Journal of Lie Theory
Volume **31** (2021) 149–168

On Hom-Pre-Lie Bialgebras

Shanshan Liu, Abdenacer Makhlouf, Lina Song

Communicated by M. Schlichenmaier

Abstract. We introduce Hom-pre-Lie bialgebras in the general framework of the cohomology theory for Hom-Lie algebras. We show that Hom-pre-Lie bialgebras, standard Manin triples for Hom-pre-Lie algebras and certain matched pairs of Hom-pre-Lie algebras are equivalent. Due to the usage of the cohomology theory, it makes us successfully study the coboundary Hom-pre-Lie bialgebras. The notion of Hom-$\mathfrak{s}$-matrix is introduced, by which we can construct Hom-pre-Lie bialgebras naturally. Finally we introduce Hom-$\mathcal{O}$-operators on Hom-pre-Lie algebras and Hom-L-dendriform algebras, by which we construct Hom-$\mathfrak{s}$-matrices.

Mathematics Subject Classification: 16T25, 17B62, 17B99.

Key Words: Hom-pre-Lie algebra, Manin triple, Hom-pre-Lie bialgebra, Hom-s-equation.

1. Introduction

For a given algebraic structure determined by a set of multiplications of various arities and a set of relations among the operations, a bialgebra structure on this algebra is obtained by a corresponding set of comultiplications together with a set of compatibility conditions between the multiplications and comultiplications. A good compatibility condition is prescribed by a rich structure theory and effective constructions. The most famous examples of bialgebras are associative bialgebras and Lie bialgebras, which have important applications in both mathematics and mathematical physics.

The notion of Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in [7] as part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map (homomorphism), and it is called Hom-Jacobi identity. Different types of Hom-algebras were introduced and widely studied. Recently, in [6], Elchinger, Lundengard, Makhlouf and Silvestrov extend the result in [7] to the case of (σ,τ)-derivations. Due to the importance of the aforementioned bialgebra theory, the bialgebra theory for Hom-algebras was deeply studied in [5, 10, 12, 17, 18, 19].

Pre-Lie algebras (also called left-symmetric algebras, quasi-associative algebras, Vinberg algebras and so on) are a class of nonassociative algebras that appeared in many fields in mathematics and mathematical physics. See the survey [4] and the references therein for more details. The notion of left-symmetric bialgebra was introduced in [2], where the author also introduced the notion of $\mathfrak{s}$-matrices to produce

ISSN 0949–5932 / \$2.50

left-symmetric bialgebras. The notion of Hom-pre-Lie algebra was introduced in [9] and play important roles in the study of Hom-Lie bialgebras and Hom-Lie 2-algebras [12, 13]. Recently, Hom-pre-Lie algebras were studied from several aspects. The geometrization of Hom-pre-Lie algebras was studied in [20]; universal α-central extensions of Hom-pre-Lie algebras were studied in [15].

The purpose of this paper is to give a systematic study of the bialgebra theory for Hom-pre-Lie algebras. Note that the notion of a Hom-pre-Lie bialgebra was already introduced in [16] under the terminology of Hom-left-symmetric bialgebra. However the bialgebra structure given in [16] does not enjoy a coboundary theory. This is also one of our motivation to study Hom-pre-Lie bialgebras that enjoy a rich structure theory. Our Hom-pre-Lie bialgebra structure enjoy the following properties:

- equivalent to a Manin triple for Hom-pre-Lie algebras as well as certain matched pair of Hom-pre-Lie algebras;
- Hom-$\mathfrak{s}$-matrices can be defined to produce Hom-pre-Lie bialgebras;
- $\mathcal{O}$-operators on Hom-pre-Lie algebras can be defined to give Hom-$\mathfrak{s}$-matrices in the semidirect product Hom-pre-Lie algebras.

The paper is organized as follows. In Section 2, we recall relevant definitions and results about matched pairs of Hom-Lie algebras and matched pairs of Hom-pre-Lie algebras. In Section 3, we introduce the notion of Hom-pre-Lie bialgebra and show that it is equivalent to Manin triples as well as matched pairs. In Section 4, we study coboundary Hom-pre-Lie bialgebras and introduce the notion of Hom-$\mathfrak{s}$-matrix, by which we can construct a Hom-pre-Lie bialgebra naturally. In Section 5, we introduce the notion of Hom-$\mathcal{O}$-operator and the notion of Hom-L-dendriform algebra, by which we can construct Hom-$\mathfrak{s}$-matrices.

Acknowledgements. We give warmest thanks to Yunhe Sheng for helpful comments.

2. Preliminaries

In this section, we briefly recall matched pairs of Hom-Lie algebras and matched pairs of Hom-pre-Lie algebras, as preparation for our later study of Hom-pre-Lie bialgebras.

Definition 2.1. ([7]) A *Hom-Lie algebra* is a triple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ consisting of a linear space $\mathfrak{g}$, a skew-symmetric bilinear map $[\cdot,\cdot]_{\mathfrak{g}} : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{g}$ and an algebra morphism $\phi_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{g}$, satisfying:

$$[\phi_{\mathfrak{g}}(x), [y,z]_{\mathfrak{g}}]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(y), [z,x]_{\mathfrak{g}}]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(z), [x,y]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \quad \forall\, x,y,z \in \mathfrak{g}. \tag{1}$$

A Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ is said to be regular if $\phi_{\mathfrak{g}}$ is invertible.

Definition 2.2. ([1, 11]) A *representation* of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ on a vector space V with respect to $\beta \in \mathfrak{gl}(V)$ is a linear map $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, such that for all $x, y \in \mathfrak{g}$, the following equalities are satisfied:

$$\rho(\phi_{\mathfrak{g}}(x)) \circ \beta = \beta \circ \rho(x), \tag{2}$$

$$\rho([x,y]_{\mathfrak{g}}) \circ \beta = \rho(\phi_{\mathfrak{g}}(x)) \circ \rho(y) - \rho(\phi_{\mathfrak{g}}(y)) \circ \rho(x). \tag{3}$$

We denote a representation by (V,β,ρ). Representations of Hom-Lie algebras can also be realized as certain homomorphisms [14]. For all $x\in\mathfrak{g}$, we define $\mathrm{ad}_x:\mathfrak{g}\to\mathfrak{g}$ by

$$\mathrm{ad}_x(y)=[x,y]_{\mathfrak{g}},\quad \forall y\in\mathfrak{g}. \tag{4}$$

Then $\mathrm{ad}:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$ is a representation of the Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ on $\mathfrak{g}$ with respect to $\phi_{\mathfrak{g}}$, which is called the adjoint representation.

Proposition 2.3. *Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ be a Hom-Lie algebra, and (V,β_V,ρ_V) and $(W,\beta_W,\rho_W,)$ its representations. Then $(V\otimes W,\beta_V\otimes\beta_W,\rho_V\otimes\beta_W+\beta_V\otimes\rho_W)$ is a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$.*

Definition 2.4. ([12]) A *matched pair of Hom-Lie algebras*, which is denoted by $(\mathfrak{g},\mathfrak{g}',\rho,\rho')$, consists of two Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ and $(\mathfrak{g}',[\cdot,\cdot]_{\mathfrak{g}'},\phi_{\mathfrak{g}'})$, together with representations $\rho:\mathfrak{g}\longrightarrow\mathfrak{gl}(g')$ and $\rho':\mathfrak{g}'\longrightarrow\mathfrak{gl}(g)$ with respect to $\phi_{\mathfrak{g}'}$ and $\phi_{\mathfrak{g}}$ respectively, such that for all $x,y\in\mathfrak{g},x',y'\in\mathfrak{g}'$, the following conditions are satisfied:

$$\begin{aligned}\rho'(\phi_{\mathfrak{g}'}(x'))[x,y]_{\mathfrak{g}} &= [\rho'(x')(x),\phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}+[\phi_{\mathfrak{g}}(x),\rho'(x')(y)]_{\mathfrak{g}} \\ &\quad +\rho'(\rho(y)(x'))(\phi_{\mathfrak{g}}(x))-\rho'(\rho(x)(x'))(\phi_{\mathfrak{g}}(y)),\end{aligned} \tag{5}$$

$$\begin{aligned}\rho(\phi_{\mathfrak{g}}(x))[x',y']_{\mathfrak{g}'} &= [\rho(x)(x'),\phi_{\mathfrak{g}'}(y')]_{\mathfrak{g}'}+[\phi_{\mathfrak{g}'}(x'),\rho(x)(y')]_{\mathfrak{g}'} \\ &\quad +\rho(\rho'(y')(x))(\phi_{\mathfrak{g}'}(x'))-\rho(\rho'(x')(x))(\phi_{\mathfrak{g}'}(y')).\end{aligned} \tag{6}$$

We define $\phi_d:\mathfrak{g}\oplus\mathfrak{g}'\longrightarrow\mathfrak{g}\oplus\mathfrak{g}'$ by

$$\phi_d(x,x')=(\phi_{\mathfrak{g}}(x),\phi_{\mathfrak{g}'}(x')), \tag{7}$$

and define a skew-symmetric bilinear map $[\cdot,\cdot]_d:\wedge^2(\mathfrak{g}\oplus\mathfrak{g}')\longrightarrow\mathfrak{g}\oplus\mathfrak{g}'$ by

$$[(x,x'),(y,y')]_d=([x,y]_{\mathfrak{g}}+\rho'(x')(y)-\rho'(y')(x),[x',y']_{\mathfrak{g}'}+\rho(x)(y')-\rho(y)(x')). \tag{8}$$

Theorem 2.5. ([12]) *With the above notations, $(\mathfrak{g}\oplus\mathfrak{g}',[\cdot,\cdot]_d,\phi_d)$ is a Hom-Lie algebra if and only if $(\mathfrak{g},\mathfrak{g}',\rho,\rho')$ is a matched pair of Hom-Lie algebras.*

Definition 2.6. ([9]) A *Hom-pre-Lie algebra* $(A,\cdot,\alpha)$ is a vector space A equipped with a bilinear product $\cdot:A\otimes A\longrightarrow A$, and $\alpha\in\mathfrak{gl}(A)$, such that for all $x,y,z\in A$, $\alpha(x\cdot y)=\alpha(x)\cdot\alpha(y)$ and the following equality is satisfied:

$$(x\cdot y)\cdot\alpha(z)-\alpha(x)\cdot(y\cdot z)=(y\cdot x)\cdot\alpha(z)-\alpha(y)\cdot(x\cdot z). \tag{9}$$

A Hom-pre-Lie algebra $(A,\cdot,\alpha)$ is said to be *regular* if α is invertible.

Let $(A,\cdot,\alpha)$ be a Hom-pre-Lie algebra. We always assume that it is regular, i.e. α is invertible. The commutator $[x,y]=x\cdot y-y\cdot x$ gives a Hom-Lie algebra $(A,[\cdot,\cdot],\alpha)$, which is denoted by A^C and called the sub-adjacent Hom-Lie algebra of $(A,\cdot,\alpha)$.

Definition 2.7. ([16]) A *representation* of a Hom-pre-Lie algebra $(A,\cdot,\alpha)$ on a vector space V with respect to $\beta\in\mathfrak{gl}(V)$ consists of a pair (ρ,μ), where $\rho:A\longrightarrow\mathfrak{gl}(V)$ is a representation of the sub-adjacent Hom-Lie algebra A^C on V

with respect to $\beta \in \mathfrak{gl}(V)$, and $\mu : A \longrightarrow \mathfrak{gl}(V)$ is a linear map, for all $x, y \in A$, satisfying:

$$\beta \circ \mu(x) = \mu(\alpha(x)) \circ \beta, \tag{10}$$

$$\mu(\alpha(y)) \circ \mu(x) - \mu(x \cdot y) \circ \beta = \mu(\alpha(y)) \circ \rho(x) - \rho(\alpha(x)) \circ \mu(y). \tag{11}$$

We denote a representation of a Hom-pre-Lie algebra $(A, \cdot, \alpha)$ by (V, β, ρ, μ). Furthermore, let $L, R : A \longrightarrow \mathfrak{gl}(A)$ be linear maps, where $L_x y = x \cdot y, R_x y = y \cdot x$. Then (A, α, L, R) is also a representation, which we call the regular representation.

Proposition 2.8. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra. For any integer s, define $L^s, R^s : A \longrightarrow \mathfrak{gl}(A)$ by*

$$L^s_x y = \alpha^s(x) \cdot y, \quad R^s_x y = y \cdot \alpha^s(x), \quad \forall x, y \in A.$$

Then (A, α, L^s, R^s) is a representation of the Hom-Lie algebra $(A, \cdot, \alpha)$.

Proof. For all $x, y, z \in A$, we have

$$L^s_{\alpha(x)} \alpha(y) = \alpha^{s+1}(x) \cdot \alpha(y) = \alpha(\alpha^s(x) \cdot y) = \alpha(L^s_x y), \tag{12}$$

which implies that $L^s_{\alpha(x)} \circ \alpha = \alpha \circ L^s_x$. Similarly, we have $R^s_{\alpha(x)} \circ \alpha = \alpha \circ R^s_x$.
By the definition of a Hom-pre-Lie algebra, we have

$$\begin{aligned} L^s_{\alpha(x)} L^s_y(z) - L^s_{\alpha(y)} L^s_x(z) &= \alpha^{s+1}(x) \cdot (\alpha^s(y) \cdot z) - \alpha^{s+1}(y) \cdot (\alpha^s(x) \cdot z) \\ &= (\alpha^s(x) \cdot \alpha^s(y)) \cdot \alpha(z) - (\alpha^s(y) \cdot \alpha^s(x)) \cdot \alpha(z) \\ &= \alpha^s([x, y]) \cdot \alpha(z) = L^s_{[x,y]} \alpha(z). \end{aligned}$$

Similarly, we have

$$R^s_{\alpha(y)} \circ R^s_x - R^s_{x \cdot y} \circ \alpha = R^s_{\alpha(y)} \circ L^s_x - L^s_{\alpha(x)} \circ R^s_y. \tag{13}$$

This finishes the proof. ■

The notion of matched pairs of Hom-pre-Lie algebras was given in [16].

