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Lie Group Approach to Grushin Operators

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Abstract. We consider a finite system $\{X_1, X_2, \ldots, X_n\}$ of complete vector fields acting on a smooth manifold M equipped with a smooth positive measure. We assume that the system satisfies Hörmander's condition and generates a finite dimensional Lie algebra of type (R). We investigate the sum of squares of the vector fields operator corresponding to this system which can be viewed as a generalisation of the notion of Grushin operators. In this setting we prove the Poincaré inequality and Li-Yau estimates for the corresponding heat kernel as well as the doubling condition for the optimal control metrics defined by the system. We discuss a surprisingly broad class of examples of the described setting.

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1. Introduction

The Grushin operators were introduced in [10] almost 50 years ago and initially were defined as a family of degenerate operators by the formula

$$L_k = -\partial_1^2 - x_1^{2k}\,\partial_2^2$$

with $k \in \mathbb{N}$ acting on the space $L^2(\mathbb{R}^2)$. Such operators provide a simple model for degenerate elliptic operators. They have attracted a lot of attention and have been generalised in many ways. A small sample of papers devoted to the Grushin operators can be found for example in [5, 8, 17, 18, 20]. (They are subelliptic operators of Hörmander type.)

One approach to the operator L_k defined above is to write it as a sum of squares of vector fields

$$L_k = -X_1^2 - X_2^2,$$

where $X_1 = \partial_1$ and $X_2 = x_1^k \partial_2$ are two vector fields on $\mathbb{R}^2$. Note that the vectors X_1 and X_2 generate a finite dimensional nilpotent Lie algebra. This observation is in a sense a starting point for our discussion in this note. This approach in which one uses representations of Lie algebras and groups to study Grushin type operators is not new. It was utilised for example in [5, 17, 19, 20] and in several other works. Earlier, the similar idea to employ representations of Lie groups to study sum-of-squares operators on manifolds was used in [7, 21]. One of the main

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aims of this note is to describe the possibly most broad framework for applying Lie group theory to investigate Grushin type operators. Our main contribution here is to observe that in the Lie group approach the doubling condition and the Poincaré inequality, see definitions (2) and (5) below, automatically hold and can be verified in a straightforward manner. We also describe a surprisingly broad class of operators which can be studied in the proposed framework.

A significant motivation for our study and the way in which we interpret it comes from seminal results described by Jerison and Sánchez-Calle in [15] and [16]. They proved the local Poincaré inequality for vector fields satisfying Hörmander's condition and local bounds for the heat kernel corresponding to the sum of squares of such vector fields. Our result can be simply stated as that: if we know in addition that the considered vector fields generate finite dimensional Lie algebra of the type (R) then the global Poincaré inequality and global heat kernels bounds are valid.

Let us recall that type (R) property was used by Guivarc'h [11] and Jenkins [14] in the celebrated characterisation of Lie groups of polynomial growth: a connected Lie group G has polynomial growth if and only if the Lie algebra $L(G) = \mathfrak{g}$ corresponding to G is of type (R). That is, if for all $X \in L(G) = \mathfrak{g}$ the operator $\mathrm{ad}(X)$ has only purely imaginary eigenvalues. Such groups are solvable-by-compact and every connected nilpotent Lie group is of type (R).

There are many other possible natural generalisations of Grushin operators which are not based on a Lie group approach to which our results are complementary. An interesting example of such generalisation was proposed by Franchi, Gutiérrez, and Wheeden in [8]. For a class of non-negative functions $\alpha(x_1) \geq 0$ defined on $\mathbb{R}^n$ contained in a strong A_∞ weight class they considered operators acting on $L^2(\mathbb{R}^{n+m})$ defined by the formula

$$L_\alpha = -\Big(\Delta_{x_1} + \alpha(x_1)\Delta_{x_2}\Big),$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, and $\Delta_{x_1}, \Delta_{x_2}$ are the Laplacians on $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^m)$ respectively.

Another direction in which Grushin operators can be generalised, which we would like to mention here, was studied in [19]. The operators considered in this paper are essentially of the form

$$L_{\delta_1,\delta_2} = -\nabla_{x_1} |x_1|^{\delta_1} \nabla_{x_1} - |x_1|^{\delta_2} \Delta_{x_2}$$

with some fixed $0 \leq \delta_1 < 2$, $0 \leq \delta_2$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, where ∇_{x_1} denotes the *gradient operator* on $L^2(\mathbb{R}^n)$ and $\Delta_{x_2} = \nabla^2_{x_2}$ is the *Laplacian* on $L^2(\mathbb{R}^m)$. The approach developed in [19] to study heat kernels theory for L_{δ_1,δ_2} was based on homogeneity of these operators and required some complex and tedious calculations.

An additional advantage of the Lie group approach which we develop in this study is that it automatically yields boundness of the corresponding Riesz transform on all L^p spaces.

