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On the Lie Pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})$

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Abstract. We investigate the structure and finite irreducible representation of a Lie H-pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})$, which is a generalization of the vector field Lie H-pseudoalgebra $W(\mathfrak{g})$ defined earlier by B. Bakalov, A. D'Andrea and V. G. Kac [*Theory of finite pseudoalgebras*, Advances in Mathematics 162(1) (2001) 1–140]. We prove that automorphisms of $W(\mathfrak{m},\pi,\mathfrak{g})$ are in one-to-one correspondence with solutions of some Maurer-Cartan equation when $\mathfrak{g}$ is a finite dimensional simple Lie algebra.

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Key Words: Lie pseudoalgebra, singular vector, Maurer-Cartan equation.

1. Introduction

Lie H-pseudoalgebras, which are multi-parameters Lie conformal algebras, were introduced in [1]. They are a kind of chiral algebra in [3]. A Lie H-pseudoalgebra is a Lie algebra in the pseudotensor category $\mu^*(H)$. This pseudotensor category has the same objects as the category $\mu^l(H)$ of all left modules over a cocommutative Hopf algebra H and the set of all pseudolinear maps between a finite family of objects $\{L_i\}_{i\in I}$ and an object M is

$$Lin(\{L_i\}_{i\in I}, M) = Hom_{H^{\otimes I}}(\boxtimes_{i\in I} L_i, H^{\otimes I}\otimes_H M),$$

where the left $H^{\otimes I}$-module $\boxtimes_{i\in I} L_i$ is equal to $\otimes_{i\in I} L_i$ as vector spaces with the action of $\otimes_{i\in I}h_i \in H^{\otimes I}$ on $\otimes_{i\in I}v_i \in \otimes_{i\in I}L_i$ given by $(\otimes_{i\in I}h_i)(\otimes_{i\in I}v_i) := \otimes_{i\in I}h_iv_i$. A Lie conformal algebra structure is then the same as a Lie pseudoalgebra over $H = C[\partial]$, with the ordinary Hopf algebra structure. Since [1] was published, pseudoalgebras have been investigated in both the mathematical and physical literature (see [4], [7], [10], [9], [8] and so on).

If $H = U(\mathfrak{g})$ is the universal enveloping algebra of finite dimensional Lie algebra $\mathfrak{g}$, the following four type Lie H-pseudoalgebras $W(\mathfrak{g})$, $S(\mathfrak{g},\chi)$, $H(\mathfrak{g},\chi,\omega)$ and $K(\mathfrak{g},\theta)$, which were introduced in [1, Section 4], are called as primitive Lie H-pseudoalgebras of vector fields. Furthermore, [1, Theorem 13,2] states that any finite simple Lie H-pseudoalgebra is isomorphic to a current Lie pseudoalgebra over a finite dimensional simple Lie algebra or one of the primitive $U(\mathfrak{g}')$-pseudoalgebras of

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vector fields, where $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$. The classification of all finite irreducible representations of $W(\mathfrak{g})$ and $S(\mathfrak{g},\chi)$ was accomplished in [2].

In this paper, we study the structure and finite irreducible representation of a Lie H-pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g}) := H\otimes\mathfrak{m}$, where $H = U(\mathfrak{g})$ is the enveloping algebra of finite-dimensional Lie algebra $\mathfrak{g}$ and $\pi : \mathfrak{m}\to\mathfrak{g}$ is an *epimorphism* of finite-dimensional Lie algebras. The pseudobracket of $W(\mathfrak{m},\pi,\mathfrak{g})$ is given by

$$\begin{aligned}[(f\otimes a)*(g\otimes b)] = \ & (f\otimes g)\otimes_H(1\otimes[a,b]) - (f\otimes g\pi(a))\otimes_H(1\otimes b)\\ & +(f\pi(b)\otimes g)\otimes_H(1\otimes a),\end{aligned} \tag{1}$$

for any $f\otimes a, g\otimes b\in W(\mathfrak{m},\pi,\mathfrak{g})$. The Lie pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})$ is a special case of [6, Example 2.5]. It yields $W(\mathfrak{g})$ when $\mathfrak{m}=\mathfrak{g}$ and $\pi = id$ and decomposes into a semi-product of $W(\mathfrak{g})$ and Cur$ker(\pi)$ when $\mathfrak{m} = \mathfrak{g}\ltimes ker(\pi)$. In all other cases, $W(\mathfrak{m},\pi,\mathfrak{g})$ is a non-trivial, but possibly non semi-direct, extension of $W(\mathfrak{g})$ by an ideal isomorphic to Cur$ker(\pi)$.

The paper is organized as follows. In Section 2, we recall some notations related to Lie pseudoalgebras, most of which can be found in [1].

In Section 3 the structure of $W(\mathfrak{m},\pi,\mathfrak{g})$ is investigated. We prove that $W(\mathfrak{m},\pi,\mathfrak{g})$ has a unique maximal ideal Cur$ker(\pi)$. From this, one can prove that $W(\mathfrak{m},\pi,\mathfrak{g})$ is semisimple if and only if $ker(\pi)$ is semisimple. In the case that $ker(\pi)$ is simple, we show that every automorphism of $W(\mathfrak{m},\pi,\mathfrak{g})$ gives a solution of a Maurer-Cartan equation. Furthermore, we determine all central extensions of $W(\mathfrak{m},\pi,\mathfrak{g})$ by the trivial module $\mathbf{k}$ through computing the second cohomology group $H^2(W(\mathfrak{m},\pi,\mathfrak{g}),\mathbf{k})$.

In Section 4, we determine all finite irreducible $W(\mathfrak{m},\pi,\mathfrak{g})$-modules, thus generalizing results from [5] and [2].

Throughout this paper, all algebras are defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. All unadorned tensor products $\otimes$ are taken over the field $\mathbf{k}$. For all other undefined concepts, we refer to [1] and [2].

2. Preliminaries

In this section we recall some basic concepts and facts related to Lie pseudoalgebras, most of which can be found in [1].

In the sequel, we always use H to denote a universal enveloping algebra $U(\mathfrak{g})$ of finite dimension Lie algebra $\mathfrak{g}$. Then H is a cocommutative Hopf algebra with a counit ε, an antipode S and a coproduct Δ, which are determined by $\varepsilon(x) = 0$, $S(x) = -x$ and $\Delta(x) = x\otimes 1 + 1\otimes x$ for any $x\in\mathfrak{g}$ respectively. We define

$$\Delta^m := (\Delta\otimes\underbrace{1\otimes\cdots\otimes 1}_{m-1})\Delta^{m-1}$$

for any positive integer m. Suppose that $\{\partial_1,\partial_2,\cdots,\partial_n\}$ is a basis of $\mathfrak{g}$ and $\mathbb{N}$ is the set of all nonnegative integers. Then H has a basis $\{\partial^M | M\in\mathbb{N}^n\}$, where $\partial^M = \frac{\partial_1^{i_1}}{i_1!}\frac{\partial_2^{i_2}}{i_2!}\cdots\frac{\partial_n^{i_n}}{i_n!}$ for any $M = (i_1,i_2,\cdots,i_n)\in\mathbb{N}^n$. The basis $\{\partial^M | M\in\mathbb{N}^n\}$ is usually called PBW basis of H. Let $|M| = i_1+i_2+\cdots+i_n$ for any $M = (i_1,i_2,\cdots,i_n)\in\mathbb{N}^n$. Then $|M|$ is called the degree of M. There is a total order defined on the PBW basis $\{\partial^M | M\in\mathbb{N}^n\}$ induced by a degree-lexicographical order of $\mathbb{N}^n$ as $\partial^{(k_1,k_2,\cdots,k_n)} > \partial^{(p_1,p_2,\cdots,p_n)}$ if either

$$\sum_{i=1}^{n} p_i > \sum_{i=1}^{n} k_i, \text{ or } \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} k_i$$