Definition 2.9. ([16]) A *matched pair of Hom-pre-Lie algebras* $(A, B, l_A, r_A, l_B, r_B)$ consists of two Hom-pre-Lie algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$, together with linear maps $l_A, r_A : A \longrightarrow \mathfrak{gl}(B)$ and $l_B, r_B : B \longrightarrow \mathfrak{gl}(A)$ such that (B, α_B, l_A, r_A) and (A, α_A, l_B, r_B) are representations, satisfying for all $x, y \in A, a, b \in B$ the following conditions:

$$\begin{aligned} r_A(\alpha_A(x))\{a, b\} = r_A(l_B(b)x)\alpha_B(a) - r_A(l_B(a)x)\alpha_B(b) + \alpha_B(a) \circ (r_A(x)b) \\ - \alpha_B(b) \circ (r_A(x)a), \end{aligned} \tag{14}$$

$$\begin{aligned} l_A(\alpha_A(x))(a \circ b) = -l_A(l_B(a)x - r_B(a)x)\alpha_B(b) + (l_A(x)a - r_A(x)a) \circ \alpha_B(b) \\ + r_A(r_B(b)x)\alpha_B(a) + \alpha_B(a) \circ (l_A(x)b), \end{aligned} \tag{15}$$

$$\begin{aligned} r_B(\alpha_B(a))[x, y] = r_B(l_A(y)a)\alpha_A(x) - r_B(l_A(x)a)\alpha_A(y) + \alpha_A(x) \cdot (r_B(a)y) \\ - \alpha_A(y) \cdot (r_B(a)x), \end{aligned} \tag{16}$$

$$\begin{aligned} l_B(\alpha_B(a))(x \cdot y) = -l_B(l_A(x)a - r_A(x)a)\alpha_A(y) + (l_B(a)x - r_B(a)x) \cdot \alpha_A(y) \\ + r_B(r_A(y)a)\alpha_A(x) + \alpha_A(x) \cdot (l_B(a)y), \end{aligned} \tag{17}$$

where $[\cdot, \cdot]$ is the Lie bracket of the sub-adjacent Hom-Lie algebra A^C and $\{\cdot, \cdot\}$ is the Lie bracket of the sub-adjacent Hom-Lie algebra B^C.

We define a bilinear operation $\diamond:\otimes^2(A\oplus B)\longrightarrow(A\oplus B)$ by

$$(x+a)\diamond(y+b):=x\cdot y+l_B(a)y+r_B(b)x+a\circ b+l_A(x)b+r_A(y)a, \tag{18}$$

and a linear map $\alpha_A\oplus\alpha_B:A\oplus B\longrightarrow A\oplus B$ by

$$(\alpha_A\oplus\alpha_B)(x+a):=\alpha_A(x)+\alpha_B(a). \tag{19}$$

The following is proved in [16].

Theorem 2.10. ([16]) *With the above notations, $(A\oplus B,\diamond,\alpha_A\oplus\alpha_B)$ is a Hom-pre-Lie algebra if and only if (A,B,l_A,r_A,l_B,r_B) is a matched pair of Hom-pre-Lie algebras.*

3. Hom-pre-Lie bialgebras

In this section, we introduce the notions of Manin triples for Hom-pre-Lie algebras and Hom-pre-Lie bialgebras. We show that Hom-pre-Lie bialgebras, standard Manin triples for Hom-pre-Lie algebras and certain matched pairs of Hom-pre-Lie algebras are equivalent.

Let (V,β,ρ,μ) be a representation of a Hom-pre-Lie algebra $(A,\cdot,\alpha)$. In the sequel, we always assume that β is invertible. For all $x\in A, u\in V,\xi\in V^*$, define $\rho^*:A\longrightarrow\mathfrak{gl}(V^*)$ and $\mu^*:A\longrightarrow\mathfrak{gl}(V^*)$ as usual by

$$\langle\rho^*(x)(\xi),u\rangle=-\langle\xi,\rho(x)(u)\rangle,\quad\langle\mu^*(x)(\xi),u\rangle=-\langle\xi,\mu(x)(u)\rangle,$$

for all $x\in A$, $\xi\in V^*$, $u\in V$. Then define $\rho^\star:A\longrightarrow\mathfrak{gl}(V^*)$ and $\mu^\star:A\longrightarrow\mathfrak{gl}(V^*)$ by

$$\rho^\star(x)(\xi):=\rho^*(\alpha(x))\big((\beta^{-2})^*(\xi)\big),\quad\mu^\star(x)(\xi):=\mu^*(\alpha(x))\big((\beta^{-2})^*(\xi)\big). \tag{20}$$

Theorem 3.1. ([8]) *Let (V,β,ρ,μ) be a representation of a Hom-pre-Lie algebra $(A,\cdot,\alpha)$. Then $(V^*,(\beta^{-1})^*,\rho^\star-\mu^\star,-\mu^\star)$ is a representation of $(A,\cdot,\alpha)$, which is called the dual representation of (V,β,ρ,μ).*

Corollary 3.2. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Lie algebra. Then $(A^*,(\alpha^{-1})^*,\mathrm{ad}^\star,-R^\star)$ is a representation of $(A,\cdot,\alpha)$, where $\mathrm{ad}=L-R$ is the adjoint representation of the sub-adjacent Hom-Lie algebra A^C.*

In the sequel, when there is a Hom-pre-Lie algebra structure on A^*, we will use $\mathcal{L}$, $\mathcal{R}$ and $\mathfrak{ad}:=\mathcal{L}-\mathcal{R}$ to denote the corresponding operations.
Before we introduce Manin triples and Hom-pre-Lie bialgebras, we give an important relation between matched pairs of Hom-pre-Lie algebras and matched pairs of the associated sub-adjacent Hom-Lie algebras.

Proposition 3.3. *Let $(A,\cdot,\alpha)$ and $(A^*,\circ,(\alpha^{-1})^*)$ be two Hom-pre-Lie algebras. Then, $(A^C,(A^*)^C,L^\star,\mathcal{L}^\star)$ is a matched pair of Hom-Lie algebras if and only if $(A,A^*,\mathrm{ad}^\star,-R^\star,\mathfrak{ad}^\star,-\mathcal{R}^\star)$ is a matched pair of Hom-pre-Lie algebras.*

Proof. We denote the bracket of the sub-adjacent Hom-pre-Lie algebra $(A^*)^C$ by $\{\cdot,\cdot\}$. For all $x,y\in A,\xi,\eta\in A^*$, we have

$$
\begin{aligned}
&\langle R^{\star}_{\alpha(\alpha^{-2}(x))}\{\xi,\eta\} - R^{\star}_{\mathfrak{ad}^{\star}_{\eta}\alpha^{-2}(x)}(\alpha^{-1})^{*}(\xi) + R^{\star}_{\mathfrak{ad}^{\star}_{\xi}\alpha^{-2}(x)}(\alpha^{-1})^{*}(\eta) - (\alpha^{-1})^{*}(\xi)\circ R^{\star}_{\alpha^{-2}(x)}\eta \\
&\quad + (\alpha^{-1})^{*}(\eta)\circ R^{\star}_{\alpha^{-2}(x)}\xi, \alpha^{2}(y)\rangle \\
&= -\langle\{\xi,\eta\}, R_{\alpha^{-2}(x)}y\rangle + \langle(\alpha^{-1})^{*}(\xi), R_{\alpha^{-1}(\mathfrak{ad}^{\star}_{\eta}\alpha^{-2}(x))}(y)\rangle - \langle(\alpha^{-1})^{*}(\eta), R_{\alpha^{-1}(\mathfrak{ad}^{\star}_{\xi}\alpha^{-2}(x))}(y)\rangle \\
&\quad - \langle\mathcal{L}_{(\alpha^{-1})^{*}(\xi)}R^{\star}_{\alpha^{-2}(x)}\eta, \alpha^{2}(y)\rangle + \langle\mathcal{L}_{(\alpha^{-1})^{*}(\eta)}R^{\star}_{\alpha^{-2}(x)}\xi, \alpha^{2}(y)\rangle \\
&= \langle -\{\xi,\eta\}, L_{y}\alpha^{-2}(x)\rangle + \langle(\alpha^{-1})^{*}(\xi), L_{y}\alpha^{-1}(\mathfrak{ad}^{\star}_{\eta}\alpha^{-2}(x))\rangle \\
&\quad - \langle(\alpha^{-1})^{*}(\eta), L_{y}\alpha^{-1}(\mathfrak{ad}^{\star}_{\xi}\alpha^{-2}(x))\rangle + \langle R^{\star}_{\alpha^{-2}(x)}\eta, \mathcal{L}^{*}_{(\alpha^{-1})^{*}(\xi)}\alpha^{2}(y)\rangle \\
&\quad - \langle R^{\star}_{\alpha^{-2}(x)}\xi, \mathcal{L}^{*}_{(\alpha^{-1})^{*}(\eta)}\alpha^{2}(y)\rangle \\
&= \langle L^{\star}_{\alpha(y)}\{\xi,\eta\}, x\rangle - \langle(\alpha^{-1})^{*}L^{*}_{y}(\alpha^{-1})^{*}(\xi), \mathfrak{ad}^{\star}_{\eta}\alpha^{-2}(x)\rangle \\
&\quad + \langle(\alpha^{-1})^{*}L^{*}_{y}(\alpha^{-1})^{*}(\eta), \mathfrak{ad}^{\star}_{\xi}\alpha^{-2}(x)\rangle - \langle\eta, R_{\alpha^{-3}(x)}\alpha^{-2}(\mathcal{L}^{*}_{(\alpha^{-1})^{*}(\xi)}\alpha^{2}(y))\rangle \\
&\quad + \langle\xi, R_{\alpha^{-3}(x)}\alpha^{-2}(\mathcal{L}^{*}_{(\alpha^{-1})^{*}(\eta)}\alpha^{2}(y))\rangle \\
&= \langle L^{\star}_{\alpha(y)}\{\xi,\eta\}, x\rangle + \langle\mathfrak{ad}_{\alpha^{*}(\eta)}\alpha^{*}(L^{*}_{y}(\alpha^{-1})^{*}(\xi)), \alpha^{-2}(x)\rangle \\
&\quad - \langle\mathfrak{ad}_{\alpha^{*}(\xi)}\alpha^{*}(L^{*}_{y}(\alpha^{-1})^{*}(\eta)), \alpha^{-2}(x)\rangle + \langle L^{*}_{\alpha^{-2}(\mathcal{L}^{\star}_{\xi}y)}\eta, \alpha^{-3}(x)\rangle - \langle L^{*}_{\alpha^{-2}(\mathcal{L}^{\star}_{\eta}y)}\xi, \alpha^{-3}(x)\rangle \\
&= \langle L^{\star}_{\alpha(y)}\{\xi,\eta\}, x\rangle + \langle\{\eta, L^{\star}_{\alpha^{-1}(y)}\alpha^{*}(\xi)\}, \alpha^{-1}(x)\rangle - \langle\{\xi, L^{\star}_{\alpha^{-1}(y)}\alpha^{*}(\eta)\}, \alpha^{-1}(x)\rangle \\
&\quad + \langle L^{\star}_{\alpha^{-3}(\mathcal{L}^{\star}_{\xi}y)}(\alpha^{2})^{*}(\eta), \alpha^{-3}(x)\rangle - \langle L^{\star}_{\alpha^{-3}(\mathcal{L}^{\star}_{\eta}y)}(\alpha^{2})^{*}(\xi), \alpha^{-3}(x)\rangle \\
&= \langle L^{\star}_{\alpha(y)}\{\xi,\eta\}, x\rangle + \langle\{\eta, \alpha^{*}(L^{\star}_{y}\xi)\}, \alpha^{-1}(x)\rangle - \langle\{\xi, \alpha^{*}(L^{\star}_{y}\eta)\}, \alpha^{-1}(x)\rangle \\
&\quad + \langle L^{\star}_{\mathcal{L}^{\star}_{\xi}y}(\alpha^{-1})^{*}(\eta), x\rangle - \langle L^{\star}_{\mathcal{L}^{\star}_{\eta}y}(\alpha^{-1})^{*}(\xi), x\rangle \\
&= \langle L^{\star}_{\alpha(y)}\{\xi,\eta\} + \{(\alpha^{-1})^{*}(\eta), L^{\star}_{y}\xi\} - \{(\alpha^{-1})^{*}(\xi), L^{\star}_{y}\eta)\} + L^{\star}_{\mathcal{L}^{\star}_{\xi}y}(\alpha^{-1})^{*}(\eta) \\
&\quad - L^{\star}_{\mathcal{L}^{\star}_{\eta}y}(\alpha^{-1})^{*}(\xi), x\rangle,
\end{aligned}
$$