Before we state our main results let us describe a couple of illustrative examples of the operators which we consider here. We find it somehow surprising that such examples can be investigated using the Lie group approach. Namely, we shall prove the doubling condition, the Gaussian two-sided bounds for the corresponding heat kernels, the Poincaré inequality and boundedness of the Riesz transform for various Grushin type operators like

$$-\Big(\partial_1^2+\partial_2^2+(x_1^2+x_2^2-1)^2\partial_3^2\Big)=-\Big(X_1^2+X_2^2+X_3^2\Big),$$

where $X_1=\partial_1$, $X_2=\partial_2$ and $X_3=(x_1^2+x_2^2-1)\partial_3$. Another example is

$$-\Big(\partial_1^2+\sin^2 x_1\partial_2^2+\cos^2 x_1\partial_3^2\Big),$$

where $X_1=\partial_1$, $X_2=\sin x_1\partial_2$ and $X_3=\cos x_1\partial_3$. We will discuss more applications in a detailed way is Section 5.

2. Main Results

Let M be a smooth manifold of dimension k endowed with a positive measure μ. In the sequel, we always assume that μ has a smooth density with respect to any coordinate map on M. By TM we denote the *tangential bundle* of M which sections are vector fields on M. We consider a finite family of smooth vector fields $\{X_1,X_2,\dots,X_n\}$. We assume that the flow $\exp(tX_i)$ generated by the vector field X_i is defined for all $t\in\mathbb{R}$ globally on the whole manifold M for all $i=1,\dots,n$. In the terminology of [9] we just say that X_i are complete vector fields. Recall that a *commutator* of two vector fields X,Y, which is also a vector field, is defined by the formula

$$[X,Y]f=XYf-YXf.$$

In the sequel, we always assume that the system $\{X_1,X_2,\dots,X_n\}$ together with all their commutators generate a finite dimensional Lie algebra $\mathfrak{g}$. It is one of our central assumptions. As a consequence of this assumption, by virtue of Corollary 1, page 113 of [9] all vector fields contained in $\mathfrak{g}$ are defined globally, that is, they are complete, see however Example 3 page 114 of [9].

In what follow, we will also assume that the vectors X_i, $i=1,\dots,n$, are skew-adjoint, which means that

$$\int_M X_if(x)g(x)d\mu(x)=-\int_M f(x)X_ig(x)d\mu(x).$$

For simplicity, we will just use the notation $X_i^*=-X_i$.

A simple calculation shows that if X and Y are skew-adjoint and $Z=[X,Y]$ then $Z^*=-Z$. It follows that if $\mathfrak{g}$ is generated as Lie algebra by a set of skew-adjoint vector fields then all its elements are skew-adjoint.

Note that if X is a complete vector field on M and $X^*=-X$ then

$$\frac{d}{dt}\int_M |f(\exp(tXx)|^2d\mu(x)=\int_M [(X+X^*)\,f(\exp(tXx)]\,\overline{f(\exp(tXx)}d\mu(x)=0$$

for any function $f\in C_c^\infty(M)$. Hence the flow $\exp(tX)$ preserves the measure μ, that is,

$$\int |f(x)|^pd\mu(x)=\int |f(\exp(tX)x)|^pd\mu(x) \tag{1}$$

for all $X\in\mathfrak{g}$, any integrable function $f\in L^p(M)$, $1\le p<\infty$ and $t\in\mathbb{R}$.

Next, recall that the system of vector fields $\{X_1, X_2, \dots, X_n\}$ is said to satisfy *Hörmander's condition* if a finite number of commutators of X_i, $i = 1, \dots, n$, linearly spans the tangent space T_xM for all $x \in M$. Recall that we assume that $\mathfrak{g}$ is a finite dimensional Lie algebra generated by the system $\{X_1, X_2, \dots, X_n\}$. Hence Hörmander's condition in our setting simply means that for every $x \in M$ the linear space corresponding to $\mathfrak{g}$ at x is equal to TM_x.

It is well-known that if the system $\{X_1, X_2, \dots, X_n\}$ satisfies Hörmander's condition then one can define on M the corresponding Carnot-Carathéodory distance, which is often also called the optimal control distance or the sub-Riemannian distance. By $d(x, y)$ we will denote this distance between any two points $x, y \in M$ and by $B(x, r)$ the open ball with respect to d with centre at x and radius r. Then we set $V(x, r) = \mu(B(x, r))$. Let us recall that we say that a metric measure space satisfies the doubling condition if there exists C such that, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \le CV(x, r). \tag{2}$$

Our study focusses on the sum of square operators corresponding to the system $\{X_1, X_2, \dots, X_n\}$ which can be defined by the formula

$$L = \sum_{i=1}^{n} X_i X_i^* = -\sum_{i=1}^{n} X_i^2. \tag{3}$$

The operator L can be precisely defined using quadratic form techniques. We define the corresponding gradient for any $f \in C_c(M)$ by the formula

$$\nabla f = (X_1 f, \dots, X_n f)$$

and set $$|\nabla f(x)|^2 = \sum_{i=1}^{n} |X_i f(x)|^2, \quad \text{so that} \quad \|\nabla f\|_2 = \|L^{1/2} f\|_2.$$

