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A Lie-Theoretic Construction of Cartan-Moser Chains

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Abstract. Let $M^3 \subset \mathbb{C}^2$ be a real-analytic Levi nondegenerate hypersurface. In the literature, Cartan-Moser chains are detected from rather advanced considerations: either from the construction of a Cartan connection associated with the CR equivalence problem; or from the construction of a formal or converging Poincaré-Moser normal form.
This note provides an alternative direct elementary construction, based on the inspection of the Lie prolongations of 5 infinitesimal holomorphic automorphisms to the space of second order jets of CR-transversal curves. Within the 4-dimensional jet fiber, the orbits of these 5 prolonged fields happen to have a simple cubic 2-dimensional degenerate exceptional orbit, the *chain locus:*

$$\Sigma_0 := \big\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 :\ x_2 = -2x_1^2 y_1 - 2y_1^3,\ y_2 = 2x_1 y_1^2 + 2x_1^3\big\}.$$

Using plain translations, we may capture all points by working *only at one point*, the origin, and computations become conceptually enlightening and simple.

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Key Words: Lie prolongations of vector fields, Cauchy-Riemann manifolds, local biholomorphic equivalences, formal and convergent normal forms.

1. Introduction

The goal of this article is to present a simplified construction of *Cartan-Moser chains*, which are certain distinguished curves in Levi nondegenerate Cauchy-Riemann (CR) manifolds of hypersurface type. We concentrate on real-analytic embedded CR manifolds, because the interaction between the *extrinsic* geometry of an ambient complex manifold X and the *intrinsic* geometry of a CR submanifold $M \subset X$ is *richer* than in an abstract seetting. Also, for the sake of intuitive clarity and for elementariness, we restrict our presentation to the 3-dimensional case. The Lie-theoretical method that we employ – which certainly has a wider scope – drastically contracts all required computations by working *only at one point*, as we shall rapidly see.

Thus, let $M^3 \subset \mathbb{C}^2$ be a $\mathcal{C}^\omega$ real hypersurface. We are interested in results of a local nature, hence we will allow to shrink neighborhoods of various points $p \in M$. If $J\colon T\mathbb{C}^2 \longrightarrow T\mathbb{C}^2$ is the standard complex structure, with $J^2 = -\mathrm{Id}$, the complex tangent bundle $T^cM := TM \cap JTM$ is J-invariant of real rank 2, hence at all point $p \in M$, the 2-planes $T^c_pM \subset T\mathbb{C}^2$ can be viewed as complex affine sublines $\mathbb{C} \subset \mathbb{C}^2$.

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Also, $T^{1,0}M := \{X - i\,JX \colon X \in T^cM\}$ and $T^{0,1}M := \{X + i\,JX \colon X \in T^cM\} = \overline{T^{1,0}}M$ are complex vector subbundles of the complexified tangent bundle $\mathbb{C} \otimes_{\mathbb{R}} TM$.

We will always assume that $M^3 \subset \mathbb{C}^2$ is *Levi nondegenerate*, namely that $T^cM + [T^cM, T^cM] = TM$, or equivalently [22]:

$$\mathbb{C} \otimes_{\mathbb{R}} TM \;=\; T^{1,0}M + T^{0,1}M + [T^{1,0}M,\, T^{0,1}M].$$

For detailed foundations, the reader may consult [22].

These "*CR bundles*" are invariant, in the sense that for any (local) biholomorphism $h \colon \mathbb{C}^2 \longrightarrow \mathbb{C}'^2$ defined in some neighborhood of M, with $M' := h(M)$ being a hypersurface of $\mathbb{C}'^2$, one has $h_*(T_p^cM) = T_{h(p)}^c M'$, and $h_*(T_p^{1,0}M) = T_{h(p)}^{1,0} M'$ as well, where, by h_*, we denote the differential of h acting both on TM and on $\mathbb{C} \otimes_{\mathbb{R}} TM$, with the convention $\overline{h}_* = h_*$, *cf.* [22]. Hence, whenever h is a (local) biholomorphism, $h|_M \colon M \longrightarrow h(M)$ realizes a CR diffeomorphism.

So by definition, biholomorphic or CR equivalences stabilize some *horizontal* 2-plane distribution T^cM, or the pair $T^{1,0}M \oplus T^{0,1}M \subset \mathbb{C}TM$. It seems that there is no reason that there should exist some *CR-transversal* structure which would also be CR-invariant. For instance, does there exist a line field $\{\ell_p\}_{p \in M}$ with $\mathbb{R} \cong \ell_p \subset T_pM$ complementing T_p^cM in $T_pM = \ell_p \oplus T_p^cM$ which would be CR invariant? *Yes*, of course in presence of some extra structure like e.g. a Riemannian metric on M – just take $\ell_p := [T_p^cM]^\perp$ –, but *no* in general, as is well known and as we will see.

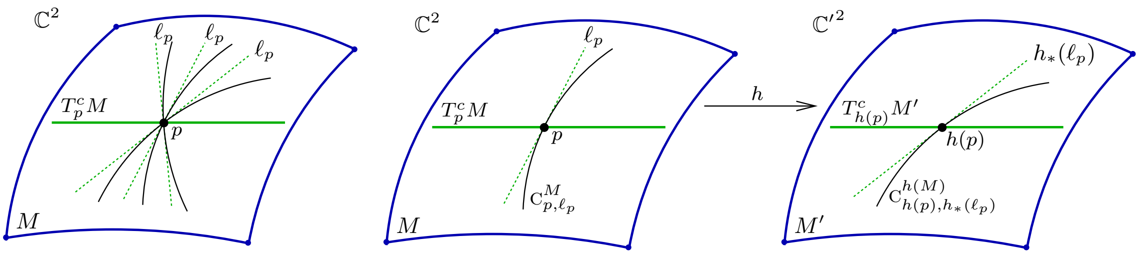

FIGURE 1: Left: representation of various chains at $p \in M$ directed by various directions $\ell_p \subset T_pM$ with $\mathbb{R}\ell_p + T_p^cM = T_pM$. Right: representation of the transfer of a chain and its direction through an ambient biholomorphism $h \colon \mathbb{C}^2 \longrightarrow \mathbb{C}'^2$, making a CR-diffeomorphism $h|_M \colon M \longrightarrow M' := h(M)$.

Élie Cartan [7, 5, 8] discovered that nevertheless, there do exist certain invariant CR-transversal curves, called *chains*, namely unparametrized curves c_{p,ℓ_p} uniquely determined at each $p \in M$ and for each line $\ell_p \ni p$ complementary to T_p^cM such that the nonzero tangent vector $\dot{\mathrm{c}}_{p,\ell_p}$ is directed by ℓ_p, but their existence always remained a bit mysterious. This unique determination is similar to that for a scalar second order ODE $\ddot{y} = H(t, y, \dot{y})$ for which a starting point $y(0)$ and a starting vector $\dot{y}(0)$ must be prescribed, but here, since M is 3-dimensional, chains are defined by a *system* of *two* scalar second order ODEs, as we now explain.

One may equip $\mathbb{C}^2$ with affine coordinates $(z, w) = (x + i\,y,\, u + i\,v)$, centered at some reference point $p_0 = 0 \in M$ so that the projection $T_0M \longrightarrow \mathbb{R}^3_{x,y,u}$ gives a local chart on M^3 near the origin and even so that $T_0M = \mathbb{C}_z \times (\mathbb{R}_u + i\,\{0\})$, whence $T_0^cM = \mathbb{C}_z \times \{0\}$. Then M can be $\mathcal{C}^\omega$ graphed as:

$$v \;=\; F(z, \overline{z}, u) \;=\; \sum_{j+k+l \geqslant 1} F_{j,k,l}\, z^j \overline{z}^k u^l \qquad\qquad (\overline{F_{k,j,l}} \;=\; F_{j,k,l})$$

Since $T_0^c M = \{u = 0\}$ within $T_0 M = \mathbb{R}^3_{x,y,u}$, any CR-transversal curve may be parametrized as $t \longmapsto (x(t), y(t), t)$ with $u(t) \equiv t$. One may show that there exist certain functions A and B such that the equations of chains write as a system:

$$\ddot{x} = A\big(t, x, y, \dot{x}, \dot{y}\big), \qquad\qquad \ddot{y} = B\big(t, x, y, \dot{x}, \dot{y}\big),$$

but the explicit expressions of A and B in terms of F and its derivatives are huge, never shown in the literature [part of the mystery]. This is because chains are considered at *every* point $p \in M$ near $p_0 = 0 \in M$, which requires hard elimination computations in the commutative differential ring with variables $\big\{F_{z^j \overline{z}^k u^l}\big\}_{j,k,l \in \mathbb{N}}$ generated by the derivatives of F. As shown in [1, 23] the explicit expression of Cartan's primary invariant $\boldsymbol{I}_{\mathsf{Cartan}}$, whose identical vanishing characterizes local biholomorphic equivalence to the Heisenberg sphere $\{v' = z'\overline{z}'\}$, is even huger.

Fortunately, we will see that thanks to plain translations $(z, w) \longmapsto (z - z_p,\, w - w_p)$, one may 'decipher' chains *only at the origin* for a family of hypersurfaces $\{M^p\}_{p \in M}$ passing through $0 \in \mathbb{C}^2$ and parametrized by all points $p \in M$ in the original hypersurface. Section 2 presents this start.

In the literature, chains are detected from rather advanced considerations:

• either from an almost complete construction of an $\{e\}$-structure or of a Cartan connection associated with the CR equivalence problem [5, 8, 23, 24];

• or from an almost complete construction of a formal or converging Moser-like normal form [13, 16, 17] for $M^3 \subset \mathbb{C}^2$ at the origin $0 \in M$.

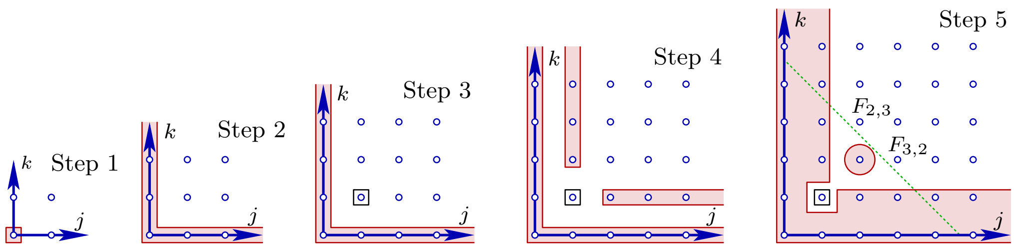

FIGURE 2: Successive annihilations (red dashed regions) of coefficient-functions $F_{j,k}(u)$ in the graphing function $v = \sum_{j,k} z^j \overline{z}^k F_{j,k}(u)$ thanks to Moser's normalization process, with $F_{1,1}(u) \equiv 1$, until first occurence of chains.

Let us comment only the second technique, which proceeds in five steps. At any reference point $p_0 = 0 \in M$, pick a curve $0 \in \gamma \subset M$ which is CR-transversal, namely $\dot{\gamma}(0) \notin T_0^c M$. Expand F in powers of $z, \overline{z}$ as:

$$v = F(z, \overline{z}, u) = \sum_{j,k} z^j \overline{z}^k F_{j,k}(u) \qquad\qquad \big(F_{j,k}(u) := \textstyle\sum_l F_{j,k,l}\, u^l\big)$$

Step 1. Straighten γ to be the u-axis, so that $F(0,0,u) \equiv 0$, that is:

$$v = \sum_{j+k \geqslant 1} z^j \overline{z}^k F_{j,k}(u).$$

Step 2. Kill all harmonic terms $z^j F_{j,0}(u)$ and $\overline{z}^k F_{0,k}(u)$, so that:

$$v = \sum_{j \geqslant 1 \text{ or } k \geqslant 1} z^j \overline{z}^k F_{j,k}(u).$$

Step 3. Normalize $F_{1,1}(u) \longmapsto 1$, using the assumption of Levi nondegeneracy, so that:

$$v \,=\, z\overline{z} + \sum_{\substack{j+k\geqslant 3\\ j\geqslant 2 \text{ or } k\geqslant 2}} z^j\overline{z}^k\, F_{j,k}(u).$$

Step 4. Absorb all $z^1\overline{z}^k\, F_{1,k}(u)$ and all $z^j\overline{z}^1\, F_{j,1}(u)$ inside $z^1\overline{z}^1$, so that:

$$v \,=\, z\overline{z} + \sum_{\substack{j\geqslant 2\\ k\geqslant 2}} z^j\overline{z}^k\, F_{j,k}(u).$$

Step 5. Kill (in some way) $F_{2,2}(u)$, so that:

$$v \,=\, z\overline{z} + z^3\overline{z}^2\, F_{3,2}(u) + z^2\overline{z}^3\, F_{2,3}(u) + \sum_{\substack{j+k\geqslant 6\\ j\geqslant 2 \text{ and } k\geqslant 2}} z^j\overline{z}^k\, F_{j,k}(u).$$

Each one of these steps requires to perform an application of the $\mathcal{C}^\omega$ implicit function theorem. Next, what about $F_{3,2}(u)$ and its conjugate $F_{2,3}(u) = \overline{F_{3,2}(u)}$**?** One (known) paradox is that it is only at an advanced stage of the progressive normalization process that one can realize that the choice of a CR-transversal curve γ should *not* be made haphazardly.

Indeed, Proposition 6 in [17, Chap. 4] states – not in the clearest thoughtful mathematical way? – : *For each direction ℓ_p transverse to T_p^cM at $p\in M$, there exists a unique (unparametrized) real analytic curve through p and tangent to that direction such that there exists some biholomorphism taking M to:*

$$v \,=\, |z|^2 + \sum_{j\geqslant 2,\ k\geqslant 2} F_{j,k}(u)\, z^j\overline{z}^k \qquad\qquad \text{with}\quad F_{3,2}(u) \,\equiv\, 0,$$

and γ to the u-axis.

What are these curves? Why do they exist? Can one get them in advance? Can one characterize them *geometrically*? Without relying on the existence of some normalizing biholomorphisms?

In fact, the proof of this Proposition 6 is the most technical and difficult to follow in [13] or in [17, Chap. 4]. One first reason is that the argumentation appears almost at the end of the normalization process, and a second reason is that it demands to perform biholomorphisms of the shape:

$$z' \,:=\, \sum_{j=0}^{\infty} z^j\, f_j(w), \qquad\qquad w' \,:=\, \sum_{j=0}^{\infty} z^j\, g_j(w),$$

with $f_0(w) \neq 0$ required not to send the curve $\{z=0\}\cap M$ to the same curve $\{z'=0\}\cap M'$ – one really has to change the CR-transversal curve! –, but this creates substantial computational obstacles.

As an alternative, we will present a construction which is elementary, simple, and requires almost no computation. Furthermore, we will work with power series in 3 variables at one point, the origin, and only up to order 5 included.

Let therefore $0\in M^3\subset\mathbb{C}^2$ be $\mathcal{C}^\omega$ Levi nondegenerate, graphed as $v = F(z,\overline{z},u)$, with $0\in M$. We assign the weights $[x] := 1 =: [y]$ and $[u] := 2 =: [v]$. It is well known that one can assume, with a *weighted* remainder, that M has equation:

$$v \,=\, z\overline{z} + \sum_{3\leqslant\delta\leqslant 5}\ \sum_{j+k+2l=\delta} F_{j,k,l}\, z^j\overline{z}^k u^l + \mathrm{O}(6).$$

Anybody with a pen or a computer will reconstitute Proposition 2.2, stating that there exists a change of holomorphic coordinates in which M becomes:

$$v \,=\, z\overline{z} + \mathrm{O}(6).$$

Next, the *key* fact is that the *ambiguity* of such a normalization up to (weighted) order 5, namely any biholomorphic equivalence:

$$v \,=\, z\overline{z} + \mathrm{O}(6) \qquad \longrightarrow \qquad v' \,=\, z'\overline{z}' + \mathrm{O}(6),$$

can be elementarily shown, by Proposition 2.3, to coincide with the expansion, up to weighted order 5, of the general isotropy group of the sphere $v = z\overline{z} \longrightarrow v' = z'\overline{z}'$ (without remainder), which is known to be:

$$z' \,=\, \frac{\lambda\,(z + \alpha\, w)}{1 - 2i\overline{\alpha}\, z - (r + i\alpha\overline{\alpha})\, w}, \qquad w' \,=\, \frac{\lambda\overline{\lambda}\, w}{1 - 2i\overline{\alpha}\, z - (r + i\alpha\overline{\alpha})\, w},$$

with $\lambda \in \mathbb{C}\backslash\{0\}$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$. (For 2-nondegenerate constant Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$, this fact becomes false, unfortunately [15].)

Then miraculously, the existence of Cartan-Moser chains amounts to just understanding how the isotropy group of the model acts on CR-transversal objects!

