not affect $\dot{\xi}$ (see (11)), the latest Type II solutions belong each one to one possible different orbit crossing $T_0\mathbb{R}^3$. Hence, there are all the possible inequivalent ones. All Type III solutions are $(\pm\lambda,0,\mp\lambda)_{(1,0,1)}$, $(0,0,\lambda)_{(1,0,0)},(\lambda,0,0)_{(0,0,1)}$, for $\lambda > 0$. They are not equivalent since belong to different strata (the last two cases) or different orbits in $\mathcal{A}_2$ (see Proposition 11.1). Of course, 0_0 is a solution. Gradations give solutions $e_{12} \in S_{(-1,0,0)}$, $e_{34} \in S_{(0,-1,0)}$, $e_{24}, e_{14} \in S_{(0,0,0)}$.

12. Conclusions and outlook

We extended Lie algebra theory methods, like root decompositions and $\mathfrak{g}$-invariant maps, to Grassmann algebras of general Lie algebras. This and G-gradations suggested simpler ways to derive of coboundary Lie bialgebras, their $\mathfrak{g}$-invariant multivectors, mCYBEs, their classification, and their structure. We classified coboundary Lie bialgebras on $\mathfrak{gl}_2$ and all three-dimensional Lie algebras. r-Matrices on $\mathfrak{so}(2,2)$, $\mathfrak{so}(3,2)$ were analysed. We plan to apply our methods to higher-dimensional coboundary Lie bialgebras and to find new tools to study non-coboundary ones.

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