Now we are able to state our main result

Theorem 2.1. *Suppose that M is a smooth manifold of dimension k and that a set of smooth, complete vector fields $\{X_1, X_2, \dots, X_n\}$ satisfies Hörmander's condition and generates a finite dimensional Lie algebra $\mathfrak{g}$ of type* (R)*. We assume in addition that the vectors X_i, $i = 1, \dots, n$, are skew-adjoint, that is, $X_i^* = -X_i$. Then the optimal control distance d satisfies the doubling condition* (2) *and the heat kernel $\tilde{h}_t$ corresponding to the operator L satisfies the upper and lower Gaussian bounds*

$$\frac{C'}{V(x, \sqrt{t})} e^{-c' d(x,y)^2/t} \le h_t(x, y) \le \frac{C}{V(x, \sqrt{t})} e^{-c d(x,y)^2/t}. \tag{4}$$

In addition, the corresponding Riesz transform is bounded for all $1 < p < \infty$, that is,

$$\|\nabla f\|_p \le C_p \|L^{1/2} f\|_p.$$

It is well-known that the two-sided Gaussian estimates (4) imply the Poincaré inequality. In fact, the Poincaré inequality and the doubling condition are equivalent to estimates (4), see [23, p. 112] or [12, Theorem 2.1, p. 20]. Hence Theorem 2.1 has the following Corollary.

Corollary 2.2. *Under the assumptions of the above theorem, the manifold M satisfies the Poincaré inequality, that is, there is a constant $C>0$ such that for any ball B*

$$\int_B |f - f_B|^2 \, d\mu \leq Cr^2 \int_B |\nabla f|^2 \, d\mu \, , \tag{5}$$

where r is the radius of B.

The considered operator L is positive definite and self-adjoint. Therefore L admits a spectral resolution $E_L(\lambda)$ and for any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ one can define the operator $F(L)$ by

$$F(L) = \int_0^\infty F(\lambda) dE_L(\lambda) \, . \tag{6}$$

By spectral theory the operator $F(L)$ is bounded on $L^2(X)$. Spectral multiplier theorems investigate under what conditions on function F the operator $F(L)$ can be extended to a bounded operator acting on the Lebesgue spaces $L^p(X)$ for some range of p, see e.g. [4, 25] for more comprehensive discussion.

Note that on any metric measure space the doubling condition (2) implies that there exist constants C, ν such that for all $\lambda \geq 1$ and $x \in X$,

$$V(x, \lambda r) \leq C\lambda^\nu V(x,r). \tag{7}$$

Now we able to formulate another consequence of Theorem 2.1. This time we note that Gaussian bounds (4) imply the following spectral multiplier result.

Corollary 2.3. *Let ν be the exponent in the doubling estimate* (7) *corresponding to the manifold M and the optimal control distance d. Suppose that $s > \nu/2$ and $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function such that*

$$\sup_{t>0} \|\eta \, \delta_t F\|_{W^{s,\infty}} < \infty, \tag{8}$$

where $\eta \in C_c^\infty(0,\infty)$ is any nontrivial cutoff, $\delta_t F(\lambda) = F(t\lambda)$ and $\|F\|_{W^{s,p}} = \|(I - d^2/dx^2)^{s/2} F\|_{L_p}$. Then, under the above assumption, $F(L)$ is weak type $(1,1)$ *and bounded on all $L^p(M)$ spaces for all $1 < p < \infty$.*

For the proof that the upper part of the Gaussian estimates (4) implies the above corollary and more detailed discussion of spectral multipliers we refer the reader to [4]. Other interesting variants of Corollary 2.3 are discussed in [25].

3. Preliminaries

Before we discuss the proof of Theorem 2.1 we need to introduce more notation, recall some known results and prove some axillary lemmata.

Recall that by Lie's third theorem there exists a unique simple connected Lie group G associated to $\mathfrak{g}$. Now, let $\tilde{X}_i$, $i = 1, \ldots, n$ be a system of left invariant vector fields on the group G corresponding to the system X_i, $i = 1, \ldots, n$. In the natural representation $\tilde{\pi}$ of $\mathfrak{g}$ it holds that $\tilde{\pi}(\tilde{X}_i) = X_i$.

By [9, Theorem 2.1 and Corollaries 1, 2, p. 113] there exists a unique action π of the group G on M such that $d\pi = \tilde{\pi}$. With some abuse of notation, we also denote by π the corresponding representation of the group G in the space of bounded operators acting on $L^2(M)$ such that, for any $f \in L^2(M)$,

$$\pi(g)f(x) = f\big(\pi(g)x\big). \tag{9}$$

Note that $\pi\big(\exp(t\tilde{X}_i)\big)f(x) = f\big(\exp(tX_i)x\big)$ for any $f \in L^2(M)$ and $x \in M$. Hence a standard argument shows that by (1), for all $g \in G$, $\pi(g)$ is an isometry acting on all $L^p(M)$ spaces for $1 \le p \le \infty$.

Following a common representation theory approach, for any function $w \in L^1(G)$ we define the operator $\pi(w)$ as the integral

$$\pi(w) = \int_G w(g)\pi(g)dg$$

with respect to the Haar measure on G.

