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Ten-Dimensional Lie Algebras with $\mathfrak{so}(3)$ Semi-Simple Factor

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Abstract. Turkowski has classified Lie algebras that have a non-trivial Levi decomposition of dimension up to and including nine. In this work the program is extended to give a partial classification of the corresponding Lie algebras in dimension ten. The key tool is the R-representation, which is the representation of the semi-simple factor by endomorphisms of the radical. The algebras studied here comprise 34 classes that have semi-simple factor $\mathfrak{so}(3)$ and three exceptions for which semi-simple factor is of dimension six. Most of the algebras have an abelian nilradical, which is probably an artifact of the low dimensions involved. The many remaining cases where the semi-simple factor is $\mathfrak{sl}(2,\mathbb{R})$ will be investigated in a different venue.

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1. Introduction

In this paper we study ten-dimensional real Lie algebras that have a non-trivial Levi decomposition. The corresponding problem has been solved in dimension up to and including nine by Turkowski [13] and we will adopt his notations throughout. We shall be content for the present to give all ten-dimensional Lie algebras except for those algebras that have a single copy of $\mathfrak{sl}(2,\mathbb{R})$ as its semi-simple factor. In fairness, algebras that have $\mathfrak{sl}(2,\mathbb{R})$ as their semi-simple part constitute the vast majority, but already there is quite an extensive list of algebras for which the semi-simple factor is $\mathfrak{so}(3)$. We intend to give a list of Lie algebras for which the semi-simple factor is $\mathfrak{sl}(2,\mathbb{R})$ at a later date.

An outline of this paper is as follows. In Section 2 we consider the problem of constructing Lie algebras that have a Levi decomposition in general. In Section 3 we consider the possible semi-simple factors that can occur in dimension ten. It will become apparent that most cases involve either $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$. In Section 4 we will consider the possible representations of $\mathfrak{so}(3)$ that can occur up to dimension seven. In Section 5, building on [13], we investigate the properties of the so-called R-representation. Most importantly, it will be explained why in the R-representation, $\mathfrak{so}(3)$ acts trivially on the complement to the nilradical inside the radical. In Section 6 we investigate which indecomposable seven-dimensional nilpotent Lie algebras are such that the one of the possible seven R-representations appears in its space of derivations. Although the classification of the six-dimensional nilpotent Lie algebras appears to be settled, it is not the case for the seven-dimensional nilpotent Lie

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algebras and accordingly, we have sought to obtain results that independent of [2]. In Section 7 we undertake the detailed study of exactly which R-representations can occur in dimension ten. Finally, in Section 8 we provide a complete list of all indecomposable Lie Algebras of dimension ten that have a non-trivial Levi decomposition and whose semi-simple part is not $\mathfrak{sl}(2,\mathbb{R})$. Following the lead given in [13], the algebras are listed as $L_{10.i}$, where 10 pertains to the dimension of the algebra and i to the ith algebra in the list. Most of the algebras that we find have an abelian nilradical, a fact which is due to the relatively small dimension of the spaces involved.

We shall supply a few words about terminology and notation. Much of the notation adopted here is based on [13]. The summation convention on repeated indices, one a subscript and one a superscript, is usually in operation and sometimes with two separate ranges of indices. We generally use S to denote a simple or occasionally semi-simple Lie algebra. We use N for a solvable Lie algebra and NR for its associated nilradical. Then a Levi decomposition is written as $S \rtimes N$. Turkowski classified nine-dimensional, indecomposable Lie algebras that have a non-trivial Levi decomposition. Actually in [13], two algebras were omitted: one is denoted as $L^*_{9,7}$ in [1] and is a "real" form corresponding to $L_{9,52}$. The second shall be denoted here as $L^*_{9,11}$ and is a semi-direct product of $\mathfrak{so}(3)$ and $\mathbb{R}^6$ coming from the irreducible 6×6 representation of $\mathfrak{so}(3)$. See [5] for more details. Turkowski denotes by R the representation of the semi-simple factor S by automorphisms of the radical N. He discerns, up to isomorphism, 63 classes of such Lie algebras and they are denoted by $L_{9,i}$ where $1 \leq i \leq 63$. As regards the other classes of low dimensional indecomposable Lie algebras, algebras of dimension less than or equal to five and the nilpotent algebras of dimension i are denoted by $A_{i,j}$ where $3 \leq i \leq 6$, and j signifies the jth algebra in the list, following the listing in [8]. The indecomposable solvable Lie algebras of dimension six that have a five-dimensional nilradical were classified by Mubarakzyanov [7] and are denoted by $g_{6,i}$ where $1 \leq i \leq 99$; see also [9] for an updated classification. The indecomposable Lie algebras of dimension six that have a four-dimensional nilradical classified by Turkowski [12] are denoted by $N_{6,i}$ where $1 \leq i \leq 40$. For much more information about low-dimensional Lie algebras in general, the reader may refer to [10]. There is also a memoir devoted to studying the invariants of the nine-dimensional, Levi decomposition algebras [1]. For abelian Lie algebras of dimension n, we usually say "Abelian" and refer to $\mathbb{R}^n$ rather than writing nA_1. The trivial representation of dimension n is denoted by nD_0. The irreducible representation of $\mathfrak{so}(3)$ of dimension n is denoted by R_n for $n > 3$ and for $n = 3$ by $\operatorname{ad}\mathfrak{so}(3)$. However, in Subsection 5.2 we shall use R_n to denote an n-dimensional irreducible representation of some simple Lie algebra, again our main concern being with $\mathfrak{so}(3)$. We shall refer to the n-dimensional Lie algebra with non-zero brackets given by

$$[e_i, e_n] = e_i, \ (1 \leq i \leq n-1)$$

as the *Milnor algebra*. The reason for so doing is that it was introduced in [6] albeit in a rather different way; Milnor described the algebra as the, unique up to isomorphism, non-abelian Lie with the property that the Lie bracket of any two elements is a linear combination of those same two elements. This algebra and small variants of it, occur repeatedly in the study of Lie algebras that have a non-trivial Levi decomposition.

The Lie algebra of dimension $2n+1$ that has brackets

$$[e_i, e_{n+j}] = \delta_{ij} e_{2n+1},\ (1 \leq i \leq j \leq n)$$

is the *Heisenberg* Lie algebra whereas the Lie algebra of dimension $2n+1$ that has brackets

$$[e_i, e_{2n+1}] = \delta_{ij} e_{n+j},\ (1 \leq i \leq j \leq n)$$

is the *anti-Heisenberg* Lie algebra. Unfortunately the Heisenberg and anti-Heisenberg coincide for $n = 1$ but in any case, that Lie algebra is denoted by H.

If ρ and σ are representations of some simple Lie algebra, for us principally $\mathfrak{so}(3)$, we write $\rho \oplus \sigma$ for the associated "diagonal" representation. At the beginning of our classification of the Levi decomposition Lie algebras appearing in Section 8, we use the notation $\mathfrak{g} \oplus \mathfrak{h}$ to denote the direct sum of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

We conclude with a few philosophical remarks. First of all, we shall see below that there are only three cases in dimension ten for which a non-trivial Levi decomposition Lie algebra can have semi-simple factor neither $\mathfrak{so}(3)$ nor $\mathfrak{sl}(2,\mathbb{R})$. Secondly, concerning such algebras with semi-simple factor $\mathfrak{so}(3)$, the smallest possible dimension for NR, that is four, does not occur. Thirdly, of the seven possible R-representations of $\mathfrak{so}(3)$ that occur in a ten-dimensional Levi decomposition algebra, see Section 4, three of the seven each give rise just to a *single* class of algebra among the 30 that we list, $L_{10.4} - L_{10.37}$ in Section 8, and another engenders just two. Fourthly and lastly, in the vast majority of cases NR is abelian; the few exceptions involve H or higher dimensional Heisenberg algebras, the anti-Heisenberg algebra, two six-dimensional indecomposable nilpotent Lie algebras and one more seven-dimensional indecomposable nilpotent known as $37D1$ in [2]. Finally, we thank the referee for making some helpful comments and suggesting that we formulate our results in a way which is independent of classifications of the seven-dimensional nilpotent Lie algebras.

2. Constructing algebras with a non-trivial Levi decomposition

Let us consider the problem of constructing a Lie algebra that has a Levi decomposition $S \rtimes N$ in general. We have the following structure equations:

$$[e_a, e_b] = C^c_{ab} e_c,\ [e_a, e_i] = C^k_{ai} e_k,\ [e_i, e_j] = C^k_{ij} e_k \tag{1}$$

where $1 \leq a, b, c, d \leq r$ and $r+1 \leq i, j, k, l \leq n$ and $\{e_a\}$ is a basis for the semi-simple subalgebra S and $\{e_i\}$ is a basis for the radical N. Calculation shows that the Jacobi identity is equivalent to the following conditions:

$$\begin{aligned} C^e_{[ab} C^d_{c]e} = 0,\ \ C^c_{ab} C^j_{ci} = C^k_{bi} C^j_{ak} - C^k_{ai} C^j_{bk}, \\ C^k_{al} C^l_{ij} = C^l_{ai} C^k_{lj} - C^l_{aj} C^k_{li},\ \ C^l_{[ij} C^m_{k]l} = 0. \end{aligned} \tag{2}$$

We interpret (2) as follows: we start with a semi-simple algebra so that the first set of conditions above is satisfied. Then the second set say that the matrices C^k_{al} make N (merely as a vector space) into an S-module. The third set say that the C^k_{al} are derivations of the Lie algebra N and the fourth of course that N is a Lie algebra. Therefore to find all possible Lie algebras of dimension n that have a *non-trivial*, that is, is not just a direct sum of S and T, Levi decomposition $S \rtimes N$ we can proceed as follows: choose a semi-simple algebra S of dimension r. Then pick any solvable algebra N of dimension $n-r$ and consider a faithful representation (should it exist) of S in N considered simply as the vector space $\mathbb{R}^{n-r}$, all of which are known and are completely reducible [3] since S is semi-simple. Finally, it only remains to check that the matrices representing S act as derivations of the Lie algebra N. In the affirmative case we have our sought after Lie algebra provided it is non-trivial; in the negative case there is no such algebra and we have to choose a different representation of S in N. If all such representations lead to a null result then there can be no Levi decomposition involving S and N.

In the case when the radical is abelian the third and fourth parts of eq. (2) are identically satisfied. Since S is semi-simple we have the following Theorem.

Theorem 2.1. *Let a semi-simple Lie algebra S have a faithful representation in $End(N)$ for some vector space N. Then there is a Lie algebra $S \rtimes N$ that has a Levi decomposition with N being an abelian radical. Conversely, every Lie algebra that has a Levi decomposition with abelian radical arises in this way. Such a Lie algebra is decomposable if and only if the representation of S, being completely reducible, contains a trivial subrepresentation.*

3. Semi-simple factor

We will consider the possible semi-simple factors that can occur in a ten-dimensional Lie algebra that has a Levi decomposition $S \rtimes N$. If S is nine-dimensional then N is one-dimensional and we will obtain a *reductive* Lie algebra and a trivial Levi decomposition. If S is eight-dimensional then N is two-dimensional. However, any eight-dimensional Lie algebra can have a non-trivial representation only in dimension at least three. There are no semi-simple algebras of dimension seven so it suffices to consider semi-simple algebras of dimension six and less which consist of $\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(3), \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(3) \oplus \mathfrak{so}(3), \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3), \mathfrak{so}(3,1)$.

Concerning the four six-dimensional semi-simple algebras, the standard representation of $\mathfrak{so}(3,1)$ is in $\mathfrak{gl}(4,\mathbb{R})$ and is irreducible. As such the radical must be abelian according to Theorem 1 [13] and we obtain a ten-dimensional Lie algebra that has a Levi decomposition with representation in $\mathfrak{gl}(5,\mathbb{R})$ of the form $\left[\begin{smallmatrix} A & x \\ 0 & 0 \end{smallmatrix}\right]$ where A is the standard representation of $\mathfrak{so}(3,1)$ in $\mathfrak{gl}(4,\mathbb{R})$ and x is arbitrary in $\mathbb{R}^4$. Similar conclusions hold for the pair of semi-simple Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ although they do not define irreducible representations in $\mathfrak{gl}(4,\mathbb{R})$. This time, since the representations contain no trivial subrepresentation, the associated radical is at least nilpotent according to Theorem 2 in [13]. However, there are only three four-dimensional nilpotent algebras up to isomorphism, that is, $H \oplus \mathbb{R}, A_{4.1}$ and $\mathbb{R}^4$. Of these three algebras, only $\mathbb{R}^4$ contains $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ among its space of derivations. As regards $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)$, there is no representation in $\mathfrak{gl}(4,\mathbb{R})$ and it does not lead to a ten-dimensional Lie algebra that has a Levi decomposition.