and there is an integer l such that $k_1 = p_1, k_2 = p_2, \cdots, k_{l-1} = p_{l-1}$ but $k_l > p_l$.
Using the PBW basis, $F^pH = span_{\mathbf{k}}\{\partial^M | \, |M| \leq p\}$, $p = 0, 1, 2, \ldots$ defines an increasing filtration on $H = U(\mathfrak{g})$. The dual algebra $X = H^*$ has an H-bimodule structure given by

$$\langle hx, f\rangle = \langle x, S(h)f\rangle, \qquad \langle xh, f\rangle = \langle x, fS(h)\rangle \tag{2}$$

for any $x \in X$ and $f, h \in H$. It was proved in [2, (2.20)] that X is isomorphic to the formal series ring $O_n = \mathbf{k}[[t_1, \cdots, t_n]]$. Then this isomorphism endows X a decreasing filtration F_pX $(p = -1, 0, 1, \cdots)$, where

$$F_pX = (t_1, t_2, \cdots, t_n)^{p+1}O_n, \; p = -1, 0, 1, \cdots. \tag{3}$$

Notice that each F_pX is the annihilation of F^pH in X. Moreover, for any $p, q \geq -1$,

$$(F_qX)(F_pX) \subset F_{q+p+1}X, \; \mathfrak{g}(F_pX) \subset F_{p-1}X, \; (F_pX)\mathfrak{g} \subset F_{p-1}X. \tag{4}$$

After recalling the notations and basic results about the Hopf algebra of $U(\mathfrak{g})$, we restate some concepts related to Lie H-pseudoalgebras introduced in [1].

Definition 2.1. A left H-module L is a Lie pseudoalgebra if there is a map $[\cdot * \cdot] \in Hom_{H^{\otimes 2}}(L \boxtimes L, H^{\otimes 2} \otimes_H L)$ satisfying

(1) *Skew-symmetry*: $[a * b] = -((12) \otimes_H 1)[b * a]$,

(2) *Jacobi identity*: $[[a * b] * c] = [a * [b * c]] - ((12) \otimes_H 1)[b * [a * c]]$

for all $a, b, c \in L$, where (12) is the permutation of first two factors in $H^{\otimes 3}$ and $[(f \otimes_H a) * (g \otimes_H b)] = (f\Delta^{n-1} \otimes g\Delta^{m-1} \otimes_H 1))[a * b]$ for any $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ and $a, b \in L$.
A left H-submodule I of a Lie pseudoalgebra L is said to be an ideal of L if $[a * b] \in H^{\otimes 2} \otimes_H I$ for any $a \in L$ and $b \in I$. A Lie pseudoalgebra L is *abelian* if $[a * b] = 0$ for any $a, b \in L$. A Lie pseudoalgebra L is said to be *simple* if L is not abelian and has only two ideals, namely, 0 and L. ■

Since we fix the Hopf algebra in the sequel, we sometimes simplify Lie H-pseudoalgebra as Lie pseudoalgebra. If $H = \mathbf{k}$, then a Lie H-pseudoalgebra is a usual Lie algebra over the field $\mathbf{k}$. A further easy example of Lie H-pseudoalgebras is a current Lie H-pseudoalgebra given by the following.

Example 2.2. Suppose that H_1 is a sub-Hopf algebra of H and L is a Lie pseudoalgebra over H_1. Let $\mathrm{Cur}_{H_1}^H(L) = H \otimes_{H_1} L$ be a free H-module with the action given by left multiplying the first factor of $\mathrm{Cur}_{H_1}^H(L)$. For any $f \otimes_{H_1} a, g \otimes_{H_1} b \in \mathrm{Cur}_{H_1}^H(L)$, define

$$[(f \otimes_{H_1} a) * (g \otimes_{H_1} b)] = \sum_i (ff_i \otimes gg_i) \otimes_H (1 \otimes_{H_1} c_i), \tag{5}$$

where $[a * b] = \sum_i f_i \otimes g_i \otimes_{H_1} c_i$. Then $\mathrm{Cur}_{H_1}^H(L)$ is a Lie pseudoalgebra via (5). In particular, if $H_1 = \mathbf{k}$, then L is a Lie algebra over $\mathbf{k}$ and $\mathrm{Cur}_{H_1}^H(L)$ is simplified as $\mathrm{Cur}L$. It is well-known that $\mathrm{Cur}_{H_1}^H(L)$ is a finite simple Lie H-pseudoalgebra if L is a finite simple Lie H_1-pseudoalgebra. ■

In addition to the above current Lie H-pseudoalgebra, there is another important simple Lie H-pseudoalgeba $W(\mathfrak{g})$ introduced in [1]. Suppose $\mathfrak{b}$ is a finite dimensional Lie algebra. In accordance with [1], we use the notation $W(\mathfrak{g}) \ltimes \mathrm{Cur}\mathfrak{b}$ to refer to the semiproduct of Lie pseudoalgebras $W(\mathfrak{g})$ and $\mathrm{Cur}\mathfrak{b}$ defined by

$$[(f \otimes a) * (g \otimes b)] = -(f \otimes ga) \otimes_H (1 \otimes b),$$

for $f \otimes a \in W(\mathfrak{g})$ and $g \otimes b \in \mathrm{Cur}\mathfrak{b}$. We generalize Lie H-pseudoalgebras $W(\mathfrak{g})$ and $W(\mathfrak{g}) \ltimes \mathrm{Cur}\mathfrak{b}$ to a Lie H-pseudoalgebra $W(\mathfrak{m}, \pi, \mathfrak{g})$ defined by the next example.

Example 2.3. Let $\pi \colon \mathfrak{m} \to \mathfrak{g}$ be an *epimorphism* of finite-dimensional Lie algebras and $H = U(\mathfrak{g})$. Set $W(\mathfrak{m}, \pi, \mathfrak{g}) = H \otimes \mathfrak{m}$. Then $W(\mathfrak{m}, \pi, \mathfrak{g})$ becomes a Lie H-pseudoalgebra with a pseudobracket defined by (1).

(a) Let $id_{\mathfrak{g}}$ be the identity map of $\mathfrak{g}$. Then $W(\mathfrak{g}, id_{\mathfrak{g}}, \mathfrak{g}) = W(\mathfrak{g})$.

(b) If $\mathfrak{m} = \mathfrak{g} \oplus ker(\pi)$ as Lie algebras, then $W(\mathfrak{m}, \pi, \mathfrak{g}) \cong W(\mathfrak{g}) \ltimes \mathrm{Cur}\ker(\pi)$.

(c) Let $\mathfrak{m}$ be a Lie algebra with a basis $\{x, y, z\}$ such that $[x, y] = y, [x, z] = z$ and $[y, z] = 0$. Then $\mathbf{k}y + \mathbf{k}z$ is an ideal of $\mathfrak{m}$. Let $\mathfrak{g} := \mathfrak{m}/(\mathbf{k}y + \mathbf{k}z)$ and $\pi \colon \mathfrak{m} \to \mathfrak{g}$ be the canonical projection. Then $W(\mathfrak{m}, \pi, \mathfrak{g})$ is a Lie conformal algebra with $[(1 \otimes x) * (1 \otimes x)] = (x \otimes 1 - 1 \otimes x) \otimes_H (1 \otimes x)$, $[(1 \otimes x) * (1 \otimes y)] = (-1 \otimes x + 1 \otimes 1) \otimes_H (1 \otimes y)$, $[(1 \otimes x) * (1 \otimes z)] = (-1 \otimes x + 1 \otimes 1) \otimes_H (1 \otimes z)$ and $[(1 \otimes y) * (1 \otimes z)] = 0$. Let $e_0 = 1 \otimes x$, $e_1 = 1 \otimes y$, $e_2 = 1 \otimes z$ and $s = x$. Then the Lie conformal algebra is a 0-type Lie conformal algebra of rank three defined in [12]. ■

