

Notes on Harish-Chandra Cells of $(\mathfrak{sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))$ -Modules

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Abstract. We fix $(G, K) = (\mathrm{Sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))$. Cells of Harish-Chandra modules partition the set of irreducible Harish-Chandra modules having the same infinitesimal character as the trivial representation. Irreducible modules in a cell form a basis of a representation of the complex Weyl group. These representations are the Harish-Chandra cells representations. The point of these notes is two-fold. We give closed formulae for the number of isomorphic cell representations. In Section 5 we give a parametrization of Harish-Chandra cells. We use our results to compute the number of Unipotent representations attached to even nilpotent orbits

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1. Introduction

Given (G, K) a symmetric pair, we consider $\mathcal{M}_\rho(\mathfrak{g}, K)$ the category of Harish-Chandra modules having the same infinitesimal character as the trivial representation. The corresponding Grothendieck group, $\mathcal{G}^{(\mathfrak{g}, K)}$, affords an action of the Weyl group of \mathfrak{g} via the coherent continuation representation. The W -structure of $(W, \mathcal{G}^{(\mathfrak{g}, K)})$ plays a key role in the study of representations of reductive groups and their invariants. Harish-Chandra cells partitions the set of irreducible modules in $\mathcal{M}_\rho(\mathfrak{g}, K)$, see [3]. Each cell \mathcal{C} , determines a W -cell representation $V^{\mathcal{C}}$, a minimal subquotient of the coherent continuation representation that is spanned by the irreducible modules in \mathcal{C} . In [18], the author studied the coherent continuation representation when G is of classical type. In particular, McGovern found a geometric parametrization of the set of Harish-Chandra cells when $(G, K) = (\mathrm{Sp}(p+q, \mathbf{C}), \mathrm{Sp}(p, \mathbf{C}) \times \mathrm{Sp}(q, \mathbf{C}))$. For this symmetric pair, Harish-Chandra cells are parametrized by nilpotent K -orbits, the associated variety of the modules in the cell.

We are interested in understanding the coherent continuation representation when $(G, K) = (\mathrm{Sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))$. In this paper we find a parametrization of Harish-Chandra cells and we give closed formulae for the number of isomorphic cell representations.

Building on work by Barbash and Vogan, we write $(W, \mathcal{G}^{(\mathrm{Sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))})$ in terms of $(W, \mathcal{G}^{(\mathrm{Sp}(p+q, \mathbf{C}), \mathrm{Sp}(p, \mathbf{C}) \times \mathrm{Sp}(q, \mathbf{C}))})$ with $p+q \leq n$. This is the content of Proposition 3.4.

In this setting, Harish-Chandra cell representations are generically reducible and are isomorphic to left cell representations of Lusztig type, see [18]. We identify a relevant subset of their irreducible constituents and we compute their symbols. Combining this combinatorial work with Proposition 3.4, we derive an explicit formula for the number of Harish-Chandra cells consisting of irreducible modules having annihilators with a given associated variety $\overline{\mathcal{O}}$, see Theorem 5.1.

The equivalence relation generated by the condition of two irreducible modules having a non-split extension defines equivalence classes of modules known as blocks. The span of representations on each block is preserved by the coherent continuation action. Thus,

$$(W, \mathcal{G}^{(\mathfrak{g}, K)}) = \bigoplus_{\{\mathcal{B}\}} (W, \mathcal{G}_{\mathcal{B}}^{(\mathfrak{g}, K)}).$$

In Section 5 we give an algorithm to compute the number of isomorphic Harish-Chandra cell representations in each block when $(\mathfrak{g}, K) = (\mathfrak{sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))$.

In Section 6 we parametrize the set of Harish-Chandra cells consisting of irreducible modules having annihilators with a given associated variety $\overline{\mathcal{O}}$, by a set consisting of triples $(\mathfrak{m}, \mathcal{O}, \mathcal{O}_{\mathbb{R}})$ where $\mathfrak{m} \subset \mathfrak{sp}(2n, \mathbf{C})$ is a Levi subalgebra; \mathcal{O} is a \mathfrak{m} nilpotent orbit, and $\mathcal{O}_{\mathbb{R}}$ is a “real form” of the dual orbit $d^{\vee}(\mathcal{O})$.

In Section 7 we explain how our formulae relates to the number of Unipotent representation attached to even nilpotent orbits.

2. Background

Notation

We work with the symmetric pairs

$$\begin{aligned} (G, K) &= (\mathrm{Sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C})) \\ (G, K_{p,q}) &= (\mathrm{Sp}(2n, \mathbf{C}), \mathrm{Sp}(p, \mathbf{C}) \times \mathrm{Sp}(q, \mathbf{C})), \text{ where } p + q = n, \text{ and} \\ (G^{\vee}, K_{p,q}^{\vee}) &= (SO(2n + 1, \mathbf{C}), S(O(p, \mathbf{C}) \times O(q, \mathbf{C}))). \end{aligned}$$

We let θ be the Cartan involution that determines K , i.e., $G^{\theta} = K$. At the Lie algebra level, we write the Cartan decomposition induced by θ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. When convenient we identify $(\mathfrak{g}/\mathfrak{k})^*$ with \mathfrak{s} . We will let \mathfrak{h} denote a maximally compact Cartan subalgebra in \mathfrak{g} , $\Delta^+(\mathfrak{g}, \mathfrak{h})$ a choice of positive roots of \mathfrak{h} in \mathfrak{g} , and \mathfrak{b} the corresponding Borel subalgebra.

The nilpotent cone of \mathfrak{g}^* is denoted by \mathcal{N}^* . Often, we use the Killing form to identify $\mathcal{N}^* \simeq \mathcal{N}$, where \mathcal{N} is the nilpotent cone in \mathfrak{g} . We let $\mathcal{N}_{\mathfrak{s}}^*$ stand for $\mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$. We write \mathcal{N}_0 for the set of complex G -nilpotent orbits; a finite set. For $\mathcal{O} = G \cdot f$, we let $A(\mathcal{O}) = A(f)$ be the component group $Z_G(f)/Z_G^0(f)$. In types C and B , component groups are abelian groups isomorphic to products of $\mathbf{Z}/2\mathbf{Z}$. We write $\widehat{A}(\mathcal{O})$ for the set of irreducible representations of $A(\mathcal{O})$. The set of nilpotent G^{\vee} -orbits is denoted by \mathcal{N}_0^{\vee} .

Parametrization of nilpotent orbits

By a partition of m we mean a set of decreasing numbers $[\mathbf{d}] = [d_1, d_2, \dots, d_k]$ with $d_1 \geq d_2 \geq \dots \geq d_k \geq 0$ and $\sum_i d_i = m$. We often identify partitions with Young diagrams. The usual partial ordering on partitions induces a partial order on the set $\mathcal{P}(m)$ of partitions of m , see for example [8].

The following bijections are well-known, see for example [8].

Proposition 2.1. (1) *There is a one-to-one correspondence between the set of partitions of $2n$ in which odd parts occur with even multiplicity and the set $Sp(2n, \mathbf{C})$ -nilpotent orbits.*

(2) *$GL(n, \mathbf{C})$ -orbits in \mathcal{N}_5^* are parametrized by signed Young diagrams such that for each fixed odd part, the number of rows that start with a $+$ equals the number of rows that start with $-$.*

Definition 2.2. We say that a nilpotent $Sp(2n, \mathbf{C})$ -orbit is special if its corresponding partition of $2n$ has an even number of even parts between any two consecutive odd parts and an even number of even parts greater than the largest odd part. We write \mathcal{N}^{sp} for the set of special nilpotent orbits.

Proposition 2.3. *If $p, q \in \mathbf{N}$, then nilpotent $K_{p,q}$ -orbits are parametrized by signed Young diagrams of signature (p, q) such that for each fixed even part, the number of rows that start with a $+$ equals the number of rows that start with $-$ and the multiplicity odd parts starting with $+$ (starting with $-$) is even (is even.)*

Proposition 2.4. (1) *The set of G^\vee -nilpotent orbits is in one-to-one correspondence with the set of partitions of $2n + 1$ in which even parts occur with even multiplicity.*

(2) *If $p, q \in \mathbf{N}$ and $p + q = 2n + 1$, then $S(O(p) \times O(q))$ -nilpotent orbits are parametrized by signed Young diagrams of signature (p, q) such that for each fixed even part, the number of rows beginning with a $+$ equals the number of rows beginning with a $-$.*

Definition 2.5. A nilpotent G^\vee -orbit is called *special* if its partition has an even number of odd parts between any two consecutive even parts and an odd number of odd parts greater than the largest even part. We write $N^{\vee, sp}$ for the set of special G^\vee -orbits.

Duality at the level of orbits

Given $[\mathbf{d}] = [d_1, d_2, \dots, d_k]$ a partition of m we set $d_i^* = \#\{j : d_j \geq i\}$ and we denote by $[\mathbf{d}^*] = [d_1^*, d_2^*, \dots, d_k^*]$ the dual partition. It is well known that when $[\mathbf{d}]$ parametrizes an special nilpotent G -orbit (G^\vee -orbit) then $[\mathbf{d}^*]$ parametrizes a nilpotent G -orbit (G^\vee -orbit, respectively). This is not the case for partitions that label non-special orbits. The notion of collapse is used to extend the assignment $[\mathbf{d}] \rightarrow [\mathbf{d}^*]$ into a duality of orbits. See for example [8, Section 6.3].

The C -collapse of $[\mathbf{d}]$, denoted $[\mathbf{d}]_C$, is the largest among the partitions $[\mu]$ of m parametrizing G -orbits and such that $[\mu] \leq [\mathbf{d}]$. The B -collapse, $[\mathbf{d}]_B$, is defined in a similar manner. The map $[\mathbf{d}] \rightarrow [\mathbf{d}^*]_C$ induces a duality

$$d : \mathcal{N}_0 \rightarrow \mathcal{N}_0^{sp}; \quad \mathcal{O}_{[\mathbf{d}]} \rightarrow \mathcal{O}_{[\mathbf{d}^*]_C}.$$

This duality is known as Spaltenstein duality.

Remark 2.6. In Section 3 we give a formula that relates the coherent continuation representation on the Grothendieck group of $(\mathfrak{sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))$ -Harish-Chandra modules in terms of the coherent continuation representation on the various Grothendieck groups of Harish-Chandra modules for $(\mathfrak{sp}(2n, \mathbf{C}), K_{p,q})$ with $p + q \leq n$. Spaltenstein duality will play an important role in Section 3. The following observation will be repeatedly used.

Assume that there exists a $(\mathfrak{sp}(2n, \mathbf{C}), K_{p,q})$ irreducible Harish-Chandra module Y with $\mathrm{AV}(\mathrm{ann}(Y)) = \overline{\mathcal{O}_{[\mathbf{d}]}}$, then all parts d_i of $[\mathbf{d}]$ occur in $[\mathbf{d}]$ an even number of times. In particular, all d_i^* are even integers. Moreover, $d(\mathcal{O}_{[\mathbf{d}]}) = \mathcal{O}_{[\mathbf{d}^*]}$ is an even orbit. ■

There is a natural order-preserving bijection between $\mathcal{N}_0^{\mathrm{sp}}$ and $\mathcal{N}_0^{\vee, \mathrm{sp}}$. This leads to a second duality of orbits, studied by Barbasch-Vogan in the setting of primitive ideals. We write this duality as

$$d^\vee : \mathcal{N}_0 \rightarrow \mathcal{N}_0^{\vee, \mathrm{sp}}. \quad (1)$$

This duality can be explicitly computed in a combinatorial manner, see [19].

Lemma 2.7. *Assume \mathcal{O} is a special nilpotent orbit of type C or B. Then,*

$$d^\vee(d(\mathcal{O})) = d(d^\vee(\mathcal{O})).$$

Proof. This is an easy consequence of the algorithm describing d^\vee in [19]. ■

Definition 2.8. For a special nilpotent orbit \mathcal{O} , the special piece $\mathcal{SP}(\mathcal{O})$ containing \mathcal{O} is the union of all nilpotent orbits $\mathcal{O}' \subset \bar{\mathcal{O}}$ which are not contained in $\bar{\mathcal{O}}_1$ for any special nilpotent orbit \mathcal{O}_1 with $\mathcal{O}_1 \subsetneq \bar{\mathcal{O}}$.

Springer correspondence

In order to be clear about conventions, we include as summary of the well-known Springer correspondence. In this subsection we assume \mathfrak{g} is of classical type. We denote by X the flag variety associated to \mathfrak{g} . For a nilpotent element $f \in \mathfrak{g}$, we write

$$X_f = \{\mathfrak{b} : \text{borel subalgebras } f \in \mathfrak{b}\}, \quad \text{and} \quad \mathcal{O}_f = G \cdot f.$$

Theorem 2.9. ([20]) *The Weyl group W and the component group $A(\mathcal{O}_f)$ act on the cohomology space $H^*(X_f, \mathbf{C})$.*

- (1) *The W and $A(\mathcal{O}_f)$ -actions commute.*
- (2) *For $\pi \in \widehat{A}(\mathcal{O}_f)$, $\sigma(\mathcal{O}_f, \pi) = (W, \mathrm{Hom}_{A(\mathcal{O}_f)}(V_\pi, H^*(X_f, \mathbf{C})))$ is either zero or irreducible.*
- (3) *If $\pi = \text{trivial}$, then $\sigma(\mathcal{O}_f, \mathbf{1}) \neq \mathbf{0}$.*
- (4) *If $\sigma \in \widehat{W}$, then there is a unique nilpotent orbit \mathcal{O}_f and a unique $\pi \in \widehat{A}(\mathcal{O}_f)$ such that $\sigma \simeq \sigma(\mathcal{O}_f, \pi)$.*

Remark 2.10. The assignment $(\mathcal{O}, \pi) \rightarrow \sigma(\mathcal{O}, \pi)$ is known as the Springer correspondence. We use the conventions,

$$\sigma(\{0\}) = \text{sign representation of } W$$

$$\sigma(\text{principal orbit}) = \text{trivial representation.}$$

A representation σ of W , is special in the sense of Lusztig if and only if $\sigma = \sigma(\mathcal{O}, \mathbf{1})$ for some special nilpotent orbit \mathcal{O} . ■

Left cells and double cells

The Weyl group W parametrizes the set of highest weight modules at infinitesimal character ρ . We write L_w for the irreducible quotient of

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_{-w\rho-\rho},$$

and we let I_w stand for the annihilator of L_w .

In the study of primitive ideals of complex semisimple Lie algebras, Joseph introduced the notion of cells of the Weyl group. These cells are equivalence classes for the relations $w \sim_L w'$ if $I(w) = I(w')$ and $w \sim_R w'$ if $I(w^{-1}) = I(w'^{-1})$. That is

$$\begin{aligned} \mathcal{C}_w^L &= \{w' \in W : I(w) = I(w')\}, \\ \mathcal{C}_w^R &= \{w' \in W : I(w^{-1}) = I(w'^{-1})\}. \end{aligned}$$

The double cell of w is given by $\mathcal{C}_w^{LR} = \{\text{smallest set generated by } \sim_L, \sim_R \text{ from } w\}$. (Joseph cells agree with the notion of cells introduced by Khazdan and Lusztig in [10].)