This 5-dimensional isotropy group has 5 generators D, R, I_1, I_2, J which are 5 linearly independent holomorphic vector fields X with $\mathsf{X} + \overline{\mathsf{X}}$ tangent to $v = z\overline{z}$. Their expressions in the intrinsic coordinates $(x, y, u) \in M^3$ read as (Section 5):

$$\begin{aligned}
\mathsf{J} + \overline{\mathsf{J}} \,&=\, (xu - x^2y - y^3)\,\partial_x + (x^3 + xy^2 + yu)\,\partial_y + \big(u^2 - (x^2 + y^2)^2\big)\,\partial_u, \\
\mathsf{I}_2 + \overline{\mathsf{I}}_2 \,&=\, (x^2 - 3y^2)\,\partial_x + (u + 4xy)\,\partial_y + (2xu - 2yx^2 - 2y^3)\,\partial_u, \\
\mathsf{I}_1 + \overline{\mathsf{I}}_1 \,&=\, (u - 4xy)\,\partial_x + (3x^2 - y^2)\,\partial_y + (-2x^3 - 2xy^2 - 2yu)\,\partial_u, \\
\mathsf{R} + \overline{\mathsf{R}} \,&=\, -\,y\partial_x + x\,\partial_y, \\
\mathsf{D} + \overline{\mathsf{D}} \,&=\, x\,\partial_x + y\,\partial_y + 2u\,\partial_u.
\end{aligned}$$

Then according to the beautiful, highly conceptional, theory of Lie [19, Chap. 25], *see* also [25, 20, 10], the action of this group on first jets $(\dot{x}(t), \dot{y}(t))$ and on second jets $(\ddot{x}(t), \ddot{y}(t))$ of curves $t \longmapsto (x(t), y(t), t)$, equipped with coordinates (x_1, y_1) and (x_2, y_2), can be understood infinitesimally by means of the *prolongations* to the second jet space $J^2_{1,2}$ of maps $\mathbb{R}^1_u \longrightarrow \mathbb{R}^2_{x,y}$, thanks to straightforward universal formulas (Sections 6 and 7). Since we work only at one point, namely above the origin, it suffices to compute the coefficients, in front of $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial y_1}$, $\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial y_2}$, of these five prolonged vector fields only for $x = y = u = 0$ (Section 7):

	∂_{x_1}	∂_{y_1}	∂_{x_2}	∂_{y_2}
$\mathsf{D}^{(2)}$	$-x_1$	$-y_1$	$-3x_2$	$-3y_2$
$\mathsf{R}^{(2)}$	$-y_1$	x_1	$-y_2$	x_2
$\mathsf{I}_1^{(2)}$	1	0	$-4x_1y_1$	$6x_1^2 + 2y_1^2$
$\mathsf{I}_2^{(2)}$	0	1	$-2x_1^2 - 6y_1^2$	$4x_1y_1$
$\mathsf{J}^{(2)}$	0	0	0	$0.$

From the first two columns that are everywhere of rank 2, it is clear that there does not exist any *invariant* CR-transversal line $\ell_0 \ni 0$ with $\ell_0 \oplus T_0^cM = T_0M$. Moreover, the action on such ℓ_0 is *transitive.*

Next, by some kind of '*algebraic miracle*' which can be verified by applying a plain Gauss pivot to the above 4×4 submatrix:

$$\begin{pmatrix} 0 & 0 & -3x_2 - 6x_1^2 y_1 - 6y_1^3 & -3y_2 + 6x_1 y_1^2 + 6x_1^3 \\ 0 & 0 & -y_2 + 2x_1 y_1^2 + 2x_1^3 & x_2 + 2x_1^2 y_1 + 2y_1^3 \\ 1 & 0 & -2x_1^2 - 6y_1^2 & 4x_1 y_1 \\ 0 & 1 & -4x_1 y_1 & 6x_1^2 + 2y_1^2 \end{pmatrix},$$

there appears to eyes (Section 7) a special surface $\Sigma_0^2 \subset \mathbb{R}^2_{x_1,y_1} \times \mathbb{R}^2_{x_2,y_2}$, graphed as shown by the (redundant by pairs) entries $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, as:

$$\Sigma_0^2 := \big\{(x_1,y_1,x_2,y_2) \in \mathbb{R}^4 \colon \ x_2 = -2x_1^2 y_1 - 2y_1^3, \ \ y_2 = 2x_1 y_1^2 + 2x_1^3\big\},$$

which is a 2-dimensional orbit of the five prolonged vector fields D^2, R^2, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$, $\mathsf{J}^{(2)}$, while the complement $\mathbb{R}^4_{x_1,y_1,x_2,y_2} \backslash \Sigma_0^2$ is a single orbit (Observation 7.1).

The existence of Σ_0^2 together with the normalizability to $v' = z'\overline{z}' + \mathrm{O}(6)$ therefore explain in an elementary manner the existence of Cartan-Moser chains above 0.

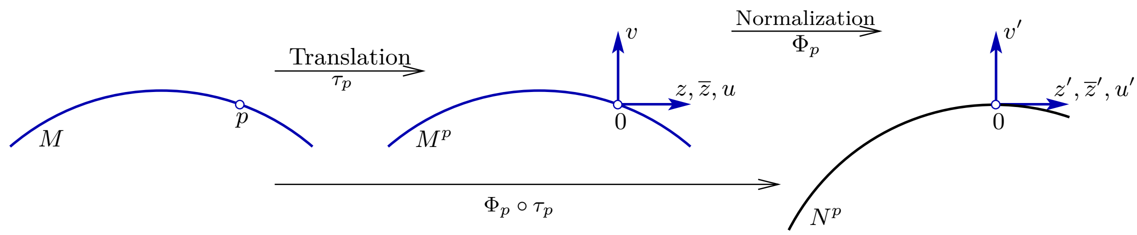

FIGURE 3: Centering (by translation) coordinates at an arbitrary point $p \in M$, and sketching what any normalization map Φ_p does near $p = 0$.

Lastly, for any Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^3$, we can define Cartan-Moser chains at any point $p \in M$ as follows: Denote the translation mapping $\tau_p \colon (M,p) \longrightarrow (M^p, 0)$ by:

$$\tau_p \colon \quad (z,w) \longmapsto \big(z - z_p, \, w - w_p\big) =: (z,w),$$

denote *any* elementary normalization map as mentioned above by:

$$\Phi_p \colon (M^p, 0) = \Big\{ v = \sum_{1 \leqslant j+k+2l \leqslant 5} F^p_{jkl} \, z^j \overline{z}^k u^l + \mathrm{O}(6) \Big\} \longrightarrow \big\{ v' = z'\overline{z}' + \mathrm{O}(6) \big\} =: (N^p, 0).$$

Recall that the action of the 5-dimensional isotropy group is transitive on 1-jets.

Given a 1-jet j_p^1 at p, using *any* normalizing map $\Phi_p \colon M^p \longrightarrow N^p$ which sends $(M^p, 0)$ to a hypersurface $(N^p, 0)$ of equation $v' = z'\overline{z}' + \mathrm{O}(6)$ *and also sends* j_p^1 to the flat 1-jet $j_0^1 = (0,0)$ at $0 \in N^p$, assign the 2-jet j_p^2 of the Moser chain at $p \in M$ associated with j_p^1 to be the inverse image of the *flat* 2-jet at $0 \in N^p$:

$$j_p^2 := \big(\Phi_p \circ \tau_p\big)^{(2)^{-1}} \big(0,0,0,0\big).$$

It is not difficult to verify that this definition provides a map $j_p^1 \longmapsto j_p^2(j_p^1)$ which is $\mathcal{C}^\omega$ on M.

Once chains are known, one can (re)start Step 1 above with the CR-transversal curve γ being a chain. Then Steps 2, 3, 4 go without modification, while in Step 5, one realizes that $F_{3,2}(u) \equiv 0$ automatically (Section 9), as a consequence of the definition of chains (Assertion 9.4).

For self-contentness and for later use in [15], although there is no originality, we perform all these steps in Section 10, 11, 12 known as Propositions 1, 2, 3, 4, 5 in [17, Chap. 4]. We conclude by stating Moser's normal form theorem in Section 13 and by proving some uniqueness property.

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2. Point normalizations of $\mathcal{C}^\omega$ hypersurfaces $M^3 \subset \mathbb{C}^2$

Consider a local real hypersurface $M^3 \subset \mathbb{C}^2$ of class (at least) $\mathcal{C}^5$. In fact, we will mainly work with $\mathcal{C}^\omega$ (real-analytic) objects, and sometimes indicate what kind of lower regularity assumptions can be afforded.

In coordinates $(z,w) = (x+i\,y,\, u+i\,v)$, assume M is graphed as $v = F(z,\overline{z},u)$, with $F \in \mathcal{C}^5$. At all points $p = (z_p, w_p) \in M$ with $v_p = F(z_p, \overline{z}_p, u_p)$, expand:

$$v \,=\, F(z,\overline{z},u) \,=\, \sum_{j+k+l\leqslant 5} \tfrac{(z-z_p)^j}{j!}\,\tfrac{(\overline{z}-\overline{z}_p)^k}{k!}\,\tfrac{(u-u_p)^l}{l!}\, F_{z^j\overline{z}^k u^l}(z_p,\overline{z}_p,u_p) + \mathrm{O}(6),$$

subtract $v - v_p$, translate coordinates $z := z - z_p$, $w := w - w_p$, and get a family of hypersurfaces $M^p \subset \mathbb{C}^3$ passing through the origin:

$$v \,=\, F^p(z,\overline{z},u) \,=\, \sum_{1\leqslant j+k+l\leqslant 5} z^j\overline{z}^k u^l\, F^p_{j,k,l} + \mathrm{O}(6),$$

namely with $F^p(0,0,0) = 0$, having coefficients $F^p_{j,k,l} := \frac{1}{j!}\,\frac{1}{k!}\,\frac{1}{l!}\,F_{z^j\overline{z}^k u^l}(z_p,\overline{z}_p,u_p)$ smoothly parametrized by p. Thanks to this, working at *only one* point, namely at the origin, we will treat *all* points $p \in M$.

Local biholomorphisms $h\colon M \longrightarrow M'$ between any two CR manifolds respect by definition complex tangent bundles $h_*(T^cM) = T^cM'$.

Problem 2.1. *Are there CR-transversal structures which are invariant under biholomorphisms?*

The goal of this note is to elaborate a simple, *Lie-theoretic* approach to this question which applies to any kind of CR structure, does not require to fully solve any equivalence problem, and does not rest on the existence of Cartan-Tanaka connections. To illustrate the process on just one advanced example, we shall show how to recover in a quite elementary way the famous *Moser chains* on Levi nondegenerate hypersurfaces $M^3 \subset \mathbb{C}^2$. Forthcoming publications will exhibit more about Lie's theoretical scope.

Since Question 2.1 is *invariant*, we are allowed to perform *normalizing* biholomorphisms in order to 'simplify' the equations $v = F^p(z,\overline{z},u)$ of our p-parametrized hypersurfaces M^p, before searching for CR-transversal structures, if any.

After an elementary biholomorphism, it is well known that one can assume:

$$v \,=\, z\overline{z} + \mathrm{O}(3).$$

This conducts to attribute weights $[z] := 1 =: [\overline{z}]$ and $[w] := 2 =: [\overline{w}]$. Up to order 5, some monomials have weight > 5, for instance $u^2 z^2$, and they will be disregarded. Thus, with a now *weighted* remainder $\mathrm{O}(6)$:

$$v \,=\, F^p(z,\overline{z},u) \,=\, z\overline{z} + \sum_{3\leqslant\delta\leqslant 5}\,\sum_{j+k+2l=\delta} F^p_{j,k,l}\, z^j\overline{z}^k u^l + \mathrm{O}(6).$$

By performing biholomorphisms of the shape $z' = z + f_{\delta-1}(z,w)$, $w' = w + g_\delta(z,w)$, with appropriate polynomials $f_{\delta-1}$, g_δ that are weighted homogeneous of degrees $\delta-1$, δ, it is not difficult to erase F_δ for $\delta = 3,4,5$.

Proposition 2.2. *Every M^p can be normalized to $v = z\overline{z} + 0 + 0 + 0 + \mathrm{O}(6)$.* ∎

Of course, such a normalizing biholomorphism is not unique.

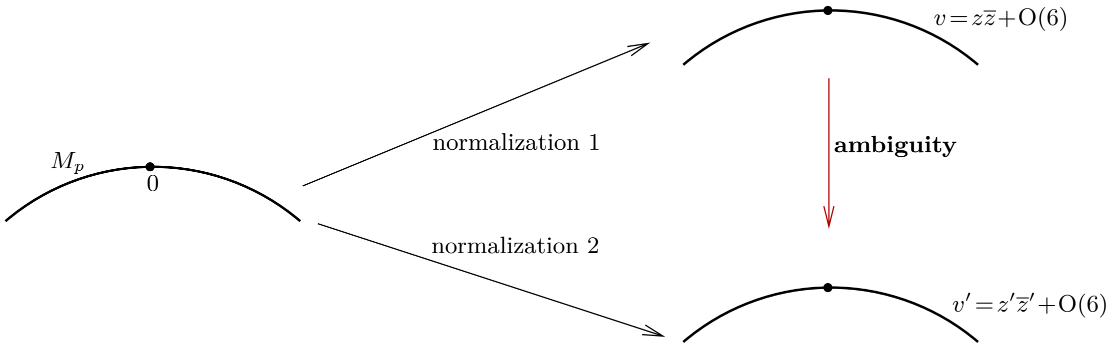


FIGURE 4: Representing two 6^{th} order normalization maps at the origin and calling '*ambiguity*' the 'difference' (composition) between them.

The next statement – whose proof is also left as an exercise[1] – determines the ambiguity transformation, which is obtained by expanding up to weight 5 included the following two fractions in which $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$ are free:

$$z' \,=\, \frac{\lambda\,(z + \alpha\, w)}{1 - 2i\overline{\alpha}\, z - (r + i\alpha\overline{\alpha})\, w}, \qquad\qquad w' \,=\, \frac{\lambda\overline{\lambda}\, w}{1 - 2i\overline{\alpha}\, z - (r + i\alpha\overline{\alpha})\, w}. \tag{1}$$

Proposition 2.3. *Every biholomorphism $z' = f(z,w) + \mathrm{O}(5)$, $w' = g(z,w) + \mathrm{O}(6)$ with f, g of weight $\leqslant 4, 5$ sending $v = z\overline{z} + \mathrm{O}(6)$ to $v' = z'\overline{z}' + \mathrm{O}(6)'$ is necessarily of the form:*

$$\begin{aligned} z' \,=\, & \lambda\, z + 2i\lambda\overline{\alpha}\, z^2 + \big(-4\lambda\overline{\alpha}^2\big)\, z^3 + \big(-8i\lambda\overline{\alpha}^3\big)\, z^4 \\ & + \lambda\alpha\, w + \big(3i\lambda\alpha\overline{\alpha} + \lambda r\big)\, zw + \big(-8\lambda\alpha\overline{\alpha}^2 + 4i\overline{\alpha}\lambda r\big)\, z^2 w \\ & + \big(\lambda\alpha r + i\lambda\alpha^2\overline{\alpha}\big)\, w^2 \end{aligned}$$

$$\begin{aligned} w' \,=\, & \lambda\overline{\lambda}\, w + 2i\lambda\overline{\lambda}\overline{\alpha}\, zw + \big(-4\lambda\overline{\lambda}\overline{\alpha}^2\big)\, z^2 w + \big(-8i\lambda\overline{\lambda}\overline{\alpha}^3\big)\, z^3 w \\ & + \big(i\lambda\overline{\lambda}\alpha\overline{\alpha} + \lambda\overline{\lambda} r\big)\, w^2 + \big(4i\lambda\overline{\lambda}\overline{\alpha} r - 4\lambda\overline{\lambda}\overline{\alpha}^2\alpha\big)\, zw^2. \end{aligned}$$

But these formulas for this stability/ambiguity group are well known!

---

[1] It turns out that all detailed proofs given later in Sections 10, 11, 12 do the job (solve the two exercises).

3. Automorphisms of the sphere $\{\mathrm{Im}\, w = z\overline{z}\}$ fixing the origin

Indeed, in $\mathbb{C}^2 \ni (z,w) = (x + i\,y,\; u + i\,v)$, consider the Heisenberg sphere:

$$v \,=\, z\overline{z},$$

which is biholomorphic, after a certain Cayley transform, to the standard 3-sphere $S^3 \subset \mathbb{C}^2$ minus one point sent to infinity. It is well-known (for details see [1], Section 3) that the 5-dimensional real Lie algebra $\mathfrak{g}^5$ of holomorphic vector fields $X = a(z,w)\,\partial_z + b(z,w)\,\partial_w$ with $a(0) = 0 = b(0)$ such that $X + \overline{X}$ is tangent to S^3_* consists of:

$$\begin{aligned}
\mathsf{D} &:= z\,\partial_z + 2w\,\partial_w,\\
\mathsf{R} &:= iz\,\partial_z,\\
\mathsf{I}_1 &:= (w + 2iz^2)\,\partial_z + 2izw\,\partial_w,\\
\mathsf{I}_2 &:= (iw + 2z^2)\,\partial_z + 2zw\,\partial_w,\\
\mathsf{J} &:= zw\,\partial_z + w^2\,\partial_w,
\end{aligned}$$

with commutator table:

	D	R	I_1	I_2	J
D	0	0	I_1	I_2	$2\,\mathsf{J}$
R	$*$	0	$-\mathsf{I}_2$	I_1	0
I_1	$*$	$*$	0	$4\,\mathsf{J}$	0
I_2	$*$	$*$	$*$	0	0
J	$*$	$*$	$*$	$*$	0

Integrating these fields, the finite equations of the istropy Lie group $G^5 = \mathsf{Iso}(0)$ are:

$$z' \,=\, \frac{\lambda\,(z + \alpha\,w)}{1 - 2i\overline{\alpha}\,z - (r + i\alpha\overline{\alpha})\,w}, \qquad\qquad w' \,=\, \frac{\lambda\overline{\lambda}\,w}{1 - 2i\overline{\alpha}\,z - (r + i\alpha\overline{\alpha})\,w},$$

where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $r \in \mathbb{R}$, as above.