A left H-module homomorphism $\psi : L \to L'$ between two Lie H-pseudoalgebras is called a homomorphism of Lie H-pseudoalgebras if, for any $a, b \in L$,

$$(1 \otimes 1 \otimes_H \psi)[a * b] = [\psi(a) * \psi(b)].$$

Example 2.4. We claim that $id_H \otimes \pi : W(\mathfrak{m}, \pi, \mathfrak{g}) \to W(\mathfrak{g})$ is a homomorphism of Lie pseudoalgebras with kernel $\mathrm{Cur}\ker(\pi)$, where $ker(\pi)$ is the kernel of π. In fact, for $a, b \in \mathfrak{m}$, we have

$$((1 \otimes 1) \otimes_H (id_H \otimes \pi))[(1 \otimes a) * (1 \otimes b)] = (1 \otimes 1) \otimes_H (1 \otimes \pi([a, b])) -$$
$$-(1 \otimes \pi(a)) \otimes_H (1 \otimes \pi(b)) + (\pi(b) \otimes 1) \otimes_H (1 \otimes \pi(a)) = [\pi(a) * \pi(b)].$$

Hence $id_H \otimes \pi$ is a homomorphism of Lie H-pseudoalgebra. It is obvious that $ker(id_H \otimes \pi) = H \otimes ker(\pi) = \mathrm{Cur}ker(\pi)$. Thus $W(\mathfrak{m}, \pi, \mathfrak{g})/\mathrm{Cur}ker(\pi) \cong W(\mathfrak{g})$. Since $W(\mathfrak{g})$ is simple, $\mathrm{Cur}\ker(\pi)$ is a maximal ideal of $W(\mathfrak{m}, \pi, \mathfrak{g})$. Furthermore, we prove that any proper ideal of $W(\mathfrak{m}, \pi, \mathfrak{g})$ is contained in $\mathrm{Cur}\ker(\pi)$ in Section 3. ■

Next, we recall representations of a Lie H-pseudoalgebra L. Suppose that M is a left H-module. Then we say M is a representation of L if there is an H-bilinear map $\cdot * \cdot \in Hom_{H^{\otimes 2}}(L \boxtimes M, H^{\otimes 2} \otimes_H M)$ such that

$$[a * b] * m = a * (b * m) - ((12) \otimes_H 1)b * (a * m)$$

for any $a, b \in L$ and any $m \in M$. A representation M of L is also called an L-module. An L-module M is said to be *finite* if it is a finitely generated left H-module. A finite L-module is also called a finite representation of L. An L-module

V is said to be trivial if $a * x = 0$ for any $a \in L$ and $v \in V$. In other words, if V is a nontrivial representation of L, then there is $a \in L$ and $v \in V$ such that $a * v \neq 0$.

Assume that N is an H-submodule of a representation M of L. Then N is called a *subrepresentation*, or *submodule*, of M if $a * x \in H^{\otimes 2} \otimes_H N$ for any $a \in L$ and $x \in N$. A nontrivial representation M of L is said to be *irreducible*, if M has only two subrepresentations, namely 0 and M. An irreducible representation of a Lie H-pseudoalgebra L is also called a *simple L-module*.

Definition 2.5. Let U and V be two representations of a Lie pseudoalgebra L. Then a left H-module homomorphism $\phi : U \to V$ is called a homomorphism of L-modules if ϕ satisfies $((1 \otimes 1) \otimes_H \phi)(a * u) = a * \phi(u)$ for any $a \in L$ and $u \in U$. ■

Suppose that L is a Lie pseudoalgebra and $\mathbf{A}(L) = X \otimes_H L$ where $X = H^*$. Then $\mathbf{A}(L)$ is a Lie algebra with the bracket

$$[x \otimes_H a,\ y \otimes_H b] = \sum (xf_i)(yg_i) \otimes_H c_i, \tag{6}$$

for $x, y \in X$ and $a, b \in L$, where $[a * b] = \sum (f_i \otimes g_i) \otimes_H c_i$.

This Lie algebra is called the *annihilation Lie algebra* of L. Moreover, $\mathbf{A}(L)$ is a left H-module with the action given by

$$h(x \otimes_H a) = hx \otimes_H a,$$

for any $h \in H$ and $x \otimes_H a \in X \otimes_H L$. With this action, we can construct semi-product $\widehat{\mathbf{A}(L)} := \mathfrak{g} \ltimes \mathbf{A}(L)$ of Lie algebras. This Lie algebra $\widehat{\mathbf{A}(L)}$ is called the extended annihilation algebra of L.

We say that an extended annihilation algebra $\widehat{\mathbf{A}(L)}$-module V is *conformal* if for any $a \in L$, $v \in V$, $x \otimes_H a \cdot v = 0$ when $x \in F_n X$ for $n \gg 0$.

We finish this section by stating the following proposition from [1, Proposition 9.1]. It is very important in the study of representations of Lie pseudoalgebras.

Proposition 2.6. *Suppose L is a Lie pseudoalgebra. Then any L-module V has a conformal $\widehat{\mathbf{A}(L)}$-module structure defined as follows. For any $h \in \mathfrak{g}$, $x \in X$, $v \in V$ and $a \in L$, if $a * v = \sum_i (f_i \otimes g_i) \otimes_H v_i$, then*

$$(h + x \otimes_H a) \cdot v := hv + \sum_i < x, S(f_i g_{i(-1)}) > g_{i(2)} v_i. \tag{7}$$

Conversely, let V be any conformal $\widehat{\mathbf{A}(L)}$-module. Then V becomes an L-module with an H-bilinear map given by

$$a * v = \sum_M (S(\partial^M) \otimes 1) \otimes_H (x_M \otimes_H a) \cdot v,$$

where $\{x_M | M \in \mathbb{N}^n\} \subseteq X$ is the dual basis to the PBW basis $\{\partial^M | M \in \mathbb{N}^n\}$ of H. Moreover, V is irreducible as an L-module if and only if it is irreducible as an $\widehat{\mathbf{A}(L)}$-module.

3. The structure of the Lie pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})$

In this section, we investigate the structure of the Lie pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})$. Unlike the Lie pseudoalgebra $W(\mathfrak{g})$ in [1], $W(\mathfrak{m},\pi,\mathfrak{g})$ is not simple if π is not injective, as it contains a non-trivial ideal Cur$ker(\pi)$. In fact, this is the unique maximal ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$.

Proposition 3.1. *Any proper ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$ is contained in Cur$ker(\pi)$. In particular, Cur$ker(\pi)$ is the unique maximal ideal of Lie pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})$.*

Proof. Suppose that $I \nsubseteq \text{Cur}ker(\pi)$ is a proper ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$. Then there exists an element $A = \sum_{M\in\Omega} \partial^M \otimes a_M \in I\setminus \text{Cur}ker(\pi)$, where $a_M \neq 0$ for any $M \in \Omega$. Let M_0 be the maximal element (under the degree-lexicographical order) in Ω and N_0 be the maximal element in the set $\{N \in \Omega | a_N \notin ker(\pi)\}$. Then $M_0 = N_0$ or $M_0 > N_0$. If $M_0 > N_0$, then $\pi(a_{M_0}) = 0$. Choosing an element $b_0 \in \mathfrak{m}$ such that $\pi(b_0) = \partial_1 \neq 0$, we have

$$\left[\sum_M (\partial^M \otimes a_M) * (1 \otimes b_0)\right] = \sum_M (\partial^M \otimes 1) \otimes_H (-\pi(a_M) \otimes b_0 + 1 \otimes [a_M, b_0]) + \sum_M (\partial^M \pi(a_M) \otimes 1) \otimes_H (1 \otimes b_0) + \sum_M (\partial^M \partial_1 \otimes 1) \otimes_H (1 \otimes a_M) \in H^{\otimes 2} \otimes_H I. \quad (8)$$