We have already defined the Carnot-Caratheodory distance on the manifold M. Now the system $\{\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n\}$ generates $\mathfrak{g}$ and we can also define the optimal control distance $\tilde{d}(g,h)$ defined on G corresponding to this system. The vectors $\tilde{X}_i$, $i = 1, \dots, n$, are group invariant so $\tilde{d}(g,h) = |g^{-1}h|$, where $|g| = \tilde{d}(g,e)$.

Note that if $\gamma(t)$ is an admissible curve on G associated with the vector fields $\tilde{X}_i$, $i = 1, \dots, n$, in the sense that for a smooth function f on G one has

$$\frac{d}{dt} f(\gamma(t)) = \sum_{i=1}^n \alpha_i(t)(\tilde{X}_i f)(\gamma(t))$$

then for any $x \in M$ the map $t \mapsto \pi(\gamma(t))x$ is an admissible curve on M in the sense that for a smooth function h on M:

$$\frac{d}{dt} h(\pi(\gamma(t))x) = \sum_{i=1}^n \alpha_i(t)(X_i h)(\pi(\gamma(t))x)\,.$$

Consequently,

$$d(x, \pi(g)x) \le |g|$$

for all $x \in M$ and $g \in G$. Similarly we denote by $\tilde{B}(g,r)$ the open ball corresponding to $\tilde{d}$ with centre at $g \in G$. Note that the volume $|\tilde{B}(g,r)|$ does not depend on g and we can define

$$\tilde{V}(r) = |\tilde{B}(e,r)| = |\tilde{B}(g,r)|\,,$$

where e is the neutral element of the group G. It is well-known that all Lie groups of polynomial growth satisfy the doubling condition (2). In this context it means that there exists $C > 0$ such that

$$\tilde{V}(2r) \le C\tilde{V}(r) \quad \text{for all } r > 0.$$

Next, we consider a function $w \in L^1(G)$ and the corresponding operator $\pi(w)$. We say that

$$\operatorname{supp} \pi(w) \subset \{(x,y) \in M^2 \colon\ d(x,y) \le R\}, \tag{10}$$

if for every open $U_i \subset M$, $f_i \in L^2(U_i, d\mu)$, $i = 1, 2$, where $R = d(U_1, U_2)$, we have

$$\langle \pi(w) f_1, f_2 \rangle = 0.$$

Using a similar approach we say that $\pi(w) \geq 0$ if

$$\langle \pi(w) f_1, f_2 \rangle \geq 0 \tag{11}$$

for all $f_i \in L^2(M, d\mu)$ such that $f_i(x) \geq 0$ for all $x \in M$ and $i = 1, 2$. Note that if the operator $\pi(w)$ has an L^∞ kernel, then condition (11) means simply that $\pi(w)(x, y) \geq 0$ almost everywhere, while (10) expresses that the kernel $\pi(w)(x, y)$ is supported in the subset $\{(x, y) \in M^2 \colon d(x, y) \leq R\}$. The definitions (10) and (11) allow us to avoid the discussion of the existence and nature of the kernel of the operator $\pi(w)$.

Lemma 3.1. *Assume that $w \in L^1(G)$ and that $\operatorname{supp} w \subset \tilde{B}(e, R)$. Then $\pi(w)$ satisfies condition* (10)*.*

Proof. Note that it follows form (3) that if $|g| < R$, where $R = d(U_1, U_2)$ and $U_i \subset M$, $f_i \in L^2(U_i, d\mu)$, $i = 1, 2$, then

$$\langle \pi(g) f_1, f_2 \rangle = 0.$$

The lemma is a straightforward consequence of the above observation. ■

Next, we denote by $\mathbb{1}_M$ the function which is identically equal 1 on M, that is, $\mathbb{1}_M(x) = 1$ for all $x \in M$.

Lemma 3.2. *Suppose that $w \in L^1(G)$ and that $\pi(w)$ is an operator defined above. Then*

$$\pi(w)\mathbb{1}_M = \mathbb{1}_M \int_G w(g) dg.$$

Moreover, for any function $w \in L^1(G)$ if $w(g) \geq 0$ for all $g \in G$ then

$$\pi(w) \geq 0,$$

that is, $\pi(w)$ satisfies condition (11)*.*

Proof. It follows from (9) and the definition of $\pi(w)$ that

$$\pi(w)\mathbb{1}_M(x) = \int_G w(g) \mathbb{1}_M(\pi(g)x) dg = \int_G w(g) dg.$$

Next note that if $f_i(x) \geq 0$ for all $x \in M$ and $i = 1, 2$, then

$$\langle \pi(g) f_1, f_2 \rangle \geq 0.$$

Hence if $w(g) \geq 0$ for all $g \in G$ then we have, as required,

$$\langle \pi(w) f_1, f_2 \rangle = \int_G w(g) \langle \pi(g) f_1, f_2 \rangle \geq 0.$$ ■