which implies that (6) $\Longleftrightarrow$ (14). We also have

$$
\begin{aligned}
&\langle -\mathfrak{ad}^{\star}_{(\alpha^{-1})^{*}(\xi)}(x\cdot y) - \mathfrak{ad}^{\star}_{L^{\star}_{x}\xi}\alpha(y) + \mathcal{L}^{\star}_{\xi}x\cdot\alpha(y) + \mathcal{R}^{\star}_{R^{\star}_{y}\xi}\alpha(x) + \alpha(x)\cdot\mathfrak{ad}^{\star}_{\xi}y, (\alpha^{-2})^{*}(\eta)\rangle \\
&= \langle x\cdot y, \mathfrak{ad}_{\xi}\eta\rangle + \langle\alpha(y), \mathfrak{ad}_{\alpha^{*}(L^{\star}_{x}\xi)}\eta\rangle - \langle\alpha(y), L^{*}_{\mathcal{L}^{\star}_{\xi}x}(\alpha^{-2})^{*}(\eta)\rangle - \langle\alpha(x), \mathcal{R}_{\alpha^{*}(R^{\star}_{y}\xi)}\eta\rangle \\
&\quad - \langle\mathfrak{ad}^{\star}_{\xi}y, L^{*}_{\alpha(x)}(\alpha^{-2})^{*}(\eta)\rangle \\
&= \langle L_{x}y, \{\xi,\eta\}\rangle + \langle\alpha(y), \{\alpha^{*}(L^{\star}_{x}\xi), \eta\}\rangle - \langle\alpha(y), L^{\star}_{\alpha^{-1}(\mathcal{L}^{\star}_{\xi}x)}\eta\rangle - \langle\alpha(x), \mathcal{L}_{\eta}\alpha^{*}(R^{\star}_{y}\xi)\rangle \\
&\quad - \langle\mathfrak{ad}^{\star}_{\xi}y, L^{\star}_{x}\eta\rangle \\
&= -\langle\alpha^{2}(y), L^{\star}_{\alpha(x)}\{\xi,\eta\}\rangle + \langle\alpha^{2}(y), \{L^{\star}_{x}\xi, (\alpha^{-1})^{*}(\eta)\}\rangle - \langle\alpha^{2}(y), (\alpha^{-1})^{*}(L^{\star}_{\alpha^{-1}(\mathcal{L}^{\star}_{\xi}x)}\eta)\rangle \\
&\quad - \langle R_{\alpha^{-1}(y)}\alpha^{-1}(\mathcal{L}^{*}_{\eta}\alpha(x)), \xi\rangle + \langle y, \mathfrak{ad}_{\alpha^{*}(\xi)}(\alpha^{2})^{*}(L^{\star}_{x}\eta)\rangle \\
&= -\langle\alpha^{2}(y), L^{\star}_{\alpha(x)}\{\xi,\eta\}\rangle + \langle\alpha^{2}(y), \{L^{\star}_{x}\xi, (\alpha^{-1})^{*}(\eta)\}\rangle - \langle\alpha^{2}(y), L^{\star}_{\mathcal{L}^{\star}_{\xi}x}(\alpha^{-1})^{*}(\eta)\rangle \\
&\quad - \langle\alpha^{2}(\mathcal{L}^{*}_{\eta}\alpha(x))\cdot\alpha^{2}(y), (\alpha^{-3})^{*}(\xi)\rangle + \langle\alpha^{2}(y), (\alpha^{-2})^{*}\mathfrak{ad}_{\alpha^{*}(\xi)}(\alpha^{2})^{*}(L^{\star}_{x}\eta)\rangle \\
&= -\langle\alpha^{2}(y), L^{\star}_{\alpha(x)}\{\xi,\eta\}\rangle + \langle\alpha^{2}(y), \{L^{\star}_{x}\xi, (\alpha^{-1})^{*}(\eta)\}\rangle - \langle\alpha^{2}(y), L^{\star}_{\mathcal{L}^{\star}_{\xi}x}(\alpha^{-1})^{*}(\eta)\rangle \\
&\quad - \langle L_{\alpha(\mathcal{L}^{\star}_{\eta}x)}\alpha^{2}(y), (\alpha^{-3})^{*}(\xi)\rangle + \langle\alpha^{2}(y), \{(\alpha^{-1})^{*}(\xi), L^{\star}_{x}\eta\}\rangle \\
&= \langle\alpha^{2}(y), -L^{\star}_{\alpha(x)}\{\xi,\eta\} + \{L^{\star}_{x}\xi, (\alpha^{-1})^{*}(\eta)\} - L^{\star}_{\mathcal{L}^{\star}_{\xi}x}(\alpha^{-1})^{*}(\eta) + L^{\star}_{\mathcal{L}^{\star}_{\eta}x}(\alpha^{-1})^{*}(\xi) \\
&\quad + \{(\alpha^{-1})^{*}(\xi), L^{\star}_{x}\eta\}\rangle,
\end{aligned}
$$

which implies that (6) $\Longleftrightarrow$ (17). Thus, we have (6) $\Longleftrightarrow$ (14) $\Longleftrightarrow$ (17). Similarly, we have (5) $\Longleftrightarrow$ (15) $\Longleftrightarrow$ (16). This finishes the proof. ∎

Now we introduce the notion of a quadratic Hom-pre-Lie algebra and the notion of Manin triple in this framework.

Definition 3.4. A *quadratic Hom-pre-Lie algebra* is a Hom-pre-Lie algebra $(A,\cdot,\alpha)$ equipped with a nondegenerate skew-symmetric bilinear form $\omega\in\wedge^2A^*$ such that for all $x,y,z\in A$, the following invariant conditions hold:

$$\omega(\alpha(x),\alpha(y)) = \omega(x,y), \tag{21}$$

$$\omega(x\cdot y,\alpha(z)) = -\omega(\alpha(y),[x,z]). \tag{22}$$

We denote a quadratic Hom-pre-Lie algebra by $(A,\cdot,\alpha,\omega)$. See [3] for more details about invariant bilinear forms on a Hom-Lie algebra.

Definition 3.5. A *Manin triple* for Hom-pre-Lie algebras is a triple $(\mathcal{A},A_1,A_2)$ in which $(\mathcal{A},\cdot,\alpha,\omega)$ is a quadratic Hom-pre-Lie algebra, $(A_1,\cdot_1,\alpha_1)$ and $(A_2,\cdot_2,\alpha_2)$ are isotropic Hom-pre-Lie sub-algebras of $\mathcal{A}$ such that

(i) $\mathcal{A}=A_1\oplus A_2$ as vector spaces,

(ii) $\alpha=\alpha_1\oplus\alpha_2$.

Two Manin triples $(\mathcal{A},A_1,A_2)$ and $(\mathcal{B},B_1,B_2)$ with bilinear forms ω_1 and ω_2 respectively are isomorphic if there exists an isomorphism of Hom-pre-Lie algebras $f:\mathcal{A}\longrightarrow\mathcal{B}$ such that

$$f(A_1)=B_1,\quad f(A_2)=B_2,\quad \omega_1(x,y)=\omega_2(f(x),f(y)),\quad \forall x,y\in\mathcal{A}. \tag{23}$$

Let $(A,\cdot,\alpha)$ and $(A^*,\circ,(\alpha^{-1})^*)$ be two Hom-pre-Lie algebras. For all $x,y\in A$, $\xi,\eta\in A^*$, define a linear map $\diamond:\otimes^2(A\oplus A^*)\rightarrow(A\oplus A^*)$ by

$$(x+\xi)\diamond(y+\eta):=x\cdot y+\mathfrak{ad}^\star(\xi)y-\mathcal{R}^\star(\eta)x+\xi\circ\eta+\mathrm{ad}^\star(x)\eta-R^\star(y)\xi. \tag{24}$$

Assume that $(A\oplus A^*,\diamond,\alpha\oplus(\alpha^{-1})^*)$ is a Hom-pre-Lie algebra, such that $(A,\cdot,\alpha)$ and $(A^*,\circ,(\alpha^{-1})^*)$ are Hom-pre-Lie subalgebras, by computation, the natural non-degenerate skew-symmetric bilinear form $\bar{\omega}$ on $A\oplus A^*$ given by

$$\bar{\omega}(x+\xi,y+\eta)=\langle\xi,y\rangle-\langle\eta,x\rangle, \tag{25}$$

is invariant. Consequently, $(A\oplus A^*,A,A^*)$ is a Manin triple, which is called the *standard Manin triple*.

Proposition 3.6. *Every Manin triple is isomorphic to the standard Manin triple.*

Proof. Let $(\mathcal{A},A_1,A_2)$ be a Manin triple with a nondegenerate skew-symmetric invariant bilinear form ω. For all $x\in A_1$, $u\in A_2$ we define a linear map $f:\mathcal{A}\longrightarrow A_1\oplus A_1^*$ by $f(x,u)=(x,\omega(u,\cdot))$. Since ω is nondegenerate, f is an isomorphism between vector spaces. Thus, f induces a Manin triple structure on $(A_1\oplus A_1^*,A_1,A_1^*)$.

First for all $x,y\in A,u,v\in A_2$, we have

$$\bar{\omega}(f(x+u),f(y+v))=\bar{\omega}(x+\omega(u,\cdot),y+\omega(v,\cdot))=\omega(u,y)-\omega(v,x)=\omega(x+u,y+v),$$

which implies that the induced bilinear form on $A_1\oplus A_1^*$ is exactly $\bar{\omega}$ given by (25). Then we assume the induced Hom-pre-Lie algebra structure on $A_1\oplus A_1^*$ is given by $(A_1\oplus A_1^*,\cdot',\alpha_{A_1}\oplus\alpha_{A_1^*})$. By (21), we obtain $\alpha_{A_1^*}=(\alpha_{A_1}^{-1})^*$.

For all $x, y \in A, \xi, \eta \in A^*$, we have

$$
\begin{aligned}
&\bar{\omega}(x \cdot' \xi, \alpha(y)) = -\bar{\omega}((\alpha^{-1})^*(\xi), [x,y]) = -\langle (\alpha^{-2})^*(\xi), L_{\alpha(x)}\alpha(y) - R_{\alpha(x)}\alpha(y)\rangle \\
&\quad = \langle L^\star_x \xi - R^\star_x \xi, \alpha(y)\rangle = \bar{\omega}(\mathrm{ad}^\star_x \xi, \alpha(y)), \quad \text{and} \\
&\bar{\omega}(x \cdot' \xi, (\alpha^{-1})^*(\eta)) = \bar{\omega}((\alpha^{-1})^*(\xi), [\eta, x]) = -\bar{\omega}(\eta \circ \xi, \alpha(x)) \qquad (26) \\
&\quad = -\langle \mathcal{R}_{(\alpha^{-1})^*(\xi)}(\alpha^{-1})^*(\eta), \alpha^2(x)\rangle = \langle (\alpha^{-1})^*(\eta), \mathcal{R}^\star_\xi x\rangle = -\bar{\omega}(\mathcal{R}^\star_\xi x, (\alpha^{-1})^*(\eta)).
\end{aligned}
$$

Thus, we have $x \cdot' \xi = \mathrm{ad}^\star_x \xi - \mathcal{R}^\star_\xi x$. Similarly, we have $\xi \cdot' x = \mathfrak{ad}^\star_\xi x - R^\star_x \xi$. Thus, we deduce that $(x+\xi)\cdot'(y+\eta) = (x+\xi)\diamond(y+\eta)$, which implies that $(A_1 \oplus A_1^*, A_1, A_1^*)$ is the standard Manin triple. ■

For a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$ and a representation (V, β, ρ), recall that a 1-cocycle δ associated to (V, β, ρ) is a linear map form $\mathfrak{g}$ to V satisfying:

$$
\delta([x,y]_\mathfrak{g}) = \rho(\phi_\mathfrak{g}(x))\delta(y) - \rho(\phi_\mathfrak{g}(y))\delta(x). \qquad (27)
$$

Definition 3.7. A pair of Hom-pre-Lie algebras $(A, \cdot, \alpha)$ and $(A^*, \circ, (\alpha^{-1})^*)$ is called a *Hom-pre-Lie bialgebra* if the following conditions hold:

(i) φ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra A^C associated to the representation $(A \otimes A, L^{-2} \otimes \alpha + \alpha \otimes \mathrm{ad}^{-2})$, where $\varphi^* : A \longrightarrow A \otimes A$ is the dual of $\circ : A^* \otimes A^* \longrightarrow A^*$, i.e. $\langle \varphi^*(x), \xi \otimes \eta\rangle = \langle x, \xi \circ \eta\rangle$.

(ii) ψ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra $(A^*)^C$ associated to the representation $(A^* \otimes A^*, \mathcal{L}^{-2} \otimes (\alpha^{-1})^* + (\alpha^{-1})^* \otimes \mathfrak{ad}^{-2})$, where the map $\psi^* : A^* \longrightarrow A^* \otimes A^*$ is the dual of $\cdot : A \otimes A \longrightarrow A^*$, i.e., $\langle \psi^*(\xi), x \otimes y\rangle = \langle \xi, x \cdot y\rangle$.

We denote a Hom-pre-Lie bialgebra by $(A, A^*, \varphi^*, \psi^*)$ or simply (A, A^*).

Now we are ready to give the main result of this section.