Each cell can be regarded as a basis of a representation of W . See for example [14, Chapter 12] or [4, Proposition 2.9]. Barbasch and Vogan proved that the representation attached to a double cell \mathcal{C}_w^{LR} contains a unique special representation $\sigma(\mathcal{O}_w, \mathbf{1}) \otimes \sigma(\mathcal{O}_w, \mathbf{1})$, [4, Corollary 2.16]. We write $V_{\mathcal{O}}^{LR}$ for the double cell representation that contains $\sigma(\mathcal{O}, \mathbf{1})$. More is true, each left cell representation contains exactly one special representation and that representation occurs with multiplicity one.

Notation: We write $V_{\mathcal{O}}^{LR}$ for the double cell representation that contains $\sigma(\mathcal{O}, \mathbf{1})$.

We set
$$\mathcal{C}_{\mathcal{O}}^{LR} = \{\sigma \in \widehat{W} : [\sigma : V_{\mathcal{O}}^{LR}] \neq 0\}.$$

Lusztig extended the Springer correspondence giving a stronger relation between \widehat{W} and nilpotent orbits. In type C_n, B_n , this theory is explained in [14, page 88]. Here, we follow the presentation in [5]. Fix \mathcal{O} , a special nilpotent orbit. To each $\mathcal{C}_{\mathcal{O}}^{LR}$, Lusztig attaches a finite group $\bar{A}(\mathcal{O})$, a quotient of the component group $A(\mathcal{O})$, and a set

$$M(\mathcal{O}) = \{(x, \xi_x) : x \in \bar{A}(\mathcal{O}) \text{ and } \xi_x \text{ irreducible representation of } Z_{\bar{A}(\mathcal{O})}(x)\}.$$

The group $\bar{A}(\mathcal{O})$ is known as Lusztig canonical quotient. For a definition see for example [5, Section 4]. In type C_n, B_n ,

$$\begin{aligned} \bar{A}(\mathcal{O}) &\simeq \bar{A}(d^{\vee}(\mathcal{O})) \text{ a product of copies of } \mathbf{Z}/2\mathbf{Z}, \text{ and} \\ M(\mathcal{O}) &= \bar{A}(\mathcal{O}) \times \widehat{\bar{A}(\mathcal{O})}. \end{aligned}$$

Theorem 2.11. (Lusztig [14], [5, Theorem 4.7]) *Assume \mathfrak{g} is of type C_n or B_n . Let \mathcal{O} be a special nilpotent orbit. There is an injective map $m: \mathcal{C}_{\mathcal{O}}^{LR} \rightarrow M(\mathcal{O})/\bar{A}(\mathcal{O}) = \{\bar{A}(\mathcal{O})\text{-orbits on } M(\mathcal{O})\}$ such that*

- (1) $m(\sigma(\mathcal{O}, \mathbf{1})) = [(\mathbf{1}, \mathbf{1})]$;
- (2) $m(\sigma(\mathcal{O}, \pi)) = [(\mathbf{1}, \pi)]$;
- (3) for each $[(x, \mathbf{1})] \in \mathbf{M}(\mathcal{O})/\bar{\mathbf{A}}(\mathcal{O})$, there exists an irreducible representation $\sigma_x \in \mathcal{C}_{\mathcal{O}}^{LR}$ with $m(\sigma_x) = [(x, \mathbf{1})]$.
- (4) There is an isomorphism $i : \bar{\mathbf{A}}(\mathcal{O}) \rightarrow \bar{\mathbf{A}}(d^\vee(\mathcal{O}))$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{O}}^{LR} & \xrightarrow{m} & M(\mathcal{O})/\bar{\mathbf{A}}(\mathcal{O}) \\ \downarrow \otimes \text{sign} & & \downarrow i \\ \mathcal{C}_{d^\vee(\mathcal{O})}^{LR} & \xrightarrow{m} & M(d^\vee(\mathcal{O}))/\bar{\mathbf{A}}(d^\vee(\mathcal{O})) \end{array}$$

commutes.

Moreover, if $\sigma(\mathcal{O}, \mathbf{1})$ occurs in a left cell representation V^L , then the number of irreducible constituents of V^L equals $|\bar{\mathbf{A}}(\mathcal{O})|$.

Theorem 2.12. ([14, Thm. 4.7]) Assume \mathfrak{g} is of type C_n or B_n . Let $\mathcal{O} = G \cdot f$ be a special nilpotent orbit.

- (1) There exists a unique left cell representation $(W, V_{\mathcal{O}}^{\text{Springer}})$, containing $\sigma(\mathcal{O}, \mathbf{1})$, isomorphic to the Springer representation $(W, H^*(X_f, \mathbf{C}))$.
- (2) There is a unique left cell representation, $V_{\mathcal{O}}^{\text{Lusztig}}$ containing $\sigma(\mathcal{O}, \mathbf{1})$ so that

$$(W, V_{\mathcal{O}}^{\text{Lusztig}}) \simeq \bigoplus_{x \in \bar{\mathbf{A}}(\mathcal{O})} \sigma_x. \quad (2)$$

For technical reasons, see [14, page 88], we will distinguish between type B_n and type C_n . Even when $W_{B_n} = W_{C_n}$ we will specify when we view a representation as a W_{B_n} and when we view it as a W_{C_n} -module.

Lemma 2.13. Let \mathcal{O} be a special nilpotent $Sp(2n, \mathbf{C})$ -orbit. Then,

$$(W_{C_n}, V_{\mathcal{O}}^{\text{Springer}} \otimes \text{sign}) \simeq (W_{C_n}, V_{d(\mathcal{O})}^{\text{Lusztig}}) \simeq (W_{B_n}, V_{d^\vee(\mathcal{O})}^{\text{Springer}}).$$

Proof. The set of left cell representations is closed under tensoring with the sign representation. Thus, $V_{\mathcal{O}}^{\text{Springer}} \otimes \text{sign}$ is a left cell representation which contains $\sigma(\mathcal{O}, \mathbf{1}) \otimes \text{sign}$. Using the notion of symbols, for example, it is not difficult to show that

$$(W_{C_n}, \sigma(\mathcal{O}, \mathbf{1}) \otimes \text{sign}) \simeq (\mathbf{W}_{C_n}, \sigma(\mathbf{d}(\mathcal{O}), \mathbf{1})) \simeq (\mathbf{W}_{B_n}, \sigma(\mathbf{d}^\vee(\mathcal{O}), \mathbf{1})).$$

Parts (2) and (4) of Theorem 2.11 yield the isomorphism

$$(W_{C_n}, V_{\mathcal{O}}^{\text{Springer}} \otimes \text{sign}) \simeq (W_{B_n}, V_{d^\vee(\mathcal{O})}^{\text{Springer}}).$$

For the identification of $(W_{C_n}, V_{\mathcal{O}}^{\text{Springer}} \otimes \text{sign})$ with $(W_{C_n}, V_{d(\mathcal{O})}^{\text{Lusztig}})$ see [14, p. 88]. ■

Remark 2.14. The isomorphism

$$(W_{C_n}, \sigma(\mathcal{O}, \mathbf{1}) \otimes \text{sign}) \simeq (\mathbf{W}_{C_n}, \sigma(\mathbf{d}(\mathcal{O}), \mathbf{1})) \simeq (\mathbf{W}_{B_n}, \sigma(\mathbf{d}^\vee(\mathcal{O}), \mathbf{1}))$$

can be verified in a combinatorial manner by using [19, Lemma 11 and Theorem 12].

Proposition 2.15. ([16, Theorem 0.4]) *Let \mathcal{O} be a special nilpotent $Sp(2n, \mathbf{C})$ -orbit. Enumerate $\mathcal{O}_1, \dots, \mathcal{O}_n$ the orbits in $\mathcal{SP}(\mathcal{O})$. Then,*

$$\sum_{\{\mathcal{O}_i \in \mathcal{SP}(\mathcal{O})\}} \sigma(\mathcal{O}_i, \mathbf{1}) \subset \mathbf{V}_{\mathcal{O}}^{Lusztig}.$$

Symbols and pairs of partitions

Assume \mathfrak{g} is of type C . We recall the parametrization of representations of W in terms of pairs of partitions and in terms of equivalence classes of symbols, see for example [11], and [6].

Proposition 2.16. *There is a one-to-one correspondence $([\lambda], [\mu]) \leftrightarrow \pi_{([\lambda], [\mu])}$ between ordered pairs of partitions $([\lambda], [\mu])$ with $|\lambda| + |\mu| = n$, and irreducible representations of W_{C_n} . If $[\mu^*]$ denotes the transpose of the partition $[\mu]$, then $\pi_{([\lambda], [\mu])} \otimes \text{sign} \simeq \pi_{([\mu^*], [\lambda^*])}$.*

Definition 2.17. A symbol of type C_n is an arrangement of non-negative integers

$$\begin{pmatrix} p_1 & p_2 & \cdots & \cdots & p_{r+1} \\ & q_1 & & q_2 & \cdots & q_r \end{pmatrix},$$

such that $p_1 < p_2 < \dots < p_{r+1}$, $q_1 < q_2 < \dots < q_r$ and

$$\sum_i (2p_i) + \sum_j (2q_j + 1) = 2n + r (2r + 1).$$

We say that a symbol of type C_n is *special* if $p_1 \leq q_1 \leq p_2 \leq q_2 \leq \dots \leq p_{r+1}$.

Theorem 2.18. ([11]) (1) *There is a bijection between \widehat{W}_{C_n} and the set of equivalence classes symbols of type C_n .*

(2) *Each pair of partitions*

$$([\lambda] = [\lambda_1, \lambda_2, \dots, \lambda_{r+1}], [\mu] = [\mu_1, \mu_2, \dots, \mu_r])$$

determines uniquely the equivalence class represented by the symbol

$$\begin{pmatrix} \lambda_{r+1} & \lambda_r + 1 & & \cdots & \lambda_2 + (r - 1) & & \lambda_1 + r \\ & \mu_r & & \mu_{r-1} + 1 & \cdots & & \mu_1 + (r - 1) \end{pmatrix}.$$

Remark 2.19. A pair $([\lambda] = [\lambda_1, \lambda_2, \dots, \lambda_{r+1}], [\mu] = [\mu_1, \mu_2, \dots, \mu_r])$ corresponds to a *special* symbol if and only if for each i , $\lambda_{i+1} \leq \mu_i \leq \lambda_i + 1$. ■

The following known tools for computing restrictions of Weyl group representations will be used. We write $W_{A_n} = S_n$ (W_n , respectively) for the the Weyl group of type A_n (the Weyl group of type C_n). If $[\mu]$ is a partition of n , the Pieri Rule gives

$$\text{ind}_{S_n \times S_m}^{S_{n+m}} (\pi([n]) \otimes \pi([\mu])) = \sum_{[\psi] \in H_n([\mu])} \pi([\psi]),$$

where $H_n([\mu])$ is the set of partitions that can be obtain from $[\mu]$ by adding n -boxes to the Young diagram of $[\mu]$ so that at most one box is added by column. See for example [21].

The Littlewood-Richardson rule gives

$$\operatorname{ind}_{S_n \times W_m}^{W_{n+m}} (\operatorname{sign} \otimes \pi([\lambda], [\mu])) = \sum_k \sum_{\substack{[\theta] \in H_k([\lambda^*]), \\ [\psi] \in H_{n-k}([\mu^*])}} \pi([\theta], [\psi]); \tag{3}$$

$$\operatorname{ind}_{S_n \times W_m}^{W_{n+m}} (\operatorname{trivial} \otimes \pi([\lambda], [\mu])) = \sum_k \sum_{\substack{[\theta^*] \in H_k([\lambda]), \\ [\psi^*] \in H_{n-k}([\mu])}} \pi([\theta], [\psi]). \tag{4}$$

3. Coherent continuation

Given (G, K) a symmetric pair, we write $\mathcal{M}_\rho(\mathfrak{g}, K)$ for the category of Harish-Chandra modules having the same infinitesimal character as the trivial representation. We view the corresponding Grothendieck group $\mathcal{G}^{(\mathfrak{g}, K)}$ as a lattice with finite basis consisting of irreducible Harish-Chandra modules with trivial infinitesimal character. The Weyl group W of \mathfrak{g} acts on $\mathcal{G}^{(\mathfrak{g}, K)}$ via the coherent continuation representation. The equivalence relation generated by the condition of two irreducible modules having a non-split extension defines equivalence classes of modules known as blocks. The span of representations on each block is preserved by the coherent continuation action. Furthermore, each block is partitioned into Harish-Chandra cells. The elements of a Harish-Chandra cell \mathcal{C}_{HC} index a basis of a subquotient of the full coherent continuation representation, see [3]. We write $V^{\mathcal{C}_{HC}}$ for the W -module generated by \mathcal{C}_{HC} .

In [18], McGovern studied the coherent continuation representation when G is of classical type. In particular, McGovern found a geometric parametrization of the set of Harish-Chandra cells in the unique block of representations for the pair $(G, K) = (\operatorname{Sp}(p + q, \mathbf{C}), \operatorname{Sp}(p, \mathbf{C}) \times \operatorname{Sp}(q, \mathbf{C}))$. For this symmetric pair, Harish-Chandra cells are parametrized by nilpotent K -orbits. That is, Harish-Chandra cells are parametrized by the associated variety of the modules in the cell, (an important invariant of irreducible modules). No such a result holds for other classical groups. Our goal is to obtain a geometric/combinatorial parametrization of Harish-Chandra cells when $(G, K) = (\operatorname{Sp}(2n, \mathbf{C}), \operatorname{GL}(n, \mathbf{C}))$.

Fix $(G, K) = (\operatorname{Sp}(2n, \mathbf{C}), \operatorname{GL}(n, \mathbf{C}))$ and write $G^\vee = \operatorname{SO}(2n + 1, \mathbf{C})$. For this symmetric pair the set of blocks $\{\mathcal{B}_r\}$ is in one to one correspondence with the set of pairs $\{(\operatorname{Sp}(2n, \mathbb{R}), \operatorname{SO}(n + 1 + r, n - r)) : 0 \leq r \leq n\}$. The coherent continuation action of W on $\mathcal{G}^{(\mathfrak{g}, K)}$ decomposes into a direct sum

$$(W, \mathcal{G}^{(\mathfrak{g}, K)}) = \bigoplus_r (W, \mathcal{G}_{\mathcal{B}_r}).$$

By Vogan’s duality ([23]), for each block \mathcal{B}_r there exists a block \mathcal{B}_r^\vee consisting of $(\mathfrak{so}(2n + 1, \mathbf{C}), S(\operatorname{O}(n + 1 - r) \times \operatorname{O}(n - r)))$ Harish-Chandra modules so that

$$(W, \mathcal{G}_{\mathcal{B}_r}) \otimes \operatorname{sign} \simeq (W, \mathcal{G}_{\mathcal{B}_r^\vee}).$$

We describe the W module structure of $(W, \mathcal{G}^{(\mathfrak{g}, K)})$ and of each $\mathcal{G}_{\mathcal{B}_r^\vee}$ in terms of the W action on various $\mathcal{G}^{(\operatorname{Sp}(p+q, \mathbf{C}), \operatorname{Sp}(p, \mathbf{C}) \times \operatorname{Sp}(q, \mathbf{C}))}$ with $p + q \leq n$ (for which a geometric parametrization of cells is known.) This is the content of Proposition 3.3 and Corollary 3.4. The proposition can viewed as a reformulation of a formula for coherent continuation by Barbasch and Vogan.

