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# On Pseudo-Riemannian Cyclic Homogeneous Manifolds of Dimension Four

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**Abstract.** We study the homogeneous structures of pseudo-Riemannian Lie groups of signature $(2,2)$. This study allows us to classify the cyclic Lie groups of the mentioned signature. We also study the cyclic pseudo-Riemannian homogeneous 4-manifolds with non-trivial isotropy.

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*Key Words:* Pseudo-Riemannian Lie groups, left-invariant cyclic metrics, homogeneous pseudo-Riemannian structures, non-trivial isotropy.

## 1. Introduction

Elie Cartan proved in [10] that a connected, simply connected and complete Riemannian manifold is a symmetric space if and only if the curvature stays constant under parallel translations. In 1958 Ambrose and Singer in [1] extended this theory and gave a characterization of homogeneous Riemannian manifolds. They proved that a connected, simply connected and complete Riemannian manifold $(M,g)$ is homogeneous if and only if there exists a tensor filed $S$ of type $(1,2)$ such that

$$\begin{aligned} &\text{(i)} \quad g(S_XY,Z) = -g(S_XZ,Y), \\ &\text{(ii)} \quad (\nabla_X R)_{YZ} = [S_X, R_{YZ}] - R_{S_XYZ} - R_{YS_XZ}, \\ &\text{(iii)} \quad (\nabla_X S)_Y = [S_X, S_Y] - S_{S_XY} \end{aligned}$$

for all $X,Y,Z \in \mathfrak{X}(M)$. Here $\nabla$ denotes the Levi-Civita connection and $R$ is the Riemannian curvature tensor of $(M,g)$.

In [21], Tricerri and Vanhecke classified Riemannian homogeneous structures into eight classes. Oubiña and Gadea defined in [14] a homogeneous pseudo-Riemannian structure on $(M,g)$ and they showed a characterization of reductive homogeneous pseudo-Riemannian manifolds. For the study of homogeneous pseudo-Riemannian structures, which has been developed successively, we may refer to the recent work [6]. Different classes were studied by several authors in more details, e.g., Kowalski and Tricerri in [18] studied more deeply the class $\mathcal{S}_2$ in dimensions less than or equal to four and the five dimensional case was considered later in [3]. In [13], Gadea and Gonález-Dávila and Oubiña gave some characterizations of cyclic and traceless cyclic

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homogeneous Riemannian manifolds and they classified simply connected cyclic homogeneous Riemannian manifolds of dimension less than or equal to four. In [12], same authors extend to the general case of the classification by Kowalski-Tricerri and Bieszk of connected and simply connected unimodular cyclic metric Lie groups for dimensions less than or equal to five [18, 3]. In the pseudo-Riemannian settings, Calvaruso and Castrillón-López [4, 5] obtained a full classification of three-dimensional homogeneous Lorentzian manifolds and so of three and four-dimensional cyclic Lorentzian metrics.

Recall that for any metric Lie group $(G, g)$, where $\mathfrak{g}$ denotes to its Lie algebra and $\nabla$ is the Levi-Civita connection, the tensor field $S_x y = \nabla_x y$, $x, y \in \mathfrak{g}$ will define a left-invariant homogeneous structure $S$ on $(G, g)$. We say that left-invariant pseudo-Riemannian metric $g$ is cyclic if the $g$-torsion of the $(-)$-connection of Cartan-Schouten $\tilde{\nabla} := \nabla - S$ is cyclic [12]. In other words, the homogeneous structure $S$ is of type $\mathcal{S}_1 \oplus \mathcal{S}_2$ in Tricerri and Vanhecke's classification [21] of homogeneous structures. Explicitly this means,

$$\mathfrak{S}_{x,y,z}\, g\left([x, y], z\right) = 0, \tag{1}$$

for all $x, y, z \in \mathfrak{g}$. Clearly, with respect to a basis $\{e_i\}$ for the Lie algebra $\mathfrak{g}$ of $G$, the cyclic condition is equivalent to requiring that

$$\mathfrak{S}^3_{i,j,k=1}\, g\left([e_i, e_j], e_k\right) = 0, \quad \text{for all indices } i, j, k. \tag{2}$$

This paper is organized in the following way. We remind some basic definitions and previous results which will be used through this study in Section 2. Cyclic pseudo-Riemannian four-dimensional examples of Lie groups with neutral signature are studied and completely classified in Section 3. Finally, the last section is devoted to the study of cyclic homogeneous pseudo-Riemannian spaces with nontrivial isotropy.

## 2. Preliminaries

Let $M$ be a connected smooth manifold of dimension $n \geq 2$. Let $g$ be a pseudo-Riemannian metric of signature $(k, n-k)$ on $M$, $\nabla$ be the Levi-Civita connection of $g$ and $R$ is the Riemann curvature tensor field. The following definition which was introduced by Gadea and Oubiña [14, 15], is basically used for the study of homogeneous pseudo-Riemannian manifolds.

**Definition 2.1.** [15] A *homogeneous pseudo-Riemannian structure* on the pseudo-Riemannian manifold $(M, g)$ is a tensor field $S$ of type $(1, 2)$ on $M$, such that the connection $\widetilde{\nabla} = \nabla - S$ satisfies

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0. \tag{3}$$

Let $V$ be a real vector space of dimension $n$ endowed with an inner product $\langle , \rangle$ of signature $(k, n-k)$. The $(V, \langle , \rangle)$ will be the model for each tangent space $T_x M$, $x \in M$, of reductive homogeneous pseudo-Riemannian manifolds of signature $(k, n-k)$. Instead of considering the tensors $S$ of $(1, 2)$ we prefer to work here with those of type $(0, 3)$ given by the isomorphism $S_{XYZ} = \langle S_X Y, Z \rangle$.

Clearly, the condition $\widetilde{\nabla} g = 0$ is equivalent to $S_{XYZ} = -S_{XZY}$.

Consider the vector space $\mathcal{S}(V)$, subspace of $\otimes^3 V^*$, determined by all the $(0,3)$-tensors on $(V, \langle , \rangle)$ having the same symmetries as a homogeneous structure, that is

$$\mathcal{S}(V) = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY}, X, Y, Z \in V\}.$$

The inner product of $V$ induces in a natural way an inner product in $\mathcal{S}(V)$, given by

$$\langle S, S' \rangle = \sum_{i,j,k=1}^{n} \varepsilon_i \varepsilon_j \varepsilon_k S_{e_i e_j e_k} S'_{e_i e_j e_k},$$

where $\{e_i\}$ is an orthonormal basis of $V$ and $\varepsilon_i = \langle e_i, e_i \rangle$.

**Theorem 2.2.** *If $dim V \geq 3$, then $\mathcal{S}(V)$ decomposes into the orthogonal direct sum of subspaces which are invariant and irreducible under the action of $O(k, n-k)$:*

$$\mathcal{S}(V) = \mathcal{S}_1(V) \oplus \mathcal{S}_2(V) \oplus \mathcal{S}_3(V), \quad \textit{where}$$

$$\mathcal{S}_1(V) = \{S \in \mathcal{S}(V) : S_{XYZ} = \langle X, Y \rangle \omega(Z) - \langle X, Z \rangle \omega(Y), \omega \in V^*\},$$

$$\mathcal{S}_2(V) = \{S \in \mathcal{S}(V) : \mathfrak{S}_{XYZ} S_{XYZ} = 0, c_{12}(S) := \sum_{i=1}^{n} \varepsilon_i S_{e_i e_i \cdot} = 0\},$$

$$\mathcal{S}_3(V) = \{S \in \mathcal{S}(V) : S_{XYZ} + S_{YXZ} = 0\}.$$

*Furthermore,*

$$\mathcal{S}_1(V) \oplus \mathcal{S}_2(V) = \{S \in \mathcal{S}(V) : \mathfrak{S}_{XYZ} S_{XYZ} = 0\},$$

$$\mathcal{S}_2(V) \oplus \mathcal{S}_3(V) = \{S \in \mathcal{S}(V) : c_{12}(S) = 0\},$$

$$\mathcal{S}_1(V) \oplus \mathcal{S}_3(V) = \left\{ S \in \mathcal{S}(V) : \begin{array}{c} S_{XYZ} + S_{YXZ} = 2\langle X, Y \rangle \omega(Z), \ \textit{and} \\ -\langle X, Z \rangle \omega(Y) - \langle Y, Z \rangle \omega(X), \ \omega \in V^* \end{array} \right\}$$

A left-invariant pseudo-Riemannian metric $g$ is said to be cyclic when admits a homogeneous pseudo-Riemannian structure $S$ of type $\mathcal{S}_1(V) \oplus \mathcal{S}_2(V)$.