Next, we define an operator $\tilde{L}$ on the group G by the formula

$$\tilde{L} = -\sum_{i=1}^{n} \tilde{X}_i^2.$$

It is well-known, see for example [12, Theorem 2.2, p. 22], that the operator $\tilde{L}$ generates a semigroup acting on all spaces $L^p(G)$ and that the corresponding convolution heat kernel $\tilde{h}_t(g,h) = \tilde{h}_t(gh^{-1})$ satisfies the two-sided Gaussian estimates (4) which can be stated in this setting as

$$\frac{c}{\tilde{V}(\sqrt{t})} e^{-\beta'|g|^2/t} \le \tilde{h}_t(g) \le \frac{C}{\tilde{V}(\sqrt{t})} e^{-\beta|g|^2/t}. \tag{12}$$

Consider next the Poisson semigroup corresponding to the operator $\widetilde{L}$, that is, $\{\exp(-t\widetilde{L}^{1/2})\}_{t\ge 0}$. By $\tilde{p}_t(g)$ we denote the convolution kernel corresponding to the Poisson semigroup. By the subordination formula $\tilde{p}_t(g)$ can be expressed by the following integral involving the heat kernel $\tilde{h}_t$:

$$\tilde{p}_t(g) = \pi^{-1/2} \int_0^\infty e^{-s} \tilde{h}_{t^2/(4s)}(g) \frac{ds}{\sqrt{s}}. \tag{13}$$

In our discussion we need some properties of the kernel $\tilde{p}_t$, which we describe in the following lemma.

Lemma 3.3. *Consider a connected Lie group G of polynomial growth, the operator $\widetilde{L}$ defined above and let $\tilde{p}_t(g)$ be the kernel corresponding to the Poisson semigroup. Then for all $t > 0$,*

$$\frac{c}{\tilde{V}(t+|g|)} \frac{t}{t+|g|} \le \tilde{p}_t(g) \le \frac{C}{\tilde{V}(t+|g|)} \frac{t}{t+|g|}. \tag{14}$$

In consequence, there exists a positive constant $C>0$ such that for all $t>0$ and $g\in G$

$$C^{-1}\tilde{p}_t(g) \le \tilde{p}_{2t}(g) \le C\tilde{p}_t(g). \tag{15}$$

Proof. The upper and lower bounds for the kernel of Poisson semigroup (14) are straightforward consequence of the subordination formula (13) and the Li-Yau estimate (12). We refer the reader to [6, Proposition 6] for details. The estimate (15) is a straightforward consequence of the doubling condition and (14). ■

Next let us denote the characteristic function of the ball $\tilde{B}(e,t)$ by $\chi_{\tilde{B}(e,t)}$. We complement Lemma 3.3 by the following observations.

Lemma 3.4. *Under the same assumptions as in Lemma 3.3 one has*

$$\frac{\chi_{\tilde{B}(e,t)}(g)}{\tilde{V}(t)} \le C\tilde{p}_t(g) \tag{16}$$

and

$$|\tilde{X}_i \tilde{p}_t(g)| \le C t^{-1} \tilde{p}_t(g) \tag{17}$$

for all $t > 0$ and $g \in G$.

Proof. Estimate (16) is a straightforward consequence of Lemma 3.3. To verify estimate (17) recall that following Gaussian estimate for the gradient of the heat kernel on Lie groups of polynomial growth were obtained by Saloff-Coste

$$\left|\tilde{X}_i \tilde{h}_t(g)\right| \le \frac{Ct^{-1/2}}{\tilde{V}(\sqrt{t})} e^{-\beta|g|^2/t},$$

see [22, Proposition 1]. Now (17) can be obtained using the subordination formula (13) and essentially the same calculations as in the proof of [6, Proposition 6]. ■

4. Proof of Theorem 2.1

Set $h_t = \pi(\tilde{h}_t)$ and $p_t = \pi(\tilde{p}_t)$. By a standard representation theory argument h_t is the heat semigroup generated by $-L$ and p_t constitutes the Poisson semigroup-generated by $-L^{1/2}$. It follows from celebrated results obtained by Hörmander in [13] that the operators $\exp(-tL^{1/2})$ and $\exp(-tL)$ have smooth kernels $p_t(x,y)$ and $h_t(x,y)$ for all $t>0$ and $(x,y)\in M^2$. Note also that

$$X_i h_t = \pi(\tilde{X}_i\tilde{h}_t) \quad \text{and} \quad X_i p_t = \pi(\tilde{X}_i\tilde{p}_t).$$

In what follow we will need the following consequence of Lemmata 3.3 and 3.4.

Corollary 4.1. *Let p_t be the kernel corresponding to the Poisson semigroup $\exp(-tL^{1/2})$ defined above. There exist constants $C>0$ and $c>0$ such that*

$$C^{-1}p_t(x,y) \le p_{2t}(x,y) \le Cp_t(x,y) \tag{18}$$

and

$$|X_i p_t(x,y)| \le ct^{-1}p_t(x,y). \tag{19}$$

Proof. As we noted above the operator p_t has a smooth kernel so (11) can be interpreted pointwise. Hence (18) is a straightforward consequence of Lemmata 3.2 and 3.3. By a similar argument, (19) follows from Lemma 3.2 and estimate (17). ■

The following lemma will be crucial in our further considerations.