Notation: In order to simplify notation we write $\mathcal{G}^{(\mathfrak{sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))} = \mathcal{G}^{\mathrm{Sp}(2n, \mathbb{R})}$, $\mathcal{G}^{\mathfrak{sp}(2p+q), \mathrm{Sp}(p, \mathbf{C}) \times \mathrm{Sp}(q, \mathbf{C}))} = \mathcal{G}^{\mathrm{Sp}(p, q)}$, and $\mathcal{G}^{SO(n+1+r, n-r)/\mathrm{Sp}(2n, \mathbb{R})}$ for $\mathcal{G}_{\mathcal{B}_V^\vee}$.

Coherent continuation representation on $\mathcal{G}^{\mathrm{Sp}(p, q)}$

The group $\mathrm{Sp}(p, q)$ admits $q + 1$ conjugacy classes of Cartan subgroups. Each conjugacy class consists of connected groups isomorphic to $\mathbf{C}^t \times T^{p+q-2t}$ having Weyl group isomorphic to $W_t \rtimes W(A_1)^t \times W_{p-t} \times W_{q-t}$.

Theorem 3.1. ([3] and [18]) *Let (p, q) be non-negative integers. Assume $q \leq p$. Then,*

- (1) $(W, \mathcal{G}^{\mathrm{Sp}(p, q)})$ is isomorphic to

$$\sum_{0 \leq t \leq q} \sum_{[\Lambda]} \mathrm{ind}_{W_{2t} \times W_{p-t} \times W_{q-t}}^W (\pi([\Lambda], [\Lambda]) \otimes \mathrm{sign} \otimes \mathrm{sign}), \tag{5}$$

where $[\Lambda]$ runs over partitions of t and sign denotes the sign representation.

- (2) If (W, σ) is an irreducible representation such that $[\sigma : \mathcal{G}^{\mathrm{Sp}(p, q)}] \neq 0$, then there exists a left cell representation of Springer type $V_{\mathcal{O}}^{\mathrm{Springer}}$ with $[\sigma : V_{\mathcal{O}}^{\mathrm{Springer}}] \neq 0$.
- (3) Let $\mathcal{C}^{\mathrm{HC}}$ be a Harish-Chandra cell. Assume Z is an irreducible Harish-Chandra module in $\mathcal{C}^{\mathrm{HC}}$ and let $\overline{\mathcal{O}} = AV(\mathrm{ann}(Z))$. Then, the cell representation (W, V^{HC}) is isomorphic to the unique complex left cell representation of Springer type in the double cell representation attached to \mathcal{O} .
- (4) The set

$$\{(W, V^{\mathrm{HC}}) \text{ Harish-Chandra cell representation isomorphic to } (W, V_{\mathcal{O}}^{\mathrm{Springer}})\}$$

is in bijection with

$$\{\mathcal{O}_{K_{p, q}}, \text{ nilpotent } \mathrm{Sp}(p) \times \mathrm{Sp}(q)\text{-orbits} : \mathcal{O} = \mathrm{Sp}(2(p+q), \mathbf{C}) \cdot \mathcal{O}_{K_{p, q}}\}.$$

Coherent continuation representation on $\mathcal{G}^{\mathrm{Sp}(2n, \mathbb{R})}$

Conjugacy classes of Cartan subgroups are indexed by pairs of integers (s, t) with $s + 2t \leq n$. Cartan subgroups in the (r, t) -class are isomorphic to $T^s \times (\mathbf{C}^*)^t \times (\mathbb{R}^*)^{n-s-2t}$ and they have 2^{n-s-2t} connected components. The Weyl groups are isomorphic to $W(A_{s-1}) \times W_t \rtimes W(A_1)^t \times W_{n-s-2t}$.

Theorem 3.2. ([18], Barbasch-Vogan(unpublished)) *Assume $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$. Let*

$$\mathcal{T} = \left\{ (s, t, a, b, [\Lambda]) : \begin{array}{l} [\Lambda] \text{ is a partition of } t \text{ and } s, t, a, b \in \mathbf{N} \\ s + 2t \leq n, a + b = n - t - 2s \end{array} \right\}.$$

Then,

- (1) $(W, \mathcal{G}^{\mathrm{Sp}(2n, \mathbb{R})})$ is isomorphic to

$$\sum_{\mathcal{T}} \mathrm{ind}_{W(A_{s-1}) \times W_{2t} \times W_a \times W_b}^W (\mathrm{sign} \otimes \pi([\Lambda], [\Lambda]) \otimes \mathrm{trivial} \otimes \mathrm{trivial}),$$

where $[\Lambda]$ runs over the partitions of t .

- (2) *Each Harish-Chandra cell in $\mathcal{G}^{Sp(2n, \mathbb{R})}$ is isomorphic to a Lusztig left cell representations.*
- (3) $(W, \mathcal{G}^{Sp(2n, \mathbb{R})}) \simeq \sum_{0 \leq r \leq n} (W, \mathcal{G}_{\mathcal{B}_r})$.

Let r be an integer so that $0 \leq r \leq n$. We find it convenient to distinguished two cases (a) $n - r$ even and (b) $n - r$ odd. We label the conjugacy classes of Cartan subgroups in $SO(n + 1 + r, n - r)$ as

$$\begin{aligned} & \{T^{a+b} \times (\mathbb{R}^*)^t \times (\mathbf{C}^*)^s : t, s \geq 0 \text{ with } t \text{ even if and only if } n - r \text{ is even} \\ & n - t - 2s \geq r, 2a = n - r - 2s - t, 2b = n + r - 2s - t\} \cup \\ & \{T^{a+b} \times (\mathbb{R}^*)^t \times (\mathbf{C}^*)^s : t, s \geq 0 \text{ with } t \text{ odd if and only if } n - r \text{ is even} \\ & n - t - 2s \geq r + 1, 2a = n - r - 2s - t - 1, 2b = n + r - 2s - t + 1\}. \end{aligned} \tag{6}$$

For an explicit description of conjugacy classes of Cartan subgroups see [22].

Proposition 3.3. *Assume r is an integer such that $0 \leq r \leq n$. If $n - r$ is even, then*

$$\begin{aligned} (W, \mathcal{G}^{SO(n+1+r, n-r)/Sp(2n, \mathbb{R})}) & \simeq \mathcal{G}^{Sp(\frac{n+r}{2}, \frac{n-r}{2})} \\ & \oplus \sum_{\substack{t \text{ even} \\ 0 < t \leq n-r}} \text{ind}_{S_t \times W_{n-t}}^{W_n} (\text{trivial} \otimes \mathcal{G}^{Sp(\frac{n+r-t}{2}, \frac{n-r-t}{2})}) \\ & \oplus \sum_{\substack{t \text{ odd} \\ t \leq n-r-1}} \text{ind}_{S_t \times W_{n-t}}^{W_n} (\text{trivial} \otimes \mathcal{G}^{Sp(\frac{n+r-t+1}{2}, \frac{n-r-t-1}{2})}). \end{aligned}$$

If $n - r$ is odd, we have

$$\begin{aligned} (W, \mathcal{G}^{SO(n+1+r, n-r)/Sp(2n, \mathbb{R})}) & \simeq \mathcal{G}^{Sp(\frac{n+r+1}{2}, \frac{n-r-1}{2})} \\ & \oplus \sum_{\substack{t \text{ even} \\ 0 < t \leq n-r-1}} \text{ind}_{S_t \times W_{n-t}}^{W_n} (\text{trivial} \otimes \mathcal{G}^{Sp(\frac{n+r-t+1}{2}, \frac{n-r-t-1}{2})}) \\ & \oplus \sum_{\substack{t \text{ odd} \\ t \leq n-r}} \text{ind}_{S_t \times W_{n-t}}^{W_n} (\text{trivial} \otimes \mathcal{G}^{Sp(\frac{n+r-t}{2}, \frac{n-r-t}{2})}). \end{aligned}$$

Proof. We assume that $n - r$ is even. The case $n - r$ odd is similar. We label conjugacy classes of Cartan subgroups for $SO(n + r + 1, n - r)$ as in (6). It follows, from work by Barbasch and Vogan, that

$$\begin{aligned} (W, \mathcal{G}^{SO(n+1+r, n-r)/Sp(2n, \mathbb{R})}) & \simeq \tag{7} \\ & \sum_{\substack{\text{Apart. of } s \\ t \text{ even}, t+2s \leq n-r}} \text{ind}_{S_t \times W_{2s} \times W_{\frac{n-r-2s-t}{2}} \times W_{\frac{n+r-2s-t}{2}}}^{W_n} (\mathbf{1} \otimes \sigma(\Lambda, \Lambda) \otimes \text{sign} \otimes \text{sign}) \oplus \\ & \sum_{\substack{\text{Apart. of } s \\ t \text{ odd}, t+2s \leq n-r+1}} \text{ind}_{S_t \times W_{2s} \times W_{\frac{n-r-2s-t-1}{2}} \times W_{\frac{n+r-2s-t+1}{2}}}^{W_n} (\mathbf{1} \otimes \sigma(\Lambda, \Lambda) \otimes \text{sign} \otimes \text{sign}). \end{aligned}$$

Using induction in stages and comparing (7) to (5) we obtain the formulae of this proposition. ■

Corollary 3.4. *The following identity is an isomorphism of W -modules.*

$$(W, \mathcal{G}^{Sp(2n, \mathbb{R})}) \simeq \sum_{(k, \ell)_{k+\ell=n}} \pi([1^k], [1^\ell]) \oplus \left[\sum_{\substack{(\hat{p}, \hat{q}): \\ \hat{p}+\hat{q}=n}} \mathcal{G}^{Sp(\hat{p}, \hat{q})} \otimes \text{sign} \right] \\ \oplus \sum_{0 < p < n} \sum_{\substack{(\hat{p}, \hat{q}): \\ \hat{p}+\hat{q}=n-p}} \text{ind}_{S_p \times W_{\hat{p}+\hat{q}}}^W \left(\text{sign} \otimes \left[\mathcal{G}^{Sp(\hat{p}, \hat{q})} \otimes \text{sign} \right] \right).$$

Proof. Coherent continuation preserves the span of irreducible modules on each block. Hence

$$(W, \mathcal{G}^{(G, K)}) = \oplus (W, \mathcal{G}_{\mathcal{B}_r}).$$

By Vogan’s duality $(W, \mathcal{G}_{\mathcal{B}_r}) \simeq (W, \mathcal{G}^{\text{SO}(n+1+r, n-r)/\text{Sp}(2n, \mathbb{R})} \otimes \text{sign})$. The Corollary follows from Proposition 3.3. We have used the well-known identity

$$\text{ind}_{S_n}^{W_n}(\text{sign}) \simeq \sum_{(k, \ell)_{k+\ell=n}} \pi([1^k], [1^\ell]). \quad \blacksquare \quad (8)$$

Example A : $G = \text{Sp}(4, \mathbb{R})$.

$\mathcal{G}^{\text{Sp}(4, \mathbb{R})}$ has three blocks $\{\mathcal{B}_r\}_{r=0}^2$, dual to blocks for $\text{SO}(3, 2)$, $\text{SO}(4, 1)$ and $\text{SO}(5)$, respectively. In the usual notation, $\rho_C = [2, 1]$ and $\rho_B = [\frac{3}{2}, \frac{1}{2}]$. By Proposition 3.3,

$$\mathcal{G}^{\text{SO}(3,2)/\text{Sp}(4, \mathbb{R})} \simeq \mathcal{G}^{\text{Sp}(1,1)} + \text{ind}_{S_2}^{W_2}(\text{trivial}) + \text{ind}_{S_1 \times W_1}^{W_2}(\text{trivial} \otimes \mathcal{G}^{\text{Sp}(1,0)}).$$

It is not difficult to show that $\text{ind}_{S_1 \times W_1}^{W_2}(\text{trivial} \otimes \mathcal{G}^{\text{Sp}(1,0)})$. We conclude that

$$\mathcal{G}^{\text{SO}(3,2)/\text{Sp}(4, \mathbb{R})} \simeq 2 \mathcal{G}^{\text{Sp}(1,1)} + \text{ind}_{S_2}^{W_2}(\text{trivial}).$$

The unique block of $\text{Sp}(1, 1)$ modules at infinitesimal character $[2, 1]$ is the spanned of four irreducible representations. There are two Harish-Chandra cells; \mathcal{C}_1 consists of the trivial representation (label it by 3), \mathcal{C}_2 consists of two discrete series representation (label them as 0 and 1) and a cohomologically induced module not in the discrete series (call it representation 2.) We single out representation 0, in the discrete series. This is cohomologically induced from the Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ with, in the usual notation, $\Delta(\mathfrak{u}) = \{-2\epsilon_2, \epsilon_1 \pm \epsilon_2, 2\epsilon_1\}$.

Observe that $(W, V_{\mathcal{C}_1}^{HC}) \simeq (W, V_{[1,1,1,1]}^{Springer})$, $(W, V_{\mathcal{C}_2}^{HC}) \simeq (W, V_{[2,2]}^{Springer})$.

$\mathcal{G}^{\text{SO}(3,2)/\text{Sp}(4, \mathbb{R})}$ is spanned by twelve irreducible modules.

- There are three Harish-Chandra cells, $\mathcal{C}_I, \mathcal{C}_{II}, \mathcal{C}_{III}$, consisting each of three irreducible modules with associated variety of their annihilators equal to the closure of the nilpotent orbit attached to the partition $[3, 1, 1]$.
- The corresponding cell representations are all isomorphic to $(W_{B_2}, V_{[3,1,1]}^{Lusztig}) \simeq (W_{C_2}, V_{[2,2]}^{Springer})$.
- Cell \mathcal{C}_I contains two cohomologically induced modules while the other two cells have exactly one cohomologically induce representation.
- If X_2, X_3 are the cohomologically induced modules in $\mathcal{C}_{II}, \mathcal{C}_{III}$, respectively then both cones $\mathbf{Q}[W]X_i$ are isomorphic to $(W, \mathcal{G}^{\text{Sp}(1,1)})$. (This can be checked with the help of the ATLAS software.)

Label X_2 as representation 4 and let 11 label the irreducible module in cell \mathcal{C}_{II} with the same annihilator as that of X_2 . Label by 6 the third representation in \mathcal{C}_{II} . Let 9 denote the trivial representation. Using the ATLAS software we verify that the map

$$\begin{aligned} \mathcal{F}_o : \mathcal{G}^{\mathrm{Sp}(1,1)} &\hookrightarrow \mathcal{G}^{\mathrm{SO}(3,2)/\mathrm{Sp}(4,\mathbb{R})} \\ 0 &\rightarrow 11, \quad 1 \rightarrow 4, \quad 2 \rightarrow 6, \quad 3 \rightarrow 9 \end{aligned}$$

is W -equivariant. ■

Example B : $G = \mathrm{Sp}(6, \mathbb{R})$.