So we know precisely the nonuniqueness (ambiguity) in Proposition 2.3. Therefore, we can pursue exploring our Question 2.1 by asking at first whether some *tangential* (order 1) CR-transversal invariant object exists.

Problem 3.1. *Is there any vector $\vec{\mathbf{v}}_0 \in T_0M^p$ not complex-tangential $\vec{\mathbf{v}}_0 \notin T_0^cM^p$ which would be invariant under biholomorphisms ?*

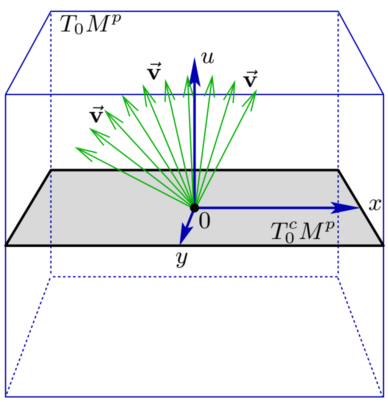


FIGURE 5: Representing horizontally the complex-tangential plane $T_0^cM^p$ of M^p at the origin within the 3-dimensional T_0M^p, and drawing various vectors $\vec{\mathbf{v}} \in T_0M^p \backslash T_0^cM^p$.

Predictably, the answer is *no*, because at order 1, the above formulas read as linear transformations:

$$z' = \lambda z + \lambda \alpha w, \qquad w' = \lambda\overline{\lambda} w,$$

and when $\alpha \in \mathbb{C}$ varies, the '*slope*' of $\vec{\mathbf{v}}_0$ changes arbitrarily. In fact, we must conceptualize carefully this intuition.

4. Lie jet theory

The historical and philosophical monograph [21] explains how near 1870 Helmholtz involuntarily '*invented*' the so-called *linearized isotropy groups*, which were theoretically understood later by Sophus Lie after finding a counterexample to Helmholtz's belief that any 'macroscopic' (local) group action can be recovered 'by integration' from its 'microscopic' (infinitesimal, linearized) behavior.

After Felix Klein's celebrated *Erlanger program*, Lie indeed developped a fantastic theory of *continuous group actions*, having in mind applications to a new '*Galois theory*' of differential equations. Lie erected a new theory of *prolongations* of group actions to jet spaces, *see* [19, Chap. 25]. Lie also conceptualized *prolongations* of infinitesimal transformations (vector fields) to jet spaces, and this is exactly what we need here!

We must work with the three *intrinsic*, *real*, coordinates (x, y, u) on M. A non CR-tangential vector $\vec{\mathbf{v}}_0 \in T_0M^p \backslash T_0^cM^p$ can be represented as the derivative $\dot{\gamma}(0) = \vec{\mathbf{v}}_0$ of some parametrized real curve passing by the origin:

$$t \longmapsto \big(x(t), y(t), u(t)\big) =: \gamma(t) \qquad\qquad (\dot{\gamma}(0) \neq 0).$$

Since $T_0^cM^p = \{u = 0\}$, we have in fact $\dot{u}(0) \neq 0$.

So we are considering local curves $\mathbb{R} \longrightarrow \mathbb{R}^2$ graphed along the (vertical!) u-axis. We can represent by putting u in the 'horizontal' place as $\big\{(u, x(u), y(u))\colon\ u \in \mathbb{R}\big\}$, with *two* graphing functions.

The associated jet space of order 2 – enough for our purposes – is equipped with further independent coordinates corresponding to $\dot{x}(u)$, $\dot{y}(u)$, $\ddot{x}(u)$, $\ddot{y}(u)$:

$$\big(u,\, x, y,\, x_1, y_1,\, x_2, y_2\big).$$

We denote the first jet space by $J^1_{1,2} \equiv \mathbb{R}^{1+2+2}$, and this second jet space by $J^2_{1,2} \equiv \mathbb{R}^{1+2+2+2}$. Any diffeomorphism $(u, x, y) \longmapsto (u', x', y')$ lifts to jet spaces of any order. The formulas rapidly become complicated ([25, 20, 10]). Lie understood this obstacle, and he linearized the formulas.

Indeed, by differentiating the prolongation to the second jet space of any one-parameter diffeomorphism $\exp(t\vec{\mathbf{v}})(u, x, y)$ obtained as the flow of a vector field $\vec{\mathbf{v}}$ on the base $\mathbb{R}^{1+2}$, Lie introduced its *prolongations* $\vec{\mathbf{v}}^{(1)}$ to $J^1_{1,2}$ and $\vec{\mathbf{v}}^{(2)}$ to $J^2_{1,2}$. A summarized presentation is available on pages 19–20 of [10].

Here, we just need to *apply* Lie's formulas. Start from a general vector field:

$$\vec{\mathbf{v}} := \xi(u, x, y)\,\frac{\partial}{\partial u} + \varphi(u, x, y)\,\frac{\partial}{\partial x} + \psi(u, x, y)\,\frac{\partial}{\partial y},$$

with smooth coefficients.

Introduce the *total differentiation operator:*

$$\mathsf{D}_u := \frac{\partial}{\partial u} + x_1 \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + x_3 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial y_2}.$$

Then the second prolongation of $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}}^{(2)} = \vec{\mathbf{v}} + \varphi_1 \frac{\partial}{\partial x_1} + \psi_1 \frac{\partial}{\partial y_1} + \varphi_2 \frac{\partial}{\partial x_2} + \psi_2 \frac{\partial}{\partial y_2},$$

has coefficients given uniquely by ([25, 20, 10]):

$$\begin{aligned}
\varphi_1 &:= \mathsf{D}_u\big(\varphi - \xi\, x_1\big) + \xi\, x_2, & \psi_1 &:= \mathsf{D}_u\big(\psi - \xi\, y_1\big) + \xi\, y_2,\\
\varphi_2 &:= \mathsf{D}_u\big(\mathsf{D}_u\big(\varphi - \xi\, x_1\big)\big) + \xi\, x_3, & \psi_2 &:= \mathsf{D}_u\big(\mathsf{D}_u\big(\psi - \xi\, y_1\big)\big) + \xi\, y_3.
\end{aligned}$$

5. Intrinsic isotropy automorphisms of the sphere

Coming back to Question 3.1, we must apply Lie's prolongation formulas within the *first* jet space to our 5 vector fields J, I_2, I_1, R, D. But these vector fields $\mathsf{X} = a(z,w)\,\partial_z + b(z,w)\,\partial_w$ were *extrinsic*, defined in $\mathbb{C}^2$, and holomorphic! Moreover, only their real parts $\frac{1}{2}\big(\mathsf{X} + \overline{\mathsf{X}}\big)$ matter!

To apply Lie's theory, we must therefore write them up in the *intrinsic* coordinates $(x,y,u) \in M^p$. We leave as an exercise to verify that the projection $\pi \colon (x,y,u,v) \longmapsto (x,y,u)$ is a chart on S^3_* for which:

$$\begin{aligned}
\pi_*\big(2\,\Re\,\mathsf{J}\big) &= (xu - x^2y - y^3)\,\partial_x + (x^3 + xy^2 + yu)\,\partial_y + \big(u^2 - (x^2+y^2)^2\big)\,\partial_u,\\
\pi_*\big(2\,\Re\,\mathsf{I}_1\big) &= (u - 4xy)\,\partial_x + (3x^2 - y^2)\,\partial_y + (-2x^3 - 2xy^2 - 2yu)\,\partial_u,\\
\pi_*\big(2\,\Re\,\mathsf{I}_2\big) &= (x^2 - 3y^2)\,\partial_x + (u + 4xy)\,\partial_y + (2xu - 2yx^2 - 2y^3)\,\partial_u,\\
\pi_*\big(2\,\Re\,\mathsf{R}\big) &= -\,y\partial_x + x\,\partial_y,\\
\pi_*\big(2\,\Re\,\mathsf{D}\big) &= x\,\partial_x + y\,\partial_y + 2u\,\partial_u.
\end{aligned}$$

We will keep the same notation for these five intrinsic vector fields.

6. Prolongation to the jet space of order 1

As we said, it suffices to work above the origin $0 \in M^p$. In fact, the projectivization $\mathbb{P}(T_0M^p) = \mathbb{P}^2$ of $T_0M^p \cong \mathbb{R}^3$ is a real projective plane. But excluding CR-tangential vectors, we are considering only $\mathbb{P}^2 \backslash \mathbb{P}^1_\infty = \mathbb{R}^2$, equipped with affine coordinates (x_1, y_1) as above.

This means that we are considering vectors $\vec{\mathbf{v}}_0 \in T_0M^p \backslash T_0^cM^p$ of coordinates $(1, x_1^0, y_1^0)$, with unit coordinate 1 along the u-axis. Though we will not work in the projective space $\mathbb{P}^2$, but only on its affine subset $\mathbb{C}^2 \subset \mathbb{P}^2$, we mention that there are homogeneous coordinates $[U_1 \colon X_1 \colon Y_1]$ on $\mathbb{P}(T_0M^p) = \mathbb{P}^2$ for which $\big[1 \colon \frac{X_1}{U_1} \colon \frac{Y_1}{U_1}\big] =: (1, x_1, y_1)$.

On the left, the Figure 6 represents this real $\mathbb{P}^2$ as a line, and on the right, as a plane. The projective line $\mathbb{P}^1_\infty$ at infinity is represented as a point, and as a square perimeter. By Lie's theory, any vector field $\vec{\mathbf{v}}$ on the base M lifts as a vector field $\vec{\mathbf{v}}^{(1)}$ on the first jet space $J^1_{1,2} = \mathbb{R}^{1+2+2}$.

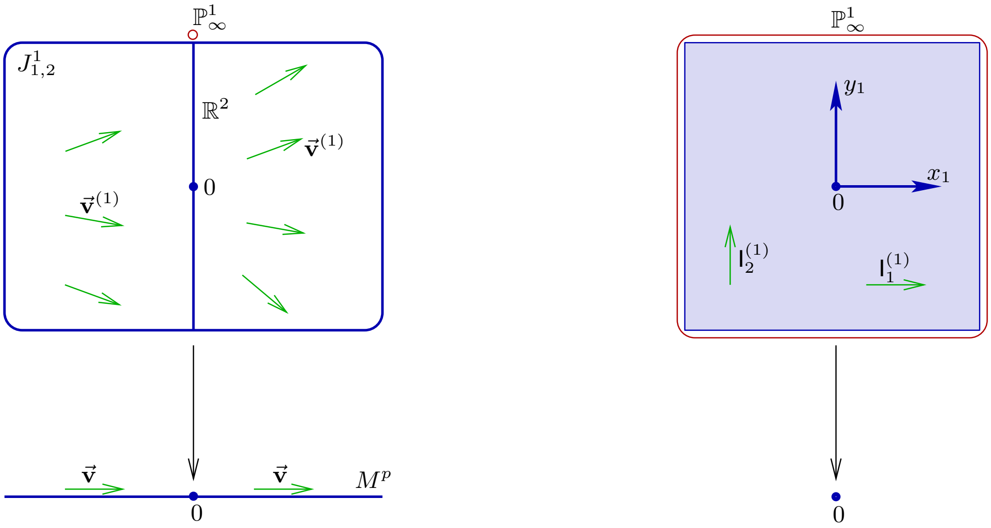


FIGURE 6: Left: representing the first prolongation of a vector field $\vec{\mathbf{v}}$ on M^p to the first jet space $J^1_{1,2}$. Right: Observing that, above the origin (only), the first prolongations $\mathsf{I}_1^{(1)}$ and $\mathsf{I}_2^{(1)}$ of I_1 and I_2 are straight (simple).

Because our five intrinsic vector fields J, I_1, I_2, R, D vanish at $u = x = y = 0$, their prolongations will automatically be tangent to the fiber $\big\{(0,0,0,x_1,y_1)\big\}$ above $(0,0,0)$ on the first jet space, a fiber which identifies with $\mathbb{R}^2 = \mathbb{P}^2 \backslash \mathbb{P}^1_\infty$.

Lie's formulas yield the very simple values of these first prolongations above the origin, namely for $x = y = u = 0$:

	∂_{x_1}	∂_{y_1}
$\mathsf{D}^{(1)}$	$-x_1$	$-y_1$
$\mathsf{R}^{(1)}$	$-y_1$	x_1
$\mathsf{I}_1^{(1)}$	1	0
$\mathsf{I}_2^{(1)}$	0	1
$\mathsf{J}^{(1)}$	0	0

Since the rank of the span of just $\mathsf{I}_1^{(1)}$ and $\mathsf{I}_2^{(1)}$ is everywhere equal to 2, the orbit is the whole fiber $\mathbb{R}^2 = \{(0,0,0,x_1,y_1)\}$, and this confirms what we already guessed, namely that *there does not exist any biholomorphically invariant CR-transversal direction* $\ell_0 \subset T_0M^p \backslash T_0^cM^p$.

So what? All this for nothing? Let us keep hope by asking

Problem 6.1. *Are there CR-transversal invariants of jet order* 2**?**

7. Prolongation to the jet space of order 2

A non CR-tangential direction $\ell_0 \subset T_0M^p \backslash T_0^cM^p$ can be represented as an order 1 jet $j_0^1 = (x_1^0, y_1^0)$. A general jet of order two then writes as $j_0^2 = \big(x_1^0, y_1^0, x_2^0, y_2^0\big)$.

Since we just saw that the stability group of the normalized equation $v = z\overline{z} + \mathrm{O}(6)$ for M_p, of dimension 5, acts *transitively* on first-order CR-transversal jets, it is clearly impossible that a *unique* second order jet be invariant under biholomorphisms. Anyway, it might be interesting to see how the second order Lie prolongations $\mathsf{R}^{(2)}$, $\mathsf{D}^{(2)}$, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$, $\mathsf{J}^{(2)}$ act on second order jets.

Lie's formulas yield the very simple values of these first prolongations above the origin, namely for $x = y = u = 0$:

	∂_{x_1}	∂_{y_1}	∂_{x_2}	∂_{y_2}
$\mathsf{D}^{(2)}$	$-x_1$	$-y_1$	$-3x_2$	$-3y_2$
$\mathsf{R}^{(2)}$	$-y_1$	x_1	$-y_2$	x_2
$\mathsf{I}_1^{(2)}$	1	0	$-4x_1y_1$	$6x_1^2+2y_1^2$
$\mathsf{I}_2^{(2)}$	0	1	$-2x_1^2-6y_1^2$	$4x_1y_1$
$\mathsf{J}^{(2)}$	0	0	0	0

The key discovery, due to Cartan and then to Moser who expressed it differently, now appears elementary. But before writing the statement, let us draw the key surface $\Sigma_0^2 \subset \mathbb{R}^2_{x_1,y_1} \times \mathbb{R}^2_{x_2,y_2}$ alluded to in the Introduction.

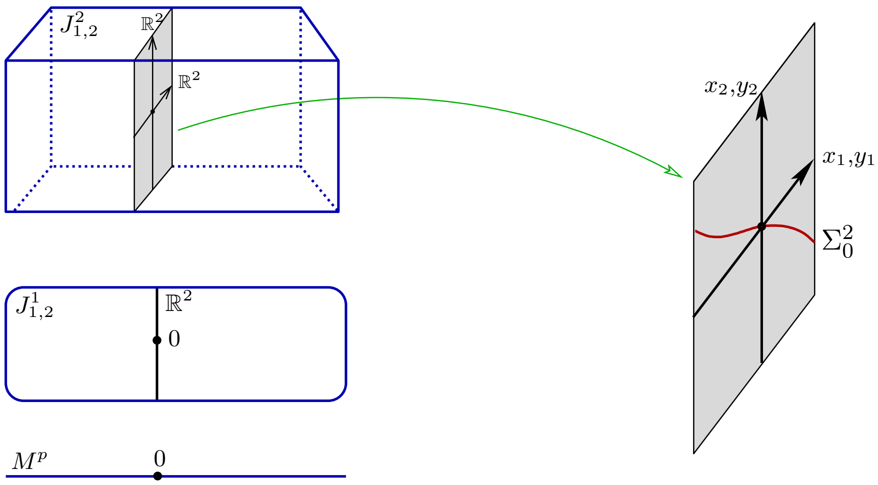


FIGURE 7: On the left, above $0 \in M^p$, we draw the first jet fiber $J^1_{1,2}\big|_0 \cong \mathbb{R}^2_{x_1,y_1}$ and the second jet fiber $J^2_{1,2}\big|_0 \cong \mathbb{R}^2_{x_1,y_1} \times \mathbb{R}^2_{x_2,y_2}$. On the right, making a zoom, collapsing twice two dimensions into one dimension, we sketch what the surface Σ_0^2 could be within $\mathbb{R}^2_{x_1,y_1} \times \mathbb{R}^2_{x_2,y_2}$, representing it abusively as a 1-curve in a 2-plane.

Lemma 7.1. *On $\mathbb{R}^4 = \mathbb{R}^2_{x_1,y_1} \times \mathbb{R}^2_{x_2,y_2}$, there exists a unique invariant 2-dimensional submanifold $\Sigma_0^2 \subset \mathbb{R}^4$, algebraic, graphed as:*

$$x_2 = -2\,x_1^2 y_1 - 2\,y_1^3, \qquad\qquad y_2 = 2\,x_1 y_1^2 + 2\,x_1^3.$$

Moreover, the complement $\mathbb{R}^4 \backslash \Sigma_0^2$ is a unique orbit under $\mathsf{D}^{(2)}$, $\mathsf{R}^{(2)}$, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$, $\mathsf{J}^{(2)}$.