Theorem 3.8. *Let $(A, \cdot, \alpha)$ and $(A^*, \circ, (\alpha^{-1})^*)$ be two Hom-pre-Lie algebras. Then the following conditions are equivalent:*

(i) (A, A^*) *is a Hom-pre-Lie bialgebra,*

(ii) $(A, A^*, \mathrm{ad}^\star, -R^\star, \mathfrak{ad}^\star, -\mathcal{R}^\star)$ *is a matched pair of Hom-pre-Lie algebras,*

(iii) $(A \oplus A^*, A, A^*)$ *is a standard Manin triple for Hom-pre-Lie algebras.*

Proof. First, we prove that (i) is equivalent to (ii). We have

$$
\begin{aligned}
&\langle -\mathrm{ad}^\star_{\alpha(x)}(\xi \circ \eta) - \mathrm{ad}^\star_{\mathcal{L}^\star_\xi x}(\alpha^{-1})^*(\eta) + L^\star_x \xi \circ (\alpha^{-1})^*(\eta) + R^\star(\mathcal{R}^\star_\eta x)(\alpha^{-1})^*(\xi) \\
&\qquad + (\alpha^{-1})^*(\xi) \circ \mathrm{ad}^\star_x \eta, \alpha^2(y)\rangle \\
&= \langle [x,y], \xi \circ \eta\rangle + \langle \mathrm{ad}_{\alpha^{-1}(\mathcal{L}^\star_\xi x)} y, (\alpha^{-1})^*(\eta)\rangle + \langle (\alpha^2)^*(L^\star_x \xi) \circ \alpha^*(\eta), y\rangle \\
&\quad - \langle R_{\alpha^{-1}(\mathcal{R}^\star_\eta x)} y, (\alpha^{-1})^*(\xi)\rangle + \langle \alpha^*(\xi) \circ (\alpha^2)^*(\mathrm{ad}^\star_x \eta), y\rangle \\
&= \langle \varphi^*[x,y], \xi \otimes \eta\rangle - \langle \mathrm{ad}_y \alpha^{-1}(\mathcal{L}^\star_\xi x), (\alpha^{-1})^*(\eta)\rangle + \langle L^\star_{\alpha^{-2}(x)}(\alpha^2)^*(\xi) \circ \alpha^*(\eta), y\rangle \\
&\quad - \langle L_y \alpha^{-1}(\mathcal{R}^\star_\eta x), (\alpha^{-1})^*(\xi)\rangle + \langle \alpha^*(\xi) \circ \mathrm{ad}^\star_{\alpha^{-2}(x)}(\alpha^2)^*(\eta), y\rangle \\
&= \langle \varphi^*[x,y], \xi \otimes \eta\rangle + \langle \mathcal{L}^\star_\xi x, (\alpha^{-1})^* \mathrm{ad}^\star_{\alpha^{-1}(y)} \alpha^*(\eta)\rangle + \langle (L^{-2})^*_{\alpha(x)} \xi \circ \alpha^*(\eta), y\rangle \\
&\quad + \langle \mathcal{R}^\star_\eta x, (\alpha^{-1})^* L^\star_{\alpha^{-1}(y)} \alpha^*(\xi)\rangle + \langle \alpha^*(\xi) \circ (\mathrm{ad}^{-2})^*_{\alpha(x)} \eta, y\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle \varphi^*[x,y], \xi\otimes\eta\rangle + \langle \mathcal{L}^\star_\xi x, \mathrm{ad}^\star_y\eta\rangle - \langle (L^{-2}_{\alpha(x)}\otimes\alpha)\varphi^*(y), \xi\otimes\eta\rangle + \langle \mathcal{R}^\star_\eta x, L^\star_y\xi\rangle \\
&\quad - \langle (\alpha\otimes \mathrm{ad}^{-2}_{\alpha(x)})\varphi^*(y), \xi\otimes\eta\rangle \\
&= \langle \varphi^*[x,y], \xi\otimes\eta\rangle - \langle x, \alpha^*(\xi)\circ(\alpha^2)^*(\mathrm{ad}^\star_y\eta)\rangle - \langle (L^{-2}_{\alpha(x)}\otimes\alpha)\varphi^*(y), \xi\otimes\eta\rangle \\
&\quad - \langle x, (\alpha^2)^*(L^\star_y\xi)\circ\alpha^*(\eta)\rangle - \langle (\alpha\otimes \mathrm{ad}^{-2}_{\alpha(x)})\varphi^*(y), \xi\otimes\eta\rangle \\
&= \langle \varphi^*[x,y], \xi\otimes\eta\rangle - \langle x, \alpha^*(\xi)\circ \mathrm{ad}^\star_{\alpha^{-2}(y)}(\alpha^2)^*(\eta)\rangle - \langle (L^{-2}_{\alpha(x)}\otimes\alpha)\varphi^*(y), \xi\otimes\eta\rangle \\
&\quad - \langle x, L^\star_{\alpha^{-2}(y)}(\alpha^2)^*(\xi)\circ\alpha^*(\eta)\rangle - \langle (\alpha\otimes \mathrm{ad}^{-2}_{\alpha(x)})\varphi^*(y), \xi\otimes\eta\rangle \\
&= \langle \varphi^*[x,y], \xi\otimes\eta\rangle - \langle x, \alpha^*(\xi)\circ (\mathrm{ad}^{-2})^*_{\alpha(y)}\eta\rangle - \langle (L^{-2}_{\alpha(x)}\otimes\alpha)\varphi^*(y), \xi\otimes\eta\rangle \\
&\quad - \langle x, (L^{-2})^*_{\alpha(y)}\xi\circ\alpha^*(\eta)\rangle - \langle (\alpha\otimes \mathrm{ad}^{-2}_{\alpha(x)})\varphi^*(y), \xi\otimes\eta\rangle \\
&= \langle \varphi^*[x,y], \xi\otimes\eta\rangle + \langle (\alpha\otimes \mathrm{ad}^{-2}_{\alpha(y)})\varphi^*(x), \xi\otimes\eta\rangle - \langle (L^{-2}_{\alpha(x)}\otimes\alpha)\varphi^*(y), \xi\otimes\eta\rangle \\
&\quad + \langle (L^{-2}_{\alpha(y)}\otimes\alpha)\varphi^*(x), \xi\otimes\eta\rangle - \langle (\alpha\otimes \mathrm{ad}^{-2}_{\alpha(x)})\varphi^*(y), \xi\otimes\eta\rangle \\
&= \langle \varphi^*[x,y] - (L^{-2}_{\alpha(x)}\otimes\alpha + \alpha\otimes \mathrm{ad}^{-2}_{\alpha(x)})\varphi^*(y) + (L^{-2}_{\alpha(y)}\otimes\alpha \\
&\quad + \alpha\otimes \mathrm{ad}^{-2}_{\alpha(y)})\varphi^*(x), \xi\otimes\eta\rangle, \qquad (28)
\end{aligned}
$$

which means that (15) holds if and only if φ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra A^C associated to the representation $(A\otimes A, L^{-2}\otimes\alpha+\alpha\otimes\mathrm{ad}^{-2})$. By Proposition 3.3, we can obtain that (15) $\Longleftrightarrow$ (16). Therefore, φ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra A^C associated to the representation $(A\otimes A, L^{-2}\otimes\alpha+\alpha\otimes\mathrm{ad}^{-2})$ if and only if (15) and (16) hold. Similarly, we can prove that ψ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra $(A^*)^C$ associated to the representation $(A^*\otimes A^*, \mathcal{L}^{-2}\otimes(\alpha^{-1})^* + (\alpha^{-1})^*\otimes\mathfrak{ad}^{-2})$ if and only if (14) and (17) hold. Thus, we obtain that condition (i) is equivalent to condition (ii).

Next, we prove that (ii) is equivalent to (iii). Let $(A, A^*, \mathrm{ad}^\star, -R^\star, \mathfrak{ad}^\star, -\mathcal{R}^\star)$ be a matched pair of Hom-pre-Lie algebras. By Theorem 2.10, $(A\oplus A^*, \diamond, \alpha\oplus(\alpha^{-1})^*)$ is a Hom-pre-Lie algebra, where "$\diamond$" is given by (24). Let $\bar{\omega}$ be the natural nondegenerate skew-symmetric bilinear form on $A\oplus A^*$ given by (25). We only need to prove that $\bar{\omega}$ satisfies the invariant conditions, it follows from straightforward computations. Thus, $(A\oplus A^*, A, A^*)$ is a standard Manin triple. Conversely, if $(A\oplus A^*, A, A^*)$ is a standard Manin triple, then it is obvious that $(A, A^*, \mathrm{ad}^\star, -R^\star, \mathfrak{ad}^\star, -\mathcal{R}^\star)$ is a matched pair of Hom-pre-Lie algebras. ■

4. Coboundary Hom-pre-Lie bialgebras

In this section, we study coboundary Hom-pre-Lie bialgebras and introduce the notion of a Hom-$\mathfrak{s}$-matrix, which gives rise to a Hom-pre-Lie bialgebra naturally.

Definition 4.1. A Hom-pre-Lie bialgebra $(A, A^*, \varphi^*, \psi^*)$ is called *coboundary* if φ^* is a 1-coboundary of the sub-adjacent Hom-Lie algebra A^C associated to the representation $(A\otimes A, L^{-2}\otimes\alpha+\alpha\otimes\mathrm{ad}^{-2})$, that is, there exists an $r\in A\otimes A$ such that

$$\varphi^*(x) = (L_x^{-2}\otimes\alpha + \alpha\otimes\mathrm{ad}_x^{-2})r, \quad \forall x\in A. \qquad (29)$$

Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra, and $r\in A\otimes A$, suppose that φ^* is a 1-coboundary of the sub-adjacent Hom-Lie algebra A^C associated to the representation $(A\otimes A, L^{-2}\otimes\alpha+\alpha\otimes\mathrm{ad}^{-2})$.

Then it is obvious that φ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra A^C associated to the representation $(A\otimes A, L^{-2}\otimes\alpha+\alpha\otimes\mathrm{ad}^{-2})$.

Proposition 4.2. *With the above notations, (A,A^*,φ^*,ψ^*) is a Hom-pre-Lie bialgebra if and only if the following two conditions are satisfied:*

(i) *$\circ : A^*\otimes A^*\longrightarrow A^*$ defines a Hom-pre-Lie algebra structure on A^*, where "$\circ$" is given by $\langle\varphi^*(x),\xi\otimes\eta\rangle=\langle x,\xi\circ\eta\rangle$.*

(ii) *ψ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra $(A^*)^C$ associated to the representation $(A^*\otimes A^*, \mathcal{L}^{-2}\otimes(\alpha^{-1})^*+(\alpha^{-1})^*\otimes\mathfrak{ad}^{-2})$, where the map $\psi^*: A^*\longrightarrow A^*\otimes A^*$ is given by $\langle\psi^*(\xi),x\otimes y\rangle=\langle\xi,x\cdot y\rangle$.*

For all $r\in A\otimes A$, the linear map $r^\sharp: A^*\longrightarrow A$ is defined by

$$\langle r^\sharp(\xi),\eta\rangle=\langle r,\xi\otimes\eta\rangle,\quad\forall\xi,\eta\in A^*. \tag{30}$$

Proposition 4.3. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Lie algebra and $\varphi^*: A\longrightarrow A\otimes A$ defined by (29). If $r\in A\otimes A$ satisfies*

$$r^\sharp\circ(\alpha^{-1})^*=\alpha\circ r^\sharp. \tag{31}$$

Then, for all $\xi,\eta\in A^*$, we have

$$\xi\circ\eta=\mathrm{ad}^\star_{r^\sharp(\xi)}\eta-R^\star_{\sigma(r)^\sharp(\eta)}\xi=\mathrm{ad}^*_{\alpha(r^\sharp(\xi))}((\alpha^{-2})^*(\eta))-R^*_{\alpha(\sigma(r)^\sharp(\eta))}((\alpha^{-2})^*(\xi)), \tag{32}$$

where $\sigma: A\otimes A\longrightarrow A\otimes A$ is the flip operator defined by $\sigma(x\otimes y)=y\otimes x$ for all $x,y\in A$. Furthermore, we have

$$r^\sharp(\alpha^*(\xi))\cdot r^\sharp(\alpha^*(\eta))-r^\sharp(\alpha^*(\xi\circ\eta))=[[r,r]](\xi,\eta), \tag{33}$$

where $[[r,r]]=[r_{12},r_{23}]-r_{13}\cdot r_{12}+r_{13}\cdot r_{23}$.

Before proving this result, let us explain the notations. Let $(A,\cdot,\alpha)$ be a Hom-pre-Lie algebra and $r=\sum_i x_i\otimes y_i\in A\otimes A$. Set

$$r_{12}=\sum_i x_i\otimes y_i\otimes\alpha,\quad r_{13}=\sum_i x_i\otimes\alpha\otimes y_i,\quad r_{23}=\sum_i\alpha\otimes x_i\otimes y_i.$$

Proof. Let $r=\sum_i x_i\otimes y_i$. Here the Einstein summation convention is used. By (29) and (31), for all $z\in A$, ξ, $\eta\in A^*$, we have

$$\begin{aligned}
&\langle z,\xi\circ\eta\rangle=\langle\varphi^*(z),\xi\otimes\eta\rangle=\langle(L_z^{-2}\otimes\alpha+\alpha\otimes\mathrm{ad}_z^{-2})(x_i\otimes y_i),\xi\otimes\eta\rangle\\
&=\langle\alpha^{-2}(z)\cdot x_i\otimes\alpha(y_i),\xi\otimes\eta\rangle+\langle\alpha(x_i)\otimes[\alpha^{-2}(z),y_i],\xi\otimes\eta\rangle\\
&=\langle\alpha^{-2}(z)\cdot x_i,\xi\rangle\langle\alpha(y_i),\eta\rangle+\langle\alpha(x_i),\xi\rangle\langle[\alpha^{-2}(z),y_i],\eta\rangle\\
&=\langle\alpha^{-2}(z)\cdot\langle\alpha(y_i),\eta\rangle x_i,\xi\rangle+\langle[\alpha^{-2}(z),\langle\alpha(x_i),\xi\rangle y_i],\eta\rangle\\
&=\langle\alpha^{-2}(z)\cdot\sigma(r)^\sharp(\alpha^*(\eta)),\xi\rangle+\langle[\alpha^{-2}(z),r^\sharp(\alpha^*(\xi))],\eta\rangle\\
&=\langle R_{\sigma(r)^\sharp(\alpha^*(\eta))}\alpha^{-2}(z),\xi\rangle-\langle\mathrm{ad}_{r^\sharp(\alpha^*(\xi))}\alpha^{-2}(z),\eta\rangle\\
&=-\langle z,(\alpha^{-2})^*R^\star_{\alpha^{-1}(\sigma(r)^\sharp(\alpha^*(\eta)))}(\alpha^2)^*(\xi)\rangle+\langle z,(\alpha^{-2})^*\mathrm{ad}^\star_{\alpha^{-1}(r^\sharp(\alpha^*(\xi)))}(\alpha^2)^*(\eta)\rangle\\
&=-\langle z,R^\star_{\sigma(r)^\sharp(\eta)}\xi\rangle+\langle z,\mathrm{ad}^\star_{r^\sharp(\xi)}\eta\rangle,
\end{aligned}$$

which implies that (32) holds.