$\mathcal{G}^{\mathrm{Sp}(6,\mathbb{R})}$ has four blocks dual to blocks for $\mathrm{SO}(4, 3)$, $\mathrm{SO}(5, 2)$, $\mathrm{SO}(6, 1)$ and $\mathrm{SO}(7)$.

$$\begin{aligned} \mathcal{G}^{\mathrm{SO}(4,3)/\mathrm{Sp}(6,\mathbb{R})} &\simeq \mathcal{G}^{\mathrm{Sp}(2,1)} + \mathrm{ind}_{S_2 \times W_1}^{W_3}(\mathbf{1} \otimes \mathcal{G}^{\mathrm{Sp}(1,0)}) + \mathrm{ind}_{S_1 \times W_2}^{W_3}(\mathbf{1} \otimes \mathcal{G}^{\mathrm{Sp}(1,1)}) \\ &\quad + \mathrm{ind}_{S_3}^{W_3}(\text{trivial}) \\ \mathcal{G}^{\mathrm{SO}(5,2)/\mathrm{Sp}(6,\mathbb{R})} &\simeq \mathcal{G}^{\mathrm{Sp}(2,1)} + \mathrm{ind}_{S_2 \times W_1}^{W_3}(\mathbf{1} \otimes \mathcal{G}^{\mathrm{Sp}(1,0)}) + \mathrm{ind}_{S_1 \times W_2}^{W_3}(\mathbf{1} \otimes \mathcal{G}^{\mathrm{Sp}(2,0)}). \end{aligned}$$

4. On the number of Harish-Chandra cells

Irreducible constituents of a relevant class of induced modules

Fix $[\ell] = [\ell_1, \ell_2, \dots, \ell_k]$, a partition of $2n$ such that the $\mathrm{Sp}(2n, \mathbf{C})$ -nilpotent orbit $\mathcal{O}_{[\ell]}$ is special. We denote by $\mathcal{C}_{\mathcal{O}_{[\ell]}}^{HC}$ be a Harish-Chandra cell consisting of $(\mathfrak{sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))$ irreducible modules with trivial infinitesimal character and so that their annihilators have associated variety equal to $\overline{\mathcal{O}_{[\ell]}}$. Our first goal is to compute the number of such cells. Each Harish-Chandra cell generates a representation (W, V^{HC}) . By part (2) of Theorem 3.2, $(W, V_{\mathcal{O}_{[\ell]}}^{HC}) \simeq (W, V_{\mathcal{O}_{[\ell]}}^{Lusztig})$. Since each left cell contains exactly one special representation with multiplicity one, it is enough to compute $[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \mathcal{G}^{\mathrm{Sp}(2n,\mathbb{R})}]$. The formula for coherent continuation in Corollary 3.4 relates $(W, \mathcal{G}^{\mathrm{Sp}(2n,\mathbb{R})})$ to various representations $(W, \mathcal{G}^{\mathrm{Sp}(p,q)} \otimes \mathrm{sign})$ with p, q integers such that $p + q \leq n$. Each irreducible modules σ , occurring in $(W, \mathcal{G}^{\mathrm{Sp}(p,q)})$ is of Springer type. Thus, by Lemma 2.13, $\sigma \otimes \mathrm{sign}$ is of Lusztig type. In particular, for $(\sigma(\mathcal{O}_{[\mathbf{c}]}, \mathbf{1})$ special of Springer type, $\sigma(\mathcal{O}_{[\mathbf{c}]}, \mathbf{1}) \otimes \mathrm{sign} \simeq \sigma(\mathcal{O}_{d([\mathbf{c}])}, \mathbf{1})$ is special of Lusztig type where $d([\mathbf{c}]) = [\mathbf{d}]$ is a partition consisting solely of even parts, see Remark 2.6.

Let $[\mathbf{d}] = [d_1, d_2, \dots, d_t]$ be a partition of $2n - 2r$ consisting solely of even parts. In this section we compute

$$[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \mathrm{ind}_{S_r \times W_{n-r}}^{W_n}(\mathrm{sign} \otimes \sigma(\mathcal{O}_{[\gamma]}, \mathbf{1}))]$$

for $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$.

Proposition 4.1. *Let $\mathcal{O}_{[\mathbf{d}]=[d_1, d_2, \dots, d_t]}$ and $\mathcal{O}_{[\ell]=[\ell_1, \ell_2, \dots, \ell_k]}$ be special nilpotent orbits. Assume $[\mathbf{d}]$ has all its parts even. Then,*

$$[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \mathrm{ind}_{S_r \times W_{n-r}}^{W_n}(\mathrm{sign} \otimes \sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}))] \neq 0$$

if and only if for some pair of positive integers (k_1, k_2) , with $k_1 + k_2 = r$, the Young diagram of $\mathcal{O}_{[\ell]}$ is obtained from that of $\mathcal{O}_{[\mathbf{d}]}$ by adding k_1 vertical dominos and k_2 horizontal dominos. The addition of dominos is subject to the constraint that at most one horizontal domino is added to a given row and no row acquires three or more boxes in the process.

Proof. Write $([\lambda] = [\lambda_1, \lambda_2, \dots, \lambda_{t+1}], [\mu] = [\mu_1, \dots, \mu_t])$ for the pair of partitions which parametrizes $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$. Thus, the symbol of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ is

$$\left(\begin{array}{ccccccc} \lambda_{t+1} & \lambda_t + 1 & \dots & \dots & \lambda_2 + (t - 1) & & \lambda_1 + t \\ & \mu_t & \mu_{t-1} + 1 & \dots & & \mu_1 + (t - 1) & \end{array} \right).$$

By Lemma A.2 $\lambda_j = \frac{d_{2j-1}}{2}$ and $\mu_j = \frac{d_{2j}}{2}$.

Let $([\theta], [\psi]) = ([\theta_1, \dots, \theta_{k+1}], [\psi_1, \dots, \psi_k])$ be the pair of partitions attached to $\sigma(\mathcal{O}_{[\ell]}, \mathbf{1})$. Since both $\mathcal{O}_{[\mathbf{d}]}$ and $\mathcal{O}_{[\ell]}$ are special orbits, for each i

$$\lambda_{i+1} \leq \mu_i \leq \lambda_i + 1, \text{ and } \theta_{i+1} \leq \psi_i \leq \theta_i + 1.$$

By the Littlewood-Richardson rule,

$$[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes \sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}))] \neq 0$$

if and only if for some positive integer c , $[\theta]$ is obtained from $[\lambda]$ by adding a number c of 1's and $[\psi]$ is obtained from $[\mu]$ by adding a number $r - c$ of 1's (subject to the constraints described in (3)). In order to relate $[\ell]$ to $[\mathbf{d}]$ we consider four possible situations. The statement of the proposition follows from a combination of the four cases listed below.

Case 1. We assume that no number 1 was added at the end of either $[\lambda]$ or $[\mu]$. Further, assume that for some index j

$$\theta_j = \lambda_j + 1 = \frac{d_{2j-1} + 2}{2}, \text{ and } \psi_j = \mu_j = \frac{d_{2j}}{2}.$$

Our first assumption guarantees that the symbols of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ and $\sigma(\mathcal{O}_{[\ell]}, \mathbf{1})$ have the same number of entries. Since $d_{2j} \leq d_{2j-1}$, $\psi_j < \theta_j$. It follows from Lemma A.2 and Lemma A.1, that ℓ_{2j} and ℓ_{2j-1} are even number. Thus, $\theta_j = \frac{\ell_{2j-1}}{2} = \frac{d_{2j-1} + 2}{2}$ and $\psi_j = \frac{\ell_{2j}}{2} = \frac{d_{2j}}{2}$. We conclude that a horizontal domino has been added to the $2j - 1$ -row of the Young diagram of $[\mathbf{d}]$.

Case 2. We assume that no number 1 was added at the end of either $[\lambda]$ or $[\mu]$. Further, assume that for some index j

$$\theta_j = \lambda_j = \frac{d_{2j-1}}{2}, \text{ and } \psi_j = \mu_j + 1 = \frac{d_{2j} + 2}{2}. \tag{9}$$

There are two possibilities. Either $\theta_j + 1 = \psi_j$ or $\psi_j < \theta_j$. If $\theta_j + 1 = \psi_j$, arguing as in Case 1, we conclude that ℓ_{2j} and ℓ_{2j-1} are both odd integers. By Lemma A.1 and (9), we have

$$\theta_j = \frac{\ell_{2j-1} - 1}{2} = \frac{d_{2j-1}}{2}, \text{ and } \psi_j = \frac{\ell_{2j} + 1}{2} = \frac{d_{2j} + 2}{2}.$$

A vertical domino was added to the Young diagram attached to $[\mathbf{d}]$.

If $\psi_j < \theta_j$, ℓ_{2j} and ℓ_{2j-1} are both even. In this case, $\ell_{2j} = d_{2j} + 2$ and $\ell_{2j-1} = d_{2j-1}$. A horizontal domino was added to Young diagram of $[\mathbf{d}]$.

Case 3. We assume that no number 1 was added at the end of either $[\lambda]$ or $[\mu]$. Further, assume that

$$\theta_j = \lambda_j + 1 = \frac{d_{2j-1} + 2}{2}, \tag{10}$$

$$\psi_j = \mu_j + 1 = \frac{d_{2j} + 2}{2}. \tag{11}$$

Our assumptions imply that $\psi_j < \theta_j$. In particular, it follows that $\theta_j = \frac{\ell_{2j-1}}{2}$ and $\psi_j = \frac{\ell_{2j}}{2}$. By (10), we conclude that one horizontal domino has been added to both the $2j-1$ -th and the $2j$ -th row of the Young diagram of $[\mathbf{d}]$.

Case 4. $([\theta], [\psi]) = ([\theta_1, \dots, \theta_{k+1}], [\psi_1, \dots, \psi_k])$ with $k = t + m$, $\theta_j = \lambda_j$, $\psi_j = \mu_j$ for $j = 1, \dots, t$, $\psi_j = 1$ for $j \geq t + 1$. The symbol of $\pi([\theta], [\psi]) = \sigma(\mathcal{O}_{[\ell]}, \mathbf{1})$ in Lemma A.5 yield the following conclusion.

- (a) In case (1) of Lemma A.5 the Young diagram of $\mathcal{O}_{[\ell]}$ is obtained from that of $\mathcal{O}_{[\mathbf{d}]}$ by adding $k - t - 1$ vertical dominos and one horizontal dominos at the end of the diagram.
- (b) In case (2) of Lemma A.5 the Young diagram of $\mathcal{O}_{[\ell]}$ is obtained from that of $\mathcal{O}_{[\mathbf{d}]}$ by adding $2m + 1$ horizontal dominos and $k - t - m - 1$ vertical dominos at the end of the diagram.
- (c) In case (3) of Lemma A.5 the Young diagram of $\mathcal{O}_{[\ell]}$ is obtained from that of $\mathcal{O}_{[\mathbf{d}]}$ by adding $2m$ horizontal dominos at the end of the diagram.
- (d) In case (4) of Lemma A.5 the Young diagram of $\mathcal{O}_{[\ell]}$ and $\mathcal{O}_{[\mathbf{d}]}$ differ by $k - t$ vertical domino tableau at the end of the first column.
- (e) In case (5) of Lemma A.5 the Young diagram of $\mathcal{O}_{[\ell]}$ is obtained from that of $\mathcal{O}_{[\mathbf{d}]}$ by adding $2m$ horizontal dominos and $k - t - m$ vertical dominos at the end of the diagram.
- (f) In case (5) of Lemma A.5 the Young diagram of $\mathcal{O}_{[\ell]}$ is obtained from that of $\mathcal{O}_{[\mathbf{d}]}$ by adding $2m$ horizontal dominos at the end of the diagram. \blacksquare

Write $V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}} = W_{\mathcal{O}_{[\mathbf{d}]}} \oplus W_{\mathcal{O}_{[\mathbf{d}]}}^c$ and $W_{\mathcal{O}_{[\mathbf{d}]}} = \bigoplus_{\widehat{\mathcal{O}} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})} \sigma(\widehat{\mathcal{O}}, \mathbf{1})$.

(This is possible in view of Proposition 2.15.)

Lemma 4.2. *Let $\mathcal{O}_{[\ell]}$ and $\mathcal{O}_{[\mathbf{d}]}$ be special nilpotent orbits. Assume that $[\mathbf{d}]$ has all its parts even. Then*

$$\left[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}}) \right] = \left[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes W_{\mathcal{O}_{[\mathbf{d}]}}) \right].$$

Proof. We argue that no irreducible constituent of $\text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes W_{\mathcal{O}_{[\mathbf{d}]}}^c)$ is special. In particular, we show that no pair of partitions $([\delta], [\tau])$ for which $[\pi([\delta], [\tau]) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes W_{\mathcal{O}_{[\mathbf{d}]}}^c)] \neq 0$ satisfies the inequalities in Remark 2.19. Assume $\pi([\delta], [\tau])$ occurs in $\text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes W_{\mathcal{O}_{[\mathbf{d}]}}^c)$. Then, for some irreducible constituent $\pi([\tilde{\lambda}], [\tilde{\mu}])$ of $W_{\mathcal{O}_{[\mathbf{d}]}}^c$ we have

$$[\pi([\delta], [\tau]) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes \pi([\tilde{\lambda}], [\tilde{\mu}])] \neq 0.$$

By part (4) of Proposition A.3, there is an index j for which $\tilde{\mu}_j < \tilde{\lambda}_{j+1} - 1$. On the other hand, by the Littlewood-Richardson rule in (3), there is an integer k so that $[\delta] \in H_k([\tilde{\lambda}])$ and $[\tau] \in H_{r-k}([\tilde{\mu}])$. It follows that the inequalities in Remark 2.19 do not hold for $([\delta], [\tau])$. \blacksquare

Proposition 4.3. *Keep the notation introduced earlier in this section. Let $\mathcal{O}_{[\ell]}$ and $\mathcal{O}_{[\mathbf{d}]}$ be special nilpotent orbits. Assume that $[\mathbf{d}]$ has all its parts even. Let $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$ and write $S_{[\gamma]}$ as in (27). Then,*

$$\left[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes \sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})) \right] \neq 0$$

if and only if

$$\left[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes \sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})) \right] \neq 0,$$

and $\ell_{2i_j} = \ell_{2i_j+1} = d_{2i_j}$ for all $i_j \in S_{[\gamma]}$.

Proof. Fix $\mathcal{O}_{[\ell]}$ a special nilpotent orbit. By Proposition 4.1, the assumptions

$$\ell_{2i_j} = \ell_{2i_j+1} = d_{2i_j} \text{ for all } i_j \in S_{[\gamma]}, \text{ and}$$

$$\left[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes \sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})) \right] \neq 0$$

hold simultaneously if and only if the Young diagram of $\mathcal{O}_{[\ell]}$ and $\mathcal{O}_{[\mathbf{d}]}$ differ by a horizontal domino on each $2i_j + 1$ -th row with $i_j \in S_{[\gamma]}$. In view of Lemma A.4, this is so if and only if the Young diagram of $\mathcal{O}_{[\ell]}$ differs from that of $\mathcal{O}_{[\gamma]}$ by a vertical domino at the end of each $(2i_j, 2i_j + 1)$ rows with $i_j \in S_{[\gamma]}$. Now, the Proposition follows Lemma A.4 and (3). ■

Before stating the main result of this subsection we introduce some notation.