Proof. Any point of $\mathbb{R}^4$ can be represented as:

$$x_2 = -2\,x_1^2 y_1 - 2\,y_1^3 + a_2, \qquad\qquad y_2 = 2\,x_1 y_1^2 + 2\,x_1^3 + b_2,$$

with some $(a_2, b_2) \in \mathbb{R}^2$. A Gauss-pivot transforms the matrix of the coefficients of the 4 vector fields $\mathsf{D}^{(2)}$, $\mathsf{R}^{(2)}$, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$ into:

$$\begin{pmatrix} 0 & 0 & -3a_2 & -3b_2 \\ 0 & 0 & -b_2 & a_2 \\ 1 & 0 & -2x_1^2 - 6y_1^2 & 4x_1y_1 \\ 0 & 1 & -4x_1y_1 & 6x_1^2 + 2y_1^2 \end{pmatrix}.$$

This matrix has maximal rank 4 if and only if $(a_2, b_2) \neq (0,0)$, and constant rank 2 for $(a_2, b_2) = (0,0)$. ∎

In other words, to every (fixed) first order jet $j_0^1 = (x_1, y_1)$ at the origin $0 \in M^p$ is associated a unique second order jet at the origin:

$$j_0^2 = \Big(x_1, y_1, \ -2x_1^2 y_1 - 2y_1^3, \ 2x_1 y_1^2 + 2x_1^3\Big),$$

and since Σ_0^2 is invariant under the stability group G^5 of $v = z\overline{z} + \mathrm{O}(6)$, this association is invariant under biholomorphic changes of coordinates.

8. Definition of Moser chains

Let us denote the translation map $\tau_p\colon (M,p) \longrightarrow (M^p, 0)$ used in Section 2 by:

$$\tau_p\colon \quad (z,w) \longmapsto \big(z - z_p, \ w - w_p\big) =: (z_0, w_0).$$

Also, taking such coordinates (z_0, w_0) around $(M^p, 0)$, let the punctual (at the origin) normalization map offered by Proposition 2.2 be:

$$\Phi_p\colon (M^p,0) = \Big\{v_0 = \sum_{1\leqslant j+k+2l\leqslant 5} F_{0jkl}^p \, z_0^j \overline{z}_0^k u_0^l + \mathrm{O}(6)\Big\} \longrightarrow \big\{v = z\overline{z} + \mathrm{O}(6)\big\} =: (N^p, 0),$$

and abbreviate:

$$\varphi := \Phi_p \circ \tau_p.$$

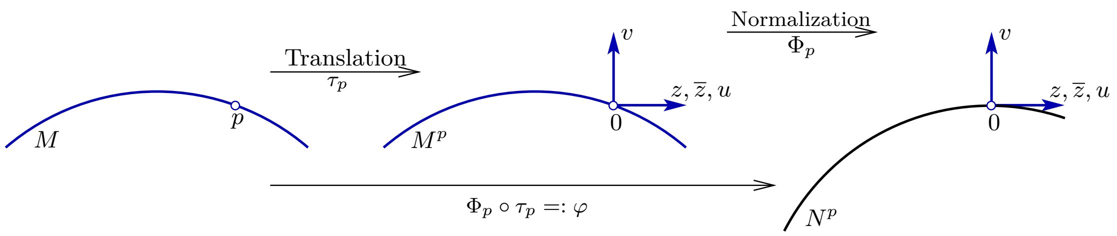


FIGURE 8: Again, represent the translation map τ_p and a normalizing map Φ_p.

As in Observation 7.1, in the 2-jet fiber above $0 \in N^p$, introduce the surface:

$$\Sigma_0 := \big\{(x_1, y_1, x_2, y_2) \in J_{N^p,0}^2 \colon \ x_2 = -2x_1^2 y_1 - 2y_1^3, \ y_2 = 2x_1 y_1^2 + 2x_1^3\big\}.$$

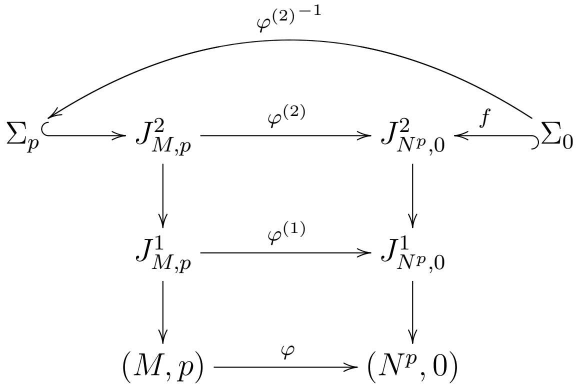
Using the second prolongation $\varphi^{(2)}$, define the 2-dimensional submanifold of $J_{M,p}^2$:

$$\Sigma_p := \varphi^{(2)^{-1}}(\Sigma_0).$$

Since $\varphi^{(1)}$ is a diffeomorphism $J_{M,p}^1 \stackrel{\sim}{\longrightarrow} J_{N^p,0}^1$, just as $\varphi^{(2)}\colon J_{M,p}^2 \stackrel{\sim}{\longrightarrow} J_{N^p,0}^2$, this Σ_p is also a graph, say of the form:

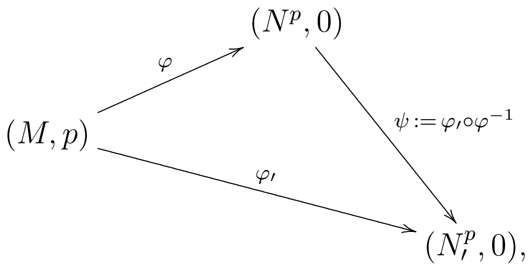
$$x_2^p = A(x_1^p, y_1^p), \qquad\qquad y_2^p = B(x_1^p, y_1^p),$$

with $(x_1^p, y_1^p, x_2^p, y_2^p) \in J_{M,p}^2$, and with two functions A, B which depend on p *and also* a priori *on the normalizing map* φ.



Lemma 8.1. *This graphed surface* $\Sigma_p \subset J^2_{M,p} \cong \mathbb{R}^4$ *is* independent *of the map* $\varphi = \Phi_p \circ \tau_p$ *normalizing* $v = F(z, \overline{z}, u)$ *near* p *to* $v = z\overline{z} + \mathrm{O}(6)$ *near* 0.

Proof. Suppose another such normalizing map is given:



with $(N^p_\prime, 0)$ also of equation $v' = z'\overline{z}' + \mathrm{O}(6)$. Define the special surface $\Sigma_0' \subset J^2_{N^p_\prime,0}$ by the *same* two graphed cubic equations $x_2' = -2{x_1'}^2 y_1' - 2{y_1'}^3$, $y_2' = 2\, x_1' {y_1'}^2 + 2\, {x_1'}^3$, and then define similarly:

$$\Sigma_p' \,:=\, \varphi_\prime^{(2)^{-1}}\big(\Sigma_0'\big).$$

Is it really true that $\Sigma_p' = \Sigma_p$**?**

Thanks to Proposition 2.3, the *relation map* $\psi := \varphi_\prime \circ \varphi^{-1}$ is a composition of flows of the five vector fields D, R, I_1, I_2, J. But because the second prolongations $\mathsf{D}^{(2)}$, $\mathsf{R}^{(2)}$, $\mathsf{I}_1^{(2)}$, $\mathsf{I}_2^{(2)}$, $\mathsf{J}^{(2)}$ of these fields are tangent to Σ_0 thanks to Observation 7.1, the map $\psi^{(2)}$ stabilizes the special surface:

$$\psi^{(2)^{-1}}\big(\Sigma_0'\big) \,=\, \Sigma_0.$$

Then as asserted:

$$\begin{aligned}\Sigma_p' \,&=\, \varphi_\prime^{(2)^{-1}}\big(\Sigma_0'\big) \,=\, \varphi_\prime^{(2)^{-1}}\big(\psi^{(2)}(\Sigma_0)\big) \,=\, \varphi_\prime^{(2)^{-1}}\Big(\big(\varphi_\prime \circ \varphi^{-1}\big)^{(2)}(\Sigma_0)\Big) \\ &=\, \underline{\varphi_\prime^{(2)^{-1}} \circ \varphi_\prime^{(2)}}_{\circ} \circ \big(\varphi^{-1}\big)^{(2)}\big(\Sigma_0\big) \,=\, \varphi^{(2)^{-1}}\big(\Sigma_0\big) \,=\, \Sigma_p.\end{aligned}$$

∎

Proposition 8.2. *There exist two* $\mathcal{C}^\omega$ *functions* A *and* B *such that* 2*-jets are invariantly associated to CR-transversal* 1*-jets as:*

$$x_2 \,=\, A\big(u, x, y, x_1, y_1\big), \qquad y_2 \,=\, B\big(u, x, y, x_1, y_1\big).$$

These functions A and B can be made explicit in terms of $\big\{F^p_{j,k,l}\big\}_{1\leqslant j+k+l\leqslant 5}$, but expressions are huge. To these two jet equations is naturally associated a system of two second order ordinary differential equations:

$$\ddot{x} \,=\, A\big(u,x,y,\dot{x},\dot{y}\big), \qquad\quad \ddot{y} \,=\, B\big(u,x,y,\dot{x},\dot{y}\big).$$

Definition 8.3. At a point $(u_p,x_p,y_p)\in M$, a *Moser chain* directed by some 1-jet $(1,x_1^p,y_1^p)$ is the unique solution $u\longmapsto (x(u),y(u))$ to the above $\mathcal{C}^\omega$ ODE system satisfying the initial conditions:

$$\big(x(u_p),\,y(u_p)\big) \,=\, (x_p,y_p) \qquad \text{and} \qquad \big(\dot{x}(u_p),\,\dot{y}(u_p)\big) \,=\, (x_1^p,y_1^p). \qquad\blacksquare$$

Equivalently, Moser chains $\big\{(u,x(u),y(u))\big\}$ are projections onto the base space $M\ni(u,x,y)$ of integral curves of the vector field on $J^1_{1,2}$:

$$\frac{\partial}{\partial u}+x_1\,\frac{\partial}{\partial x}+y_1\,\frac{\partial}{\partial y}+A\big(u,x,y,x_1,y_1\big)\,\frac{\partial}{\partial x_1}+B\big(u,x,y,x_1,y_1\big)\,\frac{\partial}{\partial y_1}.$$

Another equivalent, alternative, definition of 2-jets of Moser chains uniquely associated with 1-jets will be useful later. Recall that first prolongations $\psi^{(1)}$ of maps like $\psi=\varphi_\prime\circ\varphi$ described in Proposition 2.3 are *transitive* on 1-jets, according to Section 6.

So we can restrict considerations to normalizing maps $\varphi=\tau_p\circ\Phi_p$ which send any 1-jet j^1_p at $p\in M$ to the flat 1-jet $j^1_0=(0,0)$ at $0\in N^p$.

Definition 8.4. Given a hypersurface $M^3\subset\mathbb{C}^2$, a point $p\in M$, a 1-jet j^1_p at p, given the translation map $\tau_p\colon (M,p)\longrightarrow (M^p,0)$, and using *any* normalizing map $\Phi_p\colon M^p\longrightarrow N^p$ which sends $(M^p,0)$ to a hypersurface $(N^p,0)$ of equation $v=z\overline{z}+\mathrm{O}(6)$ *and also sends* j^1_p to the flat 1-jet $j^1_0=(0,0)$ at $0\in N^p$, assign the 2-jet j^2_p of the Moser chain at $p\in M$ associated with j^1_p to be the inverse image of the *flat* 2-jet at $0\in N^p$:

$$j^2_p \,:=\, \big(\Phi_p\circ\tau_p\big)^{(2)^{-1}}\big(0,0,0,0\big). \qquad\blacksquare$$

Thanks to the preceding reasonings, the result j^2_p is independent of the normalizing map $\Phi_p\circ\tau_p$ satisfying $(\Phi_p\circ\tau_p)^{(1)}(j^1_p)=(0,0)$, the flat 1-jet at $0\in N^p$.

9. Link of chains with $F^p_{3,2,0}$ at the origin

Once a point $p\in M$ and a CR-transversal 1-jet j^1_p at p are chosen, by known existence theorems, there is a unique local $\mathcal{C}^\omega$ curve $\gamma\colon I\longrightarrow M$ passing through p directed by j^1_p which is a Moser chain.

Because such a chain is invariant under biholomorphisms, if one wants to *normalize* the equation of a hypersurface $M^3\subset\mathbb{C}^2$, the very first natural normalization to perform is to *straighten* (to normalize) such a chain. This can be done for *any* CR-transversal curve, not necessarily a Moser chain.

Lemma 9.1. *Given any $\mathcal{C}^\omega$ curve $\gamma\colon(-1,1)\longrightarrow M$ with $\gamma(0)=p\in M$ and $\dot{\gamma}(0)\notin T^c_pM$, there exist holomorphic coordinates (z,w) centered at p with $w=u+iv$ in which M^p is graphed as $v=F^p(z,\overline{z},u)$ such that:*

$$\gamma(t) \,=\, \big(0,\,t+i\,0\big) \qquad\qquad (t\,\in\,I).$$

The (easy) proof will be written later in Section 10. So we may assume that $\{(0,u)\}$ is a chain, contained in M^p, whence $0 \equiv F^p(0,0,u)$.

In our preliminary Proposition 2.2, the existence of Moser chains was unknown. Only successive Taylor coefficients annihilations were performed. Consequently, it is necessary to restart the proof of Proposition 2.2 with the supplementary constraint to *keep invariant the straightened Moser chain* $\{(0,u)\}$.

First of all, to annihilate all monomials except $z\overline{z}$ up to weight 4 is again possible by transformations $(z,w) \longmapsto (z',w')$ sending (stabilizing) the u-axis to the u'-axis – exercise[2].

Furthermore, in weight 5, all the monomials:

$$z^5,\ \ z^4\overline{z},\ \ z\overline{z}^4,\ \ \overline{z}^5, \qquad z^3u,\ \ z^2\overline{z}u,\ \ z\overline{z}^2u,\ \ \overline{z}^3u, \qquad zu^2,\ \ \overline{z}u^2,$$

can similarly be killed without modifying the unparametrized straightened Moser chain $\{z=v=0\}$. *Only the two monomials* $z^3\overline{z}^2$ *and* $z^2\overline{z}^3$ *remain as causing troubles.* In the notations of Section 2, let us therefore formulate a

Lemma 9.2. *Every hypersurface* $0 \in M^p \subset \mathbb{C}^3$ *of equation:*

$$v_0 \,=\, F_0^p(z_0,\overline{z}_0,u_0) \qquad \textit{with} \qquad 0 \,\equiv\, F_0^p(0,0,u_0),$$

having a Moser chain straightened to be $\{(0,u_0)\}$*, can be normalized without deforming the chain being* $\{(0,u)\}$*, into a hypersurface of equation:*

$$N^p\colon \qquad v \,=\, F^p(z,\overline{z},u) \,=\, z\overline{z} + F^p_{3,2,0}\, z^3\overline{z}^2 + \overline{F}^p_{3,2,0}\, z^2\overline{z}^3 + \mathrm{O}(6). \tag{2}$$

Now, remember that Proposition 2.2 asserted that the remaining coefficient $F^p_{3,2,0}$, can be also killed. However, there is a supplementary constraint, now.

Problem 9.3. *Can one annihilate* $F^p_{3,2,0}$ *without unstraightening the chain?*

It turns out that the answer is '*no-becomes-yes*'! Indeed, for some subtle reason which lies in the definition of chains, it will soon turn out that *this coefficient* $F^p_{3,2,0}$ *needs not be annihilated*, because it will be shown to be already zero for free! Let us explain this key fact which will be very useful later in Assertion 12.2.

Lemma 9.4. *If* $F^p(z,\overline{z},u)$ *is as in* (2) *with* $0 \equiv F^p(0,0,u)$ *and with* $\{(0,u)\}$ *being a chain, then* $F^p_{3,2,0} = 0$.

Proof. Denote $h_0\colon M^p \longrightarrow N^p$ one incomplete normalizing map given by Lemma 9.2. Since h_0 sends $\{(0,u_0)\}$ to $\{(0,u)\}$, it sends the flat 1-jet $j^1_{M^p,0} = (0,0)$ to the flat 1-jet $j^1_{N^p,0} = (0,0)$. We will apply Definition 8.4 to $(N^p,0)$ with $j^1_{N^p,0} = (0,0)$.

We know by Proposition 2.2, that it is possible to continue to perform normalizations by means of a further map:

$$M^p \xrightarrow{\quad h_0 \quad} N^p \xrightarrow{\quad h \quad} N'^p,$$

in order that N'^p has equation $v' = z'\overline{z}' + \mathrm{O}(6)$.

[2] Again, it turns out that all detailed proofs given in Sections 10, 11, 12 show how to do it.