Lemma 4.2. *Let p_t be the kernel of the Poisson semigroup as in Lemma 4.1. There exists a constant $c>0$ such that, for all $x,x',y\in M$,*

$$p_t(x,y) \le p_t(x',y)\exp\Big(c\frac{d(x,x')}{t}\Big).$$

Proof. Consider an admissible curve connecting x' and x, $\gamma\colon [0,S]\to M$ parametrised with unit velocity.
Set $f(s)=p_t(\gamma(s),y)$. By (19) we obtain $|f'(s)|\le ct^{-1}f(s)$.
By a standard differential inequality argument $f(S)\le f(0)\exp(cS/t)$. Taking minimum over S for all admissible curves yields the lemma. ■

Proof of the doubling condition (2)**.**

For any $r>0$ set $q_r(x,y)=\dfrac{\pi\big(\chi_{\tilde{B}(e,r)}\big)(x,y)}{\tilde{V}(r)}$.
Lemma 3.2 and estimate (16) imply $0\le q_r(x,y)\le Cp_r(x,y)$. Moreover, by Lemma 3.1, we have $\operatorname{supp} q_r(\cdot,y)\subset B(y,r)$. Fix $y\in M$ and for any $r>0$ set

$$m_r=\sup_x q_r(x,y) \quad \text{and} \quad M_r=\sup_{x\in B(y,r)} p_r(x,y).$$

Obviously, $m_r\le CM_r$. Note also that $m_r\ge V(y,r)^{-1}$ for all $y\in M$ and $r>0$.

Indeed, by Lemma 3.2 we get $1 = \int_M q_r(x,y)\,dx \le m_r V(y,r)$.

By Lemma 4.2: $p_r(x,y) \le e^{2c} p_r(x',y)$ for all $x, x' \in B(y,r)$. Hence

$$e^{-2c} M_r \le p_r(x',y) \quad \text{for all} \quad x, x' \in B(y,r).$$

Consequently,

$$e^{-2c} M_r V(y,r) \le \int_{\mathbb{M}} p_r(x',y)\,d\mu(x') = \int_G \tilde{p}_r(g)\,dg = 1. \tag{20}$$

Thus we have proved,

$$V(y,r)^{-1} \le m_r \le C M_r \le C V(y,r)^{-1}. \tag{21}$$

Next, by (18) of Corollary 4.1, there exists a constant C independent of r such that

$$C_{-1} p_r(x,y) \le p_{2r}(x,y) \le C p_r(x,y).$$

Hence

$$M_r \sim M_{2r}. \tag{22}$$

Applying (21) for M_r and M_{2r} combined with (22) yields the doubling condition. ■

Proof of estimate (4) and boundedness of the Riesz transform.
The rest of the proof goes along standard lines. Note that

$$p_t(x,x) \le M_t \le C V(x,t)^{-1}.$$

Hence

$$\int |p_t(x,y)|^2 dy = p_{2t}(x,x) \le C V(x,t)^{-1}.$$

Now if we set

$$M_{V_t} f(x) = V(x,t) f(x),$$

then we can equivalently state the above estimates as

$$\|M_{V_t} \exp(-tL^{1/2})\|_{2\to\infty} \le C$$

for all $t > 0$. It follows that

$$\|M_{V_t} \exp(-t^2 L)\|_{2\to\infty}$$
$$\le \|M_{V_t} \exp(-tL^{1/2})\|_{2\to\infty} \times \|\exp(-t^2 L + tL^{1/2})\|_{2\to 2} \le C. \tag{23}$$

It is well-known that estimate (4) implies the upper Gaussian estimates, see [1] and [24] for some examples of many proofs of this implication available in the literature. Similarly we note that

$$\|M_{V_t} X_i \exp(-t^2 L)\|_{2\to\infty} \le C \|M_{V_t} X_i \exp(-tL^{1/2})\|_{2\to\infty} \le C/t. \tag{24}$$

A standard argument shows that (24) implies the two-sided Gaussian estimates (4) and the boundedness of Riesz transform for all $1 < p < \infty$, see for example [3, Theorem 1.1 and Corollary 2.2]. The boundedness of Riesz transform can be alternatively verified using transference techniques from [2]. ■

5. Examples and applications

5.1. Grushin type operators with coefficient x^{2k} replaced by $\omega(x)^2$ for an arbitrary polynomial ω

Consider the Euclidean plane $M = \mathbb{R}^2$ and for any function $f \in C_c^\infty(\mathbb{R}^2)$ set

$$L_\omega f = -\partial_x^2 f - \omega(x)^2 \partial_y^2 f, \tag{25}$$

where $\omega(x)$ is a polynomial in x of degree m–1. Note that L_ω can be represented as

$$L_\omega = -(Z_0^2 + Z_1^2),$$

where $\quad Z_0 f(x,y) = \partial_x f(x,y) \quad$ and $\quad Z_1 f(x,y) = \omega(x)\partial_y f(x,y)$.