Definition 4.4. Let $[\ell] = [\ell_1, \ell_2, \dots, \ell_k]$ a partition of $2n$ such that the $\text{Sp}(2n, \mathbf{C})$ -nilpotent orbit $\mathcal{O}_{[\ell]}$ is special. For a, b non-negative integers such that $0 \leq a+b < n$, define

$$\begin{aligned} S_{(a,b)}([\ell]) = \{ [\mathbf{d}] = [d_1, d_2, \dots, d_k] \in \mathcal{P}(2n - 2(a+b)) \text{ with all parts even} : \\ \text{there are } a \text{ indices } \{i_1, \dots, i_a\} \text{ for which } d_{i_t} = \ell_{i_t} - 2; \\ \text{there are } b \text{ indices } \{j_1, \dots, j_b\} \text{ for which } d_{j_t} = d_{j_t+1} = \ell_{j_t} - 1; \\ \text{and } \{i_1, \dots, i_a\} \cap \{j_1, \dots, j_b\} = \emptyset \} \end{aligned}$$

Set
$$S([\ell]) = \cup_{\{(a,b)\}} S_{(a,b)}([\ell]). \tag{12}$$

For $[\mathbf{d}] \in S([\ell])$, define

$$T_{[\mathbf{d}],[\ell]} = \left\{ i_j : \begin{array}{l} \ell_{2i_j+1} = \ell_{2i_j} = d_{2i_j}; \ d_{2i_j+1} = d_{2i_j} - 2 \text{ and such that} \\ \text{for some } \mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]}) \ \gamma_{2i_j} = \gamma_{2i_j+1} = d_{2i_j+1} + 1 \end{array} \right\}. \tag{13}$$

When $[\gamma]$ is such that $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$ and its entries $\gamma_{2i_j} = \gamma_{2i_j+1} = d_{2i_j+1} + 1$ when $i_j \in T_{[\mathbf{d}],[\ell]}$ we say that $[\gamma]$ is *linked to* $[\ell]$.

Remark 4.5. (1) If we identify partitions with Young diagrams, then the Young diagram corresponding to $[\mathbf{d}] \in S_{(a,0)}([\ell])$ differs from the Young diagram attached to $[\ell]$ by horizontal dominos. All rows of $Y([\mathbf{d}])$ have even length.

(2) The Young diagram corresponding to $[\mathbf{d}] \in S_{(0,b)}([\ell])$ differs from the Young diagram attached to $[\ell]$ by vertical dominos. ■

Theorem 4.6. *Assume $[\mathbf{d}]$ has all its parts even. Let $\mathcal{O}_{[\ell]}$ be a special nilpotent orbit. If $\left[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n}(\text{sign} \otimes \sigma_{\mathcal{O}_{[\mathbf{d}]}})\right] \neq 0$, then*

$$\left[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n}(\text{sign} \otimes V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}})\right] = \begin{cases} 0 & \text{if } |T_{[\mathbf{d}], [\ell]}| > r \\ 2^{|T_{[\mathbf{d}], [\ell]}|} & \text{if } |T_{[\mathbf{d}], [\ell]}| \leq r. \end{cases}$$

A technical lemma

Assume $[\ell]$ is a partition of $2n$ such that the corresponding $\text{Sp}(2n, \mathbf{C})$ nilpotent orbit $\mathcal{O}_{[\ell]}$ is special. Keep the notation \mathcal{C}^{HC} for a Harish-Chandra cell consisting of $(\mathfrak{sp}(2n, \mathbf{C}), \text{GL}(n, \mathbf{C}))$ irreducible modules with trivial infinitesimal character. In the next subsection we determine the cardinality of

$$HC_{[\ell]} = \{\mathcal{C}^{HC} : AV(\text{ann}(X)) = \overline{\mathcal{O}_{[\ell]}} \text{ for } X \in \mathcal{C}^{HC}\}. \quad (14)$$

Lemma 4.8 below is a preliminary result needed to prove Theorem 5.1.

Definition 4.7. Let p, q are non-negative integers and let $n = p + q$. Assume $[\mathbf{d}]$ is a partition of $2n$ consisting solely of even parts. Write $[\mathbf{d}^*]$ for the transpose partition. Let

$$m_{p,q}(\mathcal{O}_{[\mathbf{d}]}) = \#\{\mathcal{O}_{K_{p,q}}, \text{ nilpotent } \text{Sp}(p) \times \text{Sp}(q)\text{-orbits} : \mathcal{O}_{[\mathbf{d}^*]} = \text{Sp}(p+q, \mathbf{C}) \cdot \mathcal{O}_{K_{p,q}}\}.$$

Lemma 4.8. *Assume p, q are non-negative integers such that $p + q = n$. (1) Assume $[\mathbf{d}]$ is a partition of $2n$ consisting solely of even parts. Then*

$$[\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) : \mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}] = m_{p,q}(\mathcal{O}_{[\mathbf{d}]}) .$$

(2) *If $[\mathbf{d}]$ is a partition of $2n$ consisting solely of even parts and $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$,*

$$\text{then} \quad [\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1}) : \mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}] = m_{p,q}(\mathcal{O}_{[\mathbf{d}]}) .$$

Proof. By Remark 2.14, $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) = \sigma(\mathcal{O}_{[\mathbf{d}^*]}, \mathbf{1}) \otimes \text{sign}$. The first statement of this Lemma is part (3) of Theorem 3.1. For (2), write

$$\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) = \pi([\lambda], [\mu]) \quad \text{and} \quad \sigma(\mathcal{O}_{[\gamma]}, \mathbf{1}) = \pi([\tilde{\lambda}], [\tilde{\mu}])$$

as in Corollary A.4 and let S_γ be the set of indexes introduced in 27. For $i_k \in S_\gamma$, we have

$$\begin{aligned} ([\lambda], [\mu]) &= ([\cdots \lambda_{i_k} \mu_{i_k} - 1 \cdots][\cdots \mu_{i_k} \mu_{i_k+1} \cdots]) \\ ([\tilde{\lambda}], [\tilde{\mu}]) &= ([\cdots \lambda_{i_k} \mu_{i_k} \cdots][\cdots \mu_{i_k} - 1 \mu_{i_k+1} \cdots]), \end{aligned} \quad (15)$$

where if $\sum \mu_j = r$, then $\sum \tilde{\mu}_j = r - t$ with $t = |S_\gamma|$.

By Theorem 3.1, if $[\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) : \mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}] \neq 0$, then every irreducible constituent of $(W, V_{\mathcal{O}_{[\mathbf{d}^*]}^{\text{Springer}}})$ occurs in $(W, \mathcal{G}^{\text{Sp}(p,q)})$. On the other hand, by Lemma 2.13 and Proposition 2.15 $(W_{C_n}, V_{\mathcal{O}_{[\mathbf{d}^*]}^{\text{Springer}}} \otimes \text{sign}) \simeq (W_{C_n}, V_{\mathcal{O}_{[\mathbf{d}]}^{\text{Lusztig}}})$ and we have $[\sigma(\mathcal{O}_\gamma, \mathbf{1}) : V_{\mathcal{O}_{[\mathbf{d}]}^{\text{Lusztig}}}] \neq 0$ when $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$. Thus, $[\sigma(\mathcal{O}_\gamma, \mathbf{1}) : \mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}] \neq 0$.

Moreover, the multiplicities of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ and $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$ in $\mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}$ are the multiplicities of these representations in

$$\sum_{0 \leq r \leq p} \sum_{[\Lambda]} \text{ind}_{W_{2r} \times W_{p-r} \times W_{q-r}}^W (\pi([\Lambda], [\Lambda]) \otimes \text{trivial} \otimes \text{trivial}),$$

which can be computed by using (4). That is, in the notation of (4),

$$[\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) : \mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}] = \#\{\Delta \in H_{p-r}([\mu]) : \lambda \in H_{q-r}(\Delta)\}.$$

Similarly for $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$.

Each $\Delta \in H_{p-r}([\mu])$ with $\lambda \in H_{q-r}(\Delta)$ is of the form

$$\begin{aligned} \Delta &= [\mu_1 + a_1 \mu_2 + a_2 \cdots \mu_{i_k} + a_{i_k} \cdot] \\ \lambda &= [\delta_1 + b_1 \delta_2 + b_2 \cdots \delta_j + b_j \cdots] \text{ where } 0 \leq a_j \leq \mu_{j-1} - \mu_j, \\ &\text{and for } i_k \in S_\gamma \ a_{i_k+1} \leq \mu_{i_k} = \mu_{i_k+1} - 1, \ 0 \leq b_j \leq \delta_{j-1} - \delta_j, \\ &\sum a_j = p - r, \ \sum b_j = q - r. \end{aligned} \tag{16}$$

The description of parameters in (15) implies that δ_j in (16) is $\delta_j = \tilde{\mu}_j + a_j$ when $j \notin S_\gamma$; $\delta_{i_k} = \tilde{\mu}_{i_k} + a_{i_k} + 1$ when $i_k \in S_\gamma$; $\tilde{\lambda}_j = \delta_j + b_j$ when $j \notin S_\gamma$, and $\tilde{\lambda}_{i_k+1} = \delta_{i_k+1} + b_{i_k+1} + 1$ when $i_k \in S_\gamma$. Hence $\Delta \in H_{p-r+t}(\tilde{\mu})$ and $\tilde{\lambda} \in H_{q-r+t}(\Delta)$.

We conclude that $[\sigma(\mathcal{O}_\gamma, \mathbf{1}) : \mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}] \leq m_{p,q}(\mathcal{O}_{[\mathbf{d}]})$.

Conversely, assume $\tilde{\Delta} \in H_{p-r+t}(\tilde{\mu})$ and $\tilde{\lambda} \in H_{q-r+t}(\tilde{\Delta})$. Then

$$\tilde{\Delta} = [\tilde{\mu}_1 + c_1 \cdots \tilde{\mu}_{i_k} + c_{i_k} \cdots]$$

where $c_{i_k} \neq 0$ for $i_k \in S_\gamma$. Otherwise, $\tilde{\lambda}_j = \tilde{\delta}_j + b_j$ and $[b_1 \ b_j \cdots]$ would not satisfy the requirement in (15). Now $\tilde{\Delta}$ has components $\tilde{\delta}_j = \mu_j + c_j$ when $j \notin S_\gamma$, $\tilde{\delta}_{i_k} = \mu_j + a_{i_k} - 1$ when $i_k \in S_\gamma$ while $\lambda_{i_k+1} = \tilde{\delta}_{i_k+1} + b_{i_k+1} - 1$ when $i_k \in S_\gamma$ and $\lambda_j = \tilde{\delta}_j + b_j$ otherwise. Hence $\tilde{\Delta} \in H_{p-r}(\mu)$ and $\lambda \in H_{q-r}(\Delta)$. The Lemma follows. ■

$m_{p,q}(\mathcal{O}_{[\mathbf{d}]})$ in terms of Young diagrams

Let $[\mathbf{d}] = [d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_k, \dots, d_k]$ be a partition consisting of even parts. Let $Y([\mathbf{d}])$ be the Young diagram attached to the partition $[\mathbf{d}]$. Write c_1, c_2, \dots, c_t for the distinct column's length in $Y([\mathbf{d}])$ (these are the distinct entries of $[\mathbf{d}^*]$.) Each column length occurs an even number of times. In particular, $Y([\mathbf{d}])$ has $d_{k-i+1} - d_{k-i+2}$ columns of length c_i . It is elementary to show that

$$\sum_{\substack{(p,q): \\ p+q=|[\mathbf{d}]|}} m_{p,q}(\mathcal{O}_{[\mathbf{d}]}) = \begin{cases} \prod_{\text{odd}} c_j \left(\frac{d_{k-j+1} - d_{k-j+2}}{2} + 1 \right) & \text{if some } c_i \text{ is odd} \\ 1 & \text{otherwise} \end{cases} \tag{17}$$

Example C: Let $[\mathbf{d}] = [d_1, d_1, d_2] = [6, 6, 2]$. Then $\{c_1, c_2\} = \{3, 2\}$. The Young diagram $Y([\mathbf{d}])$ has two columns of length 3 and four columns of length 2.

$$\sum_{\substack{(p,q): \\ p+q=|[\mathbf{d}]|}} m_{p,q}(\mathcal{O}_{[\mathbf{d}]}) = 2.$$

There are two signed diagrams of shape $[\mathbf{d}^*] = [3, 3, 2, 2, 2]$. One such diagram parametrizes a $\text{Sp}(8) \times \text{Sp}(6)$ -nilpotent orbit, whereas the other parametrizes a $\text{Sp}(6) \times \text{Sp}(8)$ -nilpotent orbit.

5. Counting Harish-Chandra cells

Let $\mathcal{O}_{[\ell]}$ be a special $\mathrm{Sp}(2n, \mathbf{C})$ -nilpotent orbit. We keep the notation introduced in Definition 4.4. For $[\mathbf{d}] \in S([\ell])$ write $[\mathbf{d}] = [d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_k, \dots, d_k]$ and let $[c_1, \dots, c_1, c_2, \dots, c_2, \dots, c_r, \dots, c_r]$ be the length of the columns of the Young diagram $Y([\mathbf{d}])$. Let

$$S_{\text{even}}([\ell]) = \{[\mathbf{d}] \in S([\ell]) : \text{all } c_i \text{ are even}\}$$

$$S_{\text{odd}}([\ell]) = \{[\mathbf{d}] \in S([\ell]) : \text{some of the } c_i \text{ are odd}\}.$$

Theorem 5.1. *Assume $(G, K) = (\mathrm{Sp}(2n, \mathbf{C}), \mathrm{GL}(n, \mathbf{C}))$ and let $\mathcal{O}_{[\ell]}$ be a special nilpotent orbit. Then the cardinality of*

$$HC_{[\ell]} = \{\mathcal{C}^{HC} : AV(\mathrm{ann}(X)) = \overline{\mathcal{O}_{[\ell]}} \text{ for } X \in \mathcal{C}^{HC}\}$$

is given by

$$\sum_{[\mathbf{d}] \in S_{\text{even}}([\ell])} 2^{|T_{[\mathbf{d}], [\ell]}|} + \sum_{[\mathbf{d}] \in S_{\text{odd}}([\ell])} 2^{|T_{[\mathbf{d}], [\ell]}|} \prod_{\{j: c_j \text{ odd}\}} \left(\frac{d_{k-j+1} - d_{k-j+2}}{2} + 1 \right) + \delta_{[2^{2t} 1^{n-2t}]}.$$

where $\delta_{[2^{2t} 1^{n-2t}]} = \begin{cases} 1 & \text{if } [\ell] = [2^{2t} 1^{n-4t}] \text{ for some } t \leq n \\ 0 & \text{otherwise.} \end{cases}$

Proof. The cardinality of the set $HC_{[\ell]}$ is the multiplicity of the special representation $\sigma(\mathcal{O}_{[\ell]}, \mathbf{1})$ in $(W, \mathcal{G}^{\mathrm{Sp}(2n, \mathbb{R})})$. Corollary 3.4 relates $(W, \mathcal{G}^{\mathrm{Sp}(2n, \mathbb{R})})$ to various modules

$$\mathrm{ind}_{S_r \times W_{n-r}}^{W_n} (\mathrm{sign} \otimes (\mathcal{G}^{\mathrm{Sp}(\hat{p}, \hat{q})} \otimes \mathrm{sign})), \tag{18}$$

for $r \geq 0$ and $\hat{p} + \hat{q} = n - r$. As indicated in Corollary 3.4, when $r = n$ the induced representation is $\sum_{(k, \ell)_{k+\ell=n}} \pi([1^k], [1^\ell])$. The algorithm on [8, Chapter 10] identifies

$$\sum_{(k, \ell)_{k+\ell=n}} \pi([1^k], [1^\ell]) = \sum_t V_{\mathcal{O}_{[2^{2t} 1^{n-2t}]}}^{\mathrm{Lusztig}}.$$