Gadea and Oubiña proved in [15] for a two-dimensional vector space $V$, one has $\mathcal{S}(V) = \mathcal{S}_1(V)$. Consequently, any two-dimensional pseudo-Riemannian Lie group is cyclic.

In order to keep the paper self-contained, we bring some facts about three dimensional Lie groups. While Milnor in [19], studied three dimensional Riemannian Lie groups, Rahmani classified the pseudo-Riemannian counterpart, i.e., unimodular Lie groups equipped with a left-invariant Lorentzian metric of dimension three [20]. Cordero and Parker also studied three-dimensional Lie groups equipped with left-invariant Lorentzian metrics, in order to study the behavior of the Riemann curvature tensor [11]. Based on the studies [11, 20], Calvaruso studied three-dimensional homogeneous Lorentzian spaces in [4]. This study contains a classification theorem of three-dimensional Lie groups equipped with a left-invariant pseudo-Riemannian metric which we bring here.

**Theorem 2.3.** [4] *A three-dimensional connected, simply connected and complete homogeneous Lorentzian manifold $(M, g)$ is either symmetric, or $M = G$ is a three-dimensional Lie group and $g$ is left-invariant. Precisely, one of the following cases occurs:*

**(I):** *If $G$ is unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with $e_3$ time-like, such that the Lie algebra $\mathfrak{g}$ of $G$ is one of the following:*

$$\mathfrak{g}_1 : \begin{aligned} &[e_1, e_2] = \alpha e_1 - \beta e_3 \\ &[e_1, e_3] = -\alpha e_1 - \beta e_2 \\ &[e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \quad \alpha \neq 0 \end{aligned} \tag{4}$$

*In this case, $G = \widetilde{SL}(2, \mathbb{R})$ if $\beta \neq 0$, while $G = E(1,1)$ if $\beta = 0$ and $E(1,1)$ is the group of rigid motions of the Minkowski two-space.*

$$\mathfrak{g}_2 : \begin{aligned} &[e_1, e_2] = -\gamma e_1 - \beta e_3 \\ &[e_1, e_3] = -\beta e_2 + \gamma e_3, \quad \gamma \neq 0 \\ &[e_2, e_3] = \alpha e_1 \end{aligned} \tag{5}$$

*In this case, $G = \widetilde{SL}(2, \mathbb{R})$ if $\alpha \neq 0$, while $G = E(1,1)$ if $\alpha = 0$.*

$$\mathfrak{g}_3 : \begin{aligned} &[e_1, e_2] = -\gamma e_3 \\ &[e_1, e_3] = -\beta e_2, \\ &[e_2, e_3] = \alpha e_1. \end{aligned} \tag{6}$$

*According to different possibilities of the coefficients, Table 1 lists all the Lie groups $G$ which admit a Lie algebra $\mathfrak{g}_3$. Where, $\widetilde{E}(2)$ is the universal covering of the group of rigid motions in the Euclidian two-space and $H_3$ is the Heisenberg group.*

| Lie group | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| $\widetilde{SL}(2, \mathbb{R})$ | + | + | + |
| $\widetilde{SL}(2, \mathbb{R})$ | + | − | − |
| $SU(2)$ | + | + | − |
| $\widetilde{E}(2)$ | + | + | 0 |
| $\widetilde{E}(2)$ | + | 0 | − |
| $E(1,1)$ | + | − | 0 |
| $E(1,1)$ | + | 0 | + |
| $H_3$ | + | 0 | 0 |
| $H_3$ | 0 | 0 | − |
| $\mathbb{R} \bigoplus \mathbb{R} \bigoplus \mathbb{R}$ | 0 | 0 | 0 |

Table 1: 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_3$

$$\mathfrak{g}_4 : \begin{aligned} &[e_1, e_2] = -e_2 + (2\varepsilon - \beta) e_3, \quad \varepsilon = \pm 1, \\ &[e_1, e_3] = -\beta e_2 + e_3, \\ &[e_2, e_3] = \alpha e_1. \end{aligned} \tag{7}$$

*Table 2 describes all Lie groups $G$ admitting a Lie algebra $\mathfrak{g}_4$.*

**(II):** *If $G$ is non-unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with $e_3$ time-like, such that the Lie algebra of $G$ is one of the following:*

$$\mathfrak{g}_5 : \begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, e_3] = \alpha e_1 + \beta e_2, \\ &[e_2, e_3] = \gamma e_1 + \delta e_2, \quad \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0. \end{aligned} \tag{8}$$

| Lie group ($\varepsilon = 1$) | $\alpha$ | $\beta$ |
|---|---|---|
| $\widetilde{SL}(2,\mathbb{R})$ | $\neq 0$ | $\neq 1$ |
| $E(1,1)$ | $0$ | $\neq 1$ |
| $E(1,1)$ | $< 0$ | $1$ |
| $\widetilde{E}(2)$ | $> 0$ | $1$ |
| $H_3$ | $0$ | $1$ |
| Lie group ($\varepsilon = -1$) | $\alpha$ | $\beta$ |
| $\widetilde{SL}(2,\mathbb{R})$ | $\neq 0$ | $\neq -1$ |
| $E(1,1)$ | $0$ | $\neq -1$ |
| $E(1,1)$ | $> 0$ | $-1$ |
| $\widetilde{E}(2)$ | $< 0$ | $-1$ |
| $H_3$ | $0$ | $-1$ |

Table 2: 4D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_4$

$$\mathfrak{g}_6 : \begin{aligned} &[e_1, e_2] = \alpha e_2 + \beta e_3, \\ &[e_1, e_3] = \gamma e_2 + \delta e_3, \\ &[e_2, e_3] = 0, \qquad\qquad \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0. \end{aligned} \tag{9}$$

$$\mathfrak{g}_7 : \begin{aligned} &[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \\ &[e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \\ &[e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \quad \alpha + \delta \neq 0, \alpha\gamma = 0. \end{aligned} \tag{10}$$

Based on the above classification of Lorentzian Lie groups of dimension three, cyclic condition were investigated on three and four dimensional Lorentzian Lie groups in [5]. In this paper, we apply the above classification for study of neutral Lie groups of dimension four.

## 3. Four-dimensional cyclic neutral Lie groups

Lie groups of dimension four, as special classes of homogeneous manifolds, were deeply studied through Riemannian and pseudo-Riemannian geometries. In the Riemannian case, a classification of four-dimensional Riemannian Lie groups were studied by Berard-Bergery [2]. This classification of Riemannian Lie groups was used to study pseudo-Riemannian Lie groups of dimension four. In fact, Calvaruso and Zaeim showed that the class of Riemannian and pseudo-Riemannian Lie groups coincide and differences may arise from the signature of the left-invariant metric [7, 8].

**Proposition 3.1.** [8] *Every $n$-dimensional simply connected Lie group $G$ admit left-invariant metrics of any prescribed signature $(p, n-p)$. In particular, if $G$ is a four-dimensional simply connected Lie group equipped with a left-invariant metric (either Lorentzian or of neutral signature), then $G$ is a semi-direct product of $\mathbb{R}$ by a three-dimensional Lie group, also including in this description the case of directs of $\mathbb{R}$ by a non solvable Lie group. In fact, $G$ is one of the following cases:*

(i) *either one of the unsolvable direct products $\mathbb{R} \times SU(2)$ and $\mathbb{R} \times \widetilde{SL}(2,\mathbb{R})$; or*

(ii) *one of the following solvable Lie groups:*

(ii-1) *the non-trivial semi-direct products* $\mathbb{R} \ltimes E(2)$ *and* $\mathbb{R} \ltimes E(1,1)$*;*

(ii-2) *the non-nilpotent semi-direct products* $\mathbb{R} \ltimes H$*, where* $H$ *denotes the Heisenberg group;*

(ii-3) *the semi-direct products* $\mathbb{R} \ltimes \mathbb{R}^3$*.*

Due to the above proposition 3.1, the Lie algebra $\mathfrak{g}$ of $G$ is a semi-direct product of the Lie algebras $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, where $\mathfrak{r}$ is a one-dimensional Lie algebra, which acts as a derivation (possibly in a trivial way) on a three-dimensional unimodular Lie algebra $\mathfrak{h}$.