Note that the system $\{Z_0, Z_1\}$ generates m-step nilpotent Lie algebra $\mathfrak{n}$ having the *linear basis* Z_0, Z_1, ..., Z_m with the only nontrivial commuting relations

$$[Z_0, Z_j] = Z_{j+1}, \quad j = 1, 2, 3, ..., m-1.$$

Indeed, simple calculation shows that

$$Z_j f(x,y) = \omega^{(j-1)}(x)\partial_y f(x,y) \text{ for } j = 1, 2, ..., m,$$

where $\omega^{(l)}$ is the l-th derivative of ω. Thus the operator L_ω satisfies all the assumptions of Theorem 2.1.

5.2. Further generalisation of the operators L_ω

Note that one cannot formally apply the results from section 5.1 to the operator

$$L = -\big(\partial_x^2 + (x^2+1)\partial_y^2\big).$$

In this case the function $\omega(x) = \sqrt{x^2+1}$ is not a polynomial and the vector fields ∂_x and $\sqrt{x^2+1}\partial_y$ do not generate a finite dimensional Lie algebra. However the doubling condition, the two-sided Gaussian estimates, the Poincaré inequality and all results which we discuss above still hold in this setting. In fact we can use the proposed approach to investigate all operators of the form

$$L = -\Big(\partial_x^2 + (\omega_1(x)^2 + \ldots + \omega_n(x)^2)\partial_y^2\Big)$$

for any family of polynomials $\omega_1, \ldots, \omega_n$.

Indeed in this case we can consider the vector field $X_0 f(x,y) = \partial_x f(x,y)$.

Next, we put $X_k f(x,y) = \omega_k(x)\partial_y f(x,y)$ for $k = 1, \ldots, n$. Then we set

$$L = -\sum_{i=0}^{n} X_i^2.$$

It is easy to check that the system $\{X_0, X_1, \ldots, X_n\}$ generates a final dimensional nilpotent Lie algebra and satisfies Hörmander's condition.

5.3. Operators acting on $\mathbb{R}^n \times \mathbb{R}^m$

Another example which we can investigate in the proposed setting is the following operator

$$L = -\sum_{k=1}^{n} \partial^2_{x'_k} - \sum_{l=1}^{m} \sum_{i=1}^{I_l} \omega_{l,i}(x'_1, \dots, x'_n)^2 \partial^2_{x''_l},$$

where $\omega_{l,i}$ are finite order polynomials. As before it is straightforward to represent L as a sum of squares of vector fields, which generate finite dimensional nilpotent Lie algebra and satisfy Hörmander's condition.

A particular instance of a degenerate operator of this from is the operator L acting on the ambient space $\mathbb{R}^2 \times \mathbb{R} = \{(x'_1, x'_2, x''_1) \colon\ x'_1, x'_2, x''_1 \in \mathbb{R}\}$ and defined by the formula

$$L = -\Big(\partial^2_{x'_1} + \partial^2_{x'_2} + ({x'_1}^2 + {x'_2}^2 - 1)^2 \partial^2_{x''_1}\Big),$$

which we mentioned in the introduction.

5.4. An example involving non-nilpotent Lie group

Let E be the universal covering of the Lie group of motions of a plane. Topologically the group is isomorphic to $\mathbb{R}^3$ and the group action can be described by the following formula

$$(t_1, x_1, y_1)(t_2, x_2, y_2) = (t_1 + t_2, x_1 + x_2 \cos t_1 + y_2 \sin t_1, y_1 - x_2 \sin t_1 + y_2 \cos t_1).$$

The system of left-invariant vector fields on E can be described in the following way

$$\begin{aligned} T(t,x,y) &= \partial_t \\ X(t,x,y) &= -\sin t\, \partial_x + \cos t\, \partial_y \\ Y(t,x,y) &= -\cos t\, \partial_x - \sin t\, \partial_y. \end{aligned}$$

The commutator relations are given by

$$[X,Y] = 0, \quad [T,X] = Y, \quad [T,Y] = -X.$$

This group is the simplest example of a group of polynomial volume growth which is not nilpotent.

Now we are in a position to describe an interesting example of a sum of squares of vector fields operator, related to the group E. It can be written as an operator acting on $\mathbb{R} \times \mathbb{R} = \{(x,y) \colon\ x, y \in \mathbb{R}\}$ and defined by

$$L = -\Big(\partial^2_x + \sin^2 x\, \partial^2_y\Big).$$

The above operator is obtained as $L = -\tilde{T}^2 - \tilde{X}^2$ where $\tilde{T} = \partial_x$, $\tilde{X} = \sin x\, \partial_y$ and $[\tilde{T}, \tilde{X}] = \tilde{Y} = \cos x\, \partial_y$. One can check easily that the algebra generated by $\tilde{T}$, $\tilde{X}$ and $\tilde{Y}$ satisfies the same commutation relations and is isomorphic to the algebra of the group E which is spanned by X, Y and T. Note also that the system $\{X, T\}$ satisfies Hörmander's condition. It is well-known and easy to check that the Lie algebra of E is of type (R) so E is a group of polynomial growth.