Hence, if $[\ell]$ is not of the form $[2^{2t} 1^{n-2t}]$ for some integer t , it is enough to compute the multiplicity of $\sigma(\mathcal{O}_{[\ell]}, \mathbf{1})$ in (18) when $r < n$ to determine $\#H_{[\ell]}$

In the notation of Section 2, the W -irreducible constituents of $\mathcal{G}^{\mathrm{Sp}(\hat{p}, \hat{q})} \otimes \mathrm{sign}$ are of Lusztig type. That is, they occur in a left cell representation of Lusztig type, $V_{[\mathbf{d}]}^{\mathrm{Lusztig}}$ attached to a nilpotent orbit $\mathcal{O}_{[\mathbf{d}]}$ for a partition $[\mathbf{d}]$ having all its parts even. By Proposition 4.3, $\sigma(\mathcal{O}_{[\ell]}, \mathbf{1})$ occurs in $(W, \mathcal{G}^{\mathrm{Sp}(2n, \mathbb{R})})$ if and only if there exists $r > 0$ and a nilpotent orbit $\mathcal{O}_{[\mathbf{d}]}$ of $\mathfrak{sp}(2n - 2r, \mathbf{C})$ for a partition $[\mathbf{d}]$ having all its parts even so that

$$[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \mathrm{ind}_{S_r \times W_{n-r}}^{W_n} (\mathrm{sign} \otimes \sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}))] \neq 0. \tag{19}$$

When (19) holds, Proposition 4.1 and Proposition 4.3, give

$$[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \mathrm{ind}_{S_r \times W_{n-r}}^{W_n} (\mathrm{sign} \otimes (\mathcal{G}^{\mathrm{Sp}(\hat{p}, \hat{q})} \otimes \mathrm{sign}))] = \tag{20}$$

$$\sum_{[\mathbf{d}] \in S([\ell])} [\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) : \mathcal{G}^{\mathrm{Sp}(\hat{p}, \hat{q})} \otimes \mathrm{sign}] \times \#\{\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]}) : [\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \mathrm{ind}_{S_r \times W_{n-r}}^{W_n} (\mathrm{sign} \otimes \sigma(\mathcal{O}_{[\gamma]}, \mathbf{1}))] \neq 0\}.$$

Lemma 4.8 computes the first number in the product on the right hand side of equation (20). These are the integers $m_{\hat{p},\hat{q}}(\mathcal{O}_{[\mathbf{d}]})$ which are computed in terms of the length of the columns of the Young diagram attached to $[\mathbf{d}]$; see (17). The second number in the product on the right hand side of equation (20) is given in Theorem 4.6 in terms of $|T_{[\mathbf{d}],[\ell]}|$. The Theorem now follows. \blacksquare

Example D : We use Proposition 5.1 to count the number of Harish-Chandra cells of $(\mathfrak{sp}(12, \mathbf{C}), GL(6, \mathbf{C}))$ -modules with $AV(ann(V^{HC})) = \overline{\mathcal{O}_{[4,4,2,2]}}$.

$$S_{\text{even}}([\ell]) = \{[4, 4, 2, 2]; [4, 4]; [2, 2, 2, 2]; [2, 2]\};$$

$$S_{\text{odd}}([\ell]) = \{[4, 2, 2, 2]; [4, 4, 2]; [4, 2, 2]; [4, 2]; [2, 2, 2]\}.$$

One can check that for all $[\mathbf{d}] \in S([\ell])$ $2^{|T_{[\mathbf{d}],[\ell]}|} = 1$. (These numbers give information on the number of partitions $[\gamma]$ with $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$ for which $[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n}(\text{sign} \otimes \sigma(\mathcal{O}_{[\gamma]}, \mathbf{1}))] \neq 0$.)

$$Y([d_1, d_2, d_2, d_2] = [4, 2, 2, 2]) \text{ has } c_1 = 4, c_2 = 1$$

$$Y([d_1, d_1, d_2] = [4, 4, 2]) \text{ has } c_1 = 3, c_2 = 2$$

$$Y([d_1, d_2, d_2] = [4, 2, 2]) \text{ has } c_1 = 3, c_2 = 1$$

$$Y([d_1, d_2] = [4, 2]) \text{ has } c_1 = 2, c_2 = 1$$

$$Y([d_1, d_1, d_1] = [2, 2, 2]) \text{ has } c_1 = 3.$$

$$\left(\frac{d_{k-j+1} - d_{k-j+2}}{2} + 1\right) = \begin{cases} 4 & \text{for } [\mathbf{d}] = [4, 2, 2] \\ 2 & \text{for all other } [\mathbf{d}] \in S_{\text{odd}}([\ell]). \end{cases}$$

We conclude that when $(\mathfrak{g}, K) = (\mathfrak{sp}(12, \mathbf{C}), GL(6, \mathbf{C}))$ the number of Harish-Chandra cells with irreducible constituents having associated variety of their annihilator given by $\overline{\mathcal{O}_{[4,4,2,2]}}$ is sixteen.

An algorithm

In this subsection we give an algorithm to determine the number of Harish-Chandra cell with a given associated variety of the annihilator block by block.

Let $\mathcal{O}_{[\ell]}$ be a special nilpotent orbit. Write \mathcal{B}_r for the block of $Sp(2n, \mathbb{R})$ modules dual to the principal block of $SO(n + 1 + r, n - r)$. The algorithm below computes

$$\#\{V^{HC} \text{ spanned by irred. modules in } \mathcal{B}_r : AV(ann(V^{HC})) = \overline{\mathcal{O}_{[\ell]}}\}.$$

- (1) First, when $n - r$ is even, count the number $Sp(\frac{n+r}{2}) \times Sp(\frac{n-r}{2})$ nilpotent orbits of shape $[\ell^*]$ (the number of $Sp(\frac{n+r+1}{2}) \times Sp(\frac{n-r-1}{2})$ orbits of shape $[\ell^*]$ if $n - r$ is odd, resp.) This number could be zero.
- (2) Next, list the partitions $[\mathbf{d}]$ with all parts even that can be obtained from $[\ell]$ by deleting horizontal and vertical dominos according to the rule in Proposition 4.1.
- (3) For each of the partitions $[\mathbf{d}]$ in step (2), compute the transpose partition $[\mathbf{d}^*]$. If $[\mathbf{d}^*]$ is a partition of $2(n - t)$ with t even, compute the number of $Sp(\frac{n+r-t}{2}) \times Sp(\frac{n-r-t}{2})$ nilpotent orbits of shape $[\mathbf{d}^*]$ if $n - r$ is even (the number

of $\mathrm{Sp}(\frac{n+r-t+1}{2}) \times \mathrm{Sp}(\frac{n-r-t-1}{2})$ orbits of shape $[\mathbf{d}^*]$ when $n-r$ is odd). If $[\mathbf{d}^*]$ is a partition of $2(n-t)$ with t odd, compute the number of $\mathrm{Sp}(\frac{n+r-t+1}{2}) \times \mathrm{Sp}(\frac{n-r-t-1}{2})$ nilpotent orbits of shape $[\mathbf{d}^*]$ if $n-r$ is even ($\mathrm{Sp}(\frac{n+r-t}{2}) \times \mathrm{Sp}(\frac{n-r-t}{2})$ orbits of shape $[\mathbf{d}^*]$ when $n-r$ is odd).

- (4) For each partition $[\mathbf{d}]$ in (2), by listing orbits in $\mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$, compute $2^{|\mathcal{T}_{[\mathbf{d}],[\ell]}|}$.
- (5) Multiply the numbers obtained in steps (3) and (4).
- (6) If $[\ell] \neq [2^{2k}1^{2n-2k}]$, add the results obtained in (1) and (5). If $[\ell] = [2^{2k}1^{2n-2k}]$, the desired number is the sum of the numbers obtained in steps (1) and (5) plus one.

Example E : Assume $G_{\mathbb{R}} = \mathrm{Sp}(6, \mathbb{R})$ and let $[\ell] = [2, 2, 2]$. We compute the number of Harish-Chandra cells in the top block of $G_{\mathbb{R}} = \mathrm{Sp}(6, \mathbb{R})$ with $AV(\mathrm{ann}(V^{HC})) = \overline{\mathcal{O}_{[\ell]}}$. Here $r = 0$.

- Step 1 of the algorithm yields $\mathbf{1}$, as there is exactly one $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ -nilpotent orbits of shape $[3, 3]$.
- Step 2 produces the set $\{[2, 2], [2]\}$.
- Step 3 yields $\mathbf{1}$ for $[2, 2]$ (as there is only one $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ nilpotent orbit of shape $[2, 2]$) and another $\mathbf{1}$ for $[2]$ (as there is only one $\mathrm{Sp}(1)$ orbit of shape $[1, 1]$).
- In order to implement step 4 of the algorithm we first observe that $\mathcal{SP}(\mathcal{O}_{[2,2]}) = \{\mathcal{O}_{[2,2]}, \mathcal{O}_{[2,1,1]}\}$ and $\mathcal{SP}(\mathcal{O}_{[2]}) = \{\mathcal{O}_{[2]}\}$. The only orbit that contributes non-trivially to this step is $\mathcal{O}_{[2,1,1]}$ and $2^{|\mathcal{T}_{[2,2],[2,2,2]}|} = 2$.
- Step 5 gives $([2, 2], \mathbf{2}); ([2], \mathbf{1})$.
- The number of Harish-Chandra cells in \mathcal{B}_0 with $AV(\mathrm{ann}(V^{HC})) = \overline{\mathcal{O}_{[2,2,2]}}$ is $1 + 2 + 1 + 1 = 5$.

6. Parametrization of Harish-Chandra cells

Assume $[\ell]$ is a partition of $2n$ parametrizing a special $\mathrm{Sp}(2n, \mathbf{C})$ -nilpotent orbit.

We show that $HC_{[\ell]} = \{\mathcal{C}^{HC} : AV(\mathrm{ann}(X)) = \overline{\mathcal{O}_{[\ell]}} \text{ for } X \in \mathcal{C}^{HC}\}$

is in bijection with a set $G_{[\ell]}$ consisting of triples $(\mathfrak{m}, \mathcal{O}, \mathcal{O}_{\mathbb{R}})$ where $\mathfrak{m} \subset \mathfrak{sp}(2n, \mathbf{C})$ is a Levi subalgebra of the form $\mathfrak{gl}(r) \oplus \mathfrak{sp}(2n-2(r))$; \mathcal{O} is a $\mathrm{Sp}(2n-2r, \mathbf{C})$ nilpotent orbit, and $\mathcal{O}_{\mathbb{R}}$ is a “real form of $d^{\vee}(\mathcal{O})$.” The set $G_{[\ell]}$ is suggested by the counting argument in the proof of Theorem 5.1.

From $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$ nilpotent orbits to $S(\mathbf{O}(2p+1) \times \mathbf{O}(2q))$ and $S(\mathbf{O}(2p) \times \mathbf{O}(2q+1))$ nilpotent orbits

In this subsection we specify the set of orbits that enter our parametrization of $HC_{[\ell]}$. Given $\mathcal{O}_{[\ell]}$ special we let $S([\ell])$ be as in Definition 4.4.

For $[\mathbf{d}] = [d_1, \dots, d_1, d_2, \dots, d_2, \dots, d_k, \dots, d_k] \in S([\ell])$ we consider the dual orbit $d^{\vee}(\mathcal{O}_{[\mathbf{d}]})$, see (1). The set of orbits relevant to our work consists of nilpotent orbits

\mathcal{O}_K where K is either $S(O(2p + 1) \times O(2q))$ or $S(O(2p) \times O(2q + 1))$ and such that $SO(2(p + q) + 1, \mathbf{C}) \cdot \mathcal{O}_K = d^\vee(\mathcal{O}_{[\mathbf{d}]})$. We explicitly describe this set in terms of signed Young diagrams.

Let $\{c_1, c_2, \dots, c_t\}$ be the distinct column length of the Young diagram $Y([\mathbf{d}])$. That is, the columns of $Y([\mathbf{d}])$ are

$$[c_1, \dots, c_1, c_2, \dots, c_2, \dots, c_t, \dots, c_t]. \tag{21}$$

Recall that we can also view (21) as the rows of $Y([\mathbf{d}^*])$ and that c_1 contributes to (21) d_k -times, c_2 contributes $d_{k-1} - d_k$ -times, etc.

The parameter attached to the nilpotent orbit $d^\vee(\mathcal{O}_{[\mathbf{d}]})$ is

$$[c_1 + 1, \dots, c_1, c_2, \dots, c_2, \dots, c_t, \dots, c_t]_{\mathbf{B}}, \tag{22}$$

where the subscript \mathbf{B} is the \mathbf{B} -collapse of the partition.

We explicitly relate the partitions in (21) and (22).

Lemma 6.1. *We keep the notation just introduced. If the columns of $Y([\mathbf{d}])$ are as in (21), then $d^\vee(\mathcal{O}_{[\mathbf{d}]})$ is parametrized by a partition with the following properties.*

- (1) *Its last entry is one.*
- (2) *For each even part $c_{i_1}, c_{i_1}, \dots, c_{i_1}$ contributing to (21) there is a part of the form $c_{i_1} + 1, c_{i_1}, \dots, c_{i_1}, c_{i_1} - 1$ contributing to (22). The integer c_{i_1} occurs $d_{k-i_1+1} - d_{k-i_1+2} - 2$ times.*
- (3) *For each odd part $c_{j_1}, c_{j_1}, \dots, c_{j_1}$ contributing to (21) there is a part of the form $c_{j_1}, c_{i_1}, \dots, c_{j_1}, c_{j_1}$ contributing to (22). The integer c_{j_1} occurs $d_{k-j_1+1} - d_{k-j_1+2}$ times.*

Assign signs to $Y([\mathbf{d}^*])$ to obtain the signed diagram corresponding to a $K_{p,q}$ -nilpotent orbit for some pair (p, q) with $p + q = |[\mathbf{d}]|$. (That is assign alternating signs to the rows of $Y([\mathbf{d}^*])$ so that for each chunk of even length rows, half start with a $+$ sign and the other half starts with a $-$ sign; for a chunk of rows of a given odd length there is an even number of rows starting with a $+$ sign.) To each such $K_{p,p}$ orbit we associate a $S(O(2p + 1) \times O(2p))$ -nilpotent orbit with $SO(4p + 1, \mathbf{C})$ saturation equal to $d^\vee(\mathcal{O}_{[\mathbf{d}]})$. To each such $K_{p,q}$ orbit with $p > q$ we associate a $S(O(2p + 1) \times O(2q))$ -nilpotent orbit and a $S(O(2p) \times O(2q + 1))$ -nilpotent orbit with $SO(2(p + q) + 1, \mathbf{C})$ saturation equal to $d^\vee(\mathcal{O}_{[\mathbf{d}]})$. We describe this set of orbits by their parametrizing signed Young diagram.