In fact, the map $h = (z + f_4,\, w + g_5) = (z', w')$ with:

$$z' := z - i\, F^p_{3,2,0} z^2 w - \tfrac{1}{4}\, F^p_{2,3,0}\, w^2 + \mathrm{O}(5), \quad w' := w - \tfrac{i}{2}\, F^p_{3,2,0}\, z w^2 + \mathrm{O}(6), \tag{3}$$

works. Because $h = (z, w) + \mathrm{O}_{z,w}(2)$, this maps sends the flat 1-jet at $0 \in N^p$ to the flat 1-jet at $0 \in N'^p$. Then according to Definition 8.4 of a Moser chain, the 2-jet of the Moser chain at $0 \in N^p$ along $\{(0, u)\}$ – which is flat! – *must be* the inverse image, through ${h^{(2)}}^{-1}$, of the flat 2-jet at $0 \in N'^p$. Equivalently, h must send the flat 2-jet at $0 \in N^p$ to the flat 2-jet at $0 \in N'^p$.

Let us write a flat 2-jet at $0 \in N^p$ as a parametrized curve $\mathbb{R}_u \longrightarrow \mathbb{R}^2_{x,y}$:

$$x \;=\; \mathrm{O}_3(u), \qquad y \;=\; \mathrm{O}_3(u).$$

Then at $0 \in N'^p$, do we also have $x' = \mathrm{O}_3(u')$ and $y' = \mathrm{O}_3(u')$ through the map (3)?

We claim: *No if* $F^p_{3,2,0} \neq 0$!

Indeed, it comes $z = x + i\, y = \mathrm{O}_3(u)$, hence $w = u + i\, z\overline{z} + \mathrm{O}(5) = u + \mathrm{O}_3(u)$, and also $u' = u + \mathrm{O}_3(u)$ or inversely $u' + \mathrm{O}_3(u') = u$, whence:

$$x' + i\, y' \;=\; -\tfrac{1}{4}\, F^p_{3,2,0}\, {u'}^2 + \mathrm{O}_3(u). \qquad \blacksquare$$

This Lie-theoretic construction of Moser chains can be applied to any CR manifold, and the paper could certainly stop at this point. However, we would like to mention that the normalizations applied in the remainder of this paper, i.e. in the next Sections 10, 11, 12, 13, are known to be done in the general case of hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$ in any CR dimension $n \geqslant 1$ by Chern-Moser in their celebrated work [13]. More particularly, in part (d), page 246 of [13], Chern-Moser briefly concentrate on the specific case of real hypersurfaces in $\mathbb{C}^2$.

Although Chern-Moser did not mention precisely all the intermediate normalizations which are applicable in $\mathbb{C}^2$, Jacobowitz in Chapter 4 of his monograph [17] endeavoured to detect and to explain in $\mathbb{C}^2$ those appropriate normalizations.

But since, to the best of our knowledge, there is no considerable work in the literature specifying such normalizations, we hope that the rest of the paper may raise interest of readers who want to learn Chern-Moser's normalizations in the specific case of $\mathbb{C}^2$. Proofs are neither straightforward, nor elementary, because they require an intensive, repeated use of the *implicit function theorem*.

Thus, although the next results can not be regarded as new, for self-contentness reasons, and in order to prepare forthcoming works on new kinds of CR structures (cf. *e.g.* [15]), let us start to reconstitute the Chern-Moser normalization theory in $\mathbb{C}^2$, setting up fully detailed arguments readable by non-experts.

10. Chain straightening and harmonic Killing

The main feature being that Moser chains are *biholomorphically invariant*, it is natural to take them as a starting point for the process of normalization.

Let $M^3 \subset \mathbb{C}^2$ be a Levi nondegenerate hypersurface passing by the origin $0 \in M$. Since $T^c_0 M \cong \mathbb{C}$, an appropriate $\mathbb{C}$-linear transformation makes $T^c_0 M = \mathbb{C}_z \times \{0\}$ in coordinates $(z, w) \in \mathbb{C}^2$.

Our goal is to transform M into certain *normal forms*, by performing biholomorphisms fixing the origin:

$$\begin{array}{ccc} \mathbb{C}^2 \supset M^3 & \xrightarrow{\text{normalize}} & M'^3 \subset \mathbb{C}'^2, \\ (z,w) & \longrightarrow & \big(f(z,w),\, g(z,w)\big) \;=:\; \big(z',w'\big). \end{array}$$

All objects will be real analytic ($\mathcal{C}^\omega$). Thus with $w = u + i\,v$ and $w' = u' + i\,v'$, both hypersurfaces M and M' are $\mathcal{C}^\omega$-graphed as:

$$v \;=\; F\big(z, \overline{z}, u\big) \quad \text{and} \quad v' \;=\; F'\big(z', \overline{z}', u'\big).$$

We also assume $T_0^c M' = \{w' = 0\}$. Expand F as:

$$F\big(z,\overline{z},u\big) \;=\; \sum_{j+k+l \geqslant 1} F_{j,k,l}\, z^j \overline{z}^k u^l,$$

with $F_{j,k,l} \in \mathbb{C}$. Define: $\overline{F}\big(z,\overline{z},u\big) \;:=\; \sum_{j+k+l\geqslant 1} \overline{F}_{j,k,l}\, z^j\overline{z}^k u^l.$

From $\overline{v} = v$, it comes $\overline{F(z,\overline{z},u)} = F(z,\overline{z},u)$, whence:

$$\overline{F}\big(\overline{z}, z, u\big) \;\equiv\; F\big(z, \overline{z}, u\big). \tag{4}$$

Applying $\frac{1}{j!}\partial_z^j \frac{1}{k!}\partial_{\overline{z}}^k \frac{1}{l!}\,\partial_u^l$ at $(z,\overline{z},u) = (0,0,0)$ we get:

$$\overline{F}_{k,j,l} \;=\; F_{j,k,l}.$$

The hypothesis that the biholomorphism $(z,w) \longmapsto \big(f(z,w), g(z,w)\big) \;=:\; (z',w')$ fixing the origin sends M to M' expresses as a *fundamental identity:*

$$\begin{aligned} 0 \;\equiv\; & -\tfrac{1}{2i}\, g\big(z,\, u + i\,F(z,\overline{z},u)\big) + \tfrac{1}{2i}\,\overline{g}\big(\overline{z},\, u - i\,F(z,\overline{z},u)\big) + \\ & + F'\Big(f\big(z,\, u + i\,F(z,\overline{z},u)\big),\; \overline{f}\big(\overline{z},\, u - i\,F(z,\overline{z},u)\big), \\ & \qquad \tfrac{1}{2}\, g\big(z,\, u + i\,F(z,\overline{z},u)\big) + \tfrac{1}{2}\,\overline{g}\big(\overline{z},\, u - i\,F(z,\overline{z},u)\big)\Big), \end{aligned} \tag{5}$$

which holds in $\mathbb{C}\{z,\overline{z},u\}$.

According to the preceding sections, for any CR-transversal 1-jet j_0^1 at $0 \in M$, there exists a Moser chain directed by j_0^1 at 0. We let $\gamma\colon I \longrightarrow M$ with $\gamma(0) = 0$ and $0 \in I \subset \mathbb{R}$ an interval, be such a chain. In fact, the next statement is true for any local CR-transversal curve.

Lemma 10.1. *Let $\gamma\colon I \longrightarrow M$ be a local $\mathcal{C}^\omega$ curve with $\gamma(0) = 0 \in M$ and $\dot{\gamma}(0) \notin T_0^c M = \{w = 0\}$. Then there exists a biholomorphism $(z,w) \longmapsto (z',w')$ stabilizing $T_0^c M' = \{w' = 0\}$ which sends γ to the curve $\gamma\prime(t) = (0,t)$ straightened along the v'-axis.*

Notice that a third direction $\dot{\gamma}\prime(0) \in T_0 M' \backslash T_0^c M'$ implies $T_0 M' = \{u' = 0\}$.

Proof. Write: $\gamma(t) \;=\; \big(\varphi(t),\, \psi(t)\big) \;\in\; \mathbb{C} \times \mathbb{C}$.

By assumption, $\dot{\psi}(0) \neq 0$. Thus the map $z \;:=\; z' + \varphi(w')$, $w \;:=\; \psi(w')$, establishes a biholomorphism (inverse).

Similarly, the target curve writes $\gamma_\prime(t) = \big(\varphi_\prime(t), \psi_\prime(t)\big)$. Thus for all $t \in I$:

$$\varphi(t) \equiv \varphi_\prime(t) + \varphi\big(\psi_\prime(t)\big) \quad \text{and} \quad \psi(t) \equiv \psi\big(\psi_\prime(t)\big).$$

The second equation and the invertibility of ψ forces $t \equiv \psi_\prime(t)$. Replacing this in the first equation yields $0 \equiv \varphi_\prime(t)$. ∎

Consequently, the graphing function of the transformed hypersurface writes, after erasing the primes, as:

$$M\colon\ v = F\big(z, \overline{z}, u\big), \quad \text{with } F = \mathrm{O}(2) \text{ and } F(0,0,u) \equiv 0.$$

Lemma 10.2. *There exists a biholomorphism of the form:*

$$z' := z, \quad w' := w + g(z,w),$$

with $g = \mathrm{O}(2)$ *and* $g(0,w) \equiv 0$, *which transforms* $\{v = F\}$ *into* $\{v' = F'\}$ *satisfying:*

$$0 \equiv F'\big(z', 0, u'\big) \equiv F'\big(0, \overline{z}', u'\big).$$

The second vanishing follows from the first, by (4). Notice that $F'(0,0,u') \equiv 0$ is preserved.

Proof. If such a biholomorphism exists, the fundamental identity writes for it:

$$\begin{aligned} 0 \equiv & -F\big(z,\overline{z},u\big) - \tfrac{1}{2i}\, g\big(z,\, u + i\,F(z,\overline{z},u)\big) + \tfrac{1}{2i}\,\overline{g}\big(\overline{z},\, u - i\,F(z,\overline{z},u)\big) + \\ & + F'\Big(z,\, \overline{z},\, u + \tfrac{1}{2}\, g\big(z,\, u + i\,F(z,\overline{z},u)\big) + \tfrac{1}{2}\,\overline{g}\big(\overline{z},\, u - i\,F(z,\overline{z},u)\big)\Big). \end{aligned} \tag{6}$$

We want $F'(z',0,u') \equiv 0$. If this goal would be reached, putting $\overline{z} := 0$, we would deduce:

$$0 \equiv -F(z,0,u) - \tfrac{1}{2i}\, g\big(z,\, u + i\,F(z,0,u)\big) + \tfrac{1}{2i}\,\overline{g}\big(0,\, u - i\,F(z,0,u)\big) + 0. \tag{7}$$

By luck, such an equation can be used to defined $g(z,w)$ uniquely, even with the supplementary condition that the last term be identically zero.
Indeed, by $F = \mathrm{O}(2)$, the implicit function theorem enables to invert:

$$u + i\,F(z,0,u) =: \omega \iff u = \mathrm{T}(z,\omega) = \omega + \mathrm{O}(2).$$

Define therefore $g(z,\omega)$, after erasing the third term $\frac{1}{2i}\,\overline{g}$ above, by:

$$0 \equiv -F\big(z, 0, \mathrm{T}(z,\omega)\big) - \tfrac{1}{2i}\, g(z,\omega) + 0,$$

and notice then that *because* $F(0,0,u) \equiv 0$ *by assumption*, we fulfill by setting $z := 0$, :

$$0 \equiv g\big(0,\, \omega\big).$$

Thus, (7) really holds with $\frac{1}{2i}\,\overline{g} = 0$, and then coming back to $(6)\big|_{\overline{z}=0}$, we get as desired:

$$0 \equiv 0 + F'\Big(z,\, 0,\, u + \tfrac{1}{2}\, g\big(z,\, u + i\,F(z,0,u)\big)\Big).$$ ∎

11. Prenormalization

Now, erase the primes, and assume $0 \equiv F(z,0,u)$. Write:

$$v = F\big(z,\overline{z},u\big) = z\overline{z}\,F_{1,1}(u) + \sum_{\substack{j+k\geqslant 3\\ j\geqslant 1,\, k\geqslant 1}} z^j\overline{z}^k F_{j,k}(u) = z\overline{z}\,F_{1,1}(u) + z^2\overline{z}\,\big(\cdots\big) + \overline{z}^2 z\,\big(\cdots\big).$$

Since M is Levi nondegenerate at 0, after a $\mathbb{C}$-linear transformation, we make:

$$F_{1,1}(0) \,=\, 1.$$

This equality $F_{1,1}(0) = 1$ is known as Poincaré's realization of nondegenerate hypersurfaces in $\mathbb{C}^2$. It is quite crucial in the Chern-Moser normal form construction.

Lemma 11.1. *There exists a biholomorphism of the form:*

$$z' \,:=\, z\,\varphi(w), \qquad w' \,:=\, w,$$

which transforms $M = \{v = F\}$ into M' with:

$$v' \,=\, F' \,=\, z'\overline{z}' + z'^2\overline{z}'\,\big(\cdots\big) + \overline{z}'^2 z'\,\big(\cdots\big).$$

So we may normalize $F'_{1,1}(u') \equiv 1$. Notice that since $z'(\cdots) = z(\cdots)$, the preceding normalization is preserved, namely $F'(z',0,u') \equiv 0$.

Proof. Expanding:

$$\varphi\big(u + i\,F(z,\overline{z},u)\big) \,=\, \varphi\big(u + i\,z\overline{z}\,(\cdots)\big) \,=\, \varphi(u) + z\overline{z}\,\big(\cdots\big),$$

the fundamental identity writes:

$$\begin{aligned}
0 \,&\equiv\, -F\big(z,\overline{z},u\big) + F'\Big(z\,\varphi\big(u + i\,F(z,\overline{z},u)\big),\ \overline{z}\,\overline{\varphi}\big(u - i\,F(z,\overline{z},u)\big),\ u\Big)\\
&\equiv\, -z\overline{z}\,F_{1,1}(u) + z^2\overline{z}\,\big(\cdots\big) + z\overline{z}^2\,\big(\cdots\big)\\
&\quad + z\,\big(\varphi(u) + z\overline{z}\,(\cdots)\big)\,\overline{z}\,\big(\overline{\varphi(u)} + z\overline{z}\,(\cdots)\big)\,F'_{1,1}(u) + z^2\overline{z}\,\big(\cdots\big) + \overline{z}^2 z\,\big(\cdots\big)\\
&\equiv\, z\overline{z}\,\Big[-F_{1,1}(u) + \varphi(u)\,\overline{\varphi}(u)\,F'_{1,1}(u)\Big] + z^2\overline{z}\,\big(\cdots\big) + \overline{z}^2 z\,\big(\cdots\big).
\end{aligned}$$

To have $F'_{1,1}(u) \equiv 1$, it suffices to take:

$$\varphi(u) \,:=\, \sqrt{F_{1,1}(u)} \qquad\qquad \text{(remember } F_{1,1}(0) \,=\, 1\text{)},$$

which is real on the u-axis, and then to define $\varphi(w) := \varphi(u)\big|_{u:=w}$, replacing u by w in the (converging) power series of φ. ∎

Thus, erasing the primes, still with $0 \equiv F(z,0,u)$, we have:

$$v \,=\, F\big(z,\overline{z},u\big) \,=\, z\overline{z} + \sum_{\substack{j+k\geqslant 3\\ j\geqslant 1,\, k\geqslant 1}} z^j\overline{z}^k\,F_{j,k}(u).$$

Lemma 11.2. *There exists a biholomorphism of the form:*

$$z' := z + \Lambda(z,w) = z + z^2\big(\cdots\big), \qquad\qquad w' := w,$$

which transforms $M = \{v = F\}$ into M':

$$v' = F' = z'\overline{z}' + \sum_{j\geqslant 2,\, k\geqslant 2} z'^j\overline{z}'^k\, F'_{j,k}(u') = z'\overline{z}' + z'^2\overline{z}'^2\big(\cdots\big).$$

Any such biholomorphism with $z' = z + z^2(\cdots)$ preserves the already achieved normalizations.

Proof. Single out all monomials with $k = 1$:

$$\begin{aligned} v &= z\overline{z} + \sum_{j\geqslant 2} z^j\overline{z}^1\, F_{j,1}(u) + \sum_{\substack{j+k\geqslant 3\\ j\geqslant 1,\, k\geqslant 2}} z^j\overline{z}^k\, F_{j,k}(u) \\ &= \overline{z}\bigg(\underbrace{z + \sum_{j\geqslant 2} z^j\, F_{j,1}(u)}_{=:\,\Lambda(z,u)}\bigg) + \overline{z}^2\big(\cdots\big). \end{aligned}$$

Expand:
$$\begin{aligned} z' &= z + \Lambda(z,w) = z + \Lambda\big(z,\, u + i\,F(z,\overline{z},u)\big) \\ &= z + \Lambda\big(z,\, u + iz\overline{z}\,(\cdots)\big) = z + \Lambda(z,u) + z\overline{z}\,\big(\cdots\big), \end{aligned}$$

and get: $v = \overline{z}\Big(z' - z\overline{z}\,\big(\cdots\big)\Big) + \overline{z}^2\big(\cdots\big) = \overline{z}\,z' + \overline{z}^2\big(\cdots\big).$

Next, write the inverse as: $z' + z'^2\big(\cdots\big) = z' + \Lambda'(z',w') = z$, so that we have $\overline{z}^2(\cdots) = \overline{z}'^2(\cdots)$, and continue:

$$\begin{aligned} v' = v &= \overline{z}\,z' + \overline{z}^2\big(\cdots\big) = \Big(\overline{z}' + \overline{\Lambda}'\big(\overline{z}',\overline{w}'\big)\Big)\,z' + \overline{z}^2\big(\cdots\big) \\ &= \Big(\overline{z}' + \overline{z}'^2\big(\cdots\big)\Big)\,z' + \overline{z}'^2\big(\cdots\big) = z'\overline{z}' + \overline{z}'^2\big(\cdots\big). \end{aligned}$$

The remainder after $z'\overline{z}'$ being real, it must be also a multiple of z'^2. ∎

12. Complete Moser normal form for hypersurfaces $M^3 \subset \mathbb{C}^2$

Thus:
$$v = z\overline{z} + z^2\overline{z}^2\, F_{2,2}(u) + \sum_{\substack{j+k\geqslant 5\\ j\geqslant 2,\, k\geqslant 2}} z^j\overline{z}^k\, F_{j,k}(u). \tag{8}$$

Lemma 12.1. *There exists a biholomorphism of the form:*

$$z' := z\,\lambda(w), \qquad\qquad w' := w,$$

with $\lambda(u)\,\overline{\lambda(u)} \equiv 1$ and $\lambda(0) = 1$, such that the new M' has vanishing $F'_{2,2}(u') \equiv 0$:

$$v' = z'\overline{z}' + 0 + \sum_{\substack{j+k\geqslant 5\\ j\geqslant 2,\, k\geqslant 2}} z'^j\overline{z}'^k\, F'_{j,k}(u').$$

The condition $\big|\lambda(u)\big|^2 \equiv 1$ for $w = u \in \mathbb{R}$ guarantees that all the previously achieved normalizations are preserved.