It remains to examine the cases where the semi-simple part is either $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{so}(3)$. In this paper we shall be content to discuss the case $\mathfrak{so}(3)$. We shall revisit the many possibilities where the radical is $\mathfrak{sl}(2,\mathbb{R})$ in a separate venue.

4. $\boldsymbol{R}$-Representations for $\mathfrak{so}(3)$

Since we are concerned from now on only with the case where the semi-simple part of the Levi decomposition algebra is $\mathfrak{so}(3)$, we shall need to find all possible representations of $\mathfrak{so}(3)$ in $\mathfrak{gl}(7,\mathbb{R})$. They are as follows:

$$\bullet\ R_7, \quad \bullet\ R_6 \oplus D_0, \quad \bullet\ R_5 \oplus 2D_0, \quad \bullet\ R_4 \oplus 3D_0,$$
$$\bullet\ \mathrm{ad}\,\mathfrak{so}(3) \oplus 4D_0, \quad \bullet\ \mathrm{ad}\,\mathfrak{so}(3) \oplus R_4, \quad \bullet\ 2\,\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0.$$

Here $\mathrm{ad}\,\mathfrak{so}(3)$ denotes the standard, or equivalently, adjoint representation of $\mathfrak{so}(3)$ and D_0 the one-dimensional trivial representation and kD_0 signifies k copies of it. Moreover, by R_q we mean the standard irreducible representation of $\mathfrak{so}(3)$ in $\mathfrak{gl}(q,\mathbb{R})$ where $1 \leq q \leq 7$. There will be only one Lie algebra for each of R_6 and R_7 and their representations may be read off from Section 5. In Subsection 5.2 we shall also use R_q to denote an q-dimensional irreducible representation of some simple Lie algebra and not necessarily just $\mathfrak{so}(3)$.

5. $\boldsymbol{R}$-Representation

5.1. Some general results about the R-Representation

In this Section we study properties of the R-representation. First of all we adapt the following two Theorems from [13].

Theorem 5.1. *If the R-representation is irreducible on a subalgebra M of N then M is abelian.*

Proof. The derived algebra $[M,M]$ must be invariant and so if the R-representation is irreducible we can only have $[M,M]=0$. ■

Corollary 5.2. *If the R-representation is irreducible on N then the Lie algebra is uniquely determined.*

In fact given any R-representation on a vector space N there will always be a Lie algebra L with a Levi decomposition by taking N to be an abelian Lie algebra; however, usually L will be decomposable, see Theorem 2.1.

Theorem 5.3. *If the R-representation contains no trivial subrepresentation then the radical N is nilpotent.*

However, we can go further. We observe, first of all, that *any* derivation of a solvable Lie algebra maps N into the nilradical NR [4]. Secondly, for the R-representation, NR comprises an invariant subspace. Thirdly, since the R-representation is a representation of a *semi-simple* algebra, it is completely reducible as such we find:

Theorem 5.4. *If N is solvable but not nilpotent the R-representation acts trivially on the complement to NR.*

It follows that a nilpotent Lie algebra can only serve as the nilradical of a Lie algebra with a non-trivial Levi decomposition if it has a non-zero semi-simple subalgebra of derivations. As such our strategy to complete the classification, is to check all nilpotent Lie algebras of dimension between four and seven as to whether they possess such a subalgebra.

Proposition 5.5. *If $L = S \rtimes N$ engenders a non-trivial Levi decomposition then the R-representation must be faithful.*

Proof. It is easy to show from the Jacobi identity that the set of $s \in S$ such that $[s, N] = R(s)N = 0$, that is, s commutes with all $n \in N$, comprises an ideal σ in S. Since S is semi-simple it is a direct sum of semi-simple ideals and σ must appear among its summands. Hence L will be decomposable if σ is non-zero. ■

In the other direction, Turkowski calls the set of elements $n \in N$ such that $[S, n] = R(S)n = 0$ the "R-constants". Again, the following Proposition is easy to prove using the Jacobi identity.

Proposition 5.6. *The R-constants comprise a subalgebra of N.*

Proposition 5.7. *All terms in the derived and lower central series of N comprise invariant subspaces of the R-representation.*

Theorem 5.8. *In a Levi-decomposition Lie algebra $S \rtimes N$, the center Z of N defines an invariant subspace of the R-representation.*

Proof. From the Jacobi identity $[S, [N, Z]] + [Z, [S, N]] + [N, [Z, S]] = 0$. Then since $[S, N] \subset N$ we have that $[[S, Z], N] = 0$, that is, $[S, Z] \subset Z]$. ■

5.2. R-Representation: $R_q \oplus (n - q)D_0$

In this Subsection we are going to study a fairly general situation that we will apply to several different cases. Thus we shall assume that we have a Levi decomposition Lie algebra $\mathfrak{g} = S \rtimes N$ of dimension $r+n$, where S is simple of dimension r and the associated R-representation is of the form $R_q \oplus (n-q)D_0$, that is, $R_q \subset \mathfrak{gl}(q, \mathbb{R})$ and R_q denotes a q-dimensional irreducible representation of some simple Lie algebra and not necessarily just $\mathfrak{so}(3)$. We shall suppose further that the R-representation acts irreducibly on $\mathbb{R}^q$ and satisfies the conclusion of Schur's Lemma. We introduce a basis for $\mathfrak{g}$ as follows: $\{e_a, 1 \le a \le r\}$ for S, $\{e_i, r+1 \le i \le r+q\}$ for the subspace of N, abelian in view of Theorem (5.1), on which S acts irreducibly and $\{e_\alpha, q+r+1 \le \alpha \le r+n\}$ for the complementary subspace in N on which S acts trivially. Then we have the following structure equations

$$\begin{aligned}
&[e_a, e_b] = C^c_{ab} e_c,\ [e_a, e_i] = C^j_{ai} e_j, [e_i, e_j] = C^k_{ij} e_k + C^\alpha_{ij} e_\alpha, \\
&[e_i, e_\alpha] = C^j_{i\alpha} e_j + C^\beta_{i\alpha} e_\beta, [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma.
\end{aligned} \tag{3}$$

Notice that the last term in eq. (3) reflects the fact that the R-constants form a subalgebra. Now consider a derivation D of N coming from the R-representation. Then for $1 \le i < j \le n$ we have

$$\begin{aligned}
&D[e_i, e_j] = [De_i, e_j] + [e_i, De_j], D[e_i, e_\beta] = [De_i, e_\beta] + [e_i, De_\beta], \\
&D[e_\alpha, e_\beta] = [De_\alpha, e_\beta] + [e_\alpha, De_\beta].
\end{aligned} \tag{4}$$

However, $De_\alpha = 0$ and so eq. (4) reduces to

$$D[e_i, e_j] = [De_i, e_j] + [e_i, De_j], D[e_i, e_\beta] = [De_i, e_\beta]. \tag{5}$$

Now considering the three terms in eq. (5), the first is considered to be an identity expressing the fact that S acts on the subspace spanned by the $\{e_i\}$.

However, an important observation must be made here; even if the invariant subspace concerned here is not a subalgebra, nonetheless, we must still assume that the coefficients C^k_{ij} are zero, otherwise the term $C^k_{ij}e_k$ would engender an invariant subspace of an invariant space on which S is supposed to be acting irreducibly.

Concerning the second term in eq. (5) we find that

$$C^j_{i\beta}D^k_j = D^j_i C^k_{j\beta}. \tag{6}$$

and

$$D^k_j C^\alpha_{k\beta} = 0. \tag{7}$$

Since the R-representation is irreducible, eq. (7) implies that $C^\alpha_{k\beta} = 0$. Since we are assuming that the conclusion of Schur's Lemma holds, we may assert from Eq.(6) the existence of constants λ_α such that

$$[e_i, e_\alpha] = \lambda_\alpha e_i. \tag{8}$$

If some $\lambda_\alpha \neq 0$ by scaling the corresponding e_α, we may assume that $\lambda_\alpha = 1$. Eq.(8) together with

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma \tag{9}$$

now furnish the structure equations of N.

There is a variety of subcases, of which the extremes are when each $\lambda_\alpha = 1$ and when each $\lambda_\alpha = 0$. However, it is impossible to have each $\lambda_\alpha = 0$ because then the entire Lie algebra would be decomposable.

In the case where each $\lambda_\alpha = 1$, none of the e_α's are in NR; since N is solvable we have $[N, N] \subset NR$ and hence the $C^\gamma_{\alpha\beta} = 0$ in Eq.(9). It follows that $\dim NR \leq q$. Since we always have $\dim NR \geq \lceil \frac{n+1}{2} \rceil$ we find that $q \geq \lceil \frac{n+1}{2} \rceil$.

5.3. R-Representation: $R_{n-1} \oplus D_0$

We assume now that in the previous Subsection $q = n - 1$. There is only one λ_α in Eq.(8), and if it were zero, the entire Lie algebra would be decomposable. Hence, we may assume that $\lambda_\alpha = 1$, also $e_\alpha \notin NR$ and the $n-1$-dimensional subspace is in fact an abelian subalgebra. Thus the structure equations for the radical N are given by

$$[e_i, e_n] = e_i \, (1 \leq i \leq n-1) \tag{10}$$

and in conclusion the radical can only be Milnor's two-point algebra [6].

Theorem 5.9. *Suppose that a semi-simple Lie algebra S acts as $R_{n-1} \oplus D_0$ on the n-dimensional solvable algebra N and that the conclusion of Schur's Lemma is applicable to the irreducible R_{n-1} representation. Then NR is abelian and N is isomorphic to the Milnor algebra, if the Levi decomposition Lie algebra $S \rtimes N$ is not to be decomposable.*

5.4. R-Representation acting irreducibly on NR

Now we suppose that the R-representation acts irreducibly on NR and that the conclusion of Schur's Lemma is applicable to the representation. This time the R-constant subalgebra must be abelian since it is at once a subalgebra but also $[N, N] \subset NR$. As such N consists of an abelian nilradical together with an abelian complement. Referring to eq. (8), suppose that the number of λ_α's is at least two.

Then we would have two equations of the form

$$[e_i, e_\alpha] = \lambda_\alpha e_i, [e_i, e_\beta] = \lambda_\beta e_i. \tag{11}$$

However, now we see that $\lambda_\alpha e_\beta - \lambda_\beta e_\alpha$ is a central element that is not in $[\mathfrak{g}, \mathfrak{g}]$, hence $\mathfrak{g}$ is decomposable. Hence we have:

Theorem 5.10. *Suppose that a simple Lie algebra acts irreducibly on the nilradical NR of a solvable algebra N and that the codimension of NR in N is at least two and that the conclusion of Schur's Lemma is applicable to the representation on NR. Then $L = S \rtimes N$ is decomposable.*

5.5. R-Representation $R_{q-1} \oplus D_0$ on NR

We prove one last general result.

Theorem 5.11. (i) *Suppose that a simple Lie algebra acts as $R_{q-1} \oplus D_0$ on NR and that the conclusion of Schur's Lemma is applicable to the irreducible $q-1$ representation. Then NR is abelian.*

(ii) *Also $q = n-2, n-1$ or n or else the entire Lie algebra is decomposable.*

Proof. (i) According to eq. (8) we have the following structure equations for N

$$[e_i, e_\alpha] = \lambda_\alpha e_i,\ [e_i, e_{r+q}] = \lambda e_i,\ [e_{q+r}, e_\alpha] = C_\alpha e_{q+r}, [e_\alpha, e_\beta] = C_{\alpha\beta} e_{q+r}, \tag{12}$$

where $r+1 \leq i \leq r+q-1$. Now in eq. (12) we must have $\lambda = 0$ otherwise $e_{q+r} \notin NR$. Since the subspace spanned by $e_{r+1}, ..., e_{r+q-1}$ must be abelian it follows that NR is abelian.

(ii) Consider i where $r+1 \leq i \leq r+q-1$. Then $\mathrm{ad}(e_{r+\alpha})$ must have the following form:

$$\mathrm{ad}(e_{r+\alpha}) = \begin{bmatrix} \lambda & 0 & 0 & ... & 0 & 0 & ... & 0 \\ 0 & \lambda & 0 & ... & 0 & 0 & ... & 0 \\ 0 & 0 & \lambda & ... & 0 & 0 & ... & 0 \\ \vdots & & & \ddots & & \vdots & ... & \vdots \\ 0 & 0 & 0 & ... & \lambda & 0 & ... & 0 \\ * & * & * & ... & * & * & ... & * \\ 0 & 0 & 0 & ... & 0 & 0 & ... & 0 \\ \vdots & ... & \vdots & ... & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 & ... & 0 \end{bmatrix}.$$

In eq. (12) the number of such λ's is $n-q$. Now if there are two such ad-matrices we could find a linear combination of them for which the λ's are zero. Now since $e_\alpha \notin NR$ then $\mathrm{ad}(e_\alpha)$ must not be nilpotent. However, if there were three such ad-matrices there would be a linear combination of them which would be nilpotent. Hence $n-q<3$. ■

6. *N* seven-dimensional indecomposable nilpotent

We shall take up the issue now of which seven-dimensional indecomposable nilpotent Lie algebras possess a subalgebra of derivations that is isomorphic to $\mathfrak{so}(3)$. Such a subalgebra must be manifested in one of the seven representations listed in Section 4.