Case 1. Assume we start with a $K_{p,p}$ -orbit attached to a signed Young diagram of shape $Y([c_1, \dots, c_1, c_2, \dots, c_2, \dots, c_t, \dots, c_t]) = Y([\mathbf{d}^*])$. List the even length rows in decreasing order, $\{c_{i_0}, c_{i_1}, \dots\}$. The signed Young diagram of the resulting new $S(O(2p + 1) \times O(2p))$ -orbit has

- (a) for each chunk c_j, \dots, c_j with c_j odd, a chunk of rows of length c_j with the same sign distribution as that of the diagram attached to the $K_{p,p}$ -orbit;
- (b) a chunk of rows of length $c_{i_0} + 1, c_{i_0}, \dots, c_{i_0}, c_{i_0} - 1$ where the rows of length $c_{i_0} + 1$ and $c_{i_0} - 1$ start with a $+$ sign and half of the rows of length c_{i_0} start with a $-$ sign;

- (c) a chunk of rows of length $c_{i_1} + 1, c_{i_1}, \dots, c_{i_1}, c_{i_1} - 1$ where the rows of length $c_{i_1} + 1$ and $c_{i_1} - 1$ start with a $-$ sign and half of the rows of length c_{i_1} start with a $-$ sign;
- (d) the pattern of signs for even length chunk is repeated, i.e. the rows of length $c_{i_2} + 1$, and $c_{i_2} - 1$ start with a $+$ sign, and those of length $c_{i_3} + 1$, and $c_{i_3} - 1$ start with a $-$, etc.

Case 2. Assume that we start with a $K_{p,q}$ -orbit attached to a signed Young diagram of shape $Y([\mathbf{d}^*])$ and $p > q$. The new $S(O(2p + 1) \times O(2q))$ -nilpotent orbit is constructed as in Case 1.

Case 3. Assume that we start with a $K_{p,q}$ -orbit attached to a signed Young diagram of shape $Y([\mathbf{d}^*])$ and $p > q$. Then the signed Young diagram of the new $S(O(2p) \times O(2q + 1))$ -nilpotent orbit has

- (a) for each chance c_j, \cdot, c_j with c_j odd, a chunk of rows of length c_j with the same sign distribution as that of the diagram attached to the $K_{p,p}$ -orbit;
- (b) a chunk of rows of length $c_{i_0} + 1, c_{i_0}, \dots, c_{i_0}, c_{i_0} - 1$ where the rows of length $c_{i_0} + 1$ and $c_{i_0} - 1$ start with a $-$ sign and half of the rows of length c_{i_0} start with a $-$ sign;
- (c) a chunk of rows of length $c_{i_1} + 1, c_{i_1}, \dots, c_{i_1}, c_{i_1} - 1$ where the rows of length $c_{i_1} + 1$ and $c_{i_1} - 1$ start with a $+$ sign and half of the rows of length c_{i_1} start with a $-$ sign;
- (d) the pattern of signs for even length parts is repeated, i.e. the rows of length $c_{i_2} + 1$, and $c_{i_2} - 1$ start with a $-$ sign, etc.

Example F : When $[\mathbf{d}] = [4, 2]$, $[\mathbf{d}^*] = [2, 2, 1, 1]$ and the signed diagram

+	-
-	+
+	
+	

represents an $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ -nilpotent orbit. The signed Young diagrams of the corresponding $S(O(5) \times O(2))$ and $S(O(4) \times O(3))$ nilpotent orbits are

+	-	+		-	+	-
+				-		
+				+		
+				+		
-				+		

When $[\mathbf{d}] = [2, 2, 2]$, $[\mathbf{d}^*] = [3, 3]$ and the signed diagram

+	-	+
+	-	+

parametrizes an $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ -nilpotent orbit.

The signed Young diagrams of the corresponding $S(O(5) \times O(2))$ and $S(O(4) \times O(3))$ nilpotent orbits are

+	-	+
+	-	+
+		

+	-	+
+	-	+
-		

Harish-Chandra cells

Let $\mathcal{O}_{[\ell]}$ be a special nilpotent orbit. Assume $[\ell] \neq [2^{2k}1^{2n-2k}]$. Recall the set $S([\ell])$ of Definition 4.4. This is the set of partition $[\mathbf{d}]$ of $2n - 2r$ for some $r < n$ consisting solely of even parts and such that $[\sigma(\mathcal{O}_{[\ell]}, \mathbf{1}) : \text{ind}_{S_r \times W_{n-r}}^{W_n} (\text{sign} \otimes \sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}))] \neq 0$.

Definition 6.2. For $[\mathbf{d}] \in S([\ell])$ write $RF([\mathbf{d}])$ for the set consisting of all the $S(O(2p + 1) \times O(2q))$ and $S(O(2p) \times O(2q + 1))$ nilpotent orbits with $p + q = ||[\mathbf{d}]||$ attached to $Y([\mathbf{d}^*])$ by the procedure describes in subsection 6.

Remark 6.3. $\#RF([\mathbf{d}]) = \bigoplus_{\substack{\{p,q\} \\ p+q=||[\mathbf{d}]\}} [\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) : \mathcal{G}^{\text{Sp}(p,q)} \otimes \text{sign}]$. ■

Definition 6.4. Let $\mathcal{O}_{[\ell]}$ be a special nilpotent orbit. Assume $[\ell] \neq [2^{2k}1^{2n-2k}]$. Define

$$G_{[\ell]} = \left\{ (\mathfrak{m}, \mathcal{O}_{[\gamma]}, \mathcal{O}_K) : \begin{array}{l} \mathfrak{m} \subset \mathfrak{sp}(2n, \mathbf{C}) \text{ is a Levi subalgebra of the form} \\ \mathfrak{gl}(r) \oplus \mathfrak{sp}(2n - 2r); \mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]}) \text{ for } [\mathbf{d}] \in S([\ell]) \\ \text{with } ||[\mathbf{d}]|| = (2n - 2r) \text{ and } \gamma \text{ is linked to } [\ell]; \\ \mathcal{O}_K \in RF([\mathbf{d}]) \end{array} \right\}.$$

Remark 6.5. If $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$, then $d^\vee(\mathcal{O}_{[\gamma]}) = d^\vee(\mathcal{O}_{[\mathbf{d}]})$, see [5, Cor. A.3]. ■

Theorem 6.6. Let $\mathcal{O}_{[\ell]}$ be a special nilpotent orbit. Assume $[\ell] \neq [2^{2k}1^{2n-2k}]$. The set of Harish-Chandra cells

$$HC_{[\ell]} = \{ \mathcal{C}^{HC} : AV(\text{ann}(X)) = \overline{\mathcal{O}_{[\ell]}} \text{ for } X \in \mathcal{C}^{HC} \}$$

is in bijection with the set $G_{[\ell]}$ of Definition 6.4.

Proof. The Theorem follows from Theorem 5.1. The Harish-Chandra cell representations attached to cells in $HC_{[\ell]}$ are all isomorphic to the left cell representation $(W, V_{[\ell]}^{\text{Lusztig}})$. The cardinality of $HC_{[\ell]}$ is given by the multiplicity of the special representation $\sigma(\mathcal{O}_{[\ell]}, \mathbf{1})$ in $(W, \mathcal{G}^{\text{Sp}(2n, \mathbb{R})})$. By Theorem 5.1 and Remark 6.3, such multiplicity is

$$\sum_{r < n} \sum_{\substack{[\mathbf{d}] \in S([\ell]) \\ ||[\mathbf{d}]|| = 2n - 2r}} \#RF([\mathbf{d}]) \times \{ \mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]}) : \gamma \text{ is linked to } [\ell] \}. \quad \blacksquare$$

Example G : Assume $[\ell] = [2, 2]$. In this case, by Theorem 5.1, we have $|HC_{[2,2]}| = |G_{[2,2]}| + 1$ and

$$S([\ell]) = \{[2, 2]; [2]\}; \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]}) = \begin{cases} [2, 1, 1] & \text{if } [\mathbf{d}] = [2, 2] \\ [2] & \text{if } [\mathbf{d}] = [2]. \end{cases}$$

We have $\{\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]}) : [\mathbf{d}] \in S([\ell]) \text{ and } [\gamma] \text{ linked to } [\ell]\} = \{[2, 2], [2]\}$,
 $d^\vee(\mathcal{O}_{[2,2]}) = \mathcal{O}_{[3,1,1]}$ and $d^\vee(\mathcal{O}_{[2]}) = \mathcal{O}_{[1,1,1]}$.

The set $HC_{[2,2]}$ is parametrized by

$$\left\{ \left(\left(\mathfrak{sp}(4, \mathbf{C}), \mathcal{O}_{[2,2]}, \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & & \\ \hline - & & \\ \hline \end{array} \right); \left(\mathfrak{sp}(2, \mathbf{C}) \oplus \mathfrak{gl}(1, \mathbf{C}), \mathcal{O}_{[2]}, \begin{array}{|c|} \hline + \\ \hline + \\ \hline - \\ \hline \end{array} \right); \right. \\ \left. \left(\mathfrak{gl}(2, \mathbf{C}), \text{trivial}, \text{trivial} \right); \left(\mathfrak{sp}(2, \mathbf{C}) \oplus \mathfrak{gl}(1, \mathbf{C}), \mathcal{O}_{[2]}, \begin{array}{|c|} \hline + \\ \hline + \\ \hline + \\ \hline \end{array} \right) \right\}.$$

It is not difficult to see, using our counting argument, that the first triples parametrize Harish-Chandra cells in the block $\mathrm{Sp}(4, \mathbb{R})/\mathrm{SO}(3, 2)$ while the last triple corresponds to a cell in the block $\mathrm{Sp}(4, \mathbb{R})/\mathrm{SO}(4, 1)$.

There is one block of $(\mathfrak{sp}(4, \mathbf{C}), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$ Harish-Chandra modules with trivial infinitesimal character. There are two Harish-Chandra cells. To the cell parametrized by

$$\begin{array}{|c|c|} \hline + & - \\ \hline - & + \\ \hline \end{array}$$

we attach the Harish-Chandra cell of $(\mathfrak{sp}(4, \mathbf{C}), \mathrm{GL}(2))$ modules with parameter

$$\left(\mathfrak{sp}(4, \mathbf{C}), \mathcal{O}_{[2,2]}, \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & & \\ \hline - & & \\ \hline \end{array} \right).$$

To the cell containing the trivial representation for $\mathrm{Sp}(1, 1)$ we attach the Harish-Chandra cell of $(\mathfrak{sp}(4, \mathbf{C}), \mathrm{GL}(2, \mathbf{C}))$ modules containing the trivial representation. This cell has parameter equal to

$$\left(\mathfrak{sp}(2, \mathbf{C}) \oplus \mathfrak{gl}(1, \mathbf{C}), \mathcal{O}_{[2]}, \begin{array}{|c|} \hline + \\ \hline + \\ \hline - \\ \hline \end{array} \right),$$

(compare to Example A).

Example H : Assume $[\ell] = [2, 2, 2]$. In this case, by Theorem 5.1, we have $|HC_{[2,2,2]}| = |G_{[2,2,2]}| + 1$, and

$$\begin{aligned} & \{\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]}) : [\mathbf{d}] \in S([2, 2, 2]) \text{ and } [\gamma] \text{ linked to } [2, 2, 2]\} \\ & = \{[2, 2, 2], [2, 2], [2, 1, 1], [2]\}, \end{aligned}$$

and $d^\vee(\mathcal{O}_{[2,2,2]}) = \mathcal{O}_{[3,3,1]}$, $d^\vee(\mathcal{O}_{[2,2]}) = \mathcal{O}_{[3,1,1]}$, and $d^\vee(\mathcal{O}_{[2]}) = \mathcal{O}_{[1,1,1]}$.

The set $HC_{[2,2,2]}$ is parametrized by

$$\left\{ \left(\mathfrak{sp}(6, \mathbf{C}), \mathcal{O}_{[2,2,2]}, \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & - & + \\ \hline + & & \\ \hline \end{array} \right); \left(\mathfrak{sp}(6, \mathbf{C}), \mathcal{O}_{[2,2,2]}, \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & - & + \\ \hline - & & \\ \hline \end{array} \right); \right. \\ \left. \left(\mathfrak{sp}(4, \mathbf{C}) \oplus \mathfrak{gl}(1, \mathbf{C}), \mathcal{O}_{[2,2]}, \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & & \\ \hline - & & \\ \hline \end{array} \right); \right. \\ \left. \left(\mathfrak{sp}(4, \mathbf{C}) \oplus \mathfrak{gl}(1, \mathbf{C}), \mathcal{O}_{[2,1,1]}, \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + & & \\ \hline - & & \\ \hline \end{array} \right); \right. \\ \left. \left(\mathfrak{sp}(2, \mathbf{C}) \oplus \mathfrak{gl}(2, \mathbf{C}), \mathcal{O}_{[2]}, \begin{array}{|c|} \hline + \\ \hline + \\ \hline + \\ \hline \end{array} \right); \left(\mathfrak{sp}(2, \mathbf{C}) \oplus \mathfrak{gl}(2, \mathbf{C}), \mathcal{O}_{[2]}, \begin{array}{|c|} \hline + \\ \hline + \\ \hline - \\ \hline \end{array} \right) \right. \\ \left. (\mathfrak{gl}(3, \mathbf{C}), \text{trivial}, \text{trivial}) \right\}.$$

The first and fifth parameters in the set correspond to a Harish-Chandra cell in the block $\text{Sp}(6, \mathbb{R})/SO(5, 2)$.

7. Final remarks

In this section we use the results of prior sections to give a formula for the number of Unipotent representations attached to an even special orbit. For the definition of Unipotent representations see [1, Chapter 27].

We fix \mathcal{O} , an even $\text{Sp}(2n, \mathbf{C})$ nilpotent orbit. We attach to it a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u} \subset \mathfrak{sp}(2n, \mathbf{C})$ so that the simple roots in $\Delta(\mathfrak{m}, \mathfrak{h})$ are the roots with label zero in the weighted Dynkin diagram of \mathcal{O} . We write $I_{\mathfrak{p}}$ for the annihilator of the irreducible generalized Verma module induced from \mathfrak{p} with trivial infinitesimal character. Note that $AV(\text{ann}(I_{\mathfrak{p}})) = \overline{\mathcal{O}}$.

We fix $(\mathfrak{g}, K) = (\mathfrak{sp}(2n, \mathbf{C}), \text{GL}(n, \mathbf{C}))$. It follows from [1, Chapter 27], see also [7, Theorem 4.2] and [7, Corollary 3.17], that (for a fixed real form)

$$|\text{Unip}(\mathcal{O})| = \# \left\{ X : \begin{array}{l} X \text{ irreducible } (\mathfrak{g}, K) \text{ module with trivial} \\ \text{infinitesimal character } \text{ann}(X) = I_{\mathfrak{p}} \end{array} \right\}.$$

We know that the irreducible (\mathfrak{g}, K) -modules in Harish-Chandra cell at trivial infinitesimal character generate representations (W, V) of the Weyl group isomorphic to left cell representations. In particular, if the irreducible modules in a Harish-Chandra cell $\mathcal{C}_{\mathcal{O}}^{HC}$ have annihilators with associated variety $\overline{\mathcal{O}}$, then $(W, V) \simeq (W, V_{\mathcal{O}}^{Lusztig})$. By [18, Corollary 3],

$$\begin{aligned} & \# \{ X : X \text{ irreducible modules in } \mathcal{C}_{\mathcal{O}}^{HC} \text{ with } \text{ann}(X) = I_{\mathfrak{p}} \} \\ & = \dim \text{Hom}_W(V_{\mathcal{O}}^{Lusztig}, V_{I_{\mathfrak{p}}}^{left}). \end{aligned}$$

The Weyl group representation $V_{I_p}^{left}$ can be explicitly computed, see [5, Proposition 5.28]. It follows that $\dim \text{Hom}_W(V_{\mathcal{O}}^{Lusztig}, V_{I_p}^{left})$ equals the number of irreducible constituents in $V_{\mathcal{O}}^{Lusztig}$. That is $|\bar{A}(\mathcal{O})|$. Thus, $|\text{Unip}(\mathcal{O})| = |\bar{A}(\mathcal{O})| \times |HC_{\mathcal{O}}|$. The cardinality of the set $HC_{\mathcal{O}}$, that is the number of Harish-Chandra cells consisting of irreducible modules with $\bar{\mathcal{O}}$ as the associated variety of their annihilators is given in Theorem 5.1. (Barbasch computed $|\text{Unip}(\mathcal{O})|$ for a class of orbits \mathcal{O} known as smoothly cuspidal.)