Proof. Expand:

$$\begin{aligned}\lambda\big(u + i\,F(z,\overline{z},u)\big) &= \lambda\big(u + i\,z\overline{z} + z^2\overline{z}^2\,(\cdots)\big)\\ &= \lambda(u) + \lambda_u(u)\,\big[i\,z\overline{z} + z^2\overline{z}^2\,(\cdots)\big] + z^2\overline{z}^2\,\big(\cdots\big)\\ &= \lambda(u)\,\bigg(1 + \frac{\lambda_u(u)}{\lambda(u)}\,i\,z\overline{z} + z^2\overline{z}^2\,(\cdots)\bigg).\end{aligned}$$

Since we assume $\big|\lambda(u)\big|^2 \equiv 1$, i.e. $\lambda(u) = e^{i\,\varphi(u)}$ with $\varphi(u)$ real, the quotient $\frac{\lambda_u(u)}{\lambda(u)}$ is purely imaginary, hence:

$$\big|\lambda(u+i\,F)\big|^2 = 1 + 2i\,z\overline{z}\,\frac{\lambda_u(u)}{\lambda(u)} + z^2\overline{z}^2\,\big(\cdots\big).$$

Also, it is clear that $z'^j\overline{z}'^k\,(\cdots) = z^j\overline{z}^k\,(\cdots)$. Thanks to these preliminaries:

$$\begin{aligned}v' &= z'\overline{z}' + z'^2\overline{z}'^2\,F'_{2,2}(u') + z'^3\overline{z}'^2\,\big(\cdots\big) + z'^2\overline{z}'^3\,\big(\cdots\big)\\ &= \big|\lambda(u+i\,F)\big|^2\,z\overline{z} + \big|\lambda(u+i\,F)\big|^4\,z^2\overline{z}^2\,F'_{2,2}(u) + z^3\overline{z}^2\,\big(\cdots\big) + z^2\overline{z}^3\,\big(\cdots\big)\\ &= z\overline{z} + z^2\overline{z}^2\,2i\,\frac{\lambda_u(u)}{\lambda(u)} + z^3\overline{z}^3\,\big(\cdots\big)\\ &\quad + z^2\overline{z}^2\,\big(1 + z\overline{z}\,(\cdots)\big)\,F'_{2,2}(u) + z^3\overline{z}^2\,\big(\cdots\big) + z^2\overline{z}^3\,\big(\cdots\big)\\ &= z\overline{z} + z^2\overline{z}^2\,\bigg[2i\,\frac{\lambda_u(u)}{\lambda(u)} + F'_{2,2}(u)\bigg] + z^3\overline{z}^2\,\big(\cdots\big) + z^2\overline{z}^3\,\big(\cdots\big),\end{aligned}$$

and since $v' = v$ with v given by (8), an identification yields:

$$2i\,\frac{\lambda_u(u)}{\lambda(u)} + F'_{2,2}(u) \equiv F_{2,2}(u).$$

In order to annihilate $F'_{2,2}(u') := 0$, it suffices therefore to set:

$$\lambda(u) := \exp\,\bigg(\frac{1}{2i}\int_0^u F_{2,2}(t)\,dt\bigg). \qquad \blacksquare$$

Now we come to a crucial moment offering a key simplification which was prepared in advance by Assertion 9.4.

Lemma 12.2. *After having normalized:*

$$0 \equiv F_{j,0}(u) \equiv F_{0,k}(u) \equiv \underset{(j\neq 1)}{F_{j,1}(u)} \equiv \underset{(1\neq k)}{F_{1,k}(u)}, \qquad 1 \equiv F_{1,1}(u), \qquad 0 \equiv F_{2,2}(u),$$

the fact that the u-axis, contained in M, is a Moser chain, offers without any further work: $0 \equiv F_{3,2}(u) \equiv F_{2,3}(u)$.

Proof. At each point $p = (0, u_p) \in M$ with any (small) $u_p \in \mathbb{R}$ in the straightened Moser chain, the equation of M normalized up to this point and truncated after weighted order 6 writes exactly:

$$v = z\overline{z} + z^3\overline{z}^2\,F_{3,2}(u_p) + z^2\overline{z}^3\,F_{2,3}(u_p) + \mathrm{O}(6),$$

under the form considered in Assertion 9.4, which then yields $F_{3,2}(u_p) = 0 = F_{2,3}(u_p)$, this for any u_p. $\blacksquare$

Thus: $v \,=\, z\overline{z} + z^4\overline{z}^2\, F_{4,2}(u) + z^3\overline{z}^3\, F_{3,3}(u) + z^2\overline{z}^4\, F_{2,4}(u) + \sum_{\substack{j+k\geqslant 7\\ j\geqslant 2,\, k\geqslant 2}} z^j\overline{z}^k\, F_{j,k}(u).$

Lemma 12.3. *There exists a biholomorphism of the form:*

$$z' \,:=\, z\,\sqrt{\psi_w(w)}, \qquad\qquad w' \,:=\, \psi(w),$$

with $\psi(\mathbb{R}) \subset \mathbb{R}$*, with* $\psi(0) = 0$*, with* $\psi_w(0) \in \mathbb{R}_{>0}$*, such that the new* M' *has vanishing* $F'_{3,3}(u') \equiv 0$*:*

$$v' \,=\, z'\overline{z}' + z'^4\overline{z}'^2\, F'_{4,2}(u') + 0 + z'^2\overline{z}'^4\, F'_{2,4}(u') + \sum_{\substack{j+k\geqslant 7\\ j\geqslant 2,\, k\geqslant 2}} z'^j\overline{z}'^k\, F'_{j,k}(u').$$

We will see in the proof why such a biholomorphism preserves all previously achieved normalizations. The function $\psi = \psi(u)$ will be solution of the ODE:

$$\psi_{uuu}(u) \,=\, \frac{3}{2}\,\frac{\psi_{uu}^2(u)}{\psi_u(u)} - 3\,F_{3,3}(u)\,\psi_u(u).$$

Proof. More generally, we perform a biholomorphism of the form:

$$z' \,:=\, z\,\varphi(w), \quad w' \,:=\, \psi(w),$$

assuming that $\varphi(u) \in \mathbb{R}$, $\varphi(0) \neq 0$, and $\psi(u) \in \mathbb{R}$, $\psi_w(0) \in \mathbb{R}_{\neq 0}$. We let $v' = F'(z', \overline{z}', u')$ be the transformed hypersurface equation. Many computations are needed. Firstly:

$$\begin{aligned}
v' \,&=\, \Im\,\psi\big(u + i\,F\big)\\
&=\, \Im\,\Big\{\psi(u) + \psi_u(u)\,i\,F + \psi_{uu}(u)\,\frac{(i\,F)^2}{2!} + \psi_{uuu}(u)\,\frac{(i\,F)^3}{3!} + F^4\,\big(\cdots\big)\Big\}\\
&=\, \psi_u(u)\,F - \frac{1}{6}\,\psi_{uuu}(u)\,F^3 + z^4\overline{z}^4\,\big(\cdots\big)\\
&=\, \psi_u(u)\,\Big[z\overline{z} + z^4\overline{z}^2\,F_{4,2}(u) + z^3\overline{z}^3\,F_{3,3}(u) + z^2\overline{z}^4\,F_{2,4}(u) + \mathrm{O}_{z,\overline{z}}(7)\Big]\\
&\qquad\qquad - \frac{1}{6}\,\psi_{uuu}(u)\,\big[z^3\overline{z}^3 + \mathrm{O}_{z,\overline{z}}(10)\big] + z^4\overline{z}^4\,\big(\cdots\big),
\end{aligned}$$

so that no terms of order $3,\ 4,\ 5$ in $(z, \overline{z})$ are present:

$$\begin{aligned}
v' \,&=\, z\overline{z}\,\psi_u(u) + z^4\overline{z}^2\,\psi_u(u)\,F_{4,2}(u)\\
&\quad + z^3\overline{z}^3\,\big[\psi_u(u)\,F_{3,3}(u) - \tfrac{1}{6}\,\psi_{uuu}(u)\big] + z^2\overline{z}^4\,\psi_u(u)\,F_{2,4}(u) + \mathrm{O}_{z,\overline{z}}(7).
\end{aligned} \tag{9}$$

Secondly, one can convince oneself that the normalization $v' = z'\overline{z}' + z'^2\overline{z}'^2\,\big(\cdots\big)$ is preserved, so that the equation of the transformed hypersurface is:

$$v' \,=\, z'\overline{z}' + \sum_{j\geqslant 2,\, k\geqslant 2} z'^j\overline{z}'^k\,F'_{j,k}(u').$$

Thirdly, using $\varphi(u) \in \mathbb{R}$ and $F = z\overline{z} + \mathrm{O}_{z,\overline{z}}(6)$:

$$\begin{aligned}
z'\overline{z}' \,&=\, z\overline{z}\Big(\varphi(u) + \varphi_u(u)\,i\,F + \varphi_{uu}(u)\,\frac{(i\,F)^2}{2!} + F^3\,\big(\cdots\big)\Big)\cdot\\
&\qquad\cdot\Big(\varphi(u) + \varphi_u(u)\,\big(-i\,F\big) + \varphi_{uu}(u)\,\frac{(-i\,F)^2}{2!} + F^3\,\big(\cdots\big)\Big)\\
&=\, z\overline{z}\,\varphi(u)^2 + z\overline{z}\,\Big[\varphi_u(u)^2 - \varphi(u)\,\varphi_{uu}(u)\Big]\,F^2 + z\overline{z}\,F^3\,\big(\cdots\big)\\
&=\, z\overline{z}\,\varphi(u)^2 + z^3\overline{z}^3\,\big(\varphi_u(u)^2 - \varphi(u)\,\varphi_{uu}(u)\big) + \mathrm{O}_{z,\overline{z}}(8).
\end{aligned}$$

Fourthly, for every $j \geqslant 2$ and every $k \geqslant 2$:

$$\begin{aligned} z'^j \overline{z}'^k &= z^j \overline{z}^k \Big(\varphi(u) + \varphi_u(u)\, i\, F + z^2\overline{z}^2\,(\cdots)\Big)^j \cdot \\ &\qquad \cdot \Big(\varphi(u) + \varphi_u(u)\,(-i\, F) + z^2\overline{z}^2\,(\cdots)\Big)^k \\ &= z^j \overline{z}^k \Big(\varphi(u)^j + j\,\varphi(u)^{j-1}\varphi_u(u)\, i\, z\overline{z} + z^2\overline{z}^2\,(\cdots)\Big)\cdot \\ &\qquad \cdot \Big(\varphi(u)^k - k\,\varphi(u)^{k-1}\varphi_u(u)\, i\, z\overline{z} + z^2\overline{z}^2\,(\cdots)\Big) \\ &= z^j \overline{z}^k \Big(\varphi(u)^{j+k} + i\,(j-k)\,\varphi(u)^{j+k-1}\,\varphi_u(u)\, z\overline{z} + z^2\overline{z}^2\,(\cdots)\Big). \end{aligned}$$

Fifthly:

$$\begin{aligned} F'_{j,k}(u') &= F'_{j,k}\Big(\Re\,\psi\big(u + i\,F\big)\Big) = F'_{j,k}\Big(\Re\,\big[\psi(u) + \psi_u(u)\, i\, F + F^2\,(\cdots)\big]\Big) \\ &= F'_{j,k}\Big(\psi(u) + 0 + z^2\overline{z}^2\,(\cdots)\Big) = F'_{j,k}\,\big(\psi(u)\big) + z^2\overline{z}^2\,(\cdots). \end{aligned}$$

Thanks to all this:

$$\begin{aligned} F\big(z', \overline{z}', u'\big) &= z'\overline{z}' + z'^2\overline{z}'^2\, F'_{2,2}(u') + z'^3\overline{z}'^2\, F'_{3,2}(u') + z'^2\overline{z}'^3\, F'_{2,3}(u') + \\ &\quad + z'^4\overline{z}'^2\, F'_{4,2}(u') + z'^3\overline{z}'^3\, F'_{3,3}(u') + z'^2\overline{z}'^4\, F'_{2,4}(u') + \mathrm{O}_{z',\overline{z}'}(7) \\ &= z\overline{z}\,\varphi(u)^2 + z^3\overline{z}^3\,\big(\varphi_u(u)^2 - \varphi(u)\,\varphi_{uu}(u)\big) + \mathrm{O}_{z,\overline{z}}(8) + \\ &\quad + z^2\overline{z}^2\,\Big(\varphi(u)^4 + 0 + z^2\overline{z}^2\,(\cdots)\Big)\,\Big(F'_{2,2}\big(\psi(u)\big) + z^2\overline{z}^2\,(\cdots)\Big) \\ &\quad + z^3\overline{z}^2\,\Big(\varphi(u)^5 + i\,\varphi(u)^4\,\varphi_u(u)\, z\overline{z} + z^2\overline{z}^2\,(\cdots)\Big)\, F'_{3,2}\big(\psi(u)\big) \\ &\quad + z^2\overline{z}^3\,\Big(\varphi(u)^5 - i\,\varphi(u)^4\,\varphi_u(u)\, z\overline{z} + z^2\overline{z}^2\,(\cdots)\Big)\, F'_{2,3}\big(\psi(u)\big) \\ &\quad + z^4\overline{z}^2\,\varphi(u)^6\, F'_{4,2}\big(\psi(u)\big) + z^3\overline{z}^3\,\varphi(u)^6\, F'_{3,3}\big(\psi(u)\big) \\ &\quad + z^2\overline{z}^4\,\varphi(u)^6\, F'_{2,4}\big(\psi(u)\big) + \mathrm{O}_{z,\overline{z}}(7) \\ &= z\overline{z}\,\varphi(u)^2 + z^2\overline{z}^2\,\varphi(u)^4\, F'_{2,2}\big(\psi(u)\big) + z^3\overline{z}^2\,\varphi(u)^5\, F'_{3,2}\big(\psi(u)\big) \\ &\quad + z^2\overline{z}^3\,\varphi(u)^5\, F'_{2,3}\big(\psi(u)\big) + z^4\overline{z}^2\,\varphi(u)^6\, F'_{4,2}\big(\psi(u)\big) \\ &\quad + z^3\overline{z}^3\,\Big[\varphi_u(u)^2 - \varphi(u)\,\varphi_{uu}(u) + \varphi(u)^6\, F'_{3,3}\big(\psi(u)\big)\Big] \\ &\quad + z^2\overline{z}^4\,\varphi(u)^6\, F'_{2,4}\big(\psi(u)\big) + \mathrm{O}_{z,\overline{z}}(7). \end{aligned}$$

By identifying powers $z^j\overline{z}^k$ with (9), we get:

$$\begin{aligned} \psi_u(u) &\equiv \varphi(u)^2, && (1{,}1) \\ 0 &\equiv \varphi(u)^4\, F'_{2,2}\big(\psi(u)\big), && (2{,}2) \\ 0 &\equiv \varphi(u)^5\, F'_{3,2}\big(\psi(u)\big), && (3{,}2) \\ 0 &\equiv \varphi(u)^5\, F'_{2,3}\big(\psi(u)\big), && (2{,}3) \\ \psi_u(u)\, F_{4,2}(u) &\equiv \varphi(u)^6\, F'_{4,2}\big(\psi(u)\big), && (4{,}2) \\ \psi_u(u)\, F_{3,3}(u) - \tfrac{1}{6}\,\psi_{uuu}(u) &\equiv \varphi_u(u)^2 - \varphi(u)\,\varphi_{uu}(u) + \varphi(u)^6\, F'_{3,3}\big(\psi(u)\big), && (3{,}3) \\ \psi_u(u)\, F_{2,4}(u) &\equiv \varphi(u)^6\, F'_{2,4}\big(\psi(u)\big). && (2{,}4) \end{aligned}$$

Visibly, to annihilate $F'_{3,3}(u')$, it suffices to fulfill:

$$\begin{aligned}\psi_u(u) &\equiv \varphi(u)^2,\\ \psi_u(u)\,F_{3,2}(u) - \tfrac{1}{6}\,\psi_{uuu}(u) &\equiv \varphi_u(u)^2 - \varphi(u)\,\varphi_{uu}(u) + 0.\end{aligned}$$

Assuming $\psi_u(0) = 1$, choosing $\varphi(u) := \sqrt{\psi_u(u)}$, and replacing, it suffices in conclusion that ψ satisfies the solvable ODE:

$$\psi_{uuu}(u) = \frac{3}{2}\,\frac{\psi_{uu}(u)^2}{\psi_u(u)} - 3\,F_{3,3}(u)\,\psi_u(u). \qquad \blacksquare$$

In summary, we have fully reproved with expository details what is actually the equation (3.18) of Chern-Moser's celebrated work [13].