Since a semi-direct product $\widetilde{\mathfrak{h}} \rtimes \mathfrak{r}$, with $\widetilde{\mathfrak{h}}$ non-unimodular, is also isomorphic to a semi-direct product $\mathfrak{h} \rtimes \widetilde{\mathfrak{r}}$, with $\mathfrak{h}$ unimodular, so semi-direct products with a three-dimensional non-unimodular Lie algebra do not explicitly appear [5].

As mentioned above, the class of simply connected four-dimensional Lie groups of neutral signature coincides with the class of simply connected four-dimensional Riemannian Lie groups, but clearly if $g$ is a positive definite inner product on $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, the same is true for its restriction $g|_{\mathfrak{h}}$ over $\mathfrak{h}$. However, if $g$ is with neutral signature, then two different cases may occur, as $g|_{\mathfrak{h}}$ is either Lorentzian or degenerate. In particular, since $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, the restriction to $\mathfrak{h}$ is either: (a) of signature $(2,1)$, $(a^{'})$of signature $(1,2)$, or (b) degenerate
The first two cases are equivalent to each other, up to reversing the metric. We now give the following key result.

**Proposition 3.2.** [8] *Let* $\mathfrak{g}$ *denote any four-dimensional pseudo-Riemannian Lie algebra and* $g$ *be an inner product on* $\mathfrak{g}$ *of signature* $(2,2)$*. There exists a basis* $\{e_1, e_2, e_3, e_4\}$ *of* $\mathfrak{g}$*, such that* $\mathfrak{h} = \mathrm{span}\{e_1, e_2, e_3\}$ *is a there-dimensional lie algebra and* $e_4$ *acts as a derivation on* $\mathfrak{h}$ *(that is,* $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$*, where* $\mathfrak{r} = \mathrm{span}\{e_4\}$*), and with respect to* $\{e_1, e_2, e_3, e_4\}$*,* $g$ *takes one of the following forms:*

$$(a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

By this proposition, it would be sufficient to consider the following two cases:

(1) $g|_{\mathfrak{h}}$ is Lorentzian and the time-like vector $e_4$ acts as a derivation on $\mathfrak{h}$.

(2) $g|_{\mathfrak{h}}$ is degenerate and the light-like vector $e_4$ acts as a derivation on $\mathfrak{h}$.

In the following subsections we shall classify four-dimensional cyclic neutral Lie groups.

### 3.1. $g|_{\mathfrak{h}}$ is Lorentzian

The restriction of the left-invariant metric $g$ on $\mathfrak{h}$ is Lorentzian, so that the metric $g$ is described by the neutral inner product as in the case (a) of the Proposition 3.2. Consider that the time-like vector $e_4$ acts as a dervation on $\mathfrak{h}$, we define, in general, the Lie brackets $[e_i, e_4]$, $1 \leq i \leq 4$ in the following way

$$\begin{cases} [e_1, e_4] &= p_1 e_1 + p_2 e_2 + p_3 e_3, \\ [e_2, e_4] &= q_1 e_1 + q_2 e_2 + q_3 e_3, \\ [e_3, e_4] &= r_1 e_1 + r_2 e_2 + r_3 e_3. \end{cases} \tag{11}$$

for some constants $p_1, \cdots, p_3, q_1, \cdots q_3, r_1, \cdots r_3$, which in addition must satisfy the Jacobi identity

$$[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] = 0, \quad i, j, k = 1, ..., 4. \tag{12}$$

**Theorem 3.3.** *Let $G = H \rtimes \mathbb{R}$ be a connected and simply connected four-dimensional Lie group, equipped with a left-invariant metric $g$ of neutral signature, where $g|_{\mathfrak{h}}$ is Lorentzian. With respect to the basis $\{e_1, \cdots, e_4\}$ for the Lie algebra $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, $(G, g)$ is cyclic if and only if one the following cases occurs:*

(I) $\mathfrak{h} = \mathfrak{g}_1$ *with $\beta = 0$, in this case $G$ is isometric to $E(1,1) \rtimes \mathbb{R}$ with the following brackets:*

$$[e_1, e_2] = \alpha e_1, \quad [e_1, e_3] = -\alpha e_1, \quad [e_1, e_4] = A e_1,$$
$$[e_2, e_3] = \alpha e_2 + \alpha e_3, \quad [e_2, e_4] = B e_2 + B e_3, \quad [e_3, e_4] = -B e_2 - B e_3, \quad \alpha \neq 0.$$

(II) $\mathfrak{h} = \mathfrak{g}_2$ *and either $\alpha = \beta = 0$ with the following brackets:*

$$[e_1, e_2] = -\gamma e_2, \quad [e_1, e_3] = \gamma e_3, \quad [e_2, e_4] = A e_2,$$
$$[e_3, e_4] = B e_3, \quad \gamma \neq 0,$$

*or $\alpha = -2\beta \neq 0$ with the following brackets,*

$$[e_1, e_2] = -\gamma e_2 - \beta e_3, \quad [e_1, e_3] = -\beta e_2 + \gamma e_3, \quad [e_2, e_3] = -2\beta e_1 \quad \gamma \neq 0.$$

*In these cases, $G$ is isometric to $E(1,1) \rtimes \mathbb{R}$ and $\widetilde{SL}(2, \mathbb{R}) \rtimes \mathbb{R}$, respectively.*

(III) $\mathfrak{h} = \mathfrak{g}_3$ *and one of the following cases occur.*

(1) $\alpha + \beta + \gamma = 0$ *with the following brackets:*

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = -(\gamma + \beta) e_1.$$

*In this case $G$ is isometric to $\widetilde{SL}(2, \mathbb{R}) \rtimes \mathbb{R}$ or $SU(2) \rtimes \mathbb{R}$.*

(2) $\alpha + \beta = \gamma = 0$ *with the following brackets:*

$$[e_1, e_3] = -\beta e_2, \quad [e_1, e_4] = A e_1 + B e_2, \quad [e_2, e_3] = -\beta e_1,$$
$$[e_2, e_4] = B e_1 + A e_2.$$

*In this case $G$ is isometric to $E(1,1) \rtimes \mathbb{R}$.*

(3) $\alpha + \gamma = \beta = 0$ *with the following brackets:*

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_4] = A e_1 - B e_3, \quad [e_2, e_3] = -\gamma e_1,$$
$$[e_3, e_4] = B e_1 + A e_3.$$

*In this case $G$ is isometric to $\widetilde{E}(2) \rtimes \mathbb{R}$.*

(4) $\alpha = \gamma + \beta = 0$ *with the following brackets:*

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = \gamma e_2, \quad [e_2, e_4] = A e_2 - B e_3,$$
$$[e_3, e_4] = B e_2 + A e_3.$$

*In this case $G$ is isometric to $\widetilde{E}(2) \rtimes \mathbb{R}$.*

(5) $\alpha = \beta = \gamma = 0$ *with the following brackets:*

$$[e_1, e_4] = Ae_1 + Be_2 - Ce_3, \quad [e_2, e_4] = Be_1 + De_2 - Ee_3,$$
$$[e_3, e_4] = Ce_1 + Ee_2 + Fe_3.$$

*In this case $G$ is isometric to $\mathbb{R}^3 \rtimes \mathbb{R}$.*

(IV) $\mathfrak{h} = \mathfrak{g}_4$ *and* $\alpha = -2(\beta - \varepsilon) \neq 0$ *with the following brackets,*

$$[e_1, e_2] = -e_2 + (2\varepsilon - \beta)e_3, \quad [e_1, e_3] = -\beta e_2 + e_3, \quad [e_2, e_3] = 2(\varepsilon - \beta)e_1,$$

*or* $\alpha = \beta - \varepsilon = 0$ *with the following brackets:*

$$[e_1, e_2] = -e_2 + \varepsilon e_3, \qquad [e_1, e_3] = -\varepsilon e_2 + e_3,$$
$$[e_1, e_4] = \varepsilon A e_2 - Ae_3, \qquad [e_2, e_4] = A\varepsilon e_1 + Be_2 - \tfrac{\varepsilon(B-C)}{2} e_3,$$
$$[e_3, e_4] = Ae_1 + \tfrac{\varepsilon(B-C)}{2} e_2 + Ce_3.$$