6.1. R-representation R_7

It follows from Theorem 5.1 that if the R-representation is R_7, then the radical $N = \mathbb{R}^7$.

6.2. R-representation $R_6 + D_0$

It is impossible for the R-representations to be $R_6 + D_0$ according to Theorem 5.9 if N is nilpotent.

6.3. R-representation $R_5 + 2D_0$

If the R-representation is $R_5 \oplus 2D_0$, then referring to Eq.(8), we have structure equations for N of the form

$$[e_i, e_j] = C^{\alpha}_{ij} e_\alpha,\ [e_i, e_\alpha] = \lambda_\alpha e_i,\ [e_6, e_7] = ae_6 + be_7, \tag{13}$$

where $1 \leq i, j \leq 5$ and $6 \leq \alpha \leq 7$. Our immediate concern is whether N can be a seven-dimensional indecomposable nilpotent algebra. As such in eq. (13) we must have $\lambda_6 = \lambda_7 = 0$ and $a = b = 0$. Now we are left with a seven-dimensional two-step nilpotent algebra with derived algebra and center of dimension at most two. It is routine to check, using block matrices, that such a Lie algebra cannot possess an $\mathfrak{so}(3)$-type derivation. Also, according to [2] there are just two such Lie algebras $27A$ and $27B$ in his notation and each has $\mathfrak{sl}(2, \mathbb{R})$ but not $\mathfrak{so}(3)$ symmetry. In conclusion if the R-representation is $R_{n-2} \oplus 2D_0$ then N cannot be nilpotent.

6.4. R-representation ad $\mathfrak{so}(3) \oplus 4D_0$

As regards ad $\mathfrak{so}(3) \oplus 4D_0$, we are looking for a seven-dimensional indecomposable nilpotent N so we may assume that the structure equations are of the form

$$[e_i, e_j] = C^{\alpha}_{ij} e_\alpha,\ [e_\alpha, e_\beta] = C^{\gamma}_{\alpha\beta}. \tag{14}$$

Since the R-constants form a subalgebra, it must be nilpotent and there are only three up to isomorphism $A_{4.1}, H + \mathbb{R}$ or $\mathbb{R}^4$. If we begin with $A_{4.1}$ and impose the Jacobi identity we find that N must be of the form

$$[e_1, e_2] = ae_4, [e_1, e_3] = be_4, [e_2, e_3] = ce_4, [e_5, e_7] = e_4, [e_6, e_7] = e_5. \tag{15}$$

However, if we replace e_1 by $ce_1 - be_2 + ae_3$ we obtain a decomposable N.
Now suppose that the R-constant subalgebra is $H + \mathbb{R}$. Then after imposing the Jacobi identity we find that N must be of the form

$$[e_1, e_2] = ae_4 + be_7, [e_1, e_3] = ce_4 + de_7, [e_2, e_3] = ee_4 + fe_7, [e_5, e_6] = e_4. \tag{16}$$

It follows that N is two-step nilpotent and as we saw for the R-representation $R_5 + 2D_0$, such an algebra cannot possess an $\mathfrak{so}(3)$-type subalgebra of derivations. Finally if the R-constant subalgebra is abelian, we find that the brackets

$$\begin{aligned} &[e_1, e_2] = ae_4 + be_5 + ce_6 + de_7, [e_1, e_3] = ee_4 + fe_5 + ge_6 + he_7, \\ &[e_2, e_3] = ie_4 + je_5 + ke_6 + me_7 \end{aligned} \tag{17}$$

already satisfy the Jacobi identity. However, such an algebra has a four-dimensional center and therefore is decomposable.

6.5. R-representation $R_4\oplus$ ad $\mathfrak{so}(3)$

Now we consider the case where the R-representation is ad $\mathfrak{so}(3) \oplus R_4$. Here $\oplus$ stands for the diagonal representation and it corresponds to the following linear combination of matrices:

$$\begin{bmatrix} 0 & -\frac{a}{2} & -\frac{b}{2} & -\frac{c}{2} & 0 & 0 & 0 \\ \frac{a}{2} & 0 & \frac{c}{2} & -\frac{b}{2} & 0 & 0 & 0 \\ \frac{b}{2} & -\frac{c}{2} & 0 & \frac{a}{2} & 0 & 0 & 0 \\ \frac{c}{2} & \frac{b}{2} & -\frac{a}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & -b \\ 0 & 0 & 0 & 0 & -c & 0 & a \\ 0 & 0 & 0 & 0 & b & -a & 0 \end{bmatrix}. \tag{18}$$

If N is to be nilpotent but not abelian then $[N,N] \neq 0$ but $[N,N]$ must be an invariant subspace. The only possibilities are that the dimension of $[N,N]$ is four or three. In both cases we find that $[N,[N,N]] = 0$ because the only invariant subspaces are of dimension $0,3,4,7$.

However, if $\dim[N,N] = 4$, then $\dim\ Z(\mathfrak{g}) = 4$, whereas the maximum possible dimension for $\dim\ Z(\mathfrak{g})$ is 3 if $\mathfrak{g}$ is to be indecomposable. So we assume that $\dim[N,N] = 3$. We can introduce a basis $\{e_4,e_5,e_6,e_7,e_8,e_9,e_{10}\}$ for N such that $\{e_8,e_9,e_{10}\}$ spans $Z(\mathfrak{g})$ with non-zero brackets

$$[e_i,e_j] = C^{\alpha}_{ij}e_{\alpha}, \tag{19}$$

where $4 \leq i,j,k \leq 7$ and $8 \leq \alpha \leq 10$. Note that there are, in principle, 18 independent $37D1$ conditions in Eq.(19). Now if D is a derivation of N then we have

$$D^k_i C^{\alpha}_{kj} - D^k_j C^{\alpha}_{ki} = C^{\beta}_{ij} D^{\alpha}_{\beta}. \tag{20}$$

Again Eq.(20) comprises 18 independent conditions in Eq.(19). When D is chosen as each of the three matrices coming from (18) we find that there are 54 linear conditions on the structure constants C^{α}_{ij}. The solution space is one-dimensional and corresponds precisely to algebra $37D1$ in [2]. These conditions were performed using MAPLE and we will not repeat the details here.

In an appropriate basis $\{e_4,e_5,e_6,e_7,e_8,e_9,e_{10}\}$ algebra $37D1$ is given by

$$\begin{gathered} [e_4,e_5] = e_8, \quad [e_4,e_6] = e_9, \quad [e_4,e_7] = e_{10}, \\ [e_5,e_6] = -e_{10}, \quad [e_5,e_7] = e_9, \quad [e_6,e_7] = -e_8. \end{gathered} \tag{21}$$

The subalgebra of semi-simple derivations of $37D1$ is given by

$$\begin{bmatrix} 0 & s_1+s_4 & s_2+s_5 & s_3+s_6 & 0 & 0 & 0 \\ -s_1-s_4 & 0 & s_3-s_6 & -s_2+s_5 & 0 & 0 & 0 \\ -s_2-s_5 & -s_3+s_6 & 0 & s_1-s_4 & 0 & 0 & 0 \\ -s_3-s_6 & s_2-s_5 & -s_1+s_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\,s_6 & 2\,s_5 \\ 0 & 0 & 0 & 0 & 2\,s_6 & 0 & -2\,s_4 \\ 0 & 0 & 0 & 0 & -2\,s_5 & 2\,s_4 & 0 \end{bmatrix}. \tag{22}$$

We see from the matrix (22) that the subalgebra of semi-simple derivations of $37D1$ is a direct sum of R_4 and $R_4\oplus$ ad $\mathfrak{so}(3)$. Thus we certainly have both R_4 and $R_4\oplus$ ad $\mathfrak{so}(3)$ as semi-simple factors for the radical $37D1$ in a Levi decomposition algebra. However, we can also realize the R-representation 2 ad $\mathfrak{so}(3)\oplus D_0$ in four different ways by making zero precisely one of the first four rows and columns in the matrix 22; although each of these four representations are conjugate, such changes of basis cannot be done in a way that keeps the brackets of algebra $37D1$ intact and thus we obtain four mutually non-isomorphic algebras.

6.6. R-representation $R_4\oplus 3D_0$

If we are to have N nilpotent then we will have structure equations of the form

$$[e_i,e_j]=C^{\alpha}_{ij}e_{\alpha},\ [e_{\alpha},e_{\beta}]=C^{\gamma}_{\alpha\beta}e_{\gamma} \tag{23}$$

where $4\le i,j,k\le 7$ and $8\le\alpha\le 10$. The subspace spanned by $\{e_8,e_9,e_{10}\}$ must be a three-dimensional nilpotent Lie algebra. As such it can only be abelian or H. Suppose first of all that it is abelian. Then we have

$$D^k_iC^{\alpha}_{kj}-D^k_jC^{\alpha}_{ki}=0. \tag{24}$$

As such we find again that the Lie algebra is $37D1$ as was the case for the R-representation $R_4\oplus$ ad $\mathfrak{so}(3)$.

Next we suppose that our three-dimensional nilpotent Lie algebra is H. The analysis is similar to the abelian case except that we must add in the Heisenberg bracket that we write as $[e_8,e_9]=e_{10}$. We must also add some arbitrary coefficients so as to obtain an algebra that is potentially more general than $37D1$. However, when we impose the Jacobi identity we find, after appropriate normalization, that we have the seven-dimensional Heisenberg algebra.

6.7. R-representation 2 ad $\mathfrak{so}(3)\oplus D_0$

Suppose that the R-representation is 2 ad $\mathfrak{so}(3)\oplus D_0$. The only invariant subspaces are of dimension $0,1,3,4,6,7$. Since $[N,N]$ is of dimension at least two less than N, it can only be of dimension one, three or four, if N is not abelian. If dim $[N,N]=1$, then N must be the seven-dimensional Heisenberg algebra if N is not to be decomposable.

Suppose next that dim $[N,N]=4$. We may assume that $[N,N]$ is spanned by $\{e_4,e_5,e_6,e_{10}\}$. We cannot have $[N,[N,N]]=0$, because then dim $Z(N)=4$ and N would be decomposable. Thus we would find that $[N,[N,N]]$ is spanned by $\{e_4,e_5,e_6\}$ and $[N,[N,N]]=Z(N)$. Since $e_{10}\in[N,N]$ but $e_{10}\notin[N,[N,N]]$ it must be the case that $[e_7,e_{10}]=[e_8,e_{10}]=[e_9,e_{10}]=0$ and hence $e_{10}\in Z(N)$, which contradicts dim $Z(N)=3$. Thus dim $[N,N]=3$ and $[N,N]=Z(N)$.

Suppose finally that dim $[N,N]=3$. We will have the following brackets for N:

$$[e_{i+3},e_{j+3}]=C^k_{ij}e_k,[e_{i+3},e_{10}]=\lambda e_i \tag{25}$$

where $4\le i,j\le 6$ and we have applied Schur's Lemma to the second set of brackets in eq. (25). Now we may assume that $\lambda=0$ or $\lambda=1$ by scaling e_{10}. Moreover we may change basis in the subspace $[N,N]$ so that the brackets for N may be assumed to be

$$\begin{aligned}&[e_7,e_8]=e_6,\quad [e_8,e_9]=e_4,\quad [e_9,e_7]=e_5,\\&[e_7,e_{10}]=\lambda e_7,\quad [e_8,e_{10}]=\lambda e_8,\quad [e_9,e_{10}]=\lambda e_9.\end{aligned} \tag{26}$$

If in eq. (26) we have $\lambda = 0$ we obtain the seven-dimensional anti-Heisenberg algebra whereas if $\lambda = 1$ we have $37D1$.