Proposition 12.4. *Given a Levi nondegenerate $\mathcal{C}^\omega$ hypersurface $M^3 \subset \mathbb{C}^2$, for every $p \in M$ and every CR-transversal 1-jet j^1_p at p, if $\gamma_p \ni p$ denotes the unique piece of Moser chain directed by j^1_p at p, then there exist local holomorphic coordinates $(z, w = u + i\,v)$ centered at p in which γ_p is the u-axis and such that M is graphed as:*

$$v = z\overline{z} + z^4\overline{z}^2\,F_{4,2}(u) + z^2\overline{z}^4\,F_{2,4}(u) + \sum_{\substack{j+k\geqslant 7\\ j\geqslant 2,\,k\geqslant 2}} z^j\overline{z}^k F_{j,k}(u). \qquad \blacksquare$$

13. Uniqueness of Moser normal form

Starting with a $\mathcal{C}^\omega$ Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$, at any point $p \in M$, it is elementary to find holomorphic coordinates (z, w) vanishing at p in which M has equation $v = F = z\overline{z} + \mathrm{O}_{z,\overline{z},u}(3)$. Such an equation can hence freely be taken as the starting point towards a complete normalization of F.
In the preceding sections, we have in fact established the *existence* of a *normal form* for M. We can now present the known *uniqueness* statement.

Theorem 13.1. [13, 17] *Given a $\mathcal{C}^\omega$ Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$ with $0 \in M$ of the form:*

$$v = z\overline{z} + \mathrm{O}_{z,\overline{z},u}(3),$$

there exists a biholomorphism $(z, w) \longmapsto (z', w')$ fixing 0 which maps $(M, 0)$ into $(M', 0)$ of normalized equation:

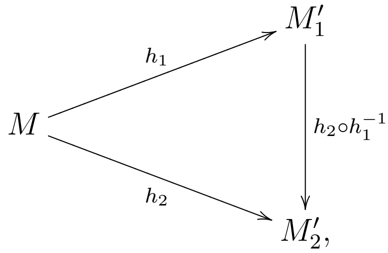
$$v' = z'\overline{z}' + F'_{4,2}(u')\,z'^4\overline{z}'^2 + F'_{2,4}(u')\,z'^2\overline{z}'^4 + z'^2\overline{z}'^2\,\mathrm{O}_{z',\overline{z}'}(3).$$

Furthermore, the map exists and is unique *if it is assumed to be of the form:*

$$\begin{aligned} z' &:= z + f(z,w), & w' &:= w + g(z,w),\\ f_z(0) &= f_w(0) = 0, & g_z(0) &= g_w(0) = \Re\,g_{ww}(0) = 0.\end{aligned}$$

Proof. By choosing a chain at $0 \in M$ whose first jet is flat, directed along the u-axis, one can verify (exercise) that all the constructions done in the preceding sections do indeed give a biholomorphism of this specific form. So our job is to establish uniqueness.

Suppose that two such normalizations $h_\iota \colon (z,w) \longmapsto (z+f_\iota,\, w+g_\iota)$, $\iota = 1,2$, are given:



with $0 = f_{\iota,z}(0) = f_{\iota,w}(0)$ and $0 = g_{\iota,z}(0) = g_{\iota,w}(0) = \operatorname{Re} g_{\iota,ww}(0)$. On $\mathbb{C}'^2 \supset M_1'$, let us take for simplicity coordinates with the same name (z,w), and coordinates (z',w') on the $\mathbb{C}'^2 \supset M_2'$.

Lemma 13.2. *Then* $h_2 \circ h_1^{-1} =: (z+f,\, w+g)$ *also satisfies* $0 = f_z(0) = f_w(0)$ *and* $0 = g_z(0) = g_w(0) = \operatorname{Re} g_{ww}(0)$.

Proof. Since both h_1 and h_2 are the identity plus $\mathrm{O}_{z,w}(2)$ terms, the same holds for $h_2 \circ h_1^{-1}$. It remains only to show $\Re\, g_{ww}(0) = 0$.

The following lemma then applies to the map $h_2 \circ h_1^{-1}$, since M_1' and M_2' are in normal form.

Lemma 13.3. *If* $(z,w) \longmapsto (z+f,\, w+g)$ *with* $f, g = \mathrm{O}_{z,w}(2)$*, maps* $v = z\overline{z} + \mathrm{O}_{z,\overline{z},u}(3)$ *to* $v' = z'\overline{z}' + \mathrm{O}_{z',\overline{z}',u'}(3)$*, then* $g_{zz}(0) = g_{zw}(0) = 0$ *and* $g_{ww}(0) \in \mathbb{R}$*, so that:* $g(z,w) \,=\, w + \frac{1}{2}\, g_{ww}(0)\, w^2 + \mathrm{O}_{z,w}(3)$.

Proof. Writing $w' = w + g = w + \alpha\, z^2 + \beta\, zw + (a+ib)\, w^2 + \mathrm{O}_{z,w}(3)$, we have:

$$
\begin{aligned}
v' \,&=\, v + \Im\,(\alpha\, z^2) + \Im\,\big(\beta\, z\,(u+iv)\big) + 2a\, uv + b\, u^2 - b\, v^2 + \mathrm{O}_{z,w}(3) \\
&=\, z\overline{z} + \Im\,(\alpha\, z^2) + \Im\,(\beta\, zu) + b\, u^2 + \mathrm{O}_{z,\overline{z},u}(3),
\end{aligned}
$$

hence using the inversion $z = z' + \mathrm{O}_{z',w'}(2)$, $w = w' + \mathrm{O}_{z',w'}(2)$, we get $\alpha = \beta = b = 0$ from:

$$
v' \,=\, z'\overline{z}' + \Im\,(\alpha\, z'^2) + \Im\,(\beta\, z'u') + b\, u'^2 + \mathrm{O}_{z',\overline{z}',u'}(3). \qquad \blacksquare
$$

Thus, the assumption $\Re\, g_{\iota,ww}(0) = 0$, $\iota = 1,2$, implies that the h_ι, are both of the form $\big(z + \mathrm{O}_{z,w}(2),\; w + \mathrm{O}_{z,w}(3)\big)$. Such a form is stable under composition and inversion, hence $h_2 \circ h_1^{-1}$ is also of this form, and in particular, one has $\Re\, g_{ww}(0) = 0$. ■

Our uniqueness goal is to obtain $h_1 = h_2$. Equivalently, $h_2 \circ h_1^{-1} = \mathrm{Id}$. This will be offered by the next independent key uniqueness statement. ■

Theorem 13.4. *If two* $\mathcal{C}^\omega$ *Levi nondegenerate hypersurfaces* $0 \in M^3 \subset \mathbb{C}^2$ *and* $0 \in M'^3 \subset \mathbb{C}'^2$ *are both in normal form:*

$$
\begin{aligned}
v \,&=\, F \,=\, z\overline{z} + z^4\overline{z}^2\, F_{4,2}(u) + z^2\overline{z}^4\, F_{2,4}(u) + \sum_{\substack{j+k\geqslant 7 \\ j\geqslant 2,\, k\geqslant 2}} z^j\overline{z}^k F_{j,k}(u), \\
v' \,&=\, F' \,=\, z'\overline{z}' + z'^4\overline{z}'^2\, F'_{4,2}(u') + z'^2\overline{z}'^4\, F'_{2,4}(u') + \sum_{\substack{j+k\geqslant 7 \\ j\geqslant 2,\, k\geqslant 2}} z'^j\overline{z}'^k F'_{j,k}(u'),
\end{aligned}
$$

and if there exists a biholomorphism $(M,0) \longrightarrow (M',0)$ *of the form:*

$$
\begin{aligned}
z' &:= z + f(z,w), & w' &:= w + g(z,w), \\
f_z(0) &= f_w(0) = 0, & g_z(0) &= g_w(0) = \Re\, g_{ww}(0) = 0,
\end{aligned}
$$

then $(f,g) \equiv (0,0)$*, and the biholomorphism is the identity.*

Proof. Equivalently, the graphing function $F = \sum_{j,k} F_{j,k}(u)\, z^j \overline{z}^k$ of M satisfies the general *prenormalization conditions:*

$$
0 \equiv F_{j,0}(u) \equiv F_{0,k}(u), \quad 0 \equiv F_{j,1}(u) \equiv F_{1,k}(u) \qquad\qquad (j,\, k \in \mathbb{N}),
$$

except of course $1 \equiv F_{1,1}(u)$, together with the *sporadic normalization conditions:*

$$
0 \equiv F_{2,2}(u) \equiv F_{3,2}(u) \equiv F_{2,3}(u) \equiv F_{3,3}(u),
$$

and the same holds about F'. Accordingly, let us introduce:

$$
S := \big\{(j,0),\ (0,k),\ (j,1),\ (1,k)\big\} \cup \big\{(2,2),\ (3,2),\ (2,3),\ (3,3)\big\}.
$$

For a general real converging power series vanishing at $(z, \overline{z}, u) = (0,0,0)$:

$$
G = \sum_{j,k,l} G_{j,k,l}\, z^j \overline{z}^k u^l \qquad\qquad (\overline{G_{k,j,l}} = G_{j,k,l}),
$$

i.e. with $G_{0,0,0} = 0$, introduce the projection:

$$
\Pi_S(G) := \sum_{(j,k)\in S} \sum_{l=0}^{\infty} G_{j,k,l}\, z^j \overline{z}^k u^l,
$$

so that: $\Pi_S(F) = z\overline{z}$ and $\Pi_S(F') = z'\overline{z}'$.
Also, reminding that granted our current assumption $\Re\, g_{ww}(0) = 0$, we already understood in Lemma 13.3 that we have in fact $g = w + \mathrm{O}_{z,w}(3)$. Next, taking integers $\nu \geqslant 3$, reminding weights $[z] = 1$, $[w] = 2$, let us decompose in weighted homogeneous components:

$$
\begin{aligned}
f(z,w) &= \sum_{j+l\geqslant 2} f_{j,l}\, z^j w^l = \sum_{\nu\geqslant 3} f_{\nu-1}, & g(z,w) &= \sum_{j+l\geqslant 3} g_{j,l}\, z^j w^l = \sum_{\nu\geqslant 3} g_\nu, \\
f_{\nu-1} &:= \sum_{j+2l=\nu-1} f_{j,l}\, z^j w^l & g_\nu &:= \sum_{j+2l=\nu} g_{j,l}\, z^j w^l.
\end{aligned}
$$

Still for any $\nu \geqslant 3$, introduce the projections:

$$
\pi_{\nu-1}(f) := f_{\nu-1}, \qquad \pi_\nu(g) := g_\nu, \qquad \pi_\nu(G) := G_\nu := \sum_{j+k+2l=\nu} G_{j,k,l}\, z^j \overline{z}^k u^l,
$$

so that:
$$
\Pi_S\big(\pi_\nu(F)\big) = 0 = \Pi_S\big(\pi_\nu(F')\big) \qquad\qquad (\nu \geqslant 3).
$$

Also, introduce:
$$
\pi^\nu := \pi_2 + \cdots + \pi_\nu.
$$

For later use, observe that for any holomorphic function $e_\mu = e_\mu(z,w)$ which is weigthed μ-homogeneous, it holds (exercise):

$$\pi^\mu\left(e_\mu\big(z,\,u+i\,[z\overline{z}+\mathrm{O}_{z,\overline{z},u}(3)]\big)\right) \;=\; e_\mu\big(z,\,u+iz\overline{z}\big). \tag{10}$$

Next, since $f = f_2 + f_3 + \cdots$ and $g = g_3 + g_4 + \cdots$, the fundamental identity writes:

$$0 \,\equiv\, -\Im\left(w+g_3+g_4+\cdots\right)+F'\big(z+f_2+f_3+\cdots,\;\overline{z}+\overline{f}_2+\overline{f}_3+\cdots,\;\Re\big(w+g_3+g_4+\cdots\big)\big),$$

identically in $\mathbb{C}\{z,\overline{z},u\}$ after replacing $(z,w) = \big(z, u+i\,F(z,\overline{z},u)\big)$.

To prove $(f,g)=(0,0)$, we may proceed progressively:

- $(f_2,g_3)=(0,0)$;   • $(f_3,g_4)=(0,0)$;
- $(f_{\mu-1},\,g_\mu)=(0,0)$ for $\mu=3,\ldots,\nu-1$ and some $\nu\geqslant 5$ implies $(f_{\nu-1},\,g_\nu)=(0,0)$.

Lemma 13.5. *One has* $(f_2,g_3)=(0,0)$.

Proof. Applying π^3 to the fundamental identity gives, using (10):

$$\begin{aligned}
0 &\equiv \pi^3\Big(-\Im\left(w+g_3\right)+F'\Big(z+f_2,\,\overline{z}+\overline{f}_2,\,\Re\left(w+g_2\right)\Big)\Big)\\
&\equiv \pi^3\Big(-v-\Im\,g_3+\left(z+f_2\right)\left(\overline{z}+\overline{f}_2\right)+F_3'\big(z,\overline{z},u\big)\Big)\\
&\equiv \pi^3\Big(-\underline{z\overline{z}}_{\circ\circ}-\underline{F_3(z,\overline{z},u)}_\circ-\Im\,g_3+\underline{z\overline{z}}_{\circ\circ}+z\overline{f}_2+\overline{z}f_2+f_2\overline{f}_2+\underline{F_3'\big(z,\overline{z},u\big)}_\circ\Big),
\end{aligned}$$

and since M and M' are normalized by assumption, with $\pi^3(f_2\overline{f}_2)\equiv 0$, it remains to show only:

$$0 \,\equiv\, \Re\Big\{i\,g_3\big(z,\,u+iz\overline{z}\big)+2\,\overline{z}\,f_2\big(z,\,u+iz\overline{z}\big)\Big\}.$$

Replacing $f_2 = f_{2,0}\,z^2 + f_{0,1}\,w$ with $f_{0,1}=0$ by assumption and then replacing $g_3 = g_{3,0}\,z^3 + g_{1,1}\,zw$, this is:

$$0 \equiv \tfrac{i}{2}\,g_{3,0}\,z^3 - \tfrac{i}{2}\,\overline{g}_{3,0}\,\overline{z}^3 + \big(f_{2,0}-\tfrac{1}{2}\,g_{1,1}\big)\,z^2\overline{z} + \big(\overline{f}_{2,0}-\tfrac{1}{2}\,\overline{g}_{1,1}\big)\,z\overline{z}^2 + \tfrac{i}{2}\,g_{1,1}\,zu - \tfrac{i}{2}\,\overline{g}_{1,1}\,\overline{z}u,$$

and starting from the end, this forces $0 = g_{1,1} = f_{2,0} = g_{3,0}$, so as asserted $0 = f_2 = g_3$. ∎

Lemma 13.6. *One has* $(f_3,g_4)=(0,0)$.