Since $\pi(a_{M_0}) = 0$, $(\partial^{M_0}\partial_1 \otimes 1) \otimes_H (1 \otimes a_{M_0}) \in H^{\otimes 2} \otimes_H I$ by (8). Thus $\partial^{M_0} \otimes a_{M_0} \in I$. Let $A_1 = A - \partial^{M_0} \otimes a_{M_0}$. Then $A_1 \in I \setminus \text{Cur}ker(\pi)$. Repeating the previous process, we get an $A_i = \sum_{M'\in\Omega'} \partial^{M'} \otimes a_{M'} \in I \setminus \text{Cur}ker(\pi)$ for some i finally such that $max\{M' | M' \in \Omega'\} = max\{N' \in \Omega' | \pi(a_{N'}) \neq 0\}$. To simplify the notation, we may assume that $A_i = A$ and $M_0 = N_0$. Then

$$[A * (1 \otimes b)] = \sum_{M\in\Omega} (\partial^M \otimes 1) \otimes_H (-\pi(a_M) \otimes b + 1 \otimes [a_M, b]) + \sum_{M\in\Omega} (\partial^M \pi(a_M) \otimes 1) \otimes_H (1 \otimes b) \in H^{\otimes 2} \otimes_H I$$

for any $b \in ker(\pi)$. Since $\pi(a_{M_0}) \neq 0$ and M_0 is the maximal element under the degree-lexicographical order, $(\partial^{M_0}\pi(a_{M_0}) \otimes 1) \otimes_H (1 \otimes b) \in H^{\otimes 2} \otimes_H I$. Hence $1 \otimes b \in I$ for any $b \in ker(\pi)$. Thus $\text{Cur}ker(\pi) \subset I$. Since $W(\mathfrak{g})$ is simple and $W(\mathfrak{m},\pi,\mathfrak{g})/\text{Cur}ker(\pi)$ is isomorphic to $W(\mathfrak{g})$, Cur$ker(\pi)$ is a maximal ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$. It forces that $\text{Cur}ker(\pi) = I$. This contradicts the assumption that $I \nsubseteq \text{Cur}ker(\pi)$. Hence I is contained in $\text{Cur}(ker\pi)$. ■

Corollary 3.2. *The Lie pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})$ is semisimple if and only if $ker(\pi)$ is semisimple. Moreover, if $W(\mathfrak{m},\pi,\mathfrak{g})$ is semisimple, then $W(\mathfrak{m},\pi,\mathfrak{g}) \cong W(\mathfrak{g}) \ltimes$ Cur$ker(\pi)$.*

Proof. Suppose that $ker(\pi)$ is semisimple. Then Cur$ker(\pi)$ is semisimple. Since $(W(\mathfrak{m},\pi,\mathfrak{g}))$ has a simple quotient, it is not solvable. Hence $\text{Rad}(W(\mathfrak{m},\pi,\mathfrak{g}))$ is a proper ideal of $(W(\mathfrak{m},\pi,\mathfrak{g}))$. Now, by Proposition 3.1, $\text{Rad}(W(\mathfrak{m},\pi,\mathfrak{g}))$ is a solvable ideal contained in Cur$ker(\pi)$. Hence $\text{Rad}(W(\mathfrak{m},\pi,\mathfrak{g})) = 0$. On the contrary, suppose that $W(\mathfrak{m},\pi,\mathfrak{g})$ is semisimple. Since $W(\mathfrak{m},\pi,\mathfrak{g})$ has the unique maximal ideal Cur$ker(\pi)$ and $W(\mathfrak{m},\pi,\mathfrak{g})/\mathrm{Cur}ker(\pi) \cong W(\mathfrak{g})$, using [1, Corollary 13.5], one can easily obtain that $W(\mathfrak{m},\pi,\mathfrak{g})$ is either isomorphic to $W(\mathfrak{g}) \ltimes \mathrm{Cur}ker(\pi)$ or $W(\mathfrak{g}) \oplus \mathrm{Cur}ker(\pi)$ as Lie pseudoalgebras. In the latter case, since $W(\mathfrak{g})$ is a proper ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$, by Proposition 3.1, $W(\mathfrak{g}) \subset \mathrm{Cur}ker(\pi)$ which is impossible. Thus $W(\mathfrak{m},\pi,\mathfrak{g}) \cong W(\mathfrak{g}) \ltimes \mathrm{Cur}ker(\pi)$. Hence Cur$ker(\pi)$ is semisimple and $ker(\pi)$ is a semisimple Lie algebra. ∎

Proposition 3.3. *Suppose that I is a proper nonempty subset of $W(\mathfrak{m},\pi,\mathfrak{g})$. Then I is an ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$ if and only if $I = H \otimes \upsilon$ for some ideal υ of $\mathfrak{m}$ contained in $ker\pi$.*

Proof. It is straightforward to check that $I = H \otimes \upsilon$ is a proper ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$ if υ is an ideal of $\mathfrak{m}$ contained in $ker\pi$. Conversely, if I is a proper ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$, then $I \subseteq \mathrm{Cur}ker(\pi)$ by Proposition 3.1.
Assume that $A = \sum_{M\in\Omega} \partial^M \otimes a_M \in I$. Similar to the proof of Proposition 3.1, we can prove that $1 \otimes a_M \in I$ for all $M \in \Omega$. Thus $I = H \otimes \upsilon$ for some space $\upsilon \subset ker(\pi)$.
For any $b_0 \in \mathfrak{m}$ and $q \in \upsilon$, we have

$$[(1 \otimes q) * (1 \otimes b_0)] = (1 \otimes 1) \otimes_H (1 \otimes [q, b_0]) + (\pi(b_0) \otimes 1) \otimes_H (1 \otimes q).$$

So $[q, b_0] \in \upsilon$. Hence υ is an ideal of $\mathfrak{m}$ contained in $ker(\pi)$. ∎

Corollary 3.4. *Suppose $I = H \otimes \upsilon$ is a proper ideal of $W(\mathfrak{m},\pi,\mathfrak{g})$. Then the proper ideal of $W(\mathfrak{m},\pi,\mathfrak{g})/I$ has the form $H \otimes \mathfrak{n}'$ where $\mathfrak{n}'$ is an ideal of $\mathfrak{m}/\upsilon$ contained in $ker(\pi)/\upsilon$. Moreover, $W(\mathfrak{m},\pi,\mathfrak{g})/I$ is semisimple if and only if $ker(\pi)/\upsilon$ is semisimple. Moreover, if $W(\mathfrak{m},\pi,\mathfrak{g})/I$ is semisimple, then $W(\mathfrak{m},\pi,\mathfrak{g})/I \cong W(\mathfrak{g}) \ltimes Cur(ker(\pi)/\upsilon)$.*

Proof. It is an easy observation of Proposition 3.3 and Corollary 3.2. ∎

Remark 3.5. Suppose π is not epimorphic and $H' = U(im(\pi))$ is the universal enveloping algebra of the image of π. Then

$$W(\mathfrak{m},\pi,\mathfrak{g})/\mathrm{Cur}ker(\pi) \cong \mathrm{Cur}_{H'}^{H} W(im(\pi)).$$

Notice that $\mathrm{Cur}_{H'}^{H} W(im(\pi))$ is still simple. Similarly, we can prove that Proposition 3.1, Corollary 3.2, Proposition 3.3 and Corollary 3.4 hold if π is not surjective. ∎

After describing all ideals of $W(\mathfrak{m},\pi,\mathfrak{g})$, we will prove that any automorphism φ of $W(\mathfrak{m},\pi,\mathfrak{g})$ is a solution of the Maurer-Cartan equation $d\Gamma + \frac{1}{2}\Gamma \wedge \Gamma = 0$ provided that $ker(\pi)$ is a simple Lie algebra in the next theorem.

Theorem 3.6. *If $ker(\pi)$ is a simple Lie algebra, then any automorphism φ of $W(\mathfrak{m},\pi,\mathfrak{g})$ is a solution of the Maurer-Cartan equation $d\Gamma + \frac{1}{2}\Gamma \wedge \Gamma = 0$, where Γ is a $ker(\pi)$-valued 1-form on the Lie algebra $\mathfrak{m}$.*

Proof. Let φ be an automorphism of $W(\mathfrak{m},\pi,\mathfrak{g})$. Then $(1 \otimes \pi)\varphi$ is a homomorphism from Cur$ker(\pi)$ to $W(\mathfrak{g})$. Since there is no non-trivial homomorphism between $W(\mathfrak{g})$ and Cur$ker(\pi)$ by [1, Theorem 13.5], $(1 \otimes \pi)\varphi = 0$ and $\varphi(\mathrm{Cur}ker(\pi)) \subseteq \mathrm{Cur}ker(\pi)$.