6.7.1. Conclusion. As a result of the investigations in this Section we conclude that the only possible seven-dimensional indecomposable nilpotent Lie algebras that could serve as the nilradical of a ten-dimensional indecomposable Levi decomposition Lie algebra are seven-dimensional Heisenberg 17, seven-dimensional anti-Heisenberg $37A$ and $37D1$ referring to the numbering in [2]. The only possible R-representations are $R_4\oplus$ ad $\mathfrak{so}(3)$, $R_4 \oplus 3D_0$ and 2 ad $\mathfrak{so}(3) \oplus D_0$. Furthermore, $R_4\oplus$ ad $\mathfrak{so}(3)$ is uniquely associated to the radical $37D1$ and $R_4 \oplus 3D_0$ to $37D1$ and 17 and 2 ad $\mathfrak{so}(3) \oplus D_0$ to 17 and $37A$ and $37D1$, the latter case in four different ways.

7. Cases by dimension of NR and attendant $\mathfrak{so}(3)$ R-representation

7.1. NR four-dimensional

The only possibilities for a four-dimensional nilradical, NR, are $A_{4.1}, H + \mathbb{R}$ or $\mathbb{R}^4$. The space of derivations of $A_{4.1}$ is solvable and so it cannot be the nilradical of a Lie algebra that has a non-trivial Levi decomposition. Concerning the case $NR = H + \mathbb{R}$, its space of derivations is ten-dimensional with semi-simple part isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, nilradical isomorphic to $A_{5.1}$ with two-dimensional abelian complement.

It remains to discuss the case $NR = \mathbb{R}^4$. If there is to be a Lie algebra that has a non-trivial Levi decomposition with semi-simple part isomorphic to $\mathfrak{so}(3)$, the only possibilities for the R-representation, restricted to NR, are ad $\mathfrak{so}(3) \oplus D_0$ or R_4. However, we may exclude the case ad $\mathfrak{so}(3) \oplus D_0$ in view of Theorem 5.11.

Finally, suppose that the R-representation, restricted to NR, is R_4. We would like to apply Theorem 5.10; unfortunately, Schur's Lemma is not applicable to R_4. Nonetheless, the space of matrices that commute with R_4 is isomorphic to $\mathbb{R}+\mathfrak{so}(3)$. We require to have three independent matrices, but if we have three we must have four and the semi-simple factor would have to be $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. It follows that there is no ten-dimensional Lie algebra with R-representation R_4 and indeed no case whatever for which the dimension of NR is four.

7.2. NR five-dimensional indecomposable

Now we suppose that the dimension of NR is five. There are six indecomposable nilpotent Lie algebras of dimension five [10]. Of these six, only $A_{5.4}$ has a subspace of derivations that is isomorphic to $\mathfrak{so}(3)$. In fact $A_{5.4}$ is the five-dimensional Heisenberg Lie algebra. We shall write it in the form

$$[e_4, e_6] = e_8, [e_5, e_7] = e_8 \tag{27}$$

in anticipation of being able to supply five extra dimensions. The space of derivations is given by

$$D = \begin{bmatrix} s_1 + t_5 & s_2 & s_5 & s_6 & 0 \\ s_3 & s_4 + t_5 & s_6 & s_7 & 0 \\ s_8 & s_9 & -s_1 + t_5 & -s_3 & 0 \\ s_9 & s_{10} & -s_2 & -s_4 + t_5 & 0 \\ t_1 & t_2 & t_3 & t_4 & 2t_5 \end{bmatrix}. \tag{28}$$

It is a 15-dimensional Levi-decomposition algebra with radical the five-dimensional Milnor Lie algebra and semi-simple factor $\mathfrak{sp}(4)$. The only possible R-representation for $\mathfrak{so}(3)$ is R_4; we obtain R_4, since it consists of skew-symmetric matrices, by putting

$s_1 = s_4 = t_1 = t_2 = t_3 = t_4 = 0, s_3 = -s_2, s_{10} = -s_7, s_8 = -s_5, s_9 = -s_6, s_7 = -s_5$.

We have the following three matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

The three matrices in (29) act as derivations on the Lie algebra given by eq. (27), giving an eight-dimensional Lie algebra, which can in fact, only be $L_{8.2}$ in [12]. The question is whether $L_{8.2}$ can be augmented by two outer derivations of $A_{5.4}$. Such derivations must commute with A, B, C since $\mathfrak{so}(3)$ acts trivially on the complement to NR in N and must be nil-independent. A basis for them is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{30}$$

Since e_8, e_9, e_{10} span the R-constants and $[N, N] \subset NR$ we can only have $[e_9, e_{10}] = ae_8$ for some a. Now do a change of basis replacing e_4, e_5, e_6, e_7, e_8 by Ce_4, Ce_5, Ce_6, Ce_7, C^2e_8, where $C \neq 0$. Then a is acted on by $\frac{1}{C^2}$ and so we may reduce a to $-1, 0, 1$. Finally we obtain a ten-dimensional Lie algebra.

7.3. NR five-dimensional decomposable

There exist three possibilities for a five-dimensional decomposable NR, that is, $A_{4.1} + \mathbb{R}$, $H \oplus \mathbb{R}^2$, $\mathbb{R}^5$. However, the derivation algebra of $A_{4.1} + \mathbb{R}$ is solvable.

7.3.1. NR $H \oplus \mathbb{R}^2$. The semi-simple part of the derivation algebra of $H \oplus \mathbb{R}^2$ consists of the direct sum of the the derivation algebras of H and $\mathbb{R}^2$, that is, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. More specifically, if we have a basis $\{e_1, e_2, e_3, e_4, e_5\}$ and non-zero bracket $[e_2, e_3] = e_1$ then the decomposition of the derivation algebra into its semi-smiple and solvable parts is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & s_1 & s_2 & 0 & 0 \\ 0 & s_3 & -s_1 & 0 & 0 \\ 0 & 0 & 0 & s_4 & s_5 \\ 0 & 0 & 0 & s_6 & -s_4 \end{bmatrix} + \begin{bmatrix} 2t_1 & t_2 & t_3 & t_4 & t_5 \\ 0 & t_1 & 0 & 0 & 0 \\ 0 & 0 & t_1 & 0 & 0 \\ 0 & t_6 & t_7 & t_8 & 0 \\ 0 & t_9 & t_{10} & 0 & t_8 \end{bmatrix}. \tag{31}$$

Thus the space of derivations cannot contain a subalgebra isomorphic to $\mathfrak{so}(3)$.

7.3.2. NR $\mathbb{R}^5$ $L_{10.6}, L_{10.7}, L_{10.8}, L_{10.9}$. The only possibilities for the R-representation restricted to NR are $R_5, R_4 \oplus D_0$ and ad $\mathfrak{so}(3) \oplus 2D_0$. However, the case of R_5 must be excluded in view of Theorem 5.10, since it does satisfy the conclusion of Schur's Lemma. As regards $R_4 \oplus D_0$, comparing the case of $A_{5.4}$ and R_4 in the previous subsection, we shall use the matrices A, B, C in (29) for the representation of $R_4 \oplus D_0$. Hence we are starting from the decomposable Lie algebra $L_{7.2} \oplus \mathbb{R}$, where $L_{7.2}$, see [13], is spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and $\mathbb{R}$ is spanned by e_8. Again we need to find a two-dimensional extension of $L_{7.2} \oplus \mathbb{R}$.

The space of 5×5 matrices that commute with $R_4 \oplus D_0$ is of the form

$$\begin{bmatrix} d & c & b & a & 0 \\ -c & d & -a & b & 0 \\ -b & a & d & -c & 0 \\ -a & -b & c & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}. \tag{32}$$

We are looking for two more vectors e_9 and e_{10} whose adjoint matrices restricted to NR, spanned by e_4, e_5, e_6, e_7, e_8, are of the form given by eq. (32). Then necessarily we have $[e_9, e_{10}] = re_8$ for some r, since the R-constants are spanned by e_8, e_9, e_{10}. We may assume that

$$-\mathrm{ad}(e_9) = \begin{bmatrix} d & 0 & 1 & 0 & 0 \\ 0 & d & 0 & 1 & 0 \\ -1 & 0 & d & 0 & 0 \\ 0 & -1 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}, \quad -\mathrm{ad}(e_{10}) = \begin{bmatrix} D & 0 & 0 & 0 & 0 \\ 0 & D & 0 & 0 & 0 \\ 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & E \end{bmatrix}. \tag{33}$$

If $D = 0$, the algebra would be decomposable so we may assume that $D \neq 0$ and by scaling $\mathrm{ad}(e_{10})$, that $D = 1$. Now we replace e_9 by $e_9 - de_{10}$ leaving e and E that cannot be further reduced but we have, in effect, made $d = 0$. Then by scaling e_8 we can reduce r to $r = 0, 1$. If $r = 0$ then we must have $e^2 + E^2 \neq 0$.

It remains for us to consider the case where the R-representation restricted to NR is $\mathrm{ad}\,\mathfrak{so}(3) \oplus 2D_0$. In this case e_4, e_5, e_6, e_7, e_8 span $NR = \mathbb{R}^5$ and the R-constants are spanned by e_7, e_8, e_9, e_{10}. A matrix that commutes with the R-representation must be of the form $\left[\begin{smallmatrix} \lambda I_3 & 0 \\ 0 & A \end{smallmatrix}\right]$, where A is 2×2. So suppose that

$$\mathrm{ad}(e_9) = \begin{bmatrix} \lambda I_3 & 0 \\ 0 & A \end{bmatrix}, \quad \mathrm{ad}(e_{10}) = \begin{bmatrix} \mu I_3 & 0 \\ 0 & B \end{bmatrix}.$$

where $\mathrm{ad}(e_9)$ and $\mathrm{ad}(e_{10})$ are restricted to NR. We put also $[e_9, e_{10}] = ge_7 + he_8$. The only conditions coming from the Jacobi identities are that the matrices A and B should commute.

Lemma 7.1. *Let $\mathfrak{g}$ be an abelian two-dimensional subalgebra of $\mathfrak{gl}(2, \mathbb{R})$. Then there exists a basis for $\mathfrak{g}$ given by $\{A, B\}$ that has one of the following three forms:*

(1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(3) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

Lemma 7.2. *By a change of basis the pair of matrices* $A = ad(e_9)$ *and* $B = ad(e_{10})$ *introduced above may be brought into one of the following six forms:*

$$(1)\ A = \begin{bmatrix} \lambda I_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} \mu I_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)\ A = \begin{bmatrix} \lambda I_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \mu I_3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$(3)\ A = \begin{bmatrix} \lambda I_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (4)\ A = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$(5)\ A = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 1 \\ 0 & -1 & a \end{bmatrix} \quad (6)\ A = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. Assume first of all that the two lower right-hand 2×2 blocks are linearly independent. Then apply Lemma(7.1) to those two blocks. Otherwise, the two lower right-hand 2×2 blocks are linearly dependent and by taking a linear combination we may assume that the first of them is zero. Then $\lambda \neq 0$ and by scaling we may assume that $\lambda = 1$ and $\mu = 0$. Then we may put the remaining lower right-hand block into real Jordan form. ∎

Now we address the issue of whether the bracket $[e_9, e_{10}] = ie_7 + je_8$ may be simplified. To that end denote the two right 2×2 blocks of $\mathrm{ad}(e_9)$ and $\mathrm{ad}(e_{10})$ by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right]$, respectively, and make a change of basis keeping the e_i's the same except that

$$e_9' = e_9 + pe_7 + qe_8, e_{10}' = e_{10} + re_7 + se_8 \tag{34}$$

where p, q, r, s are at our disposal. Under such a change of basis, the effect is only to change the bracket $[e_9, e_{10}]$ according to, on dropping the primes,

$$[e_9, e_{10}] = (ep + fq - ar - bs + i)e_7 + (gp + hq - cr - ds + j)e_8. \tag{35}$$

Thus, it will be possible to eliminate i and j provided the matrix $\left[\begin{smallmatrix} a & b & -e & -f \\ c & d & -g & -h \end{smallmatrix}\right]$ has rank two. Consulting the six cases obtained in Lemma 7.2 only the fourth case is problematic and only then for $a = 0$; in that case e_8 enters only in $[e_9, e_{10}] = ie_7 + je_8$ and if $j = 0$ the algebra is decomposable so we may redefine e_8 as $ie_7 + je_8$ giving algebra $L_{10.9}$. The fifth and sixth cases of Lemma 7.2 lead to decomposable algebras and the first three cases algebras $L_{10.6}, L_{10.7}, L_{10.8}$ for which $[e_9, e_{10}] = 0$.

7.4. NR six-dimensional indecomposable

Of the 24 six-dimensional indecomposable nilpotent Lie algebras, only four have a non-zero semi-simple subalgebra in their space of derivations. Here are those four nilpotent algebras together with a semi-simple algebra that is isomorphic to its subalgebra of semi-simple derivations:

(1) $A_{6,3}, (\mathfrak{sl}(3,\mathbb{R}))$ (2) $A_{6,4}, (\mathfrak{sl}(2,\mathbb{R}))$ (3) $A_{6,5}, (\mathfrak{so}(3,1))$ (4) $A_{6,12}, (\mathfrak{sl}(2,\mathbb{R}))$.