Moreover, for all $\theta \in A^*$, we have

$$
\begin{aligned}
&\langle \alpha(r^\sharp(\xi)) \cdot \alpha(r^\sharp(\eta)) - \alpha(r^\sharp(\xi \circ \eta)), \theta \rangle \\
&= \langle r^\sharp((\alpha^{-1})^*(\xi)) \cdot r^\sharp((\alpha^{-1})^*(\eta)), \theta \rangle - \langle \xi \circ \eta, \sigma(r)^\sharp(\alpha^*(\theta)) \rangle \\
&= \langle r^\sharp((\alpha^{-1})^*(\xi)) \cdot r^\sharp((\alpha^{-1})^*(\eta)), \theta \rangle - \langle \mathrm{ad}^*_{\alpha(r^\sharp(\xi))}((\alpha^{-2})^*(\eta)) \\
&\quad - R^*_{\alpha(\sigma(r)^\sharp(\eta))}((\alpha^{-2})^*(\xi)), \sigma(r)^\sharp(\alpha^*(\theta)) \rangle \\
&= \langle r^\sharp((\alpha^{-1})^*(\xi)) \cdot r^\sharp((\alpha^{-1})^*(\eta)), \theta \rangle + \langle (\alpha^{-2})^*(\eta), r^\sharp((\alpha^{-1})^*(\xi)) \cdot \sigma(r)^\sharp(\alpha^*(\theta)) \rangle \\
&\quad - \langle (\alpha^{-2})^*(\eta), \sigma(r)^\sharp(\alpha^*(\theta)) \cdot r^\sharp((\alpha^{-1})^*(\xi)) \rangle \\
&\quad - \langle (\alpha^{-2})^*(\xi), \sigma(r)^\sharp(\alpha^*(\theta)) \cdot \sigma(r)^\sharp((\alpha^{-1})^*(\eta)) \rangle \\
&= \langle \langle x_i, (\alpha^{-1})^*(\xi) \rangle y_i \cdot \langle x_j, (\alpha^{-1})^*(\eta) \rangle y_j, \theta \rangle \\
&\quad + \langle \langle x_i, (\alpha^{-1})^*(\xi) \rangle y_i \cdot \langle y_j, \alpha^*(\theta) \rangle x_j, (\alpha^{-2})^*(\eta) \rangle \\
&\quad - \langle \langle y_i, \alpha^*(\theta) \rangle x_i \cdot \langle x_j, (\alpha^{-1})^*(\xi) \rangle y_j, (\alpha^{-2})^*(\eta) \rangle \\
&\quad - \langle \langle y_i, \alpha^*(\theta) \rangle x_i \cdot \langle y_j, (\alpha^{-1})^*(\eta) \rangle x_j, (\alpha^{-2})^*(\xi) \rangle \\
&= \langle x_i, (\alpha^{-1})^*(\xi) \rangle \langle x_j, (\alpha^{-1})^*(\eta) \rangle \langle y_i \cdot y_j, \theta \rangle \\
&\quad + \langle x_i, (\alpha^{-1})^*(\xi) \rangle \langle y_j, \alpha^*(\theta) \rangle \langle y_i \cdot x_j, (\alpha^{-2})^*(\eta) \rangle \\
&\quad - \langle y_i, \alpha^*(\theta) \rangle \langle x_j, (\alpha^{-1})^*(\xi) \rangle \langle x_i \cdot y_j, (\alpha^{-2})^*(\eta) \rangle \\
&\quad - \langle y_i, \alpha^*(\theta) \rangle \langle y_j, (\alpha^{-1})^*(\eta) \rangle \langle x_i \cdot x_j, (\alpha^{-2})^*(\xi) \rangle \\
&= \langle \alpha(x_i) \otimes \alpha(x_j) \otimes y_i \cdot y_j - \alpha(x_i) \otimes [x_j, y_i] \otimes \alpha(y_j) \\
&\quad - x_i \cdot x_j \otimes \alpha(y_j) \otimes \alpha(y_i), (\alpha^{-2})^*(\xi) \otimes (\alpha^{-2})^*(\eta) \otimes \theta \rangle \\
&= \langle [[r, r]]((\alpha^{-2})^*(\xi), (\alpha^{-2})^*(\eta)), \theta \rangle,
\end{aligned}
$$

which implies that (33) holds. This finishes the proof. ∎

Corollary 4.4. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra. Let $r \in Sym^2(A)$ satisfying* (31) *and $[[r, r]] = 0$, then $(A^*, \circ, (\alpha^{-1})^*)$ is a Hom-pre-Lie algebra, where $\circ$ is given by* (32).

Proof. By Proposition 4.3 and $(A^*, (\alpha^{-1})^*, \mathrm{ad}^\star, -R^\star)$ a representation of a Hom-pre-Lie algebra $(A, \cdot, \alpha)$. For all $\xi, \eta, \delta \in A^*$, we have

$$
\begin{aligned}
&(\xi \circ \eta) \circ (\alpha^{-1})^*(\delta) - (\alpha^{-1})^*(\xi) \circ (\eta \circ \delta) - (\eta \circ \xi) \circ (\alpha^{-1})^*(\delta) + (\alpha^{-1})^*(\eta) \circ (\xi \circ \delta) \\
&= \mathrm{ad}^\star_{r^\sharp(\xi\circ\eta)}(\alpha^{-1})^*(\delta) - R^\star_{r^\sharp((\alpha^{-1})^*(\delta))}(\mathrm{ad}^\star_{r^\sharp(\xi)}\eta - R^\star_{r^\sharp(\eta)}\xi) \\
&\quad - \mathrm{ad}^\star_{r^\sharp((\alpha^{-1})^*(\xi))}(\mathrm{ad}^\star_{r^\sharp(\eta)}\delta - R^\star_{r^\sharp(\delta)}\eta) + R^\star_{r^\sharp(\eta\circ\delta)}(\alpha^{-1})^*(\xi) \\
&\quad - \mathrm{ad}^\star_{r^\sharp(\eta\circ\xi)}(\alpha^{-1})^*(\delta) + R^\star_{r^\sharp((\alpha^{-1})^*(\delta))}(\mathrm{ad}^\star_{r^\sharp(\eta)}\xi - R^\star_{r^\sharp(\xi)}\eta) \\
&\quad + \mathrm{ad}^\star_{r^\sharp((\alpha^{-1})^*(\eta))}(\mathrm{ad}^\star_{r^\sharp(\xi)}\delta - R^\star_{r^\sharp(\delta)}\xi) - R^\star_{r^\sharp(\xi\circ\delta)}(\alpha^{-1})^*(\eta) \\
&= \mathrm{ad}^\star_{r^\sharp(\xi\circ\eta)}(\alpha^{-1})^*(\delta) - \mathrm{ad}^\star_{r^\sharp(\eta\circ\xi)}(\alpha^{-1})^*(\delta) - \mathrm{ad}^\star_{\alpha(r^\sharp(\xi))}\mathrm{ad}^\star_{r^\sharp(\eta)}\delta + \mathrm{ad}^\star_{\alpha(r^\sharp(\eta))}\mathrm{ad}^\star_{r^\sharp(\xi)}\delta \\
&\quad - R^\star_{\alpha(r^\sharp(\delta))}L^\star_{r^\sharp(\xi)}\eta + L^\star_{\alpha(r^\sharp(\xi))}R^\star_{r^\sharp(\delta)}\eta - R^\star_{\alpha(r^\sharp(\xi))}R^\star_{r^\sharp(\delta)}\eta - R^\star_{r^\sharp(\xi\circ\delta)}(\alpha^{-1})^*(\eta) \\
&\quad + R^\star_{\alpha(r^\sharp(\delta))}L^\star_{r^\sharp(\eta)}\xi - L^\star_{\alpha(r^\sharp(\eta))}R^\star_{r^\sharp(\delta)}\xi + R^\star_{\alpha(r^\sharp(\eta))}R^\star_{r^\sharp(\delta)}\xi + R^\star_{r^\sharp(\eta\circ\delta)}(\alpha^{-1})^*(\xi) \\
&= (\mathrm{ad}^\star_{r^\sharp(\xi\circ\eta)} - \mathrm{ad}^\star_{r^\sharp(\eta\circ\xi)} - \mathrm{ad}^\star_{r^\sharp(\xi)\cdot r^\sharp(\eta) - r^\sharp(\eta)\cdot r^\sharp(\xi)})((\alpha^{-1})^*(\delta)) \\
&\quad + (R^\star_{r^\sharp(\xi)\cdot r^\sharp(\delta)} - R^\star_{r^\sharp(\xi\circ\delta)})((\alpha^{-1})^*(\eta)) - (R^\star_{r^\sharp(\eta)\cdot r^\sharp(\delta)} - R^\star_{r^\sharp(\eta\circ\delta)})((\alpha^{-1})^*(\xi)).
\end{aligned}
$$

For all $x \in A$, we have

$$
\begin{aligned}
&\langle (\xi\circ\eta)\circ(\alpha^{-1})^*(\delta) - (\alpha^{-1})^*(\xi)\circ(\eta\circ\delta) - (\eta\circ\xi)\circ(\alpha^{-1})^*(\delta) + (\alpha^{-1})^*(\eta)\circ(\xi\circ\delta), x\rangle \\
&= -\langle \mathrm{ad}^\star_{r^\sharp(\xi)\cdot r^\sharp(\eta) - r^\sharp(\xi\circ\eta)}(\alpha^{-1})^*(\delta), x\rangle + \langle \mathrm{ad}^\star_{r^\sharp(\eta)\cdot r^\sharp(\xi) - r^\sharp(\eta\circ\xi))}(\alpha^{-1})^*(\delta), x\rangle \\
&\quad + \langle R^\star_{r^\sharp(\xi)\cdot r^\sharp(\delta) - r^\sharp(\xi\circ\delta))}(\alpha^{-1})^*(\eta), x\rangle - \langle R^\star_{r^\sharp(\eta)\cdot r^\sharp(\delta) - r^\sharp(\eta\circ\delta))}(\alpha^{-1})^*(\xi), x\rangle \\
&= -\langle \mathrm{ad}^*_{\alpha(r^\sharp(\xi))\cdot\alpha(r^\sharp(\eta)) - \alpha(r^\sharp(\xi\circ\eta))}(\alpha^{-3})^*(\delta), x\rangle + \langle \mathrm{ad}^*_{\alpha(r^\sharp(\eta))\cdot\alpha(r^\sharp(\xi)) - \alpha(r^\sharp(\eta\circ\xi))}(\alpha^{-3})^*(\delta), x\rangle \\
&\quad + \langle R^*_{\alpha(r^\sharp(\xi))\cdot\alpha(r^\sharp(\delta)) - \alpha(r^\sharp(\xi\circ\delta))}(\alpha^{-3})^*(\eta), x\rangle - \langle R^*_{\alpha(r^\sharp(\eta))\cdot\alpha(r^\sharp(\delta)) - \alpha(r^\sharp(\eta\circ\delta))}(\alpha^{-3})^*(\xi), x\rangle \\
&= -\langle \mathrm{ad}^*_{[[r,r]]((\alpha^{-2})^*(\xi),(\alpha^{-2})^*(\eta))}(\alpha^{-3})^*(\delta), x\rangle + \langle \mathrm{ad}^*_{[[r,r]]((\alpha^{-2})^*(\eta),(\alpha^{-2})^*(\xi))}(\alpha^{-3})^*(\delta), x\rangle \\
&\quad + \langle R^*_{[[r,r]]((\alpha^{-2})^*(\xi),(\alpha^{-2})^*(\delta))}(\alpha^{-3})^*(\eta), x\rangle - \langle R^*_{[[r,r]]((\alpha^{-2})^*(\eta),(\alpha^{-2})^*(\delta))}(\alpha^{-3})^*(\xi), x\rangle \\
&= 0.
\end{aligned}
$$

Thus, $(A^*, \circ, (\alpha^{-1})^*)$ is a Hom-pre-Lie algebra. ■

Proposition 4.5. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra and $\varphi^* : A \longrightarrow A\otimes A$ defined by* (29)*. If $r \in A\otimes A$ satisfies* (31) *and $(A^*, \circ, (\alpha^{-1})^*)$ is a Hom-pre-Lie algebra, where $\circ$ is given by* (32)*. Then ψ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra $(A^*)^C$ associated to the representation $(A^*\otimes A^*, \mathcal{L}^{-2}\otimes(\alpha^{-1})^* + (\alpha^{-1})^*\otimes\mathfrak{ad}^{-2})$ if and only if the following equation holds:*

$$(P(x\cdot y) - P(\alpha(x))P(y))(r - \sigma(r)) = 0, \tag{34}$$

where $P(x) = L_x^{-2}\otimes\alpha + \alpha\otimes L_x^{-2}$.