*In these cases, $G$ is isometric to $\widetilde{SL}(2, \mathbb{R}) \rtimes \mathbb{R}$ and $G = H_3 \times \mathbb{R}$ respectively.*

**Proof.** Let $(G, g)$ be a four dimensional Lie group with a left-invariant metric $g$ of type (a) of the Proposition 3.2. In this case, up to isometry, $\mathfrak{h}$ will be one of the unimodular Lorentzian Lie algebras $\mathfrak{g}_1 - \mathfrak{g}_4$ classified in Theorem 2.3:

**• $\mathfrak{h}$ is of type $\mathfrak{g}_1$:**

Since $\mathfrak{g} = \mathfrak{g}_1 \rtimes \mathfrak{r}$, the general form of the Lie algebra $\mathfrak{g}$ is

$$[e_1, e_2] = \alpha e_1 - \beta e_3, \qquad [e_1, e_3] = -\alpha e_1 - \beta e_2,$$
$$[e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \quad [e_1, e_4] = p_1 e_1 + p_2 e_2 + p_3 e_3,$$
$$[e_2, e_4] = q_1 e_1 + q_2 e_2 + q_3 e_3, \quad [e_3, e_4] = r_1 e_1 + r_2 e_2 + r_3 e_3.$$

Now, using the Jacobi identity (12), we have

$$\begin{cases} 2\alpha p_2 - \beta\,(p_1 - q_2 + r_3) = 0, & 2\alpha p_3 + \beta\,(p_1 + q_2 - r_3) = 0, \\ \alpha\,(r_3 - r_2) - \beta\,(p_2 + q_1) = 0, & \alpha\,(p_2 + p_3) + \beta\,(q_3 - r_2) = 0, \\ \alpha\,(q_2 - q_3) + \beta\,(r_1 - p_3) = 0, & 2\alpha\,(q_1 + r_1) + \beta\,(p_1 - q_2 - r_3) = 0. \end{cases} \tag{13}$$

By straightforward calculations, the cyclic condition (2) now gives

$$\beta = q_1 - p_2 = p_3 + r_1 = q_3 + r_2 = 0;$$

therefore the Jacobi equations lead to

$$\begin{cases} \alpha p_2 = 0, \quad \alpha p_3 = 0, \quad \alpha\,(p_2 - p_3) = 0, \\ \alpha\,(p_2 + p_3) = 0, \quad \alpha\,(q_2 - q_3) = 0, \quad \alpha\,(r_3 + q_3) = 0. \end{cases} \tag{14}$$

Since $\alpha \neq 0$, so $p_2 = p_3 = q_1 = r_1 = 0, q_2 = q_3 = -r_2 = -r_3$ and by setting $p_1 = A, q_2 = B$ we obtain the case (I) of the statement. By means of the Theorem 2.3, we have $\mathfrak{g} = \mathfrak{e}(1,1) \rtimes \mathfrak{r}$, where $\mathfrak{e}(1,1) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

**• $\mathfrak{h}$ is of type $\mathfrak{g}_2$**

By using equation (11) and applying the Jacobi identity (12) to the Lie algebra $\mathfrak{g} = \mathfrak{g}_2 \rtimes \mathfrak{r}$ we obtain

$$\begin{cases} \alpha p_3 - \beta r_1 - \gamma q_1 = 0, \quad \alpha p2 + \beta q_1 - \gamma r_1 = 0, \quad \alpha(p_1 - q_2 - r_3) = 0, \\ \beta(q_3 - r_2) + p_1\gamma = 0, \quad \beta(p_1 + q_2 - r_3) - 2\gamma q_3 = 0, \quad \beta(p_1 - q_2 + r_3) + 2\gamma r_2 = 0, \end{cases}$$

The cyclic condition (2) yields

$$\alpha + 2\beta = q_1 - p_2 = p_3 + r_1 = q_3 + r_2 = 0.$$

Summarizing the above two sets of equations we have the following solutions:

(1) $\{\alpha = \beta = 0, p_1 = p_2 = p_3 = q_1 = q_3 = r_1 = r_2 = 0\}$. If set $q_2 = A$ and $r_3 = B$, we have the first case of the statement (II). In this case, $\mathfrak{g} = \mathfrak{e}(1,1) \rtimes \mathfrak{r}$, where $\mathfrak{e}(1,1) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

(2) $\{\alpha = -2\beta, p_1 = p_2 = p_3 = q_1 = q_2 = q_3 = r_1 = r_2 = r_3 = 0\}$ which results in the second case of the statement (II). In this case, $\mathfrak{g} = \mathfrak{sl}(2) \rtimes \mathfrak{r}$ trivially, where $\mathfrak{sl}(2) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

- **$\mathfrak{h}$ is of type $\mathfrak{g}_3$:**

By using the equation (11) and applying the Jacobi identity (12) for the Lie algebra $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ have

$$\begin{cases} \alpha p_2 + \beta q_1 = 0, \quad \alpha p_3 - \gamma r_1 = 0, \quad \beta q_3 - \gamma r_2 = 0, \\ \gamma(r_3 - p_1 - q_2) = 0, \quad \alpha(p_1 - q_2 - r_3) = 0, \quad \beta(p_1 - q_2 + r_3) = 0. \end{cases}$$

Now, by straightforward computations, the cyclic condition (2) gives

$$\gamma + \alpha + \beta = q_1 - p_2 = p_3 + r_1 = q_3 + r_2 = 0.$$

Solving together the above two systems of equations yields the following five sets of solutions.

(1) $\{\alpha + \beta + \gamma = 0,\ p_1 = p_2 = p_3 = q_1 = q_2 = q_3 = r_1 = r_2 = r_3 = 0\}$. Thus we obtain the case (III-1) of the statement. In this case $\mathfrak{g} = \mathfrak{sl}(2) \rtimes \mathfrak{r}$ or $\mathfrak{g} = \mathfrak{su}(2) \rtimes \mathfrak{r}$, where $\mathfrak{sl}(2) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

(2) $\{\alpha + \beta = \gamma = 0,\ p_1 = q_2,\ p_2 = q_1,\ p_3 = q_3 = r_1 = r_2 = r_3 = 0\}$. If we set $p_1 = A,\ p_2 = B$ the case (III-2) of the statement will be deduced. By means of Theorem 2.3, $\mathfrak{g} = \mathfrak{e}(1,1) \rtimes \mathfrak{r}$, where $\mathfrak{e}(1,1) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

(3) $\{\alpha + \gamma = \beta = 0,\ p_1 - r_3 = p_3 + r_1 = p_2 = q_1 = q_2 = q_3 = r_2 = 0\}$. If we set $p_1 = A,\ r_1 = B$, we have the case (III-3) of the statement. Due to the Theorem 2.3, $\mathfrak{g} = \mathfrak{e}(2) \rtimes \mathfrak{r}$, where $\mathfrak{e}(2) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

(4) $\{\alpha = \beta + \gamma = 0,\ q_2 - r_3 = q_3 + r_2 = p_1 = p_2 = p_3 = q_1 = r_1 = 0\}$. If we set $q_2 = A, r_2 = B$, the case (III-4) of the statement will be deduced. According to the Theorem 2.3, $\mathfrak{g} = \mathfrak{e}(2) \rtimes \mathfrak{r}$, where $\mathfrak{e}(2) = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

(5) $\{\alpha = \beta = \gamma = 0,\ p_2 - q_1 = p_3 + r_1 = q_3 + r_2 = 0\}$. We set $p_1 = A,\ p_2 = B$, $r_1 = C,\ q_2 = D,\ r_2 = E,\ r_3 = F$ and we get the case (III-5) of the statement. By Theorem 2.3, $\mathfrak{g} = \mathfrak{r}^3 \rtimes \mathfrak{r}$, where $\mathfrak{r}^3 = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$.