Hence only $A_{6.3}$ and $A_{6.5}$ can have R-representations of $\mathfrak{so}(3)$.

7.4.1. $NR = A_{6.3}, L_{10.10}$. We take $NR{=}A_{6.3}$ in the form $[e_4, e_5]{=}e_9$, $[e_4, e_6]{=}{-}e_8$, $[e_5, e_6] = e_7$. The space of derivations of $A_{6.3}$ is given by

$$\begin{bmatrix} s_1 & s_2 & s_3 & 0 & 0 & 0 \\ s_4 & s_5 & s_6 & 0 & 0 & 0 \\ s_7 & s_8 & s_9 & 0 & 0 & 0 \\ s_{10} & s_{11} & s_{12} & s_5 + s_9 & -s_4 & -s_7 \\ s_{13} & s_{14} & s_{15} & -s_2 & s_1 + s_9 & -s_8 \\ s_{16} & s_{17} & s_{18} & -s_3 & -s_6 & s_1 + s_5 \end{bmatrix}. \tag{36}$$

In consequence we find that the only possible way to obtain a representation of $\mathfrak{so}(3)$ is ad $\mathfrak{so}(3)\oplus$ ad $\mathfrak{so}(3)$. Thereafter we need another derivation that commutes with the ad $\mathfrak{so}(3)\oplus$ ad $\mathfrak{so}(3)$ representation. The only possibility is

$$\begin{bmatrix} \lambda I_3 & 0 & 0 \\ 0 & 2\lambda I_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\lambda \neq 0$ we may assume that $\lambda = 1$ and we obtain a ten-dimensional algebra.

7.4.2. $NR = A_{6.5}, L_{10.11}$. In [13] the Lie algebra listed as $L_{9.4}$ has radical $A_{6.5}$. We need to see if $L_{9.4}$ has a one-dimensional extension. To that end we note that the space of semi-simple derivations of $A_{6.5}$ is given by:

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & 0 & 0 \\ -s_2 & s_1 & s_4 & -s_3 & 0 & 0 \\ s_5 & s_6 & -s_1 & s_2 & 0 & 0 \\ s_6 & -s_5 & -s_2 & -s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{37}$$

and the solvable complement is given by:

$$\begin{bmatrix} t_1 & t_2 & 0 & 0 & 0 & 0 \\ -t_2 & t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_1 & -t_2 & 0 & 0 \\ 0 & 0 & t_2 & t_1 & 0 & 0 \\ t_3 & t_4 & t_5 & t_6 & 2t_1 & -2t_2 \\ t_7 & t_8 & t_9 & t_{10} & 2t_2 & 2t_1 \end{bmatrix}. \tag{38}$$

The only way to get a representation of $\mathfrak{so}(3)$ is to take $s_1 = 0$, $s_5 = -s_3$, $s_6 = -s_4$ giving R_4. This copy of $\mathfrak{so}(3)$ and $A_{6.5}$ give $L_{9.4}$. The centralizer of the representation R_4 is given by

$$\begin{bmatrix} u_6 & u_7 & -u_8 & -u_1 & 0 & 0 \\ -u_7 & u_6 & u_1 & -u_8 & 0 & 0 \\ u_8 & -u_1 & u_6 & -u_7 & 0 & 0 \\ u_1 & u_8 & u_7 & u_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_5 & u_4 \\ 0 & 0 & 0 & 0 & u_3 & u_2 \end{bmatrix}. \tag{39}$$

We are looking for a new basis vector e_{10}: we must have that $\mathrm{ad}(e_{10})$ restricted to $A_{6.5}$ is an outer derivation of $A_{6.5}$ and must commute with R_4. A basis for such matrices is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}. \tag{40}$$

We obtain a ten-dimensional algebra by taking a linear combination of these two matrices.

7.5. NR six-dimensional decomposable

Up to isomorphism, there are ten six-dimensional decomposable nilpotent Lie algebras: $A_{5.1}\oplus\mathbb{R}$, $A_{5.2}\oplus\mathbb{R}$, $A_{5.3}\oplus\mathbb{R}$, $A_{5.4}\oplus\mathbb{R}$, $A_{5.5}\oplus\mathbb{R}$, $A_{5.6}\oplus\mathbb{R}$, $A_{4.1}\oplus\mathbb{R}^2$, $H\oplus\mathbb{R}^3$, $H\oplus H$, $\mathbb{R}^6$. Among the first six of these algebras, only $A_{5.1}\oplus\mathbb{R}$ and $A_{5.4}\oplus\mathbb{R}$ have a non-zero semi-simple subalgebra of derivations. However, the semi-simple part of the derivation algebra of $A_{5.1}\oplus\mathbb{R}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and so is not of interest. Of the four remaining algebras, each has a non-zero semi-simple subalgebra of derivations; however, in the cases of $A_{4.1}\oplus\mathbb{R}^2$ and $H\oplus H$ the subalgebra of derivations is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, respectively, and so those two cases may be ignored. We consider the three cases $A_{5.4}\oplus\mathbb{R}, H\oplus\mathbb{R}^3, \mathbb{R}^6$ in turn.

7.5.1. $NR = A_{5.4}\oplus\mathbb{R}, L_{10.12}$**.** We take its non-zero brackets as $[e_5,e_7]=e_4$, $[e_6,e_8]=e_4$ and suppose that $A_{5.4}\oplus\mathbb{R}$ has a basis consisting of e_4,e_5,e_6,e_7,e_8,e_9. The derivation algebra of $A_{5.4}\oplus\mathbb{R}$ may be described as follows. The semi-simple part is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & C & -A^t & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where A,B,C,D are 2×2, B and C are symmetric and the overall matrix is 6×6. It is isomorphic to $\mathfrak{sp}(4)$. The only way to obtain a representation of $\mathfrak{so}(3)$ is to take A skew-symmetric and $C=-B^t$ and furthermore, B to have trace zero giving R_4. Compare eq. (28) above.

The solvable complement of the derivation algebra of $A_{5.4}\oplus\mathbb{R}$ is given by

$$\begin{bmatrix} 2t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ 0 & t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_1 & 0 \\ 0 & t_7 & t_8 & t_9 & t_{10} & t_{11} \end{bmatrix} \tag{41}$$

where $t_1,t_2,\ldots,t_{11}$ are arbitrary.

The subspace spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ produces a direct sum of $L_{8.2}$ and $\mathbb{R}$. In order to obtain $L_{8.2}$ in the form given here it is necessary to change the basis for the radical by mapping e_4, e_5, e_6, e_7, e_8 to $e_8, e_5, e_7, -e_4, e_6$ compared with the form given in [12]. We shall take its brackets in the form

$$\begin{gathered}[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_1, e_5] = \tfrac{1}{2}e_8, \quad [e_1, e_6] = \tfrac{1}{2}e_7, \quad [e_1, e_7] = -\tfrac{1}{2}e_6,\\ [e_1, e_8] = -\tfrac{1}{2}e_5, \quad [e_2, e_3] = e_1, \quad [e_2, e_5] = \tfrac{1}{2}e_7, \quad [e_2, e_6] = -\tfrac{1}{2}e_8,\\ [e_2, e_7] = -\tfrac{1}{2}e_5, \quad [e_2, e_8] = \tfrac{1}{2}e_6, \quad [e_3, e_5] = -\tfrac{1}{2}e_6, \quad [e_3, e_6] = \tfrac{1}{2}e_5,\\ [e_3, e_7] = -\tfrac{1}{2}e_8, \quad [e_3, e_8] = \tfrac{1}{2}e_7, \quad [e_5, e_7] = e_4, \quad [e_6, e_8] = e_4.\end{gathered}$$

Now we wish to introduce a new basis vector e_{10}: it will commute with e_1, e_2, e_3. The effect of $-\mathrm{ad}(e_{10})$ on the subspace spanned by $e_4, e_5, e_6, e_7, e_8, e_9$ is given by matrix (41) except that we must remember that we have changed the basis. We know also that $\mathrm{ad}(e_{10})$ commutes with the R-representation. It follows that the matrix of the form(41) is diagonal apart from the entry t_6. We may scale one of the diagonal entries to unity, but the other remains non-zero but otherwise arbitrary. We have a bracket of the form $[e_9, e_{10}] = ae_4 + be_9$. Now if $b \neq 0$ and $b \neq 2$ we can assume that $a = 0$. If $b = 2$ we can reduce a to 0 or 1. Finally if $b = 0$ the algebra is decomposable.

7.5.2. $NR = H \oplus \mathbb{R}^3, L_{10.13}$. The algebra of derivations is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & s_1 & s_2 & 0 & 0 & 0\\ 0 & s_3 & -s_1 & 0 & 0 & 0\\ 0 & 0 & 0 & s_4 & s_5 & s_6\\ 0 & 0 & 0 & s_7 & s_8 & s_9\\ 0 & 0 & 0 & s_{10} & s_{11} & -s_4 - s_8 \end{bmatrix} + \begin{bmatrix} 2t_1 & t_2 & t_3 & t_4 & t_5 & t_6\\ 0 & t_1 & 0 & 0 & 0 & 0\\ 0 & 0 & t_1 & 0 & 0 & 0\\ 0 & t_7 & t_8 & t_{13} & 0 & 0\\ 0 & t_9 & t_{10} & 0 & t_{13} & 0\\ 0 & t_{11} & t_{12} & 0 & 0 & t_{13} \end{bmatrix}. \tag{42}$$

The semi-simple subalgebra is isomorphic to block diagonal $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$. The only way to obtain a representation of $\mathfrak{so}(3)$ is to take zero (2×2 zero matrix) for the $\mathfrak{sl}(2,\mathbb{R})$-factor and to take $\mathfrak{so}(3)$ in place of $\mathfrak{sl}(3,\mathbb{R})$. We obtain a direct sum of H spanned by e_4, e_5, e_6 with bracket $[e_5, e_6] = e_4$ and $L_{6.1}$ [13] spanned by $e_1, e_2, e_3, e_7, e_8, e_9$.

We now attempt to extend this direct sum by a one-dimensional subspace spanned by e_{10}. Since $\mathrm{ad}(e_{10})$ is a derivation of $H \oplus \mathbb{R}^3$, its restriction to NR is of the form given by the second matrix in (42). However, each of the brackets $[e_4, e_{10}]$, $[e_5, e_{10}]$, $[e_6, e_{10}]$ must be a linear combination of e_4, e_5, e_6 since the R-constants are spanned by e_4, e_5, e_6, e_{10} and each such bracket must be in NR. Thus, we may assume in (42) that $t_7 = t_8 = t_9 = t_{10} = t_{11} = t_{12} = 0$.

Now consider the Jacobi identity involving e_3, e_7, e_{10}. We find that

$$\begin{aligned} 0 &= [[e_3, e_7], e_{10}] + [[e_{10}, e_3], e_7] + [[e_7, e_{10}], e_3]\\ &= [e_8, e_{10}] + [t_4 e_4 + t_{13} e_7, e_3] = t_5 e_4 + t_{13} e_8 - t_{13} e_8. \end{aligned}$$

Hence $t_5 = 0$. Similarly, using e_1, e_8, e_{10} we deduce $t_6 = 0$ and from e_2, e_9, e_{10} that $t_4 = 0$. We may further make a transformation of the form

$$e_5{}' = \lambda e_4 + e_5, e_6{}' = \mu e_4 + e_6 \tag{43}$$

and by choosing λ and μ appropriately, we may assume that $t_2 = t_3 = 0$. Finally, by scaling we may suppose that $t_{13} = 1$. The algebra is completely determined, indecomposable and contains no parameters.

7.5.3. $NR = \mathbb{R}^6, L_{10.14}, L_{10.15}, ..., L_{10.26}$. If we assume that NR is spanned by $e_4, e_5, e_6, e_7, e_8, e_9$ then as regards just N itself, $\mathrm{ad}\, e_{10}$ is an arbitrary 7×7 matrix, subject to the conditions that its bottom row and last column should be zero. Of the seven possible representations of $\mathfrak{so}(3)$, see Section 4, R_7 and $\mathrm{ad}\,\mathfrak{so}(3) \oplus R_4$ are excluded because the representation must act trivially on e_{10}. We have already characterized those Levi-decomposition algebras for which the said representations are, respectively, $R_6 \oplus D_0$ and $R_5 \oplus 2D_0$. Each of the remaining three representations, $R_4 \oplus 3D_0$, $\mathrm{ad}\,\mathfrak{so}(3) \oplus 4D_0, 2$ $\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0$ is possible.