Proof. It follows from straightforward computations. ■

Definition 4.6. Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra. Assume that $r \in A\otimes A$ is symmetric and satisfies (31). Then the equation $[[r, r]] = 0$ is called the *Hom-$\mathfrak{s}$-equation* in $(A, \cdot, \alpha)$ and r is called a *Hom-$\mathfrak{s}$-matrix.* A *triangular Hom-pre-Lie bialgebra* is a coboundary Hom-pre-Lie bialgebra, in which r is a Hom-$\mathfrak{s}$-matrix.

Theorem 4.7. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra and r a Hom-$\mathfrak{s}$-matrix. Then $(A, A^*, \varphi^*, \psi^*)$ is a Hom-pre-Lie bialgebra, where $\varphi^* : A \longrightarrow A\otimes A$ is defined by* (29) *and ψ^* is the dual of the multiplication $\cdot$ in A.*

Proof. By Corollary 4.4, $(A^*, \circ, (\alpha^{-1})^*)$ is a Hom-pre-Lie algebra, where $\circ$ is given by (32). By Proposition 4.5, ψ^* is a 1-cocycle of the sub-adjacent Hom-Lie algebra $(A^*)^C$ associated to the representation

$$(A^*\otimes A^*, \mathcal{L}^{-2}\otimes(\alpha^{-1})^* + (\alpha^{-1})^*\otimes\mathfrak{ad}^{-2}).$$

By Proposition 4.2, $(A, A^*, \varphi^*, \psi^*)$ is a Hom-pre-Lie bialgebra. ■

5. Hom-$\mathcal{O}$-operators, Hom-L-dendriform algebras and Hom-$\mathfrak{s}$-matrices

In this section, we introduce Hom-$\mathcal{O}$-operators on Hom-pre-Lie algebras and Hom-L-dendriform algebras, by which we construct Hom-$\mathfrak{s}$-matrices.

Definition 5.1. Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra and (V, β, ρ, μ) be a representation of $(A, \cdot, \alpha)$. A linear map $T : V \longrightarrow A$ is called a *Hom-$\mathcal{O}$-operator* if for all $u, v \in V$, the following equalities are satisfied

$$
\begin{aligned}
T\circ\beta &= \alpha\circ T, &(35)\\
T(u)\cdot T(v) &= T(\rho(T(\beta^{-1}(u)))(v) + \mu(T(\beta^{-1}(v)))(u)). &(36)
\end{aligned}
$$

Proposition 5.2. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Lie algebra and $r\in A\otimes A$ is symmetric. Then r satisfies (31) and $[[r,r]]=0$ if and only if $r^\sharp\circ(\alpha^{-1})^*$ is a Hom-$\mathcal{O}$-operator associated to the representation $(A^*,(\alpha^{-1})^*,\mathrm{ad}^\star,-R^\star)$.*

Proof. Straightforward by Proposition 4.3. ■

Definition 5.3. A *Hom-L-dendriform algebra* $(A,\triangleright,\triangleleft,\alpha)$ is a vector space A equipped with two bilinear products $\triangleright,\triangleleft : A\otimes A\longrightarrow A$ and $\alpha\in\mathfrak{gl}(V)$, such that for all $x,y,z\in A$, $\alpha(x\triangleright y)=\alpha(x)\triangleright\alpha(y)$, $\alpha(x\triangleleft y)=\alpha(x)\triangleleft\alpha(y)$, and the following equalities are satisfied

$$(x\triangleright y)\triangleright\alpha(z)+(x\triangleleft y)\triangleright\alpha(z)+\alpha(y)\triangleright(x\triangleright z)$$
$$-(y\triangleleft x)\triangleright\alpha(z)-(y\triangleright x)\triangleright\alpha(z)-\alpha(x)\triangleright(y\triangleright z)=0, \quad (37)$$

$$(x\triangleright y)\triangleleft\alpha(z)+\alpha(y)\triangleleft(x\triangleright z)+\alpha(y)\triangleleft(x\triangleleft z)-(y\triangleleft x)\triangleleft\alpha(z)-\alpha(x)\triangleright(y\triangleleft z)=0. \quad (38)$$

Proposition 5.4. *Let $(A,\triangleright,\triangleleft,\alpha)$ be a Hom-L-dendriform algebra.*

(1) *The bilinear product $\bullet : A\otimes A\longrightarrow A$ given by*

$$x\bullet y=x\triangleright y+x\triangleleft y,\quad \forall x,y\in A, \quad (39)$$

defines a Hom-pre-Lie algebra. $(A,\bullet,\alpha)$ is called the associated horizontal Hom-pre-Lie algebra of $(A,\triangleright,\triangleleft,\alpha)$ and $(A,\triangleright,\triangleleft,\alpha)$ is called a compatible Hom-L-dendriform algebra structure on the Hom-pre-Lie algebra $(A,\bullet,\alpha)$.

(2) *The bilinear product $\cdot : A\otimes A\longrightarrow A$ given by*

$$x\cdot y=x\triangleright y-y\triangleleft x,\quad \forall x,y\in A, \quad (40)$$

defines a Hom-pre-Lie algebra. $(A,\cdot,\alpha)$ is called the associated vertical Hom-pre-Lie algebra of $(A,\triangleright,\triangleleft,\alpha)$ and $(A,\triangleright,\triangleleft,\alpha)$ is called a compatible Hom-L-dendriform algebra structure on the Hom-pre-Lie algebra $(A,\cdot,\alpha)$.

(3) *Both $(A,\bullet,\alpha)$ and $(A,\cdot,\alpha)$ have the same sub-adjacent Hom-Lie algebra A^C defined by*

$$[x,y]=x\triangleright y+x\triangleleft y-y\triangleleft x-y\triangleright x,\quad \forall x,y\in A. \quad (41)$$

Proof. Straightforward. ■

Proposition 5.5. *Assume that A is a vector space with two bilinear products $\triangleright,\triangleleft : A\otimes A\longrightarrow A$. Then*

(1) *$(A,\triangleright,\triangleleft,\alpha)$ is a Hom-L-dendriform algebra if and only if $(A,\bullet,\alpha)$ is a Hom-pre-Lie algebra and $(A,\alpha,L_\triangleright,R_\triangleleft)$ is a representation of $(A,\bullet,\alpha)$.*

(2) *$(A,\triangleright,\triangleleft,\alpha)$ is a Hom-L-dendriform algebra if and only if $(A,\cdot,\alpha)$ is a Hom-pre-Lie algebra and $(A,\alpha,L_\triangleright,-L_\triangleleft)$ is a representation of $(A,\cdot,\alpha)$.*

Proof. We only prove condition (1). If $(A,\triangleright,\triangleleft,\alpha)$ is a Hom-L-dendriform algebra, then, for all $x,y\in A$, we have

$$L_\triangleright(\alpha(x))\alpha(y)=\alpha(x)\triangleright\alpha(y)=\alpha(x\triangleright y)=\alpha(L_\triangleright(x)y), \quad (42)$$

which implies that $L_{\triangleright}(\alpha(x))\circ\alpha=\alpha\circ L_{\triangleright}(x)$. Similarly, we have $R_{\triangleleft}(\alpha(x))\circ\alpha=\alpha\circ R_{\triangleleft}(x)$. For all $x,y,z\in A$, by (37), we have

$$\begin{aligned}
&L_{\triangleright}([x,y])\alpha(z)-L_{\triangleright}(\alpha(x))L_{\triangleright}(y)z+L_{\triangleright}(\alpha(y))L_{\triangleright}(x)z\\
&=[x,y]\triangleright\alpha(z)-\alpha(x)\triangleright(y\triangleright z)+\alpha(y)\triangleright(x\triangleright z)\\
&=(x\triangleright y)\triangleright\alpha(z)+(x\triangleleft y)\triangleright\alpha(z)-(y\triangleright x)\triangleright\alpha(z)-(y\triangleleft x)\triangleright\alpha(z)\\
&\quad-\alpha(x)\triangleright(y\triangleright z)+\alpha(y)\triangleright(x\triangleright z)\\
&=\ 0,
\end{aligned}$$

which implies that

$$L_{\triangleright}([x,y])\circ\alpha=L_{\triangleright}(\alpha(x))L_{\triangleright}(y)-L_{\triangleright}(\alpha(y))L_{\triangleright}(x).$$

Similarly, we have

$$R_{\triangleleft}(\alpha(y))\circ R_{\triangleleft}(x)-R_{\triangleleft}(x\bullet y)\circ\alpha=R_{\triangleleft}(\alpha(y))\circ L_{\triangleright}(x)-L_{\triangleright}(\alpha(x))\circ R_{\triangleleft}(y).$$

Thus $(A,\alpha,L_{\triangleright},R_{\triangleleft})$ is a representation of the Hom-pre-Lie algebra $(A,\bullet,\alpha)$. The converse part can be proved similarly. We omit details. The proof is finished. ■

Proposition 5.6. *Let $(A,\triangleright,\triangleleft,\alpha)$ be a Hom-L-dendriform algebra. Define two bilinear products $\triangleright^t,\triangleleft^t:A\otimes A\longrightarrow A$ by*

$$x\triangleright^t y=x\triangleright y,\quad x\triangleleft^t y=-y\triangleleft x,\quad \forall x,y\in A. \tag{43}$$

Then $(A,\triangleright^t,\triangleleft^t,\alpha)$ is a Hom-L-dendriform algebra. The associated horizontal Hom-pre-Lie algebra of $(A,\triangleright^t,\triangleleft^t,\alpha)$ is the associated vertical Hom-pre-Lie algebra $(A,\cdot,\alpha)$ of $(A,\triangleright,\triangleleft,\alpha)$ and the associated vertical Hom-pre-Lie algebra of $(A,\triangleright^t,\triangleleft^t,\alpha)$ is the associated horizontal Hom-pre-Lie algebra $(A,\bullet,\alpha)$ of $(A,\triangleright,\triangleleft,\alpha)$, that is,

$$\bullet^t=\cdot,\quad \cdot^t=\bullet.$$

Proof. Straightforward. ■

Definition 5.7. Let $(A,\triangleright,\triangleleft,\alpha)$ be a Hom-L-dendriform algebra. The Hom-L-dendriform algebra $(A,\triangleright^t,\triangleleft^t,\alpha)$ given by (43) is called the transpose of $(A,\triangleright,\triangleleft,\alpha)$.

For brevity, we only give the study of vertical Hom-pre-Lie algebras.

Theorem 5.8. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Lie algebra and (V,β,ρ,μ) be a representation of $(A,\cdot,\alpha)$. Suppose that $T:V\longrightarrow A$ is a Hom-$\mathcal{O}$-operator. Then there exists a Hom-L-dendriform algebra structure on V defined by*

$$u\triangleright v=\rho(T(\beta^{-1}(u)))v,\quad u\triangleleft v=-\mu(T(\beta^{-1}(u)))v,\quad \forall u,v\in V. \tag{44}$$

Proof. First by $T\circ\beta=\alpha\circ T$ and (V,β,ρ,μ) is a representation of $(A,\cdot,\alpha)$, for all $u,v\in V$, we have

$$\begin{aligned}
\beta(u\triangleright v)&=\beta(\rho(T(\beta^{-1}(u)))v)=\rho(\alpha(T(\beta^{-1}(u))))\beta(v)=\rho(T(u))\beta(v)\\
&=\beta(u)\triangleright\beta(v).
\end{aligned}\tag{45}$$

Similarly, we have $\beta(u\triangleleft v)=\beta(u)\triangleleft\beta(v)$. Furthermore, by (36) and (V,β,ρ,μ) being a representation of $(A,\cdot,\alpha)$, for all $u,v,w\in V$, we have

$$\begin{aligned}
&(u\triangleright v)\triangleright\beta(w)+(u\triangleleft v)\triangleright\beta(w)+\beta(v)\triangleright(u\triangleright w)-(v\triangleleft u)\triangleright\beta(w)-(v\triangleright u)\triangleright\beta(w)\\
&\quad-\beta(u)\triangleright(v\triangleright w)\\
&=\rho(T(\beta^{-1}(u)))v\triangleright\beta(w)-\mu(T(\beta^{-1}(u)))v\triangleright\beta(w)+\beta(v)\triangleright\rho(T(\beta^{-1}(u)))w\\
&\quad+\mu(T(\beta^{-1}(v)))u\triangleright\beta(w)-\rho(T(\beta^{-1}(v)))u\triangleright\beta(w)-\beta(u)\triangleright\rho(T(\beta^{-1}(v)))w\\
&=\rho(T(\beta^{-1}(\rho(T(\beta^{-1}(u)))v)))\beta(w)-\rho(T(\beta^{-1}(\mu(T(\beta^{-1}(u)))v)))\beta(w)\\
&\quad+\rho(T(v))\rho(T(\beta^{-1}(u)))w+\rho(T(\beta^{-1}(\mu(T(\beta^{-1}(v)))u)))\beta(w)\\
&\quad-\rho(T(\beta^{-1}(\rho(T(\beta^{-1}(v)))u)))\beta(w)-\rho(T(u))\rho(T(\beta^{-1}(v)))w\\
&=\rho(\alpha^{-1}(T(u)\cdot T(v)))\beta(w)-\rho(\alpha^{-1}(T(v)\cdot T(u)))\beta(w)\\
&\quad+\rho(T(v))\rho(T(\beta^{-1}(u)))w-\rho(T(u))\rho(T(\beta^{-1}(v)))w\\
&=0,
\end{aligned}$$

which implies that (37) holds. Similarly, we have

$$(u\triangleright v)\triangleleft\beta(w)+\beta(v)\triangleleft(u\triangleright w)+\beta(v)\triangleleft(u\triangleleft w)-(v\triangleleft u)\triangleleft\beta(w)-\beta(u)\triangleright(v\triangleleft w)=0, \quad (46)$$

which implies that (38) holds. This finishes the proof. ■

Corollary 5.9. *With the above conditions. T is a homomorphism from the associated vertical Hom-pre-Lie algebra of $(V,\triangleright,\triangleleft,\beta)$ to the Hom-pre-Lie algebra $(A,\cdot,\alpha)$. Moreover, $T(V)=\{T(u)|u\in V\}\subset A$ is a Hom-pre-Lie subalgebra of $(A,\cdot,\alpha)$ and there is an induced Hom-L-dendriform algebra structure on $T(V)$ given by*

$$T(u)\triangleright T(v)=T(u\triangleright v),\quad T(u)\triangleleft T(v)=T(u\triangleleft v),\quad \forall u,v\in V. \quad (47)$$