- **$\mathfrak{h}$ is of type $\mathfrak{g}_4$:**

Since $\mathfrak{g} = \mathfrak{g}_4 \rtimes \mathfrak{r}$, so by using the equation (11) and applying the Jacobi identity (12) we obtain

$$\begin{cases} \alpha p_2 + \beta q_1 - r_1 = 0, \ \ \alpha(q_2 + r_3 - p_1) = 0, \ \ \beta(q_2 - p_1 - r_3) - 2r_2 = 0, \\ \alpha p_3 - \beta r_1 + 2\varepsilon r_1 - q_1 = 0, \quad \beta(q_3 - r_2) + p_1 + 2\varepsilon r_2 = 0, \\ \beta(p_1 - q_2 + r_3) + 2r_2 = 0, \quad (\beta - 2\varepsilon)(p_1 - r_3 + q_2) - 2q_3 = 0. \end{cases} \tag{15}$$

Also, the cyclic equation (2) gives

$$q_1 - p_2 = p_3 + r_1 = q_3 + r_2 = -2\varepsilon + 2\beta + \alpha = 0.$$

If we solve the equations of Jacobi together with the cyclic equations we get two sets of solutions as follows.

(1) $\{\alpha + 2\beta - 2\varepsilon = 0\}$, which gives the first case of the statement (IV). Due to Theorem 2.3, $\mathfrak{g} = \mathfrak{sl}(2) \rtimes \mathfrak{r}$, where $\mathfrak{sl}(2) = \mathrm{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \mathrm{span}\{e_4\}$.

(2) $\{\alpha = \beta - \varepsilon = 0,\ p_1 = p_3 + r_1 = p_2 - \varepsilon r_1 = q_1 - \varepsilon r_1 = q_3 + \frac{\varepsilon}{2}(q_2 - r_3) = r_2 - \frac{\varepsilon}{2}(q_2 - r_3) = 0\}$. If we set $r_1 = A$, $q_2 = B$, $r_3 = C$, we obtain the second case of the statement (IV). By Theorem 2.3, $\mathfrak{g} = \mathfrak{h}_3 \rtimes \mathfrak{r}$, where $\mathfrak{h}_3 = \mathrm{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \mathrm{span}\{e_4\}$. ■

### 3.2. $g|_{\mathfrak{h}}$ is degenerate

By means of the Proposition 3.2, in order to complete the study of four-dimensional Lie groups of neutral signature, we need to investigate the case where the restriction of the metric $g$ on $\mathfrak{h}$ is degenerate. In this case, the metric $g$ is determined by the case (b) of the proposition 3.2, according to the basis $\{e_1, e_2, e_3, e_4\}$. Differently from the case (a) in the previous section, the procedure for this case is to study different possibilities on dimension of the derived Lie algebra $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$, where the brackets corresponding to $[\mathfrak{h}, e_4]$ are in general deduced from the equation (11). We note here that we can restrict our study to the case where $\mathfrak{g}' = \mathfrak{h}$, in fact, if $\dim \mathfrak{g}' = \dim[\mathfrak{g}, \mathfrak{g}] < 3$, this means that at least to independent vectors are acting as derivations on $\mathfrak{g}$. Since $g$ is with neutral signature, there exist no two independent degenerate vectors and so we can choose a vector which is time-like and acts as derivation of $\mathfrak{g}$. Thus, in fact we are working in one of the cases on the previous subsection.

**(1)** $\dim \mathfrak{h}' = 0$**:**

In this case, $\mathfrak{h} = \mathfrak{r}^3$ is abelian and so the only non-vanishing Lie brackets are given by (11). The Jacobi identity holds trivially. Moreover, the cyclic condition (2) yields that the metric $g$ is cyclic if and only if $r_1 = r_2 = p_2 + q_1 = 0$. Therefore, after substituting $p_1 = A, q_1 = B, p_3 = C, q_2 = D, q_3 = E, r_3 = F$, the Lie algebra is completely described by:

$$[e_1, e_4] = Ae_1 - Be_2 + Ce_3, \quad [e_2, e_4] = Be_1 + De_2 + Ee_3, \quad [e_3, e_4] = Fe_3. \tag{16}$$

**(2)** $\dim \mathfrak{h}' = 1$**:**

In this case, $\mathfrak{h}' = \mathfrak{h}_3$ is the three-dimensional Heisenberg Lie algebra. Assume $\mathfrak{h}' = \mathrm{span}\{X\}$. We can write $X = V + \lambda e_3$ for a real constant $\lambda$, where $V$ is space-like (resp. time-like) and $e_3 \perp V$ is null. One of the following cases occurs.

**(a):** $V\neq 0$**:** Depending on the sign of the vector $V$, one of the following cases occurs.

- If the vector $V$ is space-like, we can set $e_1 = \frac{X}{\|X\|}$, and complete the basis of $\mathfrak{h}$ with a time-like unit vector $e_2$ and the null vector $e_3$, such that

$$g|_{\mathfrak{h}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

  and the Lie algebra $\mathfrak{h}$ will be

$$[e_1, e_2] = \alpha e_1, \quad [e_1, e_3] = \beta e_1, \quad [e_2, e_3] = \mu e_1.$$

  We complete the Lie algebra $\mathfrak{h}$ with the brackets of the equation (11). Now the cyclic condition (2) together with the Jacobi identity (12) will result three sets of solutions.

  (1) $\{\mu = 0,\ q_3 + \frac{\alpha}{\beta} q_2 = p_2 = p_3 = q_1 = r_1 = r_2 = r_3 = 0,\ \beta \neq 0\}$. If we set $p_1 = A,\ q_2 = B$, then the Lie algebra $\mathfrak{g}$ is

$$[e_1, e_2] = \alpha e_1,\ [e_1, e_3] = \beta e_1,\ [e_1, e_4] = A e_1,\ [e_2, e_4] = B e_2 - \frac{\alpha}{\beta} B e_3. \tag{18}$$

  (2) $\{\alpha = \beta = \mu = 0, p_2 + q_1 = r_1 = r_2 = 0\}$, which is obviously is not acceptable since $\dim \mathfrak{h}' = 0$ in this case.

  (3) $\{\beta = \mu = 0, p_2 = p_3 = q_1 = q_2 = r_1 = r_2 = 0\}$, which is not acceptable since $\dim \mathfrak{g}' < 3$ in this case.

- If the vector $V$ is time-like, we assume $e_2 = \frac{X}{\|X\|}$, which is time-like and complete the basis of $\mathfrak{h}$ with a space-like unit vector $e_1$ and the null vector $e_3$, such that the Lie algebra $\mathfrak{h}$ will be

$$[e_1, e_2] = \alpha e_2, \quad [e_1, e_3] = \beta e_2, \quad [e_2, e_3] = \mu e_2$$

  The cyclic condition (2) gives $\beta = 0,\ r_1 = r_2 = q_1 + p_2 = 0$, and applying the Jacobi identity we have the following possible solutions:

  (1) $\{\beta = 0,\ p_2 = p_3 - \frac{\alpha}{\mu} p_1 = q_1 = q_3 = r_1 = r_2 = r_3 = 0\}$, which by substituting $p_1 = A,\ q_2 = B$, gives

$$[e_1, e_2] = \alpha e_2,\ [e_1, e_4] = A e_1 + \frac{\alpha}{\mu} A e_3,\ [e_2, e_3] = \mu e_2,\ [e_2, e_4] = B e_2. \tag{19}$$

  (2) $\{\alpha = \beta = \mu = 0, p_2 + q_1 = r_1 = r_2 = 0\}$, which is not valid since $\dim \mathfrak{h}' = 0$ in this case.

  (3) $\{\beta = \mu = 0, p_1 = p_2 = q_1 = q_3 = r_1 = r_2 = 0\}$, which in not valid since $\dim \mathfrak{g}' < 3$ in this case.

**(b):** $V = 0$**:** In this case, there exists an orthogonal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{h}$ such that $g|_{\mathfrak{h}}$ is completely determined by $g(e_1, e_1) = -g(e_2, e_2) = 1$, and

$$[e_1, e_2] = \alpha e_3, \quad [e_1, e_3] = \beta e_3, \quad [e_2, e_3] = \mu e_3.$$

By using the equation (11) the cyclic condition yields

$$r_1 + \beta = \mu - r_2 = \alpha + q_1 + p_2 = 0.$$

Now by Jacobi identity (12) we get the following two sets of solutions:

(1) $\{\alpha+q_1+p_2=\beta=\mu=0,\ p_1+q_2-r_3=r_1=r_2=0\}$, so by setting $q_1=-A$, $p_2=-B,\ r_3=C,\ q_2=-D,\ p_3=E,\ q_3=F,$ we have

$$[e_1,e_2]=\alpha e_3,\quad [e_1,e_4]=(A+B)e_1+(C-\alpha)e_2+De_3,$$
$$[e_2,e_4]=-Ce_1-Be_2+Ee_3,\quad [e_3,e_4]=Ae_3. \tag{20}$$

(2) $\{\alpha=\beta=\mu=0,\ p_2+q_1=r_1=r_2=0\}$, which is not acceptable since we get $\dim\mathfrak{h}'=0$ in this case.