In the case of 2 $\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0$, starting from the nine-dimensional algebra 9.10 it is required to add e_{10}. On N, the adjoint matrix $\mathrm{ad}(e_{10})$ must be of the form

$$\begin{bmatrix} aI & bI & 0 \\ cI & dI & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where I denotes the 3×3 identity matrix. Comparing with the real Jordan normal form of the 2×2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and using the fact that we can scale e_{10} we may reduce to three cases.

As regards the representation $R_4 \oplus 3D_0$, we must choose a matrix that commutes with $\mathrm{ad}(e_1)$, $\mathrm{ad}(e_2)$ and $\mathrm{ad}(e_7)$ restricted to the subspace spanned by $\{e_4, e_5, e_6, e_7\}$ on which $\mathfrak{so}(3)$ acts irreducibly, noting that Schur's Lemma is not applicable. Furthermore, the subspace spanned by $\{e_8, e_9, e_{10}\}$ is an arbitrary solvable three-dimensional subalgebra. Consulting [8], this can be reduced to three subcases involving up to three parameters.

Finally, concerning the representation $\mathrm{ad}\,\mathfrak{so}(3) \oplus 4D_0$, we must choose a multiple of the identity on the subspace spanned by $\{e_4, e_5, e_6\}$ on which $\mathfrak{so}(3)$ acts irreducibly. Furthermore, the subspace spanned by $\{e_7, e_8, e_9, e_{10}\}$ is an arbitrary solvable four-dimensional subalgebra with three-dimensional abelian ideal. Consulting [8], this can be reduced to six subcases involving up to three parameters.

7.6. NR seven dimension indecomposable

The only possibilities for seven-dimensional indecomposable nilpotent Lie algebras that have a non-trivial semi-simple space of derivations are the Heisenberg 17, anti-Heisenberg $37A$ and $37D1$ algebras as we found in Section 6, the numbering coming from Gong's list [2]. In Section 6 we supplied the algebra of derivations of $37D1$. Concerning 17, its space of derivations is a 28-dimensional Levi-decomposition algebra. Its semi-simple factor is $\mathfrak{sp}(6)$ and its radical is the seven-dimensional Milnor algebra. Inside $\mathfrak{sp}(6)$ we can find only R_4 and 2 ad $\mathfrak{so}(3)$ as a representation of $\mathfrak{so}(3)$. Concerning $37A$, its algebra of derivations is again a Levi-decomposition algebra again with the seven-dimensional Milnor algebra as its radical. The semi-simple factor is $2\,\mathfrak{sl}(3)$ and we can find only $2 \operatorname{ad} \mathfrak{so}(3)$ as a representation of $\mathfrak{so}(3)$.

7.6.1. NR seven dimension decomposable

We have the following seven possible cases to consider.

$$(1)\ A_{6.i} \oplus \mathbb{R},\ 1 \le i \le 22 \quad (2)\ A_{5.i} \oplus \mathbb{R}^2,\ 1 \le i \le 6 \quad (3)\ A_{4.1} \oplus \mathbb{R}^3$$
$$(4)\ A_{4.1} \oplus H \quad (5)\ H \oplus \mathbb{R}^4 \quad (6)\ H \oplus H \oplus \mathbb{R} \quad (7)\ \mathbb{R}^7$$

Concerning $A_{6.i} \oplus \mathbb{R}$, the only cases for which there can be a non-trivial R-representation of $\mathfrak{so}(3)$ are for $i = 3$ and $i = 5$. Concerning $A_{5.i} \oplus \mathbb{R}$, the only case for which there can be a non-trivial R-representation of $\mathfrak{so}(3)$ is for $i = 4$. For these three cases as well as $A_{4.1} \oplus \mathbb{R}^3$ and $H + R^4$ any R-representation of $\mathfrak{so}(3)$ acts trivially on one of the factors in the direct sum decomposition. It follows that the entire ten-dimensional Lie algebra is decomposable. As regards $A_{4.1} \oplus H$ and $H \oplus H \oplus \mathbb{R}$, there can be no non-trivial R-representation of $\mathfrak{so}(3)$. It remains only to consider the case $NR = \mathbb{R}^7$.

7.6.2. $NR = \mathbb{R}^7, L_{10.36}, L_{10.37}$**.** In this case if the R-representation contains any factor of D_0, then the ten-dimensional Lie algebra would be decomposable according to Theorem 2.1. However, both the representations R_7 and $\operatorname{ad} \mathfrak{so}(3) \oplus R_4$ are possible.

8. Ten-dimensional Indecomposable Lie Algebras having non-trivial Levi decomposition and semi-simple part not $\mathfrak{sl}(2, \mathbb{R})$.

8.1. S neither $\mathfrak{so}(3)$ nor $\mathfrak{sl}(2, \mathbb{R})$, NR four-dimensional decomposable

8.1.1. $S = \mathfrak{so}(3,1), N = NR = \mathbb{R}^4$

$$\begin{aligned} L_{10.1}:\ & [e_1,e_2]=e_3,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_1,e_6]=-e_5,\ [e_1,e_8]=e_9,\\ & [e_1,e_9]=-e_8,\ [e_2,e_3]=e_1,\ [e_2,e_4]=-e_6,\ [e_2,e_6]=e_4,\ [e_2,e_7]=-e_9,\\ & [e_2,e_9]=e_7,\ [e_3,e_4]=e_5,\ [e_3,e_5]=-e_4,\ [e_3,e_7]=e_8,\ [e_3,e_8]=-e_7,\\ & [e_4,e_5]=-e_3,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_{10},\ [e_4,e_{10}]=e_7,\ [e_5,e_6]=-e_1,\\ & [e_5,e_8]=e_{10},\ [e_5,e_{10}]=e_8,\ [e_6,e_9]=e_{10},\ [e_6,e_{10}]=e_9. \end{aligned}$$

8.1.2. $S = \mathfrak{so}(3) \oplus \mathfrak{so}(3), N = NR = \mathbb{R}^4$

$$\begin{aligned} L_{10.2}:\ & [e_1,e_2]=e_3,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_1,e_6]=-e_5,\ [e_1,e_8]=e_9,\\ & [e_1,e_9]=-e_8,\ [e_2,e_3]=e_1,\ [e_2,e_4]=-e_6,\ [e_2,e_6]=e_4,\ [e_2,e_7]=-e_9,\\ & [e_2,e_9]=e_7,\ [e_3,e_4]=e_5,\ [e_3,e_5]=-e_4,\ [e_3,e_7]=e_8,\ [e_3,e_8]=-e_7,\\ & [e_4,e_5]=e_3,\ [e_4,e_6]=-e_2,\ [e_4,e_7]=e_{10},\ [e_4,e_{10}]=-e_7,\ [e_5,e_6]=e_1,\\ & [e_5,e_8]=e_{10},\ [e_5,e_{10}]=-e_8,\ [e_6,e_9]=e_{10},\ [e_6,e_{10}]=-e_9. \end{aligned}$$

8.1.3. $S = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}), N = NR = \mathbb{R}^4$

$$\begin{aligned} L_{10.3}:\ & [e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_7]=e_7,\ [e_1,e_8]=-e_8,\\ & [e_2,e_3]=e_1,\ [e_2,e_8]=e_7,\ [e_3,e_7]=e_8,\ [e_4,e_5]=2e_5,\ [e_4,e_6]=-2e_6,\\ & [e_4,e_9]=e_9,\ [e_4,e_{10}]=-e_{10},\ [e_5,e_6]=e_4,\ [e_5,e_{10}]=e_9,\ [e_6,e_9]=e_{10}. \end{aligned}$$

8.2. $S = \mathfrak{so}(3), NR$ **five-dimensional indecomposable**

8.2.1. $NR = A_{5.4}$**,** $R -$ **representation** $R_4 \oplus 3D_0$**.**

$$L_{10.4}: \quad [e_1,e_2]=e_3,\ [e_1,e_3]=-e_2,\ [e_1,e_4]=\tfrac{1}{2}e_5,\ [e_1,e_5]=-\tfrac{1}{2}e_4,\ [e_1,e_6]=\tfrac{1}{2}e_7,$$
$$[e_1,e_7]=-\tfrac{1}{2}e_6,\ [e_2,e_3]=e_1,\ [e_2,e_4]=\tfrac{1}{2}e_6,\ [e_2,e_5]=-\tfrac{1}{2}e_7,\ [e_2,e_6]=-\tfrac{1}{2}e_4,$$
$$[e_2,e_7]=\tfrac{1}{2}e_5,\ [e_3,e_4]=\tfrac{1}{2}e_7,\ [e_3,e_5]=\tfrac{1}{2}e_6,\ [e_3,e_6]=-\tfrac{1}{2}e_5,\ [e_3,e_7]=-\tfrac{1}{2}e_4,$$
$$[e_4,e_6]=e_8,\ [e_4,e_9]=-e_6,\ [e_4,e_{10}]=e_4,\ [e_5,e_7]=e_8,\ [e_5,e_9]=-e_7,$$
$$[e_5,e_{10}]=e_5,\ [e_6,e_9]=e_4,\ [e_6,e_{10}]=e_6,\ [e_7,e_9]=e_5,\ [e_7,e_{10}]=e_7,$$
$$[e_8,e_{10}]=2e_8,\ [e_9,e_{10}]=ae_8,\ (a=-1,0,1).$$

8.3. $S = \mathfrak{so}(3), NR$ **five-dimensional decomposable**

8.3.1. $NR = \mathbb{R}^5,\ R -$ **representation** $R_4 \oplus 3D_0$

$$L_{10.5}: \quad [e_1,e_2]=e_3,\ [e_1,e_3]=-e_2,\ [e_1,e_4]=\tfrac{1}{2}e_7,\ [e_1,e_5]=\tfrac{1}{2}e_6,\ [e_1,e_6]=-\tfrac{1}{2}e_5,$$
$$[e_1,e_7]=-\tfrac{1}{2}e_4,\ [e_2,e_3]=e_1,\ [e_2,e_4]=\tfrac{1}{2}e_5,\ [e_2,e_5]=-\tfrac{1}{2}e_4,\ [e_2,e_6]=\tfrac{1}{2}e_7,$$
$$[e_2,e_7]=-\tfrac{1}{2}e_6,\ [e_3,e_4]=\tfrac{1}{2}e_6,\ [e_3,e_5]=-\tfrac{1}{2}e_7,\ [e_3,e_6]=-\tfrac{1}{2}e_4,\ [e_3,e_7]=\tfrac{1}{2}e_5,$$
$$[e_4,e_9]=-e_6,\ [e_4,e_{10}]=e_4,\ [e_5,e_9]=-e_7,\ [e_5,e_{10}]=e_5,\ [e_6,e_9]=e_4,$$
$$[e_6,e_{10}]=e_6,\ [e_7,e_9]=e_5,\ [e_7,e_{10}]=e_7,\ [e_8,e_9]=ae_8,\ [e_8,e_{10}]=be_8,$$
$$[e_9,e_{10}]=ce_8,\ (c=0 \text{ or } 1,\ a^2+b^2+c^2\neq 0).$$

8.3.2. $NR = \mathbb{R}^5,\ R -$ **representation** $\mathbf{ad}\,\mathfrak{so}(3) \oplus 4D_0$

$$L_{10.6}: \quad [e_1,e_2]=e_3,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_1,e_6]=-e_5,\ [e_2,e_3]=e_1,$$
$$[e_2,e_4]=-e_6,\ [e_2,e_6]=e_4,\ [e_3,e_4]=e_5,\ [e_3,e_5]=-e_4,\ [e_4,e_9]=ae_4,$$
$$[e_5,e_9]=ae_5,\ [e_6,e_9]=ae_6,\ [e_7,e_9]=e_7,\ [e_4,e_{10}]=be_4,\ [e_5,e_{10}]=be_5,$$
$$[e_6,e_{10}]=be_6,\ [e_8,e_{10}]=e_8,\ (ab\neq 0).$$

$$L_{10.7}: \quad [e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_4,e_9]=ae_4,$$
$$[e_5,e_9]=ae_5,\ [e_6,e_9]=ae_6,\ [e_7,e_9]=e_7,\ [e_8,e_9]=e_8,\ [e_4,e_{10}]=be_4,$$
$$[e_5,e_{10}]=be_5,\ [e_6,e_{10}]=be_6,\ [e_7,e_{10}]=-e_8,\ [e_8,e_{10}]=e_7,\ (a^2+b^2\neq 0).$$

$$L_{10.8}: \quad [e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_4,e_9]=ae_4,$$
$$[e_5,e_9]=ae_5,\ [e_6,e_9]=ae_6,\ [e_7,e_9]=e_7,\ [e_8,e_9]=e_8,\ [e_4,e_{10}]=e_4,$$
$$[e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_8,e_{10}]=e_7.$$

$$L_{10.9}: \quad [e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_4,e_9]=e_4,$$
$$[e_5,e_9]=e_5,\ [e_6,e_9]=e_6,\ [e_7,e_{10}]=e_7,\ [e_9,e_{10}]=e_8.$$