Theorem 5.10. *Let $(A,\cdot,\alpha)$ be a Hom-pre-Lie algebra. Then there exists a compatible Hom-L-dendriform algebra structure on $(A,\cdot,\alpha)$ such that $(A,\cdot,\alpha)$ is the associated vertical Hom-pre-Lie algebra if and only if there exists an invertible Hom-$\mathcal{O}$-operator T associated to a representation (V,β,ρ,μ).*

Proof. Let T be an invertible Hom-$\mathcal{O}$-operator associated to a representation (V,β,ρ,μ). By Theorem 5.8, Corollary 5.9 and (35), there exists a Hom-L-dendriform algebra on $T(V)$ given by

$$x\triangleright y=T(\rho(T(\beta^{-1}(u)))T^{-1}(y))=T(\rho(\alpha^{-1}(x))T^{-1}(y)). \quad (48)$$

Similarly, we have $x\triangleleft y=-T(\mu(\alpha^{-1}(x))T^{-1}(y))$. Moreover, by (36), we have

$$x\triangleright y-y\triangleleft x=T(\rho(\alpha^{-1}(x))T^{-1}(y)+\mu(\alpha^{-1}(y))T^{-1}(x))=x\cdot y. \quad (49)$$

Conversely, Let $(A,\triangleright,\triangleleft,\alpha)$ be a Hom-L-dendriform algebra and $(A,\cdot,\alpha)$ be the associated vertical Hom-pre-Lie algebra. By Theorem 5.5, $(A,\alpha,L_\triangleright,-L_\triangleleft)$ is a representation of $(A,\cdot,\alpha)$ and $\alpha:A\longrightarrow A$ is a Hom-$\mathcal{O}$-operator of $(A,\cdot,\alpha)$ associated to $(A,\alpha,L_\triangleright,-L_\triangleleft)$. ■

In the sequel, we give the relation between Hessian structures and Hom-L-dendriform algebras.

Definition 5.11. ([8]) A Hessian structure on a regular Hom-pre-Lie algebra $(A, \cdot, \alpha)$ is a symmetric nondegenerate 2-cocycle $\mathcal{B} \in Sym^2(A^*)$, i.e., $\partial_T \mathcal{B} = 0$, satisfying $\mathcal{B} \circ (\alpha \otimes \alpha) = \mathcal{B}$. More precisely, we have for all $x, y, z \in A$

$$\mathcal{B}(\alpha(x), \alpha(y)) = \mathcal{B}(x, y), \tag{50}$$

$$\mathcal{B}(x \cdot y, \alpha(z)) - \mathcal{B}(\alpha(x), y \cdot z) = \mathcal{B}(y \cdot x, \alpha(z)) - \mathcal{B}(\alpha(y), x \cdot z). \tag{51}$$

Let A be a vector space, for all $\mathcal{B} \in Sym^2(A^*)$, the linear map $\mathcal{B}^\sharp : A \longrightarrow A^*$ is given by

$$\langle \mathcal{B}^\sharp(x), y \rangle = \mathcal{B}(x, y), \quad \forall x, y \in A. \tag{52}$$

Theorem 5.12. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra with a Hessian structure $\mathcal{B}$. Then there exists a compatible Hom-L-dendriform algebra structure on $(A, \cdot, \alpha)$ given by*

$$\mathcal{B}(x \triangleright y, z) = -\mathcal{B}(y, [\alpha^{-1}(x), \alpha^{-2}(z)]), \;\; \mathcal{B}(x \triangleleft y, z) = -\mathcal{B}(y, \alpha^{-2}(z) \cdot \alpha^{-1}(x)), \tag{53}$$

for all $x, y, z \in A$.

Proof. By (50) and (52), we obtain $(\mathcal{B}^\sharp)^{-1} \circ (\alpha^{-1})^* = \alpha \circ (\mathcal{B}^\sharp)^{-1}$. Thus, we have

$$(\mathcal{B}^\sharp)^{-1} \circ (\alpha^{-1})^* \circ (\alpha^{-1})^* = \alpha \circ (\mathcal{B}^\sharp)^{-1} \circ (\alpha^{-1})^*. \tag{54}$$

For all $x, y, z \in A$, $\xi, \eta, \gamma \in A^*$, setting $x = \alpha((\mathcal{B}^\sharp)^{-1}(\xi))$, $y = \alpha((\mathcal{B}^\sharp)^{-1}(\eta))$, and $z = \alpha((\mathcal{B}^\sharp)^{-1}(\gamma))$, we have

$$\begin{aligned}
&\langle (\mathcal{B}^\sharp)^{-1}((\alpha^{-1})^*(\xi)) \cdot (\mathcal{B}^\sharp)^{-1}((\alpha^{-1})^*(\eta)) - (\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\mathrm{ad}^\star((\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\alpha^*(\xi)))\eta \\
&\quad - R^\star((\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\alpha^*(\eta)))\xi), (\alpha^{-2})^*(\gamma) \rangle \\
&= \langle x \cdot y - (\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\mathrm{ad}^\star_{(\mathcal{B}^\sharp)^{-1}(\xi)}\eta - R^\star_{(\mathcal{B}^\sharp)^{-1}(\eta)}\xi), (\alpha^{-2})^*(\gamma) \rangle \\
&= \langle x \cdot y, \mathcal{B}^\sharp(\alpha(z)) \rangle - \langle \mathrm{ad}^\star_{\alpha^{-1}(x)}\mathcal{B}^\sharp(\alpha^{-1}(y)) - R^\star_{\alpha^{-1}(y)}\mathcal{B}^\sharp(\alpha^{-1}(x)), z \rangle \\
&= \mathcal{B}(\alpha(z), x \cdot y) + \langle \mathcal{B}^\sharp(\alpha^{-1}(y)), [\alpha^{-2}(x), \alpha^{-2}(z)] \rangle - \langle \mathcal{B}^\sharp(\alpha^{-1}(x)), \alpha^{-2}(z) \cdot \alpha^{-2}(y) \rangle \\
&= \mathcal{B}(\alpha(z), x \cdot y) + \mathcal{B}([x, z], \alpha(y)) - \mathcal{B}(\alpha(x), z \cdot y) \\
&= 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(\mathcal{B}^\sharp)^{-1}((\alpha^{-1})^*(\xi)) \cdot (\mathcal{B}^\sharp)^{-1}((\alpha^{-1})^*(\eta)) \\
&\quad = (\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\mathrm{ad}^\star((\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\alpha^*(\xi)))\eta - R^\star((\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\alpha^*(\eta)))\xi).
\end{aligned} \tag{55}$$

By (54) and (55), we deduce that $(\mathcal{B}^\sharp)^{-1} \circ (\alpha^{-1})^*$ is a Hom-$\mathcal{O}$-operator associated to the representation $(A^*, (\alpha^{-1})^*, \mathrm{ad}^\star, -R^\star)$. By Theorem 5.10, there is a compatible Hom-L-dendriform algebra structure on A defined by

$$\begin{aligned}
\mathcal{B}(x \triangleright y, z) &= \mathcal{B}((\mathcal{B}^\sharp)^{-1}(\alpha^{-1})^*(\mathrm{ad}^\star_{\alpha^{-1}(x)}\alpha^*(\mathcal{B}^\sharp(y))), z) = \langle \mathrm{ad}^\star_{\alpha^{-1}(x)}\alpha^*(\mathcal{B}^\sharp(y)), \alpha^{-1}(z) \rangle \\
&= -\langle \mathcal{B}^\sharp(y), [\alpha^{-1}(x), \alpha^{-2}(z)] \rangle = -\mathcal{B}(y, [\alpha^{-1}(x), \alpha^{-2}(z)]).
\end{aligned}$$

Similarly, we have $\mathcal{B}(x \triangleleft y, z) = -\mathcal{B}(y, \alpha^{-2}(z) \cdot \alpha^{-1}(x))$. The proof is finished. ■

Next we consider the semi-direct product Hom-pre-Lie algebra $A \ltimes_{(\rho^\star-\mu^\star,-\mu^\star)} V^*$. Any linear map $T : V \longrightarrow A$ can be viewed as an element $\bar{T} \in \otimes^2(A \oplus V^*)$ via

$$\bar{T}(\xi + u, \eta + v) = \langle T(u), \eta\rangle, \quad \forall \xi + u, \eta + v \in A^* \oplus V, \tag{56}$$

then $r = \bar{T} + \sigma(\bar{T})$ is symmetric.

Theorem 5.13. *Let $(A, \cdot, \alpha)$ be a Hom-pre-Lie algebra, (V, β, ρ, μ) be a representation of $(A, \cdot, \alpha)$ and $T : V \longrightarrow A$ be a linear map satisfying $T \circ \beta = \alpha \circ T$. Then $r = \bar{T} + \sigma(\bar{T})$ is a Hom-$\mathfrak{s}$-matrix in the Hom-pre-Lie algebra $A \ltimes_{(\rho^\star-\mu^\star,-\mu^\star)} V^*$ if and only if $T \circ \beta$ is a Hom-$\mathcal{O}$-operator.*

Proof. Let $\{v_1, \dots, v_n\}$ be a basis of V and $\{v_1^*, \dots, v_n^*\}$ be its dual basis. It is obvious that $\bar{T}$ can be expressed by $\bar{T} = T(v_i) \otimes v_i^*$. Here the Einstein summation convention is used. Therefore, we can write $r = T(v_i)\otimes v_i^* + v_i^* \otimes T(v_i)$. Then we have

$$\begin{aligned}
r_{12} \cdot r_{23} = &-\alpha(T(v_i)) \otimes \mu^\star(T(v_j))v_i^* \otimes (\beta^{-1})^*(v_j^*)\\
&+ (\beta^{-1})^*(v_i^*) \otimes T(v_i) \cdot T(v_j) \otimes (\beta^{-1})^*(v_j^*)\\
&+ (\beta^{-1})^*(v_i^*) \otimes \rho^\star(T(v_i))v_j^* \otimes \alpha(T(v_j))\\
&- (\beta^{-1})^*(v_i^*) \otimes \mu^\star(T(v_i))v_j^* \otimes \alpha(T(v_j)),\\
-r_{23} \cdot r_{12} = &-\alpha(T(v_j)) \otimes \rho^\star(T(v_i))v_j^* \otimes (\beta^{-1})^*(v_i^*)\\
&+ \alpha(T(v_j)) \otimes \mu^\star(T(v_i))v_j^* \otimes (\beta^{-1})^*(v_i^*)\\
&- (\beta^{-1})^*(v_j^*) \otimes T(v_i) \cdot T(v_j) \otimes (\beta^{-1})^*(v_i^*)\\
&+ (\beta^{-1})^*(v_j^*) \otimes \mu^\star(T(v_j))v_i^* \otimes \alpha(T(v_i)),\\
-r_{13} \cdot r_{12} = &-T(v_i) \cdot T(v_j) \otimes (\beta^{-1})^*(v_j^*) \otimes (\beta^{-1})^*(v_i^*)\\
&- \rho^\star(T(v_i))v_j^* \otimes \alpha(T(v_j)) \otimes (\beta^{-1})^*(v_i^*)\\
&+ \mu^\star(T(v_i))v_j^* \otimes \alpha(T(v_j)) \otimes (\beta^{-1})^*(v_i^*)\\
&+ \mu^\star(T(v_j))v_i^* \otimes (\beta^{-1})^*(v_j^*) \otimes \alpha(T(v_i)),\\
r_{13} \cdot r_{23} = &-\alpha(T(v_i)) \otimes (\beta^{-1})^*(v_j^*) \otimes \mu^\star(T(v_j))v_i^*\\
&+ (\beta^{-1})^*(v_i^*) \otimes \alpha(T(v_j)) \otimes \rho^\star(T(v_i))v_j^*\\
&- (\beta^{-1})^*(v_i^*) \otimes \alpha(T(v_j)) \otimes \mu^\star(T(v_i))v_j^*\\
&+ (\beta^{-1})^*(v_i^*) \otimes (\beta^{-1})^*(v_j^*) \otimes T(v_i) \cdot T(v_j).
\end{aligned}$$