**(3)** $\dim\mathfrak{h}'=2$ :

In this case, either $\mathfrak{h}=\mathfrak{e}(1,1)$ or $\mathfrak{h}=\mathfrak{e}(2)$. Suppose that $\mathfrak{h}=\mathrm{span}\{X_1,X_2\}$, where $X_i=V_i+\lambda_i e_3$, where $V_1$ is space-like, $V_2$ is time-like and $e_3$ is null, orthogonal to $V_1$ and $V_2$ and $\lambda_i$ are arbitrary real constants. We consider the following possibilities.

**(a)** $V_1$ and $V_2$ are linearly independent: Since $V_1$ is space-like and $V_2$ is time-like, there exists orthonormal vectors $e_1$ and $e_2$, such that $\mathfrak{h}'=\mathrm{span}\{X_1,X_2\}=\mathrm{span}\{e_1,e_2\}$. With respect to the orthogonal basis $\{e_1,e_2,e_3\}$ of $\mathfrak{h}$, we then have

$$[e_1,e_2]=a_1e_1+a_2e_2,\quad [e_1,e_3]=b_1e_1+b_2e_2,\quad [e_2,e_3]=c_1e_1+c_2e_2,$$

which is equipped with the left-invariant metric (17). By means of the cyclic condition we obtain

$$r_1=r_2=q_1+p_2=c_1+b_2=0,$$

then, however, when we apply the Jocobi identity, all the obtained solutions are incompatible with either $\dim\mathfrak{g}'=3$ or $\dim\mathfrak{h}'=2$. In fact, since $\dim\mathfrak{g}'=3$ we assume that $p_3^2+q_3^2+r_3^2\neq 0$. In this case, by means of $\dim\mathfrak{h}'=2$ all of the solutions will fail. For example, one of the solutions is,

$$\{p_1+\frac{a_1q_1}{a_2}=p_2+q_1=p_3=q_2-\frac{a_2q_1}{a_1}=q_3=r_1=r_2=b_1=b_2=c_1=c_2=0\},$$

so if we set $a_1=A,\ a_2=B,\ q_1=C,\ r_3=D,$ the full Lie algebra will be

$$[e_1,e_2]=Ae_1+Be_2,\ [e_1,e_4]=-\frac{AC}{B}e_1-Ce_2,\ [e_2,e_4]=Ce_1+\frac{BC}{A}e_2,\ [e_3,e_4]=De_3,$$

which clearly contradicts the condition $\dim\mathfrak{h}'=2$.

**(b)** $V_1$ and $V_2$ are linearly dependent: In this case, we can assume that either $V_1\neq 0$ or $V_2\neq 0$ and then one of the following cases may happen:

- If $V_1\neq 0$, we choose $\{V_1,e_3\}$ as a basis for $\mathfrak{h}'$. We consider $e_1=\frac{V_1}{\|V_1\|}$, and time-like vector $e_2$, orthogonal to both $e_1$ and $e_3$ such that $g|_{\mathfrak{h}}$ is described by the equation (17). Moreover, we have

  $$[e_1,e_2]=a_1e_1+a_3e_3,\quad [e_1,e_3]=b_1e_1+b_3e_3,\quad [e_2,e_3]=c_1e_1+c_3e_3.$$

  The cyclic condition (2) yields

  $$c_1=b_3+r_1=c_3-r_2=a_3+q_1+p_2=0.$$

  Similar to the previous case, the Jacobi identity together with the above cyclic equations does not give any solutions compatible with $\dim\mathfrak{g}'=3$ and $\dim\mathfrak{h}'=2$. Therefore, this case cannot occur.

- If $V_2 \neq 0$ we choose $\{V_2, e_3\}$ as a basis for $\mathfrak{h}'$. We consider $e_2 = \frac{V_2}{\|V_2\|}$, and space-like vector $e_1$, orthogonal to both $e_2$ and $e_3$. Again, we do not obtain any solution compatible with the conditions $\dim \mathfrak{g}' = 3$ and $\dim \mathfrak{h}' = 2$. Therefore, this case cannot occur.

**(4)** $\dim \mathfrak{h}' = 3$**:**

In this case, since $\mathfrak{h}' = \mathfrak{h}$, we have either $\mathfrak{h} = \mathfrak{sl}(2)$ or $\mathfrak{h} = \mathfrak{su}(2)$. We can suppose that $e_3 \in \mathfrak{h}$ is orthogonal to $\mathfrak{h}$ itself. In order to study different possibilities, we consider the map $\mathrm{ad}_{e_3} : \mathfrak{h} \to \mathfrak{h}$, which is necessarily of rank 2, since $\mathfrak{h}' = \mathfrak{h}$. Besides $0$, $\mathrm{ad}_{e_3}$ has either two real eigenvalues or two conjugate complex eigenvalues. In addition, if we write $e_3 = [X_1, X_2]$, we have

$$\mathrm{ad}_{e_3} = \mathrm{ad}_{X_1} \circ \mathrm{ad}_{X_2} - \mathrm{ad}_{X_2} \circ \mathrm{ad}_{X_1},$$

so that $\mathrm{tr}(\mathrm{ad}_{e_3}) = 0$. We then have one of the following possible cases.

**(a)** Eigenvalues of $\mathrm{ad}_{e_3}$ are $0, \lambda \neq 0$ and $-\lambda$: We choose $e_1$ and $e_2$ unit eigenvectors, such that (rescaling $e_3$ if needed) the Lie algebra of $\mathfrak{h}$ is described as

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = \lambda e_1, \quad [e_2, e_3] = -\lambda e_2,$$

with the left-invariant metric $$g = \begin{pmatrix} 1 & k & 0 & 0 \\ k & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad (21)$$

where $k$ is an arbitrary real constant. Imposing the cyclic condition, we then find

$$2\lambda k = 0, \; kr_2 + r_1 = 0, \; kr_1 - r_2 = 0, \; 1 + q_1 + k(q_2 - q_1) + p_2 = 0.$$

Since $\lambda \neq 0$, the above system reduces to $k = r_1 = r_2 = 0 = p_2 + q_1 + 1 = 0$. Imposing the Jacobi identity to $\mathfrak{g}$, we get $\lambda = 0$, which is a contradiction. Hence, this case cannot occur.

**(b)** Eigenvalues of $\mathrm{ad}_{e_3}$ are $0, i\beta$ and $-i\beta$, with $\beta \neq 0$: We choose $e_1$ and $e_2$ (unitary) Jordan vectors, so by rescaling $e_3$ if needed we have

$$[e_1, e_2] = \beta e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \beta e_1,$$

equipped with the left-invariant metric (21). Imposing the cyclic condition for $\mathfrak{g}$, we have $kr_2 + r_1 = kr_1 - r_2 = \beta + q_1 + k(q_2 - p_1) + p2 = 0$. Applying the Jacobi equation (12), we get $\beta = 0$ which is a contradiction. Thus, this case cannot happen.

According to the above study of different cases, when $g|_{\mathfrak{h}}$ is degenerate, we proved the following theorem.

**Theorem 3.4.** *Let $G = H \rtimes \mathbb{R}$ be a connected and simply connected four-dimensional Lie group, equipped with a left-invariant neutral metric $g$ such that $g|_H$ is degenerate. If $g$ is cyclic, then we can choose a basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebra $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{r}$, such $\mathfrak{h} = \mathrm{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \mathrm{span}\{e_4\}$ and with respect to $\{e_i\}$, the metric is described as in case (b) of the Proposition 3.2, and one of the following holds:*

(I) *$G = \mathbb{R}^3 \rtimes \mathbb{R}$, with brackets as in* (16).

(II) *$G = H_3 \rtimes \mathbb{R}$, with brackets as in* (18), (19) *or as in* (20).