8.4. $S = \mathfrak{so}(3), NR$ **six-dimensional indecomposable**

8.4.1. $NR = A_{6.3},\ R -$ **representation** $\mathbf{2\,ad}\,\mathfrak{so}(3) \oplus D_0$

$$L_{10.10}: \quad [e_1,e_2]=e_3,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_1,e_6]=-e_5,\ [e_1,e_8]=e_9,$$
$$[e_1,e_9]=-e_8,\ [e_2,e_3]=e_1,\ [e_2,e_4]=-e_6,\ [e_2,e_6]=e_4,\ [e_2,e_7]=-e_9,$$
$$[e_2,e_9]=e_7,\ [e_3,e_4]=e_5,\ [e_3,e_5]=-e_4,\ [e_3,e_7]=e_8,\ [e_3,e_8]=-e_7,$$

$[e_4, e_5] = e_9,\ [e_4, e_6] = -e_8,\ [e_4, e_{10}] = e_4,\ [e_5, e_6] = e_7,\ [e_5, e_{10}] = e_5,$
$[e_6, e_{10}] = e_6,\ [e_7, e_{10}] = 2e_7,\ [e_8, e_{10}] = 2e_8,\ [e_9, e_{10}] = 2e_9\,.$

8.4.2. $NR = A_{6.5},\ R-$ **representation** $R_4 \oplus 3D_0$

$L_{10.11}:$ $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_4] = \frac{1}{2}e_7,\ [e_1, e_5] = \frac{1}{2}e_6,\ [e_1, e_6] = -\frac{1}{2}e_5,$
$[e_1, e_7] = -\frac{1}{2}e_4,\ [e_2, e_3] = e_1,\ [e_2, e_4] = \frac{1}{2}e_5,\ [e_2, e_5] = -\frac{1}{2}e_4,\ [e_2, e_6] = \frac{1}{2}e_7,$
$[e_2, e_7] = -\frac{1}{2}e_6, [e_3, e_4] = \frac{1}{2}e_6, [e_3, e_5] = -\frac{1}{2}e_7, [e_3, e_6] = -\frac{1}{2}e_4, [e_3, e_7] = \frac{1}{2}e_5,$
$[e_4, e_6] = e_8,\ [e_4, e_7] = e_9,\ [e_4, e_{10}] = ae_4 - be_5,\ [e_5, e_6] = -e_9,\ [e_5, e_7] = e_8,$
$[e_5, e_{10}] = ae_5 + be_4,\ [e_6, e_{10}] = ae_6 + be_7,\ [e_7, e_{10}] = ae_7 - be_6,$
$[e_8, e_{10}] = 2ae_8 + 2be_9,\ [e_9, e_{10}] = 2ae_9 - 2be_8,\ (a^2 + b^2 \neq 0)\,.$

8.5. $S = \mathfrak{so}(3), NR$ **six-dimensional decomposable**

8.5.1. $NR = A_{5.4} \oplus \mathbb{R},\ R-$ **representation** $R_4 \oplus 3D_0$

$L_{10.12}:$ $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_5] = \frac{1}{2}e_8,\ [e_1, e_6] = \frac{1}{2}e_7,\ [e_1, e_7] = -\frac{1}{2}e_6,$
$[e_1, e_8] = -\frac{1}{2}e_5,\ [e_2, e_3] = e_1,\ [e_2, e_5] = \frac{1}{2}e_7,\ [e_2, e_6] = -\frac{1}{2}e_8,\ [e_2, e_7] = -\frac{1}{2}e_5,$
$[e_2, e_8] = \frac{1}{2}e_6,\ [e_3, e_5] = -\frac{1}{2}e_6,\ [e_3, e_6] = \frac{1}{2}e_5,\ [e_3, e_7] = -\frac{1}{2}e_8,\ [e_3, e_8] = \frac{1}{2}e_7,$
$[e_5, e_7] = e_4,\ [e_6, e_8] = e_4],\ [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,$
$[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ae_4 + be_9,$
$(b \neq 0, 2,\ a = 0;\ b = 2,\ a = 0, 1)\,.$

8.5.2. $NR = H \oplus \mathbb{R}^3,\ R-$ **representation ad** $\mathfrak{so}(3) \oplus 4D_0$

$L_{10.13}:$ $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_8] = e_9,\ [e_1, e_9] = -e_8,\ [e_2, e_3] = e_1,$
$[e_2, e_7] = -e_9,\ [e_2, e_9] = e_7,\ [e_3, e_7] = e_8,\ [e_3, e_8] = -e_7,\ [e_4, e_{10}] = 2e_4,$
$[e_5, e_6] = e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,$
$[e_9, e_{10}] = e_9\,.$

8.5.3. $NR = \mathbb{R}^6,\ R-$ **representation** $R_6 \oplus D_0$

$L_{10.14}:$ $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_6] = -e_8,\ [e_1, e_7] = -e_9,\ [e_1, e_8] = e_6,$
$[e_1, e_9] = e_7,\ [e_2, e_3] = e_1,\ [e_2, e_4] = -e_8,\ [e_2, e_5] = -e_9,\ [e_2, e_8] = e_4,$
$[e_2, e_9] = e_5,\ [e_3, e_4] = -e_6,\ [e_3, e_5] = -e_7,\ [e_3, e_6] = e_4,\ [e_3, e_7] = e_5,$
$[e_4, e_{10}] = e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,$
$[e_9, e_{10}] = e_9\,.$

8.5.4. $NR = \mathbb{R}^6,\ R-$ **representation** $R_5 \oplus 2D_0$

$L_{10.15}:$ $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_4] = \frac{1}{2}e_7,\ [e_1, e_5] = -\frac{1}{2}e_6,$
$[e_1, e_6] = 2e_5 - e_8,\ [e_1, e_7] = -2e_4,\ [e_1, e_8] = 3e_6,\ [e_2, e_3] = e_1,\ [e_2, e_4] = \frac{1}{2}e_6,$
$[e_2, e_5] = \frac{1}{2}e_7,\ [e_2, e_6] = -2e_4,\ [e_2, e_7] = -2e_5 - e_8,\ [e_2, e_8] = 3e_7,$
$[e_3, e_4] = 2e_5,\ [e_3, e_5] = -2e_4,\ [e_3, e_6] = e_7,\ [e_3, e_7] = -e_6,\ [e_4, e_{10}] = e_4,$
$[e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ae_9,$
$(a \neq 0)\,.$

8.5.5. $NR = \mathbb{R}^6,\ R-$ **representation** $R_4 \oplus 3D_0$

$L_{10.16}:$ $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_4] = \frac{1}{2}e_7,\ [e_1, e_5] = \frac{1}{2}e_6,\ [e_1, e_6] = -\frac{1}{2}e_5,$
$[e_1, e_7] = -\frac{1}{2}e_4,\ [e_2, e_3] = e_1,\ [e_2, e_4] = \frac{1}{2}e_5,\ [e_2, e_5] = -\frac{1}{2}e_4,\ [e_2, e_6] = \frac{1}{2}e_7,$

$[e_2,e_7]=-\frac{1}{2}e_6,\ [e_3,e_4]=\frac{1}{2}e_6,\ [e_3,e_5]=-\frac{1}{2}e_7,\ [e_3,e_6]=-\frac{1}{2}e_4,$
$[e_3,e_7]=\frac{1}{2}e_5,\ [e_4,e_{10}]=ae_4-be_6,\ [e_5,e_{10}]=ae_5-be_7,\ [e_6,e_{10}]=be_4+ae_6,$
$[e_7,e_{10}]=be_5+ae_7, [e_8,e_{10}]=ce_8,\ [e_9,e_{10}]=ce_9+e_8\ (a=1 \text{ or } b=1).$

$L_{10.17}:$ $[e_1,e_2]=e_3,\ [e_1,e_3]=-e_2,\ [e_1,e_4]=\frac{1}{2}e_7,\ [e_1,e_5]=\frac{1}{2}e_6,\ [e_1,e_6]=-\frac{1}{2}e_5,$
$[e_1,e_7]=-\frac{1}{2}e_4,\ [e_2,e_3]=e_1,\ [e_2,e_4]=\frac{1}{2}e_5,\ [e_2,e_5]=-\frac{1}{2}e_4,\ [e_2,e_6]=\frac{1}{2}e_7,$
$[e_2,e_7]=-\frac{1}{2}e_6,\ [e_3,e_4]=\frac{1}{2}e_6,\ [e_3,e_5]=-\frac{1}{2}e_7,\ [e_3,e_6]=-\frac{1}{2}e_4,$
$[e_3,e_7]=\frac{1}{2}e_5,\ [e_4,e_{10}]=ae_4-be_6,\ [e_5,e_{10}]=ae_5-be_7,$
$[e_6,e_{10}]=be_4+ae_6,\ [e_7,e_{10}]=be_5+ae_7,\ [e_8,e_{10}]=ce_8,\ [e_9,e_{10}]=de_9,$
$(a=1 \text{ or } b=1,\ cd\neq 0).$

$L_{10.18}:$ $[e_1,e_2]=e_3,\ [e_1,e_3]=-e_2,\ [e_1,e_4]=\frac{1}{2}e_7,\ [e_1,e_5]=\frac{1}{2}e_6,\ [e_1,e_6]=-\frac{1}{2}e_5,$
$[e_1,e_7]=-\frac{1}{2}e_4,\ [e_2,e_3]=e_1,\ [e_2,e_4]=\frac{1}{2}e_5,\ [e_2,e_5]=-\frac{1}{2}e_4,\ [e_2,e_6]=\frac{1}{2}e_7,$
$[e_2,e_7]=-\frac{1}{2}e_6, [e_3,e_4]=\frac{1}{2}e_6, [e_3,e_5]=-\frac{1}{2}e_7, [e_3,e_6]=-\frac{1}{2}e_4, [e_3,e_7]=\frac{1}{2}e_5,$
$[e_4,e_{10}]=ae_4-be_6, [e_5,e_{10}]=ae_5-be_7, [e_6,e_{10}]=be_4+ae_6, [e_7,e_{10}]=be_5+ae_7,$
$[e_8,e_{10}]=ce_8+de_9,\ [e_9,e_{10}]=-de_8+ce_9,\ (a=1 \text{ or } b=1,\ d\neq 0).$

8.5.6. $NR=\mathbb{R}^6,\ R-$ **representation ad** $\mathfrak{so}(3)\oplus 4D_0$

$L_{10.19}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$
$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_4,e_{10}]=e_4,$
$[e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=ae_7,\ [e_8,e_{10}]=e_7+ae_8,$
$[e_9,e_{10}]=be_9\ (b\neq 0).$

$L_{10.20}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$
$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_4,e_{10}]=e_4,$
$[e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=ae_7,\ [e_9,e_{10}]=e_8\ (a\neq 0).$

$L_{10.21}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$
$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_4,e_{10}]=e_4,$
$[e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=ae_7,\ [e_8,e_{10}]=e_7+ae_8,$
$[e_9,e_{10}]=e_8+ae_9.$

$L_{10.22}:$ $[e_1,e_2]=e_3,\ [e_1,e_3]=-e_2,\ [e_1,e_5]=e_6,\ [e_1,e_6]=-e_5,\ [e_2,e_3]=e_1,$
$[e_2,e_4]=-e_6,\ [e_2,e_6]=e_4,\ [e_3,e_4]=e_5,\ [e_3,e_5]=-e_4,\ [e_4,e_{10}]=e_4,$
$[e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=ae_7,\ [e_8,e_{10}]=be_8,\ [e_9,e_{10}]=ce_9$
$(abc\neq 0).$

$L_{10.23}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$
$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_4,e_{10}]=e_4,$
$[e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=ae_7,\ [e_8,e_{10}]=be_8-ce_9,$
$[e_9,e_{10}]=ce_8+be_9\ (abc\neq 0).$

8.5.7. $NR=\mathbb{R}^6,\ R-$ **representation 2 ad** $\mathfrak{so}(3)\oplus D_0$

$L_{10.24}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$
$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_1,e_8]=e_9,$

$[e_2,e_7]=-e_9,\ [e_3,e_7]=e_8,\ [e_1,e_9]=-e_8,\ [e_2,e_9]=e_7,\ [e_3,e_8]=-e_7,$
$[e_4,e_{10}]=e_4,\ [e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=ae_7,\ [e_8,e_{10}]=ae_8,$
$[e_9,e_{10}]=ae_9\,.$