Now, φ induces an automorphism $\hat{\varphi}$ of $W(\mathfrak{m},\pi,\mathfrak{g})/\text{Cur}ker(\pi)$, where

$$\hat{\varphi}(l+\text{Cur}ker(\pi)) = \varphi(l) + \text{Cur}ker(\pi)$$

for $l \in W(\mathfrak{m},\pi,\mathfrak{g})$. Since $W(\mathfrak{m},\pi,\mathfrak{g})/\text{Cur}ker(\pi)$ is isomorphic to $W(\mathfrak{g})$ and the only non-zero endomorphism of $W(\mathfrak{g})$ is the identity map by [1, Theorem 13.7], $\hat{\varphi}$ is the identity map of $W(\mathfrak{m},\pi,\mathfrak{g})/\text{Cur}ker(\pi)$. More explicitly, for any $a \in \mathfrak{m}$, there is an element $Y_a \in \text{Cur}ker(\pi)$ such that

$$\varphi(1\otimes a) = (1\otimes a) + Y_a.$$

From [1, Theorem 13.6], we obtain that $\varphi_{|\text{Cur}ker(\pi)} = 1\otimes\rho$ for some automorphism ρ of $ker(\pi)$.
We claim that $Y_a \in \mathbf{k}\otimes ker(\pi)$ for any $a \in \mathfrak{m}$. In fact, if $a \in ker(\pi)$, then $\varphi(1\otimes a) = (1\otimes\rho)(1\otimes a) = 1\otimes\rho(a) \in \mathbf{k}\otimes ker(\pi)$. Thus $Y_a = 1\otimes(\rho(a)-a) \in \mathbf{k}\otimes ker(\pi)$. Next, we assume that $\pi(a) \neq 0$ and $Y_a = \sum_M \partial^M \otimes K_M$ for some $K_M \in ker(\pi)$. Let $b \in ker(\pi)$. Then

$$\begin{aligned}
&[\varphi(1\otimes a) * \varphi(1\otimes b)] = [((1\otimes a)+Y_a) * (1\otimes\rho(b))]\\
&= (1\otimes 1)\otimes_H (1\otimes[a,\rho(b)]) - (1\otimes\pi(a))\otimes_H(1\otimes\rho(b))\\
&\quad + \sum_M (\partial^M\otimes 1)\otimes_H(1\otimes[K_M,\rho(b)])
\end{aligned}$$

and

$$(1\otimes 1\otimes_H\varphi)[(1\otimes a)*(1\otimes b)] = (1\otimes 1)\otimes_H(1\otimes\rho([a,b])) - (1\otimes\pi(a))\otimes_H(1\otimes\rho(b)).$$

Since $[\varphi(1\otimes a)*\varphi(1\otimes b)] = (1\otimes 1\otimes_H\varphi)[(1\otimes a)*(1\otimes b)]$, we have

$$\sum_{M,|M|>0} (\partial^M\otimes 1)\otimes_H(1\otimes[K_M,\rho(b)]) = 0.$$

Thus $[K_M,\rho(b)] = 0$ for any $|M|>0$ and any $b \in ker(\pi)$. As ρ is an automorphism of $ker(\pi)$, K_M is contained in the center of the Lie algebra $ker(\pi)$. Since $ker(\pi)$ is simple, $K_M = 0$ for all $|M|>0$. By now, we have completed the proof of the previous claim, that is, $\varphi(1\otimes a) = 1\otimes a + 1\otimes Y_a$ for any $a\in\mathfrak{m}$, where $Y_a \in ker(\pi)$.
To complete our proof, assume that $a,b\in\mathfrak{m}$. Then

$$\begin{aligned}
&(1\otimes 1\otimes_H\varphi)[(1\otimes a)*(1\otimes b)]\\
&= (1\otimes 1)\otimes_H(1\otimes([a,b]+Y_{[a,b]})) - (1\otimes\pi(a))\otimes_H(1\otimes(b+Y_b))\\
&\quad + (\pi(b)\otimes 1)\otimes_H(1\otimes(a+Y_a))\\
&= [\varphi(1\otimes a)*\varphi(1\otimes b)]\\
&= (1\otimes 1)\otimes_H(1\otimes([a,b]+[a,Y_b]+[Y_a,b]+[Y_a,Y_b])) - (1\otimes\pi(a))\otimes_H(1\otimes b)\\
&\quad + (\pi(b)\otimes 1)\otimes_H(1\otimes a) + (\pi(b)\otimes 1)\otimes_H(1\otimes Y_a) - (1\otimes\pi(a))\otimes_H(1\otimes Y_b).
\end{aligned}$$

Comparing the term $(1\otimes 1)\otimes_H 1\otimes *$ in the two sides of the above equation, we get

$$Y_{[a,b]} = [Y_a,Y_b] + [a,Y_b] - [b,Y_a]. \tag{9}$$

Regarding $ker\pi$ as an $\mathfrak{m}$-module, we can define a $ker(\pi)$-valued 1-form Γ_0 via

$$\Gamma_0 : \mathfrak{m} \to ker\pi, \qquad a \mapsto Y_a.$$

Then (9) implies that Γ_0 is a solution to the equation $d\Gamma + \frac{1}{2}\Gamma\wedge\Gamma = 0$. ■

In the remainder of this section, we will determine all central extensions of $W(\mathfrak{m},\pi,\mathfrak{g})$ by the trivial representation $\mathbf{k}$. To achieve our purpose, we just need to measure the second cohomological group $H^2(W(\mathfrak{m},\pi,\mathfrak{g}),\mathbf{k})$. Recall that a central extension of $W(\mathfrak{m},\pi,\mathfrak{g})$ by trivial coefficients is a Lie pseudoalgebra $W(\mathfrak{m},\pi,\mathfrak{g})^e$, where $W(\mathfrak{m},\pi,\mathfrak{g})^e = W(\mathfrak{m},\pi,\mathfrak{g}) \oplus \mathbf{k}$ as direct sum of left H-modules. In addition, for $x,y \in W(\mathfrak{m},\pi,\mathfrak{g})$ and $u,v \in \mathbf{k}$, the pseudobracket $\hat{*}$ on $W(\mathfrak{m},\pi,\mathfrak{g})^e$ is defined via

$$[(x+u)\hat{*}(y+v)] = [x*y] + (\beta(x,y)\otimes 1)\otimes_H 1,$$

for some $\beta : W(\mathfrak{m},\pi,\mathfrak{g})^{\otimes 2} \to H$. Obviously, β has the following properties

$$\beta(hx,y) = h\beta(x,y), \quad \beta(x,hy) = \beta(x,y)S(h), \quad \beta(x,y) = -S(\beta(y,x)),$$

for any $x,y \in W(\mathfrak{m},\pi,\mathfrak{g})$, $h \in H$, where S is the antipode of the Hopf algebra H. Since $W(\mathfrak{m},\pi,\mathfrak{g})$ is a free H-module, β is determined by its value on basis. For simplicity, we use a and b to denote the element $1\otimes a, 1\otimes b \in W(\mathfrak{m},\pi,\mathfrak{g})$ in $\beta(1\otimes a, 1\otimes b)$ respectively. Thus, $\beta(a,b)$ means $\beta(1\otimes a, 1\otimes b)$.