By direct computation, we have

$$\begin{aligned}
[[r, r]] = &\langle(\beta^{-1})^*(v_i^*), v_m\rangle v_m^* \otimes T(v_i) \cdot T(v_j) \otimes \langle(\beta^{-1})^*(v_j^*), v_n\rangle v_n^*\\
&+ \langle(\beta^{-1})^*(v_i^*), v_m\rangle v_m^* \otimes \langle\rho^\star(T(v_i))v_j^*, v_n\rangle v_n^* \otimes \alpha(T(v_j))\\
&- \alpha(T(v_j)) \otimes \langle\rho^\star(T(v_i))v_j^*, v_m\rangle v_m^* \otimes \langle(\beta^{-1})^*(v_i^*), v_n\rangle v_n^*\\
&- \langle(\beta^{-1})^*(v_j^*), v_m\rangle v_m^* \otimes T(v_i) \cdot T(v_j) \otimes \langle(\beta^{-1})^*(v_i^*), v_n\rangle v_n^*\\
&- T(v_i) \cdot T(v_j) \otimes \langle(\beta^{-1})^*(v_j^*), v_m\rangle v_m^* \otimes \langle(\beta^{-1})^*(v_i^*), v_n\rangle v_n^*\\
&- \langle\rho^\star(T(v_i))v_j^*, v_m\rangle v_m^* \otimes \alpha(T(v_j)) \otimes \langle(\beta^{-1})^*(v_i^*), v_n\rangle v_n^*\\
&+ \langle\mu^\star(T(v_i))v_j^*, v_m\rangle v_m^* \otimes \alpha(T(v_j)) \otimes \langle(\beta^{-1})^*(v_i^*), v_n\rangle v_n^*
\end{aligned}$$

$$
\begin{aligned}
&+\langle\mu^\star(T(v_j))v_i^*,v_m\rangle v_m^*\otimes\langle(\beta^{-1})^*(v_j^*),v_n\rangle v_n^*\otimes\alpha(T(v_i))\\
&-\alpha(T(v_i))\otimes\langle(\beta^{-1})^*(v_j^*),v_m\rangle v_m^*\otimes\langle\mu^\star(T(v_j))v_i^*,v_n\rangle v_n^*\\
&+\langle(\beta^{-1})^*(v_i^*),v_m\rangle v_m^*\otimes\alpha(T(v_j))\otimes\langle\rho^\star(T(v_i))v_j^*,v_n\rangle v_n^*\\
&-\langle(\beta^{-1})^*(v_i^*),v_m\rangle v_m^*\otimes\alpha(T(v_j))\otimes\langle\mu^\star(T(v_i))v_j^*,v_n\rangle v_n^*\\
&+\langle(\beta^{-1})^*(v_i^*),v_m\rangle v_m^*\otimes\langle(\beta^{-1})^*(v_j^*),v_n\rangle v_n^*\otimes T(v_i)\cdot T(v_j)\\
=\ &v_m^*\otimes\langle v_i^*,\beta^{-1}(v_m)\rangle\langle v_j^*,\beta^{-1}(v_n)\rangle T(v_i)\cdot T(v_j)\otimes v_n^*\\
&-v_m^*\otimes v_n^*\otimes\langle v_i^*,\beta^{-1}(v_m)\rangle\langle v_j^*,\rho(T(\beta^{-1}(v_i)))\beta^{-2}(v_n)\rangle\alpha(T(v_j))\\
&+\langle v_j^*,\rho(T(\beta^{-1}(v_i)))\beta^{-2}(v_m)\rangle\langle v_i^*,\beta^{-1}(v_n)\rangle\alpha(T(v_j))\otimes v_m^*\otimes v_n^*\\
&-v_m^*\otimes\langle v_j^*,\beta^{-1}(v_m)\rangle\langle v_i^*,\beta^{-1}(v_n)\rangle T(v_i)\cdot T(v_j)\otimes v_n^*\\
&-\langle v_j^*,\beta^{-1}(v_m)\rangle\langle v_i^*,\beta^{-1}(v_n)\rangle T(v_i)\cdot T(v_j)\otimes v_m^*\otimes v_n^*\\
&+v_m^*\otimes\langle v_j^*,\rho(T(\beta^{-1}(v_i)))\beta^{-2}(v_m)\rangle\langle v_i^*,\beta^{-1}(v_n)\rangle\alpha(T(v_j))\otimes v_n^*\\
&-v_m^*\otimes\langle v_j^*,\mu(T(\beta^{-1}(v_i)))\beta^{-2}(v_m)\rangle\langle v_i^*,\beta^{-1}(v_n)\rangle\alpha(T(v_j))\otimes v_n^*\\
&-v_m^*\otimes v_n^*\otimes\langle v_i^*,\mu(T(\beta^{-1}(v_j)))\beta^{-2}(v_m)\rangle\langle v_j^*,\beta^{-1}(v_n)\rangle\alpha(T(v_i))\\
&+\langle v_j^*,\beta^{-1}(v_m)\rangle\langle v_i^*,\mu(T(\beta^{-1}(v_j)))\beta^{-2}(v_n)\rangle\alpha(T(v_i))\otimes v_m^*\otimes v_n^*\\
&-v_m^*\otimes\langle v_i^*,\beta^{-1}(v_m)\rangle\langle v_j^*,\rho(T(\beta^{-1}(v_i)))\beta^{-2}(v_n)\rangle\alpha(T(v_j))\otimes v_n^*\\
&+v_m^*\otimes\langle v_i^*,\beta^{-1}(v_m)\rangle\langle v_j^*,\mu(T(\beta^{-1}(v_i)))\beta^{-2}(v_n)\rangle\alpha(T(v_j))\otimes v_n^*\\
&+v_m^*\otimes v_n^*\otimes\langle v_i^*,\beta^{-1}(v_m)\rangle\langle v_j^*,\beta^{-1}(v_n)\rangle T(v_i)\cdot T(v_j)\\
=\ &v_m^*\otimes T(\beta^{-1}(v_m))\cdot T(\beta^{-1}(v_n))\otimes v_n^*\\
&-v_m^*\otimes v_n^*\otimes T\beta(\rho(T(\beta^{-2}(v_m)))\beta^{-2}(v_n))\\
&+T\beta(\rho(T(\beta^{-2}(v_n)))\beta^{-2}(v_m))\otimes v_m^*\otimes v_n^*\\
&-v_m^*\otimes T(\beta^{-1}(v_n))\cdot T(\beta^{-1}(v_m))\otimes v_n^*\\
&-T(\beta^{-1}(v_n))\cdot T(\beta^{-1}(v_m))\otimes v_m^*\otimes v_n^*\\
&+v_m^*\otimes T\beta(\rho(T(\beta^{-2}(v_n)))\beta^{-2}(v_m))\otimes v_n^*\\
&-v_m^*\otimes T\beta(\mu(T(\beta^{-2}(v_n)))\beta^{-2}(v_m))\otimes v_n^*\\
&-v_m^*\otimes v_n^*\otimes T\beta(\mu(T(\beta^{-2}(v_n)))\beta^{-2}(v_m))\\
&+T\beta(\mu(T(\beta^{-2}(v_m)))\beta^{-2}(v_n))\otimes v_m^*\otimes v_n^*\\
&-v_m^*\otimes T\beta(\rho(T(\beta^{-2}(v_m)))\beta^{-2}(v_n))\otimes v_n^*\\
&+v_m^*\otimes T\beta(\mu(T(\beta^{-2}(v_m)))\beta^{-2}(v_n))\otimes v_n^*\\
&+v_m^*\otimes v_n^*\otimes T(\beta^{-1}(v_m))\cdot T(\beta^{-1}(v_n))\\
=\ &v_m^*\otimes(T(\beta^{-1}(v_m))\cdot T(\beta^{-1}(v_n))-T\beta(\rho(T(\beta^{-2}(v_m)))\beta^{-2}(v_n)\\
&-\mu(T(\beta^{-2}(v_n)))\beta^{-2}(v_m))\otimes v_n^*+v_m^*\otimes v_n^*\otimes(T(\beta^{-1}(v_m))\cdot T(\beta^{-1}(v_n))\\
&-T\beta(\rho(T(\beta^{-2}(v_m)))\beta^{-2}(v_n)-\mu(T(\beta^{-2}(v_n)))\beta^{-2}(v_m))\\
&-(T(\beta^{-1}(v_n))\cdot T(\beta^{-1}(v_m))-T\beta(\rho(T(\beta^{-2}(v_n)))\beta^{-2}(v_m)\\
&-\mu(T(\beta^{-2}(v_m)))\beta^{-2}(v_n))\otimes v_m^*\otimes v_n^*-v_m^*\otimes(T(\beta^{-1}(v_n))\cdot T(\beta^{-1}(v_m))\\
&-T\beta(\rho(T(\beta^{-2}(v_n)))\beta^{-2}(v_m)-\mu(T(\beta^{-2}(v_m)))\beta^{-2}(v_n))\otimes v_n^*,
\end{aligned}
$$

which implies that $[[r,r]]=0$ if and only if for all $u,v\in V$,

$$T\beta(\beta^{-2}(u))\cdot T\beta(\beta^{-2}(v))$$
$$= T\beta(\rho(T\beta(\beta^{-1}(\beta^{-2}(u))))\beta^{-2}(v)+\mu(T\beta(\beta^{-1}(\beta^{-2}(v))))\beta^{-2}(u)). \qquad (57)$$

Furthermore, since $T\circ\beta=\alpha\circ T$, it is obvious that $T\circ\beta$ satisfies $(T\circ\beta)\circ\beta=\alpha\circ(T\circ\beta)$. Thus, r is a Hom-$\mathfrak{s}$-matrix in the Hom-pre-Lie algebra $A\ltimes_{(\rho^\star-\mu^\star,-\mu^\star)}V^*$ if and only if $T\circ\beta$ is a Hom-$\mathcal{O}$-operator. ■

Corollary 5.14. *Let $(A,\triangleright,\triangleleft,\alpha)$ be a Hom-L-dendriform algebra. Then $r=v_i\otimes v_i^*+v_i^*\otimes v_i$ is a Hom-$\mathfrak{s}$-matrix in the associated vertical Hom-pre-Lie algebra $A\ltimes_{(L_\triangleright^\star+L_\triangleleft^\star,L_\triangleleft^\star)}A^*$.*

Proof. By Proposition 5.5 and Proposition 3.1, $(A^*,(\alpha^{-1})^*,L_\triangleright^\star+L_\triangleleft^\star,L_\triangleleft^\star)$ is a dual representation of the associated vertical Hom-pre-Lie algebra $(A,\cdot,\alpha)$. Moreover, $\alpha=\mathrm{Id}\circ\alpha: \mathrm{A}\longrightarrow\mathrm{A}$ is a Hom-$\mathcal{O}$-operator associated to the representation $(A,\alpha,L_\triangleright,-L_\triangleleft)$. By Theorem 5.13, $r=v_i\otimes v_i^*+v_i^*\otimes v_i$ is a Hom-$\mathfrak{s}$-matrix. ■

References

[1] F. Ammar, Z. Ejbehi, A. Makhlouf: *Cohomology and deformations of Hom-algebras,* J. Lie Theory 21/4 (2011) 813–836.

[2] C. Bai: *Left-symmetric bialgebras and analogue of the Classical Yang-Baxter equation*, Commun. Contemp. Math. 10/2 (2008) 221–260.

[3] S. Benayadi, A. Makhlouf: *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms*, J. Geom. Phys. 76 (2014) 38–60.

[4] D. Burde: *Left-symmetric algebras and pre-Lie algebras in geometry and physics*, Cent. Eur. J. Math. 4 (2006) 323–357.

[5] L. Cai, Y. Sheng: *Purely Hom-Lie bialgebras*, Sci. China Math. 61/9 (2018) 1553–1566.

[6] O. Elchinger, K. Lundengard, A. Makhlouf, S. Silvestrov: *Brackets with (σ,τ)-derivations and (p,q)-deformations of Witt and Virasoro algebras*, Forum Math. 28/4 (2016) 657–673.

[7] J. Hartwig, D. Larsson, S. Silvestrov: *Deformations of Lie algebras using σ-derivations*, J. Algebra 295 (2006) 314–361.

[8] S. Liu, L. Song, T. Rong: *Representations and cohomologies of regular Hom-pre-Lie algebras*, J. Algebra Appl. 19/8 (2020), art. no. 2050149.

[9] A. Makhlouf, S. Silvestrov: *Hom-algebra structures*, J. Gen. Lie Theory Appl. 2/2 (2008) 51–64.

[10] A. Makhlouf, S. Silvestrov: *Hom-algebras and Hom-coalgebras*, J. Algebra Appl. 9/4 (2010) 1–37.

[11] Y. Sheng: *Representations of Hom-Lie algebras*, Algebras Representation Theory 15/6 (2012) 1081–1098.

[12] Y. Sheng, C. Bai: *A new approach to hom-Lie bialgebras*, J. Algebra 399 (2014) 232–250.

[13] Y. Sheng, D. Chen: *Hom-Lie 2-algebras*, J. Algebra 376 (2013) 174–195.

[14] Y. Sheng, Z. Xiong: *On hom-Lie algebras*, Linear Multilinear Algebra 63/12 (2015) 2379–2395.

[15] B. Sun, L. Chen, X. Zhou: *On universal α-central extensions of Hom-pre-Lie algebras*, arXiv:1810.09848 (2018).

[16] Q. Sun, H. Li: *On parakähler hom-Lie algebras and hom-left-symmetric bialgebras*, Comm. Algebra 45/1 (2017) 105–120.

[17] Y. Tao, C. Bai, L. Guo: *Another approach to Hom-Lie bialgebras via Manin triples*, Comm. Algebra 7 (2020) 3109–3132.

[18] D. Yau: *The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A 42/16 (2009) 165202, 12 pp.

[19] D. Yau: *The classical Hom-Yang-Baxter equation and Hom-Lie bialgebras*, Int. Electron. J. Algebra 17 (2015) 11–45.

[20] Q. Zhang, H. Yu, C. Wang: *Hom-Lie algebroids and hom-left-symmetric algebroids*, J. Geom. Physics 116 (2017) 187–203.

Shanshan Liu, Lina Song, Dept. of Mathematics, Jilin University, Changchun 130012, P. R. China; shanshan18@mails.jlu.edu.cn, songln@jlu.edu.cn.

Abdenacer Makhlouf, IRIMAS – Dép. de Mathématiques, University of Haute Alsace, Mulhouse, France; abdenacer.makhlouf@uha.fr.

Received February 9, 2020
and in final form July 27, 2020