## 4. Homogeneous spaces with nontrivial isotropy

Following the study of cyclic Lorentzian Lie groups in [5] and cyclic neutral Lie groups in the previous sections, in order to complete the classification of the cyclic homogeneous manifolds of dimension four, we shall now consider the homogeneous manifolds with non-trivial isotropy.

A reductive homogeneous pseudo-Riemannian manifold $(M = G/H, g)$, with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is called *cyclic* if it satisfies

$$\mathfrak{S}_{x,y,z}\langle [x,y]_{\mathfrak{m}}, z\rangle = 0, \quad x,y,z \in \mathfrak{m}, \tag{22}$$

where $\langle\cdot,\cdot\rangle$ is the $\mathrm{Ad}(H)$-invariant inner product induced by $g$.

Consider the four-dimensional homogeneous space $(M, g)$ with the Lie group $G$ of isometries acting transitively on $M$. Let $H = G_x$ the stabilizer of an arbitrary point $x \in M$ and $(\mathfrak{g}, \mathfrak{h})$ be the pair of Lie algebras corresponding to the pair $(G, H)$ of Lie groups. Note that the pair $(\mathfrak{g}, \mathfrak{h})$ locally uniquely defines the homogeneous space $(M, g)$ and invariant inner products on $\mathfrak{g}$ are in one to one correspondence with the class of metrics $g$ on $M$, up to isometry. Four-dimensional pseudo-Riemannian homogeneous spaces were classified by Komrakov in [16, 17], by considering different Lie subalgebras $\mathfrak{h}$ of $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ and $\mathfrak{so}(2,2)$, $(\mathfrak{so}(p,q), p+q = 4)$. We note here that Komrakov's classification theorem contains *all* of the possible presentations $G/H$ of four-dimensional homogeneous pseudo-Riemannian manifolds, thus, considering all cases is enough to specify all of the cyclic homogeneous four-manifolds.

The factor space $\mathfrak{m} = \frac{\mathfrak{g}}{\mathfrak{h}}$ is a subspace of $\mathfrak{g}$, complementary to $\mathfrak{h}$. The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropic representation:

$$\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m}), \rho(x)(y) = [x,y]_{\mathfrak{m}}, \ \textit{for all } x \in \mathfrak{g}, y \in \mathfrak{m}.$$

A bilinear form $B$ on $\mathfrak{m}$ is invariant if $\rho(x)^t \circ B + B \circ \rho(x) = 0$, for all $x \in \mathfrak{h}$. In particular, every invariant symmetric nondegenerate bilinear form $B$ on $\mathfrak{m}$, corresponds to one class of the pseudo-Riemannian metrics $g$ on the homogeneous space $M = \frac{G}{H}$, up to an isometry.

Due to the notation of [17], a homogeneous space of type $n.m^k : q$ is the one corresponding to the $q$-th pair $(\mathfrak{g}, \mathfrak{h})$ of type $n.m^k$, where $n = \dim(\mathfrak{h})$ $(n = 1, ..., 6)$, $m$ is the number of the complex subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{so}(4, \mathbb{C})$ and $k$ is the number of the real form of $\mathfrak{h}^{\mathbb{C}}$.

As cyclic homogeneous manifolds are considered through of reductive examples, we firstly exclude the non-reductive cases, i.e. those examples which do not satisfy the relation $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. Thus, we exclude the following cases from the Komrakov's classification theorem [17].

| Isotropy dimension | Non-reductive case |
|---|---|
| 1 | $1.3^1.1$, $1.4^1.1 - 4$ |
| 2 | $2.2^1.1$, $2.5^1.1 - 2$, $2.5^2.1$ |
| 3 | $3.2^1.1 - 2$, $3.2^2.1$ |

Table 3: Non-reductive pseudo-Riemannian four-manifolds.

We assume $\{e_j\}$ and $\{u_i\}$ be the bases for $\mathfrak{h}$ and $\mathfrak{m}$ respectively.

| Case | Invariant metric | Symmetric | Cyclic Condition | Signa-ture |
|---|---|---|---|---|
| $1.1^1:8$ | $2a\theta^1\theta^3+2b\theta^2\theta^4,\quad ab\neq 0$ | ✗ | ✓ | N |
| $1.1^2:2$ | $a\theta^1\theta^1+b\theta^2\theta^2+2c\theta^2\theta^4+$ $+a\theta^3\theta^3+d\theta^4\theta^4,$ $a(c^2-bd)\neq 0$ | $p(p-1)=0$ | ✓ | N/L |
| $1.1^2:11$ | $a\theta^1\theta^1+b\theta^2\theta^2+a\theta^3\theta^3+$ $+b\theta^4\theta^4,\ ab\neq 0$ | ✗ | ✓ | N |
| $1.3^1:5$ | $-2a\theta^1\theta^4+2a\theta^2\theta^3+b\theta^3\theta^3+$ $+2c\theta^3\theta^4+d\theta^4\theta^4,$ $a\neq 0$ | $\mu=\lambda=0,\ or$ $b+d=\mu-2$ $=\lambda=0$ | $\lambda=\mu+2=0$ | N |
| $1.3^1:9$ | as above | $b\lambda(\lambda+1)=0$ | $\lambda=-\frac{1}{3}$ | N |
| $1.3^1:12$ | as above | $b=\lambda-\mu\pm1=0,\ or$ $\mu=\lambda\pm1=0,\ or$ $\mu-\frac{1}{2}=\lambda+\frac{1}{2}=0,\ or$ $\mu-\frac{1}{2}=\lambda-\frac{3}{2}=0,\ or$ $\lambda=\mu\pm1=0$ | $\lambda+3\mu-1=0$ | N |
| $1.3^1:13$ | as above | ✗ | $\lambda=-\frac{1}{2}$ | N |
| $1.3^1:14$ | as above | ✗ | $\lambda=1$ | N |
| $1.3^1:23$ | as above | ✗ | ✓ | N |
| $1.3^1:24$ | as above | $\lambda=0,\ or$ $b-4d=\lambda-2=0$ | $\lambda=\frac{2}{5}$ | N |
| $1.3^1:25$ | as above | $\lambda=0,\ or$ $b+4d=\lambda-2=0$ | $\lambda=\frac{2}{5}$ | N |
| $1.3^1:30$ | as above | $c=0,\ \lambda=\mu=1$ | $\lambda=\mu=-1$ | N |
| $1.4^1:6-7$ | $-2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+$ $+2c\theta^3\theta^4+d\theta^4\theta^4,\ ad\neq 0$ | ✗ | ✓ | N/L |
| $1.4^1:10$ | as above | $p=r=0,\ or$ $p+1=r=0$ | ✓ | N/L |
| $1.4^1:12$ | as above | $r=0$ | ✓ | N/L |
| $2.5^1:3$ | $2a\theta^1\theta^3+2a\theta^2\theta^4+$ $+b\theta^3\theta^3,\ a\neq 0$ | $g=-h=2,\ k=0$ | $h=0$ | N |
| $2.5^1:4$ | as above | $4g+2h-h^2=0$ | $h=0$ | N |
| $2.5^2:2$ | $2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+$ $+a\theta^4\theta^4,\ a\neq 0$ | $p+r^2=s=0$ | $r=0$ | L |
| $3.3^1:1$ | $2a\theta^1\theta^3+2a\theta^2\theta^4+b\theta^3\theta^3,\ a\neq 0$ | $p=0$ | ✓ | N |
| $3.3^2:1$ | $2a\theta^1\theta^3+a\theta^2\theta^2+b\theta^3\theta^3+$ $a\theta^4\theta^4,\ a\neq 0$ | $p=0$ | ✓ | L |

Table 4: Cyclic four-dimensional pseudo-Riemannian homogeneous space with non-trivial isotropy. In the table, ✓(resp. ✗) means that the condition is always (resp. never) satisfied and “L” (resp. “N”, “N/L”) means the Lorentzian (resp. neutral, Lorentzian or neutral) signature.

**Theorem 4.1.** *Let $(\frac{G}{H}, g)$ be a reductive pseudo-Riemannian four-dimensional homogeneous space with non-trivial isotropy, equipped with an invariant metric $g$. Then $(\frac{G}{H}, g)$ is cyclic, if and only if either is symmetric or it belongs to one of the following cases of Table 4.*

**Proof.** Since the proof is based on the case-by-case study of reductive homogeneous four dimensional manifolds with non-trivial isotropy from [17], we summarize below the explicit calculations for the cases $1.1^1.1$ and $2.5^2.2$ and other examples studied by similar arguments.