5.5. An example of Grushin type operator acting on compact manifolds

The above example can be modified so that one can construct an instance of Grushin type operators acting on compact manifolds. To that end, consider the torus which is a product of two circles $\Pi^2 = \{(\theta_1, \theta_2)\colon\ \theta_1, \theta_2 \in \Pi\}$. We consider the operator

$$L = -\Big(\partial_{\theta_1}^2 + \sin^2\theta_1\ \partial_{\theta_2}^2\Big).$$

In this example $L = X^2 + Y^2$ where

$$X = \partial_{\theta_1}, \quad Y = \sin\theta_1\ \partial_{\theta_2}, \quad [X, Y] = T = \cos\theta_1\ \partial_{\theta_2}.$$

One can verify in a standard way that X, Y and T generates three dimensional Lie algebra of type (R) which coincides with the Lie algebra considered in Section 5.4.

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References

[1] S. Boutayeb, T. Coulhon, A. Sikora: *A new approach to pointwise heat kernel upper-bounds on doubling metric measure space*, Adv. Math. 270 (2015) 302–374.

[2] R. Coifman, G. Weiss: *Transference Methods in Analysis*, CBMS Regional Conference Series in Mathematics 31, American Mathematical Society, Providence (1977).

[3] T. Coulhon, A. Sikora: *Riesz meets Sobolev*, Colloq. Math. 118 (2010) 685–704.

[4] X. T. Duong, E. M. Ouhabaz, A. Sikora: *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Analysis 196 (2002) 443–485.

[5] J. Dziubański, K. Jotsaroop: *On Hardy and BMO Spaces for Grushin Operator*, J. Fourier Analysis Appl. 22 (2016) 954–995.

[6] J. Dziubański, M. Preisner: *Hardy spaces for semigroups with Gaussian bounds*, Ann. Mat. Pura Appl. 197 (2018) 965–987.

[7] G. B. Folland, E. M. Stein: *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. 27 (1974) 429–522.

[8] B. Franchi, C. E. Gutiérrez, R. L. Wheeden: *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Part. Diff. Equations 19 (1994) 523–604.

[9] V. V. Gorbatsevich, A. L. Onishchik, E. B. Vinberg: *Lie Groups and Lie Algebras. I: Foundations of Lie Theory and Lie Transformation Groups*, reprint of the 1993 translation, Lie Groups and Lie Algebras I, Encyclopaedia Math. Sci. 20, Springer, Berlin (1993).

[10] V. V. Grushin: *A certain class of hypoelliptic operators*, Mat. Sb. (N.S.) 83 (1970) 456–473.

[11] Y. Guivarc'h: *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France 101 (1973) 353–379.

[12] P. Gyrya, L. Saloff-Coste: *Neumann and Dirichlet Heat Kernels in Inner Uniform Domains*, Astérisque 336, Société Mathématique de France, Paris (2011).

[13] L. Hörmander: *Hypoelliptic second order differential equations*, Acta Math. 119 (1967) 147–171.

[14] J. W. Jenkins: *A characterization of growth in locally compact groups*, Bull. Amer. Math. Soc. 79 (1973) 103–106.

[15] D. Jerison: *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. 53 (1986) 503–523.

[16] D. S. Jerison, A. Sánchez-Calle: *Estimates for the heat kernel for a sum of squares of vector fields*, Ind. Univ. Math. J. 35 (1986) 835–854.

[17] K. Jotsaroop, P. Sanjay, S. Thangavelu: *Riesz transforms and multipliers for the Grushin operator*, J. Anal. Math. 119 (2013) 255–273.

[18] A. Martini, D. Müller: *A sharp multiplier theorem for Grushin operators in arbitrary dimensions*, Rev. Mat. Iberoam. 30 (2014) 1265–1280.

[19] D. W. Robinson, A. Sikora: *Analysis of degenerate elliptic operators of Grušin type*, Math. Zeitschrift 260 (2008) 475–508.

[20] C. Rockland: *Hypoellipticity on the Heisenberg group-representation-theoretic criteria*, Trans. Amer. Math. Soc. 240 (1978) 1–52.

[21] L. P. Rothschild, E. M. Stein: *Hypoelliptic differential operators and nilpotent groups*, Acta Math. 137 (1976) 247–320.

[22] L. Saloff-Coste: *Analyse sur les groupes de Lie à croissance polynômiale*, Ark. Mat. 28 (1990) 315–331.

[23] L. Saloff-Coste: *Aspects of Sobolev Inequalities*, LMS Lecture Notes Series 289, Cambridge University Press, Cambridge (2002).

[24] A. Sikora: *Riesz transform, Gaussian bounds and the method of wave equation*, Math. Zeitschrift 247(3) (2004) 643–662.

[25] A. Sikora, L. X. Yan, X. H. Yao: *Sharp spectral multipliers for operators satisfying generalized Gaussian estimates*, J. Funct. Analysis 266 (2014) 368–409.

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