Theorem 3.7. *Suppose that $ker(\pi)$ is a finite-dimensional simple Lie algebra. Then*

$$H^2(W(\mathfrak{m},\pi,\mathfrak{g}),\mathbf{k}) = \begin{cases} 0, & dim(\mathfrak{g}) > 1, \\ \mathbf{k}\oplus\mathbf{k}, & dim(\mathfrak{g}) = 1. \end{cases}$$

Proof. For any $a \notin ker(\pi)$, $b,c \in ker(\pi)$,

$$\begin{aligned} &[[(1\otimes a)\hat{*}(1\otimes b)]\hat{*}(1\otimes c)] \\ &= [[(1\otimes a)*(1\otimes b)]*(1\otimes c)] + (\Delta(\beta([a,b],c))\otimes 1)\otimes_H 1 \\ &\qquad - (1\otimes\pi(a)\otimes 1)(\Delta(\beta(b,c))\otimes 1)\otimes_H 1, \end{aligned} \tag{10}$$

$$\begin{aligned} &[(1\otimes a)\hat{*}[(1\otimes b)\hat{*}(1\otimes c)]] \\ &= [(1\otimes a)*[(1\otimes b)*(1\otimes c)] + (\beta(a,[b,c])\otimes 1\otimes 1)\otimes_H 1. \end{aligned} \tag{11}$$

$$\begin{aligned} &[(1\otimes b)\hat{*}[(1\otimes a)\hat{*}(1\otimes c)]] \\ &= [(1\otimes b)*[(1\otimes a)*(1\otimes c)]] + (\beta(b,[a,c])\otimes 1\otimes 1)\otimes_H 1 \\ &\qquad - (1\otimes 1\otimes\pi(a))(\beta(b,c)\otimes 1\otimes 1)\otimes_H 1. \end{aligned} \tag{12}$$

Since

$$\begin{aligned} &[[(1\otimes a)\hat{*}(1\otimes b)]\hat{*}(1\otimes c))] \\ &= [(1\otimes a)\hat{*}[(1\otimes b)\hat{*}(1\otimes c)]] - ((12)\otimes_H 1)[(1\otimes b)\hat{*}[(1\otimes a)\hat{*}(1\otimes c)]], \end{aligned}$$

(10)–(12) implies that

$$\begin{aligned} &\Delta(\beta([a,b],c)) - (1\otimes\pi(a))\Delta(\beta(b,c)) \\ &= \beta(a,[b,c])\otimes 1 - 1\otimes\beta(b,[a,c]) - \pi(a)\otimes\beta(b,c) - 1\otimes\beta(b,c)\pi(a). \end{aligned} \tag{13}$$

As $\beta(b,c)$, $\beta([a,b],c)$, $\beta(b,[a,c])$ belong to $\mathfrak{g}$ by [1, Proposition 15.8], we get that $\pi(a)\otimes\beta(b,c) = \beta(b,c)\otimes\pi(a)$. If $\dim(\mathfrak{g}) > 1$, then $\beta(b,c) = 0$ and $\beta(a,[b,c]) = 0$. Since $ker(\pi)$ is simple, $[ker(\pi), ker(\pi)] = ker(\pi)$. Thus $\beta(a,b) = 0$ for any $b \in ker(\pi)$. Hence

$$H^2(W(\mathfrak{m},\pi,\mathfrak{g}),\mathbf{k}) = H^2(W(\mathfrak{g}),\mathbf{k}) = 0$$

when $\dim(\mathfrak{g}) > 1$ by [1, Theorem 15.2]. If $\dim(\mathfrak{g}) = 1$, then $\mathfrak{g} = \mathbf{k}\pi(a)$.

Thus, $$\beta([a,b],c)=\beta(a,[b,c])=-\beta(b,[a,c]) \tag{14}$$
by (13). Since a acts on $\ker(\pi)$ as a derivation of $\ker(\pi)$, there exists some $a'\in ker(\pi)$ such that $[a,b]=[a',b]$ for any $b\in ker(\pi)$. Now (14) implies that
$$\beta(a,[b,c])=\beta([a,b],c)=\beta([a',b],c)=-\beta([b,a'],c)=\beta(a',[b,c]). \tag{15}$$
The last equation follows from the fact that the restriction of β to $\mathrm{Cur}ker(\pi)$ is an invariant bilinear form with values in $\mathfrak{g}$. Since $[ker(\pi),ker(\pi)]=ker(\pi)$, $\beta(a,b)=\beta(a',b)$ for all $b\in ker(\pi)$ by (15). In other words, the value of $\beta(a,\mathrm{Cur}ker(\pi))$ is completely determined by that of $\beta(\mathrm{Cur}ker(\pi),\mathrm{Cur}ker(\pi))$. In this case, we have $H^2(W(\mathfrak{m},\pi,\mathfrak{g}),\mathbf{k})\cong H^2(W(\mathfrak{g}),\mathbf{k})\oplus H^2(\mathrm{Cur}ker(\pi),\mathbf{k})=\mathbf{k}\oplus\mathbf{k}$ by [1, Theorem 15.2]. ■

4. Representations of $W(\mathfrak{m},\pi,\mathfrak{g})$

In this section, we study representations of $W(\mathfrak{m},\pi,\mathfrak{g})$. First, we deal with the special case that $\mathfrak{m}=\mathfrak{g}\oplus ker(\pi)$ as Lie algebras. To simplify notation, we use $\mathfrak{n}$ to denote $ker(\pi)$ and $W(\mathfrak{m},\mathfrak{n},\mathfrak{g})$ to denote $W(\mathfrak{m},\pi,\mathfrak{g})$ in this special case. Explicitly, $W(\mathfrak{m},\mathfrak{n},\mathfrak{g})\cong W(\mathfrak{g})\ltimes \mathrm{Cur}\mathfrak{n}$ where the action of $W(\mathfrak{g})$ on $\mathrm{Cur}(\mathfrak{n})$ is given by:
$$[(f\otimes a)*(g\otimes b)]=-(f\otimes ga)\otimes_H(1\otimes b),$$
for $f\otimes a\in W(\mathfrak{g})$ and $g\otimes b\in \mathrm{Cur}\mathfrak{n}$.
For convenience, we use $\mathfrak{L}$ to denote the extended annihilation Lie algebra
$$\mathbf{A}(\widehat{W(\mathfrak{m},\mathfrak{n},\mathfrak{g})})=\mathfrak{g}\ltimes(X\otimes_H W(\mathfrak{m},\mathfrak{n},\mathfrak{g}))\ \text{ of }\ W(\mathfrak{m},\mathfrak{n},\mathfrak{g}).$$
The filtration of X gives a filtration on $\widehat{\mathfrak{L}}:=\mathfrak{g}\ltimes\mathfrak{L}$ as follows. Set $\widehat{\mathfrak{L}}_{-1}=\widehat{\mathfrak{L}}$, $\widehat{\mathfrak{L}}_p=\mathfrak{L}_p$ for $p\geq 0$ and
$$\mathfrak{L}_p=F_pX\otimes_H(H\otimes\mathfrak{g})+F_{p-1}X\otimes_H(H\otimes\mathfrak{n})=F_pX\otimes\mathfrak{g}+F_{p-1}X\otimes\mathfrak{n}, \tag{16}$$
for $p=-1,0,1,\cdots$. It is easy to check that
$$[\mathfrak{L}_p,\mathfrak{L}_q]\subset\mathfrak{L}_{p+q},\quad \text{for } p,q\geq -1, \tag{17}$$
$\mathfrak{L}_{-1}=\mathfrak{L}$ and $\mathfrak{L}/\mathfrak{L}_0\cong\mathbf{k}\otimes\mathfrak{g}\cong\mathfrak{g}$, $\mathfrak{L}_0/\mathfrak{L}_1\cong\mathfrak{g}^*\otimes\mathfrak{g}\oplus\mathfrak{n}$ as vector spaces. Let $\{\partial_1,\partial_2,\cdots,\partial_n\}$ be a basis of $\mathfrak{g}$ and $\{x^i\}_{i=1}^n\subseteq\mathfrak{g}^*$ be the dual basis of $\{\partial_i\}_{i=1}^n$. Suppose $e_i^j\in\mathfrak{gl}(\mathfrak{g})$ $(1\leq i,j\leq n)$ are linear transformations of $\mathfrak{g}$ satisfying $e_i^j\partial_t=\delta_t^j\partial_i$ for $t=1,2,\cdots,n$. Then the mapping
$$\varsigma:\mathfrak{g}\otimes\mathfrak{g}^*\to\mathfrak{gl}(\mathfrak{g}),\partial_i\otimes x^j\mapsto e_i^j$$
is an isomorphism of vector spaces. The isomorphism ς conveys the Lie algebra structure of $\mathfrak{gl}(\mathfrak{g})$ to the vector space $\mathfrak{g}\otimes\mathfrak{g}^*$. Let $\phi:\mathfrak{L}_0/\mathfrak{L}_1\to\mathfrak{g}\otimes\mathfrak{g}^*\oplus\mathfrak{n}$ be the mapping defined by
$$x\otimes a\mod\mathfrak{L}_1\mapsto -b\otimes(x\bmod F_1X),\quad \text{for } x\in F_0X,\ a\in\mathfrak{g}.$$
$$y\otimes b\mod\mathfrak{L}_1\mapsto(y\mod F_0X)\otimes b,\quad \text{for } y\in F_{-1}X,\ b\in\mathfrak{n}.$$
Then ϕ is an isomorphism of Lie algebras by [2, Corollary 3.1].
Let $W=\mathbf{A}(W(\mathfrak{g}))=X\otimes_H(H\otimes\mathfrak{g})=X\otimes\mathfrak{g}$ be the annihilation Lie algebra of $W(\mathfrak{g})$. Then W is a subalgebra of $\mathfrak{L}$ with filtration $W_p:=W\cap\mathfrak{L}_p$ for all $p\geq -1$. It is well-known that W is a linearly compact simple Lie algebra, and all derivations of W are inner. Set $\widehat{W}:=\mathfrak{g}\ltimes W\subset\widehat{\mathfrak{L}}$. Let $r:\mathfrak{g}\to W$ be the homomorphism of Lie algebras such that $[\partial,w]=[r(\partial),w]$ for any $\partial\in\mathfrak{g}$ and any $w\in W$.