$L_{10.25}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$
$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_1,e_8]=e_9,$
$[e_2,e_7]=-e_9,\ [e_3,e_7]=e_8,\ [e_1,e_9]=-e_8,\ [e_2,e_9]=e_7,\ [e_3,e_8]=-e_7,$
$[e_4,e_{10}]=e_4,\ [e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=e_4+e_7,$
$[e_8,e_{10}]=e_5+e_8,\ [e_9,e_{10}]=e_6+e_9\,.$

$L_{10.26}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,$
$[e_3,e_4]=e_5,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4,\ [e_1,e_8]=e_9,$
$[e_2,e_7]=-e_9,\ [e_3,e_7]=e_8,\ [e_1,e_9]=-e_8,\ [e_2,e_9]=e_7,\ [e_3,e_8]=-e_7,$
$[e_4,e_{10}]=ae_4-e_7,\ [e_5,e_{10}]=ae_5-e_8,\ [e_6,e_{10}]=ae_6-e_9,\ [e_7,e_{10}]=e_4+ae_7,$
$[e_8,e_{10}]=e_5+ae_8,\ [e_9,e_{10}]=e_6+ae_9\,.$

8.6. $S=\mathfrak{so}(3), NR$ seven-dimensional indecomposable

8.6.1. $NR=$ seven-dimensional Heisenberg, $R-$ representation $R_4\oplus 3D_0$

$L_{10.27}:$ $[e_1,e_2]=e_3,\ [e_1,e_3]=-e_2,\ [e_1,e_4]=\frac{1}{2}e_7,\ [e_1,e_5]=\frac{1}{2}e_6,$
$[e_1,e_6]=-\frac{1}{2}e_5,\ [e_1,e_7]=-\frac{1}{2}e_4,\ [e_2,e_3]=e_1,\ [e_2,e_4]=\frac{1}{2}e_5,$
$[e_2,e_5]=-\frac{1}{2}e_4,\ [e_2,e_6]=\frac{1}{2}e_7,\ [e_2,e_7]=-\frac{1}{2}e_6,\ [e_3,e_4]=\frac{1}{2}e_6,$
$[e_3,e_5]=-\frac{1}{2}e_7,\ [e_3,e_6]=-\frac{1}{2}e_4,\ [e_3,e_7]=\frac{1}{2}e_5,\ [e_4,e_5]=e_{10},$
$[e_6,e_7]=-e_{10},\ [e_8,e_9]=e_{10}\,.$

8.6.2. $NR=$ seven-dimensional Heisenberg, $R-$ representation 2 ad $\mathfrak{so}(3)\oplus D_0$

$L_{10.28}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_1,e_6]=-e_5,$
$[e_1,e_8]=e_9,\ [e_1,e_9]=-e_8,\ [e_2,e_4]=-e_6,\ [e_2,e_6]=e_4,\ [e_2,e_7]=-e_9,$
$[e_2,e_9]=e_7,\ [e_3,e_4]=e_5,\ [e_3,e_5]=-e_4,\ [e_3,e_7]=e_8,\ [e_3,e_8]=-e_7,$
$[e_4,e_7]=e_{10},\ [e_5,e_8]=e_{10},\ [e_6,e_9]=e_{10}\,.$

8.6.3. $NR=$ seven-dimensional anti-Heisenberg, $R-$ representation 2 ad $\mathfrak{so}(3)\oplus D_0$

$L_{10.29}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_5]=e_6,\ [e_1,e_6]=-e_5,$
$[e_1,e_8]=e_9,\ [e_1,e_9]=-e_8,\ [e_2,e_4]=-e_6,\ [e_2,e_6]=e_4,\ [e_2,e_7]=-e_9,$
$[e_2,e_9]=e_7,\ [e_3,e_4]=e_5,\ [e_3,e_5]=-e_4,\ [e_3,e_7]=e_8,\ [e_3,e_8]=-e_7,$
$[e_7,e_{10}]=e_4,\ [e_8,e_{10}]=e_5,\ [e_9,e_{10}]=e_6\,.$

8.6.4. $NR=$ seven-dimensional $37D1$, $R-$ representation $R_4\oplus$ ad $\mathfrak{so}(3)$

$L_{10.30}:$ $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_4]=-\frac{1}{2}e_5,\ [e_1,e_5]=\frac{1}{2}e_4,$
$[e_1,e_6]=\frac{1}{2}e_7,\ [e_1,e_7]=-\frac{1}{2}e_6,\ [e_1,e_9]=e_{10},\ [e_1,e_{10}]=-e_9,\ [e_2,e_4]=-\frac{1}{2}e_6,$
$[e_2,e_5]=-\frac{1}{2}e_7,\ [e_2,e_6]=\frac{1}{2}e_4,\ [e_2,e_7]=\frac{1}{2}e_5,\ [e_2,e_8]=-e_{10},\ [e_2,e_{10}]=e_8,$
$[e_3,e_4]=-\frac{1}{2}e_7,\ [e_3,e_5]=\frac{1}{2}e_6,\ [e_3,e_6]=-\frac{1}{2}e_5,\ [e_3,e_7]=\frac{1}{2}e_4,\ [e_3,e_8]=e_9,$
$[e_3,e_9]=-e_8,\ [e_4,e_5]=e_8,\ [e_4,e_6]=e_9,\ [e_4,e_7]=e_{10},\ [e_5,e_6]=-e_{10},$
$[e_5,e_7]=e_9,\ [e_6,e_7]=-e_8\,.$

8.6.5. $NR =$ **seven-dimensional** $37D1$, $R-$ **representation** $R_4 \oplus 3D_0$

$L_{10.31}$: $[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2,\ [e_1, e_4] = -\frac{1}{2}e_5,\ [e_1, e_5] = \frac{1}{2}e_4,$
$[e_1, e_6] = -\frac{1}{2}e_7,\ [e_1, e_7] = \frac{1}{2}e_6,\ [e_2, e_4] = -\frac{1}{2}e_6,\ [e_2, e_5] = \frac{1}{2}e_7,\ [e_2, e_6] = \frac{1}{2}e_4,$
$[e_2, e_7] = -\frac{1}{2}e_5,\ [e_3, e_4] = \frac{1}{2}e_7,\ [e_3, e_5] = \frac{1}{2}e_6,\ [e_3, e_6] = -\frac{1}{2}e_5,$
$[e_3, e_7] = -\frac{1}{2}e_4,\ [e_4, e_5] = e_8,\ [e_4, e_6] = e_9,\ [e_4, e_7] = e_{10},\ [e_5, e_6] = -e_{10},$
$[e_5, e_7] = e_9,\ [e_6, e_7] = -e_8.$

8.6.6. $NR =$ **seven-dimensional** $37D1$, $R-$ **representation 2 ad** $\mathfrak{so}(3) \oplus D_0$

$L_{10.32}$: $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_5] = e_6,\ [e_1, e_6] = -e_5,\ [e_1, e_8] = e_9,$
$[e_1, e_9] = -e_8,\ [e_2, e_3] = e_1,\ [e_2, e_4] = -e_6,\ [e_2, e_6] = e_4,\ [e_2, e_7] = -e_9,$
$[e_2, e_9] = e_7,\ [e_3, e_4] = e_5,\ [e_3, e_5] = -e_4,\ [e_3, e_7] = e_8,\ [e_3, e_8] = -e_7,$
$[e_7, e_8] = e_6,\ [e_7, e_9] = -e_5,\ [e_7, e_{10}] = e_4,\ [e_8, e_9] = e_4,\ [e_8, e_{10}] = e_5,$
$[e_9, e_{10}] = e_6.$

$L_{10.33}$: $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_5] = e_6,\ [e_1, e_6] = -e_5,\ [e_1, e_8] = e_9,$
$[e_1, e_9] = -e_8,\ [e_2, e_3] = e_1,\ [e_2, e_4] = e_6,\ [e_2, e_6] = -e_4,\ [e_2, e_7] = e_9,$
$[e_2, e_9] = -e_7,\ [e_3, e_4] = -e_5,\ [e_3, e_5] = e_4,\ [e_3, e_7] = -e_8,\ [e_3, e_8] = e_7,$
$[e_7, e_8] = e_6,\ [e_7, e_9] = -e_5,\ [e_7, e_{10}] = e_4,\ [e_8, e_9] = e_4,\ [e_8, e_{10}] = e_5,$
$[e_9, e_{10}] = e_6.$

$L_{10.34}$: $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_5] = -e_6,\ [e_1, e_6] = e_5,\ [e_1, e_8] = -e_9,$
$[e_1, e_9] = e_8,\ [e_2, e_3] = e_1,\ [e_2, e_4] = -e_6,\ [e_2, e_6] = e_4,\ [e_2, e_7] = -e_9,$
$[e_2, e_9] = e_7,\ [e_3, e_4] = -e_5,\ [e_3, e_5] = e_4,\ [e_3, e_7] = -e_8,\ [e_3, e_8] = e_7,$
$[e_7, e_8] = e_6,\ [e_7, e_9] = -e_5,\ [e_7, e_{10}] = e_4,\ [e_8, e_9] = e_4,\ [e_8, e_{10}] = e_5,$
$[e_9, e_{10}] = e_6.$

$L_{10.35}$: $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_5] = -e_6,\ [e_1, e_6] = e_5,\ [e_1, e_8] = -e_9,$
$[e_1, e_9] = e_8,\ [e_2, e_3] = e_1,\ [e_2, e_4] = e_6,\ [e_2, e_6] = -e_4,\ [e_2, e_7] = e_9,$
$[e_2, e_9] = -e_7,\ [e_3, e_4] = e_5,\ [e_3, e_5] = -e_4,\ [e_3, e_7] = e_8,\ [e_3, e_8] = -e_7,$
$[e_7, e_8] = e_6,\ [e_7, e_9] = -e_5,\ [e_7, e_{10}] = e_4,\ [e_8, e_9] = e_4,\ [e_8, e_{10}] = e_5,$
$[e_9, e_{10}] = e_6.$

8.7. $S = \mathfrak{so}(3), NR$ seven-dimensional decomposable

8.7.1. $NR = \mathbb{R}^7$, $R-$ **representation** R_7

$L_{10.36}$: $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_1, e_4] = \frac{1}{4}e_7,\ [e_1, e_5] = -\frac{1}{4}e_6,\ [e_1, e_6] = e_9 + 6e_5,$
$[e_1, e_7] = -e_8 - 6e_4,\ [e_1, e_8] = \frac{5}{2}e_7,\ [e_1, e_9] = 3e_{10} - \frac{5}{2}e_6,\ [e_1, e_{10}] = -2e_9,$
$[e_2, e_3] = e_1,\ [e_2, e_4] = \frac{1}{4}e_6,\ [e_2, e_5] = \frac{1}{4}e_7,\ [e_2, e_6] = e_8 - 6e_4,$
$[e_2, e_7] = e_9 - 6e_5,\ [e_2, e_8] = -3e_{10} - \frac{5}{2}e_6,\ [e_2, e_9] = -\frac{5}{2}e_7,\ [e_2, e_{10}] = 2e_8,$
$[e_3, e_4] = 3e_5,\ [e_3, e_5] = -3e_4,\ [e_3, e_6] = 2e_7,\ [e_3, e_7] = -2e_6,\ [e_3, e_8] = e_9,$
$[e_3, e_9] = -e_8.$

8.7.2. $NR = \mathbb{R}^7$, $R-$ representation $R_4 \oplus \mathbf{ad}\,\mathfrak{so}(3)$

$$L_{10.37}: [e_1,e_2]=e_3,\ [e_1,e_3]=-e_2,\ [e_1,e_4]=\tfrac{1}{2}e_7,\ [e_1,e_5]=\tfrac{1}{2}e_6,\ [e_1,e_6]=-\tfrac{1}{2}e_5,$$
$$[e_1,e_7]=-\tfrac{1}{2}e_4,\ [e_1,e_9]=e_{10},\ [e_1,e_{10}]=-e_9,\ [e_2,e_3]=e_1,\ [e_2,e_4]=\tfrac{1}{2}e_5,$$
$$[e_2,e_5]=-\tfrac{1}{2}e_4,\ [e_2,e_6]=\tfrac{1}{2}e_7,\ [e_2,e_7]=-\tfrac{1}{2}e_6,\ [e_2,e_8]=-e_{10},\ [e_2,e_{10}]=e_8,$$
$$[e_3,e_4]=\tfrac{1}{2}e_6,\ [e_3,e_5]=-\tfrac{1}{2}e_7,\ [e_3,e_6]=-\tfrac{1}{2}e_4,\ [e_3,e_7]=\tfrac{1}{2}e_5,\ [e_3,e_8]=e_9,$$
$$[e_3,e_9]=-e_8\,.$$

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