**Case $\mathbf{1.1^1.1}$.** In this case, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the reductive homogeneous space $\frac{G}{H}$ is generated by the following non-zero brackets:

$$[e_1, u_1] = u_1, \qquad [e_1, u_3] = -u_3, \qquad [u_1, u_3] = [u_2, u_4] = u_2, \qquad [u_3, u_4] = u_3,$$

where $\mathfrak{h} = \mathrm{span}\{e_1\}$ and $\mathfrak{m} = \mathrm{span}\{u_1, u_2, u_3, u_4\}$. By applying the isotropy representation, the invariant metrics with respect to $\{u_i\}$ will be

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix}, a(c^2 - bd) \neq 0,$$

that is equivalent to

$$g = 2a\theta^1\theta^3 + b\theta^2\theta^2 + 2c\theta^2\theta^4 + d\theta^4\theta^4, a(c^2 - bd) \neq 0,$$

where $a, b, c, d$ are arbitrary real constants, and $\{\theta_i\}$ denotes the dual basis of $\{u_i\}$. We apply the well known Koszul formula to describe the Levi-Civita connection $\nabla$ of $g$. Setting $\Lambda_i = \nabla_{u_i}$, with respect to basis $\{u_i\}$ we find

$$\mathbf{\Lambda_1} = \begin{pmatrix} 0 & -\frac{b}{2a} & 0 & \frac{a-c}{2a} \\ 0 & 0 & \frac{bd+ac-c^2}{2(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{ab}{2(bd-c^2)} & 0 \end{pmatrix}, \quad \mathbf{\Lambda_2} = \begin{pmatrix} -\frac{b}{2a} & 0 & 0 & 0 \\ 0 & \frac{cb}{bd-c^2} & 0 & \frac{db}{bd-c^2} \\ 0 & 0 & \frac{b}{2a} & 0 \\ 0 & -\frac{b^2}{bd-c^2} & 0 & -\frac{cb}{bd-c^2} \end{pmatrix},$$

$$\mathbf{\Lambda_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{-bd+ac+c^2}{2(bd-c^2)} & 0 & 0 & 0 \\ 0 & \frac{b}{2a} & 0 & \frac{a+c}{2a} \\ -\frac{ab}{2(bd-c^2)} & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{\Lambda_4} = \begin{pmatrix} \frac{a-c}{2a} & 0 & 0 & 0 \\ 0 & \frac{c^2}{bd-c^2} & 0 & \frac{dc}{bd-c^2} \\ 0 & 0 & -\frac{a-c}{2a} & 0 \\ 0 & -\frac{cb}{bd-c^2} & 0 & -\frac{c^2}{bd-c^2} \end{pmatrix}. \tag{23}$$

The curvature tensor is calculated by the relation $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$, therefore, if set $R_{ij} = R(u_i, u_j)$ we have

$$\mathbf{R_{12}} = \begin{pmatrix} 0 & -\frac{b^2(2a^2+bd-c^2)}{4a^2(bd-c^2)} & 0 & -\frac{b(bad+2a^2c-c^2a+cbd-c^3)}{4a^2(bd-c^2)} \\ 0 & 0 & \frac{(bd-ac-c^2+2a^2)b}{4a(bd-c^2))} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b^2}{4(bd-c^2)} & 0 \end{pmatrix},$$

$$\mathbf{R_{13}} = \begin{pmatrix} -\frac{b(-3bd+3c^2+a^2)}{4a(bd-c^2)} 0 & 0 & 0 \\ 0 & -\frac{cb}{2(bd-c^2)} & 0 & -\frac{db}{2(bd-c^2)} \\ 0 & 0 & \frac{b(-3bd+3c^2+a^2)}{4a(bd-c^2))} & 0 \\ 0 & \frac{b^2}{2(bd-c^2)} & 0 & \frac{cb}{2(bd-c^2)} \end{pmatrix},$$

$$\mathbf{R_{14}} = \begin{pmatrix} 0 & -\frac{b(c^2a+2a^2c-bad+cbd-c^3)}{4a^2(bd-c^2)} & 0 & -\frac{c^2a^2+ba^2d+bdc^2-c^4}{4a^2(bd-c^2)} \\ 0 & 0 & \frac{-bad+cbd+a^2c-c^3)}{4a(bd-c^2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{b(a+c)}{4(bd-c^2)} & 0 \end{pmatrix},$$

$$\mathbf{R_{23}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{(bd+ac+2a^2-c^2)b}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & \frac{b^2(2a^2+bd-c^2)}{4a^2(bd-c^2)} & 0 & \frac{b(c^2a+2a^2c-bad+cbd-c^3)}{4a^2(bd-c^2)} \\ \frac{b^2}{4(bd-c^2)} & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{R_{24}} = \begin{pmatrix} \frac{b}{2a} & 0 & 0 & 0 \\ 0 & -\frac{cb}{bd-c^2} & 0 & -\frac{bd}{bd-c^2} \\ 0 & 0 & -\frac{b}{2a} & 0 \\ 0 & \frac{b^2}{bd-c^2} & 0 & \frac{bc}{bd-c^2} \end{pmatrix},$$

$$\mathbf{R_{34}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{bad+cbd+a^2c-c^3}{4a(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{b(bad+2a^2c-c^2a+cbd-c^3)}{4a^2(bd-c^2)} & 0 & -\frac{c^2a^2+ba^2d+bdc^2-c^4}{4a^2(bd-c^2)} \\ \frac{(a-c)b}{4(bd-c^2)} & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

By using the equations (23) and (24), the space is locally symmetric if and only if $bd + a^2 - c^2 = 0$. We now apply the cyclic condition (22) and we get

$$a = -c, \quad b = 0.$$

As we see, the cyclic condition yields locally symmetry and thus we do not report this case in Table 4.

**Case 2.5².2.** In this case, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the reductive homogeneous space $\frac{G}{H}$ is generated by the following non-zero brackets:

$$[e_1, u_2] = u_1, \quad [e_1, u_3] = -u_2, \quad [e_2, u_3] = u_4, \quad [e_2, u_4] = -u_1,$$
$$[u_1, u_3] = u_1, \quad [u_2, u_3] = A, \quad [u_2, u_4] = 2ru_1, \quad [u_3, u_4] = B$$

where $A = (p+s)e_1 + re_2 + u_2 - 2ru_4$, $B = -re_1 + (p-s)e_2 - 2ru_2 - u_4$, $r \geq 0$, $s \geq 0$ and $\mathfrak{h} = \text{span}\{e_1, e_2\}$ and $\mathfrak{m} = \text{span}\{u_1, ..., u_4\}$. Direct calculations due to the isotropy representation yield the following invariant metrics with respect to the basis $\{u_i\}$:

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad a \neq 0,$$

that is equivalent to $g = 2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3$, $a \neq 0$.

Direct computations give the non-zero components of the Levi-Civita connection and the curvature tensor as follows:

$$\mathbf{\Lambda_1} = \begin{pmatrix} 0 \end{pmatrix}, \qquad \mathbf{\Lambda_2} = \begin{pmatrix} 0 & -1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \end{pmatrix},$$

$$\mathbf{\Lambda_3} = \begin{pmatrix} -1 & 0 & -\frac{b}{a} & 0 \\ 0 & 0 & 0 & -r \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & 0 \end{pmatrix}, \quad \mathbf{\Lambda_4} = \begin{pmatrix} 0 & -r & 0 & -1 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{25}$$

$$\mathbf{R_{23}} = \begin{pmatrix} 0 & -r^2-p-s & 0 & 0 \\ 0 & 0 & r^2+p+s & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{26}$$
$$\mathbf{R_{34}} = \begin{pmatrix} 0 & 0 & 0 & r^2+p-s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r^2-p+s & 0 \end{pmatrix}.$$

Referring to the above equations (25) and (26), the space is locally symmetric if and only if $p + r^2 = s = 0$. Since $a \neq 0$, applying the cyclic condition (22) yields $r = 0$. We note here that also symmetric reductive homogeneous manifolds are cyclic homogeneous but symmetric spaces may not be cyclic due to the current reductive decomposition which is considering. ■

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