Then $\overline{\partial} = \partial - r(\partial) \in \widehat{W}$ for $\partial \in \mathfrak{g}$. Suppose that $\overline{\mathfrak{g}} = (id - r)(\mathfrak{g})$ and $N_{\mathfrak{L}} = \mathfrak{L}_0 \oplus \overline{\mathfrak{g}}$. Then $\overline{\mathfrak{g}} \leqslant \widehat{W}$ is isomorphic to $\mathfrak{g}$. With the above preparations, we have the following propositions.

Proposition 4.1. *Suppose that* $A = \partial + 1 \otimes \partial - \overline{\partial} \in \widehat{\mathfrak{L}} = \mathfrak{g} \ltimes (X \otimes \mathfrak{m})$ *for* $\partial \in \mathfrak{g}$. *Then* $A \in \mathfrak{L}_0$ *and the adjoint action of* A *on* $\mathfrak{g} \cong \mathfrak{L}/\mathfrak{L}_0$ *is equal to the action of* $ad\partial$ *on* $\mathfrak{g} \cong \mathfrak{L}/\mathfrak{L}_0$.

Proof. This is similar to [2, Lemma 3.3]. ∎

Proposition 4.2. *For every* $p \geq 0$, *the normalizer of* $\mathfrak{L}_p$ *in* $\widehat{\mathfrak{L}}$ *is equal to* $N_{\mathfrak{L}}$.

Proof. This is similar to [2, Proposition 3.3]. ∎

Recall that an $N_{\mathfrak{L}}$-module V is said to be conformal if every $v \in V$ is killed by some $\mathfrak{L}_p$ for $p \gg 0$.

Proposition 4.3. *The subalgebra* $\mathfrak{L}_1 \subset N_{\mathfrak{L}}$ *acts trivially on any irreducible finite dimensional conformal* $N_{\mathfrak{L}}$*-module. Irreducible finite-dimensional conformal* $N_{\mathfrak{L}}$*-modules correspond to irreducible finite dimensional conformal modules over* $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g}) \cong N_{\mathfrak{L}}/\mathfrak{L}_1$.

Proof. This is similar to [2, Proposition 3.4]. ∎

Let V be a representation of $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$ and $singV = \{v \in V | \mathfrak{L}_1 \cdot v = 0\}$. We call elements of $singV$ singular vectors of V. It is easy to see that $singV$ is an $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g}) \cong N_{\mathfrak{L}}/\mathfrak{L}_1$-module. The action of this module determines a Lie algebra homomorphism $\rho_{sing} : \mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g}) \to \mathfrak{gl}(singV)$. ρ_{sing} satisfies the following two conditions

$$\rho_{sing}(\partial)v = \overline{\partial} \cdot v, \ \rho_{sing}(\beta)v = (1 \otimes \beta) \cdot v, \quad \text{for } \partial \in \mathfrak{g}, \ \beta \in \mathfrak{n}, \tag{18}$$

and

$$\rho_{sing}(e_i^j)v = -(x^j \otimes \partial_i) \cdot v, \ i = 1, 2, \cdots, n, \ j = 1, 2, \cdots, n. \tag{19}$$

Proposition 4.4. *Suppose* V *is a non-trivial finite* $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$*-module. Then* $singV \neq \{0\}$ *and* $singV/kerV$ *is finite dimensional, where* $kerV = \{v \in V | \mathfrak{L} \cdot v = 0\}$.

Proof. This is similar to [2, Theorem 6.1]. ∎

Theorem 4.5. *Let* V *be a finite* $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$*-module. For every singular vector* $v \in singV$, *the action of* $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$ *on* v *is given by*

$$(1 \otimes \beta) * v = (1 \otimes 1) \otimes_H \rho_{sing}(\beta)v, \quad \textit{for } \beta \in \mathfrak{n}, \tag{20}$$

and

$$\begin{aligned}(1 \otimes \partial_i) * v = &\sum_{j=1}^{n} (\partial_j \otimes 1) \otimes_H \rho_{sing}(e_i^j)v - (1 \otimes 1) \otimes_H \partial_i v \\ &+ (1 \otimes 1) \otimes_H \rho_{sing}(\partial_i + ad\partial_i)v, \quad 1 \leq i \leq n.\end{aligned} \tag{21}$$

Proof. Let $\beta \in \mathfrak{n}$ and $v \in singV$. Then

$$(1 \otimes \beta) * v = (1 \otimes 1) \otimes_H (1 \otimes \beta) \cdot v. \tag{22}$$

by Proposition 2.6 and (16). Thus, by (18) and (16),

$$(1 \otimes \beta) * v = (1 \otimes 1) \otimes_H \rho_{sing}(\beta)v. \tag{23}$$

Similarly, for $\alpha \in \mathfrak{g}$, we have

$$(1 \otimes \partial) * v = \sum_{j=1}^{n} (-\partial_j \otimes 1) \otimes_H (x^j \otimes \partial) \cdot v + (1 \otimes 1) \otimes_H (1 \otimes \partial) \cdot v. \tag{24}$$

Then Proposition 4.1 implies that

$$(\partial_i + 1 \otimes \partial_i - \overline{\partial}_i) \cdot v = \rho_{sing}(ad\partial_i)v \tag{25}$$

for $1 \le i \le n$. Combining (18),(19),(25), we get (21). ■

Corollary 4.6. *Suppose that V is a finite $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module. If U is a submodule of the $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-module $singV$, then HU is a $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module. Thus $V = HU$ if V is an irreducible $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module.*

Proof. From (20)–(21), we obtain that $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g}) * U \subset H \otimes H \otimes_H (HU)$. Thus, $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g}) * HU \subset H \otimes H \otimes_H (HU)$. Consequently, HU is a submodule of the $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module V. ■

Then, we will prove that every finite $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module can be controlled by some irreducible finite-dimensional $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-module. For this purpose, we define a $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module $\mathbf{V}(R)$ by a $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-module. Suppose that R is a finite-dimensional $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-module with action ρ_R and $\mathbf{V}(R) = H \otimes R$, where $\mathbf{V}(R)$ is a free H-module with the action given by left multiplying on the first factor. Define an action of $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$ on $\mathbf{V}(R)$ via

$$(1 \otimes \beta) * (1 \otimes a) = (1 \otimes 1) \otimes_H (1 \otimes \rho_R(\beta)a), \quad \text{for } \beta \in \mathfrak{n},\ a \in R, \tag{26}$$

and

$$(1 \otimes \partial_i) * (1 \otimes a) = \sum_{j=1}^{n} (\partial_j \otimes 1) \otimes_H (1 \otimes \rho_R(e_i^j)a) -$$
$$-(1 \otimes 1) \otimes_H (\partial_i \otimes a) + (1 \otimes 1) \otimes_H (1 \otimes \rho_R(\partial_i + ad\partial_i)a), \tag{27}$$

for $1 \le i \le n,\ a \in R$. It is easy to check that $\mathbf{V}(R)$ is a $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module with the action given by (26) and (27). Using the module $\mathbf{V(R)}$, we have the following result.