Proof. Applying now π^4 to the fundamental identity, taking into account that F and F' are normalized, we compute:

$$\begin{aligned}
0 &\equiv \pi^4\bigg(-\Im\left(w+0+g_4\right)+\left(z+0+f_3\right)\left(\overline{z}+0+\overline{f}_3\right)\\
&\qquad\qquad +\sum_{3\leqslant\mu\leqslant 4}F_\mu'\Big(z+0+f_3,\;\overline{z}+0+\overline{f}_3,\;\Re\left(w+0+g_4\right)\Big)\bigg)\\
&\equiv \pi^4\bigg(-z\overline{z}-\underline{F_3(z,\overline{z},u)}_\circ-\underline{F_4(z,\overline{z},u)}_\circ+\Re\left(i\,g_4\right)+z\overline{z}+z\overline{f}_3+\overline{z}f_3\\
&\qquad\qquad +\underline{f_3\overline{f}_3}_\circ+\underline{F_3'\big(z,\overline{z},u\big)}_\circ+\underline{F_4'\big(z,\overline{z},u\big)}_\circ\bigg)\\
&\equiv \pi^4\Big(\Re\Big\{i\,g_4\big(z,\,u+i[z\overline{z}+\mathrm{O}_{z,\overline{z},u}(3)]\big)+2\,\overline{z}\,f_3\big(z,\,u+i[z\overline{z}+\mathrm{O}_{z,\overline{z},u}(3)]\big)\Big\}\Big)\\
&\equiv \Re\Big\{i\,g_4\big(z,\,u+i\,z\overline{z}\big)+2\,\overline{z}\,f_3\big(z,\,u+i\,z\overline{z}\big)\Big\}.
\end{aligned}$$

Replacing $f_3 = f_{3,0}\, z^3 + f_{1,1}\, zw$ and $g_4 = g_{4,0}\, z^4 + g_{2,1}\, z^2 w + g_{0,2}\, w^2$ with $\Re\, g_{0,2} = 0$ by assumption (or even $g_{0,2} = 0$, but only null real part will suffice), this is:

$$
\begin{aligned}
0 \equiv\ & \tfrac{i}{2}\, g_{4,0}\, z^4 - \tfrac{i}{2}\, \overline{g}_{4,0}\, \overline{z}^4 + \big(f_{3,0} - \tfrac{1}{2}\, g_{2,1}\big)\, z^3 \overline{z} + \big(\overline{f}_{3,0} - \tfrac{1}{2}\, \overline{g}_{2,1}\big)\, z \overline{z}^3 \\
& + \big(i\, f_{1,1} - i\, \overline{f}_{1,1} - \tfrac{i}{2}\, g_{0,2} + \tfrac{i}{2}\, \overline{g}_{0,2}\big)\, z^2 \overline{z}^2 + \tfrac{i}{2}\, g_{2,1}\, z^2 u - \tfrac{i}{2}\, \overline{g}_{2,1}\, \overline{z}^2 u \\
& + \big(f_{1,1} + \overline{f}_{1,1} - g_{0,2} - \overline{g}_{0,2}\big)\, z \overline{z} u + \big(\tfrac{i}{2}\, g_{0,2} - \tfrac{i}{2}\, \overline{g}_{0,2}\big)\, u^2,
\end{aligned}
$$

and starting from the end, since $g_{0,2}$ is purely imaginary, this forces $0 = g_{0,2}$, then $f_{1,1} + \overline{f}_{1,1} = 0$, then $0 = g_{2,1}$, then $0 = f_{1,1}$, then $0 = f_{3,0}$, and lastly $0 = g_{4,0}$, so as asserted $0 = f_3 = g_4$. ■

Now, we discuss the induction vanishing process. We assume $(f_{\mu-1},\, g_\mu) = (0, 0)$ for $\mu = 3, \dots, \nu - 1$ and some $\nu \geqslant 5$, and we want to have $(f_{\nu-1},\, g_\nu) = (0, 0)$. At first, it is not difficult to verify (left to the reader) that, then:

$$
F'_\mu\big(z, \overline{z}, u\big) \equiv F_\mu\big(z, \overline{z}, u\big) \qquad\qquad (\mu = 3, \dots, \nu - 1).
$$

Using this, the fundamental identity then reads:

$$
\begin{aligned}
0 &\equiv \pi^\nu\Big(-\Im\,(w + g_\nu) + (z + f_{\nu-1})\,(\overline{z} + \overline{f}_{\nu-1}) + \sum_{3 \leqslant \mu \leqslant \nu} F'_\mu\Big(z + f_{\nu-1},\, \overline{z} + \overline{f}_{\nu-1},\, u + \Re\, g_\nu\Big)\Big) \\
&\equiv \pi^\nu\Big(-\underline{z\overline{z}}_\circ - \sum_{3 \leqslant \mu \leqslant \nu-1} \underline{F_\mu\big(z, \overline{z}, u\big)}_{\circ\circ} - F_\nu\big(z, \overline{z}, u\big) - \Im\, g_\nu + \underline{z\overline{z}}_\circ + z\overline{f}_{\nu-1} + \overline{z} f_{\nu-1} \\
&\qquad + \underline{f_{\nu-1}\overline{f}_{\nu-1}}_\circ + \sum_{3 \leqslant \mu \leqslant \nu-1} \underline{F'_\mu\big(z, \overline{z}, u\big)}_{\circ\circ} + F'_\nu\big(z, \overline{z}, u\big)\Big) \\
&\equiv \pi^\nu\Big(\Re\Big\{i\, g_\nu\big(z,\, u + i\,[z\overline{z} + \mathrm{O}_{z,\overline{z},u}(3)]\big) + 2\,\overline{z}\, f_{\nu-1}\big(z,\, u + i\,[z\overline{z} + \mathrm{O}_{z,\overline{z},u}(3)]\big)\Big\} \\
&\qquad - F_\nu\big(z, \overline{z}, u\big) + F'_\nu\big(z, \overline{z}, u\big)\Big) \\
&\equiv \Re\Big\{i\, g_\nu\big(z,\, u + iz\overline{z}\big) + 2\,\overline{z}\, f_{\nu-1}\big(z,\, u + iz\overline{z}\big)\Big\} - F_\nu\big(z, \overline{z}, u\big) + F'_\nu\big(z, \overline{z}, u\big).
\end{aligned}
$$

Now, we project further this equation by applying to it $\Pi_S(\bullet)$. Since F and F' are in normal form, we obtain, still for any $\nu \geqslant 5$:

$$
0 \equiv \Pi_S\Big(\Re\Big\{i\, g_\nu\big(z,\, u + iz\overline{z}\big) + 2\,\overline{z}\, f_{\nu-1}\big(z,\, u + iz\overline{z}\big)\Big\}\Big) - 0 + 0.
$$

This is a linear system of equations in the coefficients $g_{j',l'}$ of g_ν and $f_{j',l'}$ of $f_{\nu-1}$. Instead of solving this linear system for any fixed $\nu \geqslant 5$ (the cases $\nu = 3, 4$ have been done above), we will solve in one stroke *all such systems for any* $\nu \geqslant 3$, and this will simplify our job, especially by lightening a bit the combinatorics.

In any case, by taking the coefficients of all the monomials $z^j \overline{z}^k u^l$ with $(j, k) \in S$ and $j + k + 2l = \nu$, we know that there exist linear forms $L_{j,k,l}$ such that the above system writes:

$$
0 = L_{j,k,l}\Big(\big\{f_{j',l'}\big\}_{j'+2l'=\nu-1},\ \big\{g_{j',l'}\big\}_{j'+2l'=\nu}\Big),
$$

a system that we may abbreviate as:

$$
(\mathsf{E}_\nu): \qquad\qquad 0 = L_{j,k,l}\big(f_{\bullet,\bullet},\, g_{\bullet,\bullet}\big) \qquad\qquad ((j, k) \in S,\ j + k + 2l = \nu)
$$

From now on, $\nu \geqslant 3$, so we incorporate $\nu = 3, 4$ in the discussion.

On the other hand, by considering the complete $f = f_2 + f_3 + \cdots$ and the complete $g = g_3 + g_4 + \cdots$, we can introduce the analog ‘complete’ linear system:

$$0 \;\equiv\; \Pi_S\Big(\Re\Big\{i\,g\big(z,\,u+iz\overline{z}\big)+2\,\overline{z}\,f\big(z,\,u+iz\overline{z}\big)\Big\}\Big),$$

which, similarly, after extracting the coefficients of all monomials $z^j\overline{z}^ku^l$ with $(j,k)\in S$ and any $l\in\mathbb{N}$, can be abbreviated as:

$$(\mathsf{E}):\qquad 0 \;=\; L_{j,k,l}\big(f_{\bullet,\bullet},\,g_{\bullet,\bullet}\big)\qquad\big((j,k)\,\in\,S,\ l\,\in\,\mathbb{N}\big).$$

The key and elementary observation is that, because $u+iz\overline{z}$ is 2-homogeneous, the full system (E) *splits in the linear subsystems* (E_ν) *having separate unknowns* $\big(f_{\nu-1},\,g_\nu\big)$:

$$(\mathsf{E}) \;=\; (\mathsf{E}_3)\cup(\mathsf{E}_4)\cup\cdots\cup(\mathsf{E}_\nu)\cup\cdots.$$

Therefore:

$$\Big((\mathsf{E})\;\Longrightarrow\;(f,g)=(0,0)\Big)\quad\Longleftrightarrow\quad\Big((\mathsf{E}_\nu)\;\Longrightarrow\;\big(f_{\nu-1},g_\nu\big)=(0,0)\quad\text{for all }\nu\geqslant 3\Big).$$

Thus, we are left with establishing the following main technical statement, which will close the proof of Theorem 13.4. ■

Theorem 13.7. *Let $f(z,w)$ and $g(z,w)$ be holomorphic of weights $\geqslant 2$ and $\geqslant 3$, namely $f=f_2+f_3+\cdots$ and $g=g_3+g_4+\cdots$, and with:*

$$0 \;=\; f_w(0),\qquad\qquad 0 \;=\; \Re\,g_{ww}(0).$$

If for all $(j,k)\in S$ and all $l\in\mathbb{N}$:

$$0 \;=\; \big[z^j\overline{z}^ku^l\big]\left(\Re\Big\{i\,g\big(z,\,u+iz\overline{z}\big)+2\,\overline{z}\,f\big(z,\,u+iz\overline{z}\big)\Big\}\right),$$

then $(f,g)\equiv(0,0)$.

Proof. The key simplification is to gather all powers u^l in the linear system so as to deal with finitely many functions of the CR-transversal variable u.

Indeed, given a holomorphic function $e=e(w)$, we may expand:

$$e\big(u+i\,z\overline{z}\big) \;=\; e(u)+e_w(u)\,i\,z\overline{z}+e_{ww}(u)\,\tfrac{1}{2!}\,\big(i\,z\overline{z}\big)^2+e_{www}(u)\,\tfrac{1}{3!}\,\big(i\,z\overline{z}\big)^3+\cdots,$$

and we will write $e'(u)$, $e''(u)$, $e'''(u)$, etc., instead of $e_w(u)$, $e_{ww}(u)$, $e_{www}(u)$, etc. Thus:

$$\begin{aligned}
f\big(z,\,u+i\,z\overline{z}\big) \;&=\; \sum_{k\geqslant 0} z^k\,f_k\big(u+i\,z\overline{z}\big)\\
&= \sum_{k\geqslant 0} z^k\,\Big[f_k(u)+f_k'(u)\,i\,z\overline{z}+f_k''(u)\,\tfrac{1}{2!}\,\big(i\,z\overline{z}\big)^2+f_k'''(u)\,\tfrac{1}{3!}\,\big(i\,z\overline{z}\big)^3+\cdots\Big],
\end{aligned}$$

and similarly:

$$\begin{aligned}
g\big(z,\,u+i\,z\overline{z}\big) \;&=\; \sum_{k\geqslant 0} z^k\,g_k\big(u+i\,z\overline{z}\big)\\
&= \sum_{k\geqslant 0} z^k\,\Big[g_k(u)+g_k'(u)\,i\,z\overline{z}+g_k''(u)\,\tfrac{1}{2!}\,\big(i\,z\overline{z}\big)^2+g_k'''(u)\,\tfrac{1}{3!}\,\big(i\,z\overline{z}\big)^3+\cdots\Big],
\end{aligned}$$

Hence our zero equation is:

$$\begin{aligned}
0 &\equiv 2\,\Re\Big\{2\,\overline{z}\,f\big(z,\,u+i\,z\overline{z}\big)+i\,g\big(z,\,u+i\,z\overline{z}\big)\Big\} \\
&\equiv 2\,\overline{z}\,f+2\,z\,\overline{f}+i\,g-i\,\overline{g} \\
&\equiv \sum_{k\geqslant 0}\Big(2\,f_k\,z^k\overline{z}+2\,i\,f_k'\,z^{k+1}\overline{z}^2-f_k''\,z^{k+2}\overline{z}^3-\tfrac{i}{3}\,f_k'''\,z^{k+3}\overline{z}^4+\cdots\Big) \\
&\quad+\sum_{k\geqslant 0}\Big(2\,\overline{f}_k\,z\overline{z}^k-2\,i\,\overline{f}_k'\,z^2\overline{z}^{k+1}-\overline{f}_k''\,z^3\overline{z}^{k+2}+\tfrac{i}{3}\,\overline{f}_k'''\,z^4\overline{z}^{k+3}+\cdots\Big) \\
&\quad+\sum_{k\geqslant 0}\Big(i\,g_k\,z^k-g_k'\,z^{k+1}\overline{z}-\tfrac{i}{2}\,g_k''\,z^{k+2}\overline{z}^2+\tfrac{1}{6}\,g_k'''\,z^{k+3}\overline{z}^3+\cdots\Big) \\
&\quad+\sum_{k\geqslant 0}\Big(-\,i\,\overline{g}_k\,\overline{z}^k-\overline{g}_k'\,z\overline{z}^{k+1}+\tfrac{i}{2}\,\overline{g}_k''\,z^2\overline{z}^{k+2}+\tfrac{1}{6}\,\overline{g}_k'''\,z^3\overline{z}^{k+3}+\cdots\Big),
\end{aligned}$$

where the common argument of all f_k, f_k', f_k'', g_k, g_k', g_k'', g_k''' is $u\in\mathbb{R}$.

We are thus capturing the coefficients $[z^j\overline{z}^k](\bullet)$ of this identity, not anymore all $[z^j\overline{z}^k u^l](\bullet)$. This means that we are extracting identities satisfied by functions of u.

Let us therefore list the coefficients of $z^j\overline{z}^k$, indicating plainly (j,k). Note that we can restrict the considerations to only $j\geqslant k$, since the above zero equation is real.

FIGURE 9: Two infinite red-shaded families of coefficients $f_3(u)\equiv f_4(u)\equiv\cdots\equiv 0$ and $g_2(u)\equiv g_3(u)\equiv\cdots\equiv 0$ easily shown to vanish identically.

Firstly, we extract the coefficients of z^k with $k\geqslant 2$ and of $z^k\overline{z}$ for $k\geqslant 3$:

$$\begin{aligned}
0 &= i\,g_k, && (k^{\geqslant 2},0) \\
0 &= 2\,f_k-g_{k-1}'. && (k^{\geqslant 3},1)
\end{aligned}$$

So $g_k(u)\equiv 0$ for all $k\geqslant 2$ and $f_k(u)\equiv 0$ for all $k\geqslant 3$, and therefore:

$$f = f_0(w)+z\,f_1(w)+z^2\,f_2(w), \qquad\qquad g = g_0(w)+z\,g_1(w).$$

Next, we extract the remaining coefficients of $z^j\overline{z}^k$, and we get 7 equations:

$$\begin{aligned}
0 &= i\,g_0-i\,\overline{g}_0, && (0,0) \\
0 &= 2\,\overline{f}_0+i\,g_1, && (1,0) \\
0 &= 2\,f_1+2\,\overline{f}_1-g_0'-\overline{g}_0', && (1,1) \\
0 &= 2\,f_2-2i\,\overline{f}_0'-g_1', && (2,1) \\
0 &= 2i\,f_1'-2i\,\overline{f}_1'-\tfrac{i}{2}\,g_0''+\tfrac{i}{2}\,\overline{g}_0'', && (2,2) \\
0 &= 2i\,f_2'-\overline{f}_0''-\tfrac{i}{2}\,g_1'', && (3,2) \\
0 &= -\,f_1''-\overline{f}_1''+\tfrac{1}{6}\,g_0'''-\tfrac{1}{6}\,\overline{g}_0'''. && (3,3)
\end{aligned}$$

Now, since:

$$\begin{aligned} 0 &= f(0) &\iff&& f_0(0) &= 0, & 0 &= g(0) &\iff&& g_0(0) &= 0,\\ 0 &= f_z(0) &\iff&& f_1(0) &= 0, & 0 &= g_z(0) &\iff&& g_1(0) &= 0,\\ 0 &= f_w(0) &\iff&& f_0'(0) &= 0, & 0 &= g_w(0) &\iff&& g_0'(0) &= 0, \end{aligned}$$

and since: $$0 = \Re\, g_{ww}(0) \iff \Re\, g_0''(0) = 0,$$

the assumptions of the theorem are equivalent to the ones formulated in the next statement, which will finish everything. ∎

Lemma 13.8. *If five functions* f_0, f_1, f_2, g_0, g_1 *of the real variable* $u \in \mathbb{R}$ *with:*

$$\begin{aligned} 0 &= f_0(0) = f_0'(0), & 0 &= g_0(0) = g_0'(0) = \Re\, g_0''(0),\\ 0 &= f_1(0), & 0 &= g_1(0), \end{aligned}$$

satisfy the above 7 *linear ordinary differential equations, then they all vanish identically:* $0 \equiv f_0(u) \equiv f_1(u) \equiv f_2(u)$, $0 \equiv g_0(u) \equiv g_1(u)$.

Proof. From $(0,0)$, solve $\overline{g}_0 := g_0$. From $(1,0)$, solve $g_1 := 2i\,\overline{f}_0$. Then the five remaining equations become:

$$0 = 2\, f_1(u) + 2\,\overline{f}_1(u) - 2\, g_0'(u), \tag{1,1}$$

$$0 = 2\, f_2(u) - 4i\,\overline{f}_0'(u), \tag{2,1}$$

$$0 = 2i\, f_1'(u) - 2i\,\overline{f}_1'(u), \tag{2,2}$$

$$0 = 2i\, f_2'(u), \tag{3,2}$$

$$0 = -\, f_1''(u) - \overline{f}_1''(u). \tag{3,3}$$

From $(3,2)$, we see $f_2 = \alpha \in \mathbb{C}$ is constant. From $(2,1)$ at $u = 0$, since $f_0'(0) = 0$ by assumption, we get $\alpha = 0$. So $f_2(u) \equiv 0$ in $(2,1)$ gives $f_0(u) \equiv 0$ too. Thus $g_1(u) \equiv 0$ as well.

From $(3,3)$ and $\frac{d}{du}(2,2)$, it comes $f_1''(u) \equiv 0$, and since $f_1(0) = 0$ by assumption, $f_1(u) = c\,u$ with $c \in \mathbb{R}$ by $(2,2)$. From $(1,1)$, it comes $g_0'(u) = 2\,c\,u$, and since $\Re\, g_0''(0) = 0$, we get $c = 0$. Thus $f_1(u) \equiv 0$.

From $g_0'(u) \equiv 0$ and $g_0(0) = 0$ by assumption, we get $g_0(u) \equiv 0$. This concludes everything. ∎

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