Theorem 4.7. *If V is a finite irreducible $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-module, then V is a quotient module of $\mathbf{V}(R)$, where R is an irreducible finite-dimensional $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-module.*

Proof. Since $singV \neq \{0\}$ is finite dimensional by Proposition 4.4, we can choice R to be a non-zero irreducible $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-submodule of $singV$. Now by Corollary 4.6, $V = HR$. Consequently, the map

$$p : \mathbf{V}(R) = H \otimes R \to HR = V, \quad h \otimes a \mapsto ha$$

is surjective. Comparing (26),(27) with (20) and (21), one can easily prove that p is an epimorphism of $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-modules. ■

Next, we prove that $\mathbf{V}(R)$ is irreducible under some additional conditions.

Lemma 4.8. *If R is an irreducible finite-dimensional $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-module and $\mathfrak{n}$ acts non-trivially on R, then $sing\mathbf{V}(R) = \mathbf{k} \otimes R$.*

Proof. Obviously, $\mathbf{k} \otimes R \subset sing\mathbf{V}(R)$. Let $u = \sum_{M \in \Omega} \partial^M \otimes v_M$ be an element of $sing\mathbf{V}(R)$. Let M_0 be the maximal element (under the degree-lexicographical order) of Ω. Suppose $\rho_R(a)v_{M_0} = 0$ for any $a \in \mathfrak{n}$. Then $\mathfrak{n}$ acts trivially on the $\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{m}$-submodule $U(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g})v_{M_0}$ of R. Since R is assumed to be irreducible, $U(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g})v_{M_0} = R$. This is impossible. Hence we can find an element $b \in \mathfrak{n}$ such that $\rho_R(b)v_{M_0} \neq 0$. Now, (26) forces that $|M_0| = 0$. This implies that $sing\mathbf{V}(R) \subset \mathbf{k} \otimes R$. ■

Proposition 4.9. *If R is an irreducible finite-dimensional $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{g})$-module and $\mathfrak{n}$ acts non-trivially on R, then $\mathbf{V}(R)$ is irreducible.*

Proof. Suppose M is a non-zero $W(\mathfrak{m}, \mathfrak{n}, \mathfrak{g})$-submodule of $\mathbf{V}(R)$. Then $singM \subset sing\mathbf{V}(R) = \mathbf{k} \otimes R$ by Lemma 4.8. Since R is irreducible, $singM = \mathbf{k} \otimes R$. Thus $\mathbf{V}(R) = HsingM \subset M$. Hence $M = \mathbf{V}(R)$. ■

Finally, we are going to determine all finite irreducible $W(\mathfrak{m}, \pi, \mathfrak{g})$-modules, where $\pi : \mathfrak{m} \to \mathfrak{g}$ is an epimorphism of any two finite-dimensional Lie algebras with pseudobracket (1). Before that, we give an important lemma.

Lemma 4.10. *Any non-trivial irreducible finite representation of $W(\mathfrak{m}, \pi, \mathfrak{g})$ factors through a quotient isomorphic to a semi-direct product $W(\mathfrak{g}) \ltimes Cur\mathfrak{n}'$, where $\mathfrak{n}'$ is a reductive quotient of $\mathfrak{n}$ whose center is of dimension at most one.*

Proof. Suppose V is an irreducible finite representation of $W(\mathfrak{m}, \pi, \mathfrak{g})$. Then there is a homomorphism of Lie pseudoalgebras $\phi\colon W(\mathfrak{m}, \pi, \mathfrak{g}) \to gc(V)$ (see [1, Proposition 10.2]). Hence $L := W(\mathfrak{m}, \pi, \mathfrak{g})/ker(\phi)$ acts faithfully and irreducibly on V. On the other hand, by Proposition 3.3, $ker(\phi) = H \otimes \upsilon$ for some ideal υ of $\mathfrak{m}$ contained in $ker(\pi)$. Let $\mathfrak{n}'$ be the quotient $ker(\pi)/\upsilon$. If L is semisimple, by Corollary 3.4, $L \cong W(\mathfrak{g}) \ltimes \mathrm{Cur}\mathfrak{n}'$ where $\mathfrak{n}'$ is semisimple in this case. If L is not semisimple, by virtue of [1, Theorem 14.2(ii)],

$$L \cong L/\mathrm{Rad}(L) \ltimes \mathrm{Rad}(L) \subset W(\mathfrak{g}) \ltimes \mathrm{Cur}\mathfrak{gl}(R)$$

such that $\mathrm{Rad}(L)$ is an H-module of rank one, where R is a finite dimensional subspace of V. Since $\mathrm{Rad}(L) = \mathrm{CurRad}(\mathfrak{n}')$ by Corollary 3.4, $\mathrm{dimRad}(\mathfrak{n}') = 1$ and $L/\mathrm{Rad}(L) \cong W(\mathfrak{g}) \ltimes \mathrm{Cur}\mathfrak{s}$ where $\mathfrak{s}$ is the Lie algebra $\mathfrak{n}'/\mathrm{Rad}(\mathfrak{n}')$. Thus $\mathfrak{s}$ is semisimple or 0 which completes the proof. ■

Theorem 4.11. *Let $\mathfrak{t} = \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g} \oplus ker(\pi)$ be a direct sum of three Lie algebras. Any non-trivial finite irreducible $W(\mathfrak{m}, \pi, \mathfrak{g})$-module V is isomorphic to one of the following cases:*

(1) *If $Curker(\pi)$ acts non-trivially on V, then $V \cong \mathbf{V}(R)$, where R is an irreducible finite dimensional $\mathfrak{t}$-module such that $ker(\pi)$ acts non-trivially on R.*

(2) *If $Curker(\pi)$ acts trivially on V, then V is a finite irreducible module of $W(\mathfrak{g})$.*

Proof. The second claim is clear and the classification of finite irreducible $W(\mathfrak{g})$-modules was accomplished in [2]. We focus on the first claim. By Lemma 4.10, any non-trivial finite irreducible $W(\mathfrak{m}, \pi, \mathfrak{g})$-module V factors through a quotient isomorphic to a semi-direct product $W(\mathfrak{g}) \ltimes \mathrm{Cur}\mathfrak{n}'$, where $\mathfrak{n}'$ is a reductive quotient of $ker(\pi)$ with a projection map ϕ. Let R be an irreducible $\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}\oplus\mathfrak{n}'$-submodule of $singV$. Then $V = HR$ by Corollary 4.6. By Theorem 4.7, V is a non-zero quotient of $\mathbf{V}(R)$. On the other hand, if $\mathrm{Cur}ker(\pi)$ acts non-trivially on V, $\mathfrak{n}'$ acts non-trivially on R. Thus $\mathbf{V}(R)$ is irreducible by Proposition 4.9. This implies that $V \cong \mathbf{V}(R)$. Finally, define $pr : \mathfrak{t} \to \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g} \oplus \mathfrak{n}'$ by $pr(A, b, c) = (A, b, \phi(c))$ for $A \in \mathfrak{gl}(\mathfrak{g})$, $b \in \mathfrak{g}$ and $c \in ker(\pi)$. Then pr is a surjection of Lie algebras and R admits an irreducible $\mathfrak{t}$-module structure induced by pr. ∎

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