Example I : In Example D we computed $HC_{\mathcal{O}_{[4,4,2,2]}} = 16$. One can verify that $|\bar{A}(\mathcal{O}_{[4,4,2,2]})| = 4$. Thus, $|\text{Unip}(\mathcal{O}_{[4,4,2,2]})| = 64$.

A. Appendix

Much of our work relies in understanding the structure of $(W_{C_n}, V_{\mathcal{O}_{[\mathbf{d}]}}^{Lusztig})$. For the convenience of the reader, in this appendix we include detailed computations of symbols parametrizing irreducible constituents of $(W_{C_n}, V_{\mathcal{O}_{[\mathbf{d}]}}^{Lusztig})$ for $[\mathbf{d}]$ a partition consisting solely of even parts. Proposition 2.16 below is used in our counting arguments in Section 4.

Lemma A.1. *Let $[\mathbf{d}] = [d_1, d_2, \dots, d_t]$ parametrize a special $Sp(2n, \mathbf{C})$ nilpotent orbit. Write $([\lambda] = [\lambda_1, \dots, \lambda_{k+1}], [\mu] = [\mu_1, \dots, \mu_k])$ for the pair of partitions attached to the special representation $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$.*

- (1) *If $t = 2s + 1$, then $k = s$ and $\lambda_{s+1} = \frac{d_{2s+1}}{2}$.*
- (2) *If $t = 2s$, then $k = s$ and $\lambda_{s+1} = 0$.*
- (3) *The integer d_{2i} is even if and only if d_{2i-1} is even.*
- (4) *If d_{2s-2j} is an odd integer, then $d_{2s-2j} = d_{2s-2j-1}$ and*

$$\lambda_{s-j} = \frac{d_{2s-2j-1} - 1}{2}; \quad \mu_{s-j} = \frac{d_{2s-2j} + 1}{2} = \lambda_{s-j} + 1.$$

- (5) *If d_{2s-2j} is an even number and $d_{2s-2j+1}$ is odd, then the symbol of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ has in its first row entries*

$$\dots a = \frac{d_{2s-2j+1} - 1}{2} + j \frac{d_{2s-2j-1}}{2} + (j+1) \dots$$

The second row of the corresponding symbol contains

$$\dots a = \frac{d_{2s-2j+2} + 1}{2} + (j-1) \frac{d_{2s-2j}}{2} + j \dots$$

Proof. By assumption $[\mathbf{d}]$ parametrizes a special nilpotent orbit. Since $\mathcal{O}_{[\mathbf{d}]}$ is a nilpotent orbit of type C_n , each odd part in $[\mathbf{d}]$ occurs an even number of times. By Definition 2.2, if d_{i-1} is even and d_i is odd, then i is an odd integer. These observations imply parts (1), (2) and (3) of the lemma. The computation of symbols is done by applying the algorithm in [8, Section 10]. \blacksquare

Lemma A.2. (1) Assume $[\mathbf{d}] = [d_1, d_2, \dots, d_{2s+1}]$ has all its parts even. The special representation $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ is parametrized by the pair of partitions

$$([\lambda] = [\frac{d_1}{2}, \frac{d_3}{2}, \dots, \frac{d_{2s-1}}{2}, \frac{d_{2s+1}}{2}], [\mu] = [\frac{d_2}{2}, \frac{d_4}{2}, \dots, \frac{d_{2s-2}}{2}, \frac{d_{2s}}{2}]).$$

The corresponding symbol is

$$\begin{pmatrix} \frac{d_{2s+1}}{2} & & \frac{d_{2s-1}}{2} + 1 & & \dots & & \frac{d_1}{2} + s \\ & \frac{d_{2s}}{2} & & \frac{d_{2s-2}}{2} + 1 & \dots & \frac{d_2}{2} + (s-1) & \end{pmatrix}.$$

(2) If $[\mathbf{d}] = [d_1, d_2, \dots, d_{2s}]$ has all its parts even then, the special representation $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ is parametrized by the pair of partitions

$$([\lambda] = [\frac{d_1}{2}, \frac{d_3}{2}, \dots, \frac{d_{2s-3}}{2}, \frac{d_{2s-1}}{2}], [\mu] = [\frac{d_2}{2}, \frac{d_4}{2}, \dots, \frac{d_{2s-2}}{2}, \frac{d_{2s}}{2}]).$$

The corresponding symbol is

$$\begin{pmatrix} 0 & & \frac{d_{2s-1}}{2} + 1 & & \frac{d_{2s-3}}{2} + 2 & & \dots & & \frac{d_1}{2} + s \\ & \frac{d_{2s}}{2} & & \frac{d_{2s-2}}{2} + 1 & \dots & \dots & \frac{d_2}{2} + (s-1) & \end{pmatrix}.$$

Proof. We follow the algorithm described in [8, Section 10]. We assume that $[\mathbf{d}] = [d_1, d_2, \dots, d_t]$ with $t = 2s + 1$ odd. The case $t = 2s$ can be treated in a similar manner. The symbol of $\sigma(\mathcal{O}_{[d_1, d_2, \dots, d_{2s+1}]}, \mathbf{1})$ is computed by (a) setting $e_i = d_{2s+2-i} + (i - 1)$, (b) building two increasing sequences from the set $\{e_i\}$ (one consisting of the even e_i 's, the other is formed from the odd e_i 's), and (c) defining the symbol

$$\begin{pmatrix} p_1 & p_2 & \dots & \dots & p_{r+1} \\ & q_1 & & q_2 \dots & q_r \end{pmatrix} \tag{23}$$

where $p_i = \frac{e_i}{2}$ if e_i is even and $q_i = \frac{e_i-1}{2}$ if e_i is odd. Under our assumptions e_i is even if and only if i is odd. ■

Proposition A.3. Assume $[\mathbf{d}] = [d_1, d_2, \dots, d_t]$ has all its parts even. Let $V_{\mathcal{O}_{[\mathbf{d}]}}^{Lusztig}$ be the Lusztig left cell representation containing $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$. Parametrize $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ by the pair of partitions $([\lambda] = [\lambda_1, \dots, \lambda_{t+1}], [\mu] = [\mu_1, \dots, \mu_t])$. Write

$$\begin{aligned} & \begin{pmatrix} p_1 & p_2 & \dots & p_{r+1} \\ & q_1 & q_2 & \dots & q_r \end{pmatrix} \\ & = \begin{pmatrix} \lambda_{t+1} & \lambda_t + 1 & \dots & & \lambda_1 + t + 1 \\ & \mu_t & \mu_{t-1} + 1 & \dots & \mu_1 + t \end{pmatrix} \end{aligned} \tag{24}$$

for its symbol.

- (1) $(W_{C_n}, V_{\mathcal{O}_{[\mathbf{d}]}}^{Lusztig})$ is multiplicity free.
- (2) If $[\pi: V_{\mathcal{O}_{[\mathbf{d}]}}^{Lusztig}] \neq 0$ for $\pi \in \widehat{W}_{C_n}$, then the symbol of π is obtained from the symbol (24) by exchanging entries (p_i, q_i) with $p_i \neq q_i$.

- (3) If the pair of partitions $([\tilde{\lambda}] = [\tilde{\lambda}_1, \dots, \tilde{\lambda}_{t+1}], [\tilde{\mu}] = [\tilde{\mu}_1, \dots, \tilde{\mu}_t])$ parametrizes π , a constituent of $V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}}$, then there is a set of indexes S such that $\tilde{\mu}_j = \lambda_{j+1}$, $\tilde{\lambda}_{j+1} = \mu_j$ when $j \in S$, and $\tilde{\mu}_k = \mu_k$, $\tilde{\lambda}_{k+1} = \lambda_{k+1}$ when $k \notin S$.
- (4) Assume π is irreducible and $[\pi : V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}}] \neq 0$. Then $\pi \simeq \sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$ for some orbit $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$ if and only if π is parametrized by a pair of partitions $([\tilde{\lambda}] = [\tilde{\lambda}_1, \dots, \tilde{\lambda}_{t+1}], [\tilde{\mu}] = [\tilde{\mu}_1, \dots, \tilde{\mu}_t])$ such that $\tilde{\mu}_i \geq \tilde{\lambda}_{i+1} - 1$ for all i .

Proof. For classical groups, left cell representations are always multiplicity free, see [14, Chapter 20]. As $\mathcal{O}_{[\mathbf{d}]}$ is a special orbit, if $\{p_1, \dots, p_{r+1}, q_1, \dots, q_r\}$ are the entries in the symbol of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$, then

$$p_1 \leq q_1 \leq p_2 \leq \dots \leq q_t \leq p_{t+1}. \tag{25}$$

By Lemma A.2, for each index i ,

$$(p_{i+1}, q_i) = \left(\frac{d_{2(s-i)+1}}{2} + i, \frac{d_{2(s-i)+2}}{2} + (i - 1) \right).$$

Thus, $p_{i+1} \neq q_i$. In other words, if q_i occurs on the first row of the symbol, then $q_i = p_i$. Select the collection of pairs of integers (p_j, q_j) with $j = 1, \dots, t$. The resulting list is an admissible pairing in the sense of [6, Section 13.2]. The symbols of the irreducible constituents of $V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}}$ are of the form

$$\begin{pmatrix} r_1 & r_2 & \cdots & \cdots & r_{t+1} \\ & s_1 & & s_2 & \cdots & s_t \end{pmatrix} \tag{26}$$

where as sets we have $\{r_1, \dots, r_{t+1}, s_1, \dots, s_t\} = \{p_1, \dots, p_{t+1}, q_1, \dots, q_t\}$, and the list $\{s_1, \dots, s_t\}$ contains exactly one number from the t -selected pairs; see [6, Section 13.2]. Our statement (2) about the shape of the symbols of irreducible constituents follows. For (3), we use the bijection in Theorem 2.18 to translate the description of symbols of constituents of $V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}}$ to the language of pairs of partitions.

In order to prove part (4) we assume that $[\mathbf{d}] = [d_1, d_2, \dots, d_{2s+1}]$, i.e., $t = 2s + 1$ is odd. The case $t = 2s$ can be treated in a similar manner. The proof is a computation that depends on the shape of the orbits $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$, and on the algorithm that attaches to $[\mathbf{d}] = [d_1, d_2, \dots, d_{2s+1}]$ a symbol (the pair of partitions parametrizing $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$ can be read from the corresponding symbol.) The description of $\mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$ follows from [8, Lemma 6.3.9]. The recipe to compute relevant symbols was used in the proof of Lemma A.2, see [8, Chapter 10].

Let $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$. We compute the symbol of $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$. By Proposition 2.15, we have $[\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1}) : V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}}] \neq 0$. Thus, by Part (2) of Proposition A.3, the symbol of $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$ is obtained from that of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ by exchanging some pairs (p_k, q_k) . We need to identify this set of pairs. It follows from the proof of [8, Lemma 6.3.9] that

- (a) $\mathcal{O}_{[\gamma]} = \mathcal{O}_{[\gamma_1, \gamma_2, \dots, \gamma_{2s+1}]}$ has $\gamma_{2s+1} \neq 0$, and
- (b) there exists a set of indexes S_γ such that

$$\begin{aligned} d_{2j} &= d_{2j+1} + 2 \text{ when } j \in S_\gamma \\ \gamma_{2j+1} &= d_{2j+1} + 1; \gamma_{2j} = d_{2j} - 1 \text{ when } j \in S_\gamma \\ \gamma_{2k} &= d_{2k}; \gamma_{2k+1} = d_{2k+1} \text{ when } k \notin S_\gamma. \end{aligned} \tag{27}$$

The symbol of $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$ is computed from the set $\{\widehat{e}_k = \gamma_{2s+2-k} + (k-1)\}$. Note that when $j \in S_\gamma$

$$\begin{aligned} \widehat{e}_{2s+2-2j} &= \gamma_{2j} + (2s+1-2j) = d_{2j} + (2s-2j), \text{ an even integer} \\ \widehat{e}_{2s+2-(2j+1)} &= \gamma_{2j+1} + (2s-2j) = d_{2j+1} + (2s-2j+1), \text{ an odd integer.} \end{aligned}$$

Hence, when $j \in S_\gamma$, the symbol of $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$ has

$$\widehat{p}_{s-j+1} = \frac{d_{2j}}{2} + (s-j), \quad \text{and} \quad \widehat{q}_{s-j+1} = \frac{d_{2j+1}}{2} + (s-j).$$

In other words, the symbol of $\sigma(\mathcal{O}_{[\gamma]}, \mathbf{1})$ is obtained from that of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ by exchanging $(\widehat{p}_{s-j+1}, \widehat{q}_{s-j+1})$ with $j \in S_\gamma$. We relate symbols to pairs of partitions as in Theorem 2.18, to conclude that $\lambda_{k+1} = \lambda_{k+1}$, $\tilde{\mu}_k = \mu_k$ when $k \notin S_\gamma$, and $\tilde{\lambda}_{j+1} = \frac{d_{2j}}{2}$, $\tilde{\mu}_j = \frac{d_{2j+1}}{2} = \frac{d_{2j}}{2} - 1$ when $j \in S_\gamma$. Now, Remark 2.19 applied to $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$, implies that $\tilde{\mu}_k \geq \tilde{\lambda}_{k+1}$ when $k \notin S_\gamma$ and $\tilde{\mu}_j = \tilde{\lambda}_{j+1} - 1$ when $j \in S_\gamma$.

We reverse this argument to show that the non-special irreducible constituents $\pi(\tilde{\lambda}, \tilde{\mu})$ of $V_{\mathcal{O}_{[\mathbf{d}]}}^{\text{Lusztig}}$ having $\tilde{\mu}_j \geq \tilde{\lambda}_{j+1} - 1$ are isomorphic to $\sigma(\mathcal{O}_{[m_1, m_2, \dots, m_t]}, \mathbf{1})$ for some $[m_1, m_2, \dots, m_t]$ satisfying (27). By [8, Lemma 6.3.9] we conclude that $\mathcal{O}_{[m_1, m_2, \dots, m_t]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$. ■

Corollary A.4. *Assume $[\mathbf{d}]$ has all its parts even. Let $\mathcal{O}_{[\gamma]} \in \mathcal{SP}(\mathcal{O}_{[\mathbf{d}]})$. Write $[\gamma] = [\gamma_1, \gamma_2, \dots, \gamma_m]$. If*

$$\begin{aligned} \sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) &= \pi([\lambda_1, \dots, \lambda_{t+1}], [\mu_1, \dots, \mu_t]) \text{ and} \\ \sigma(\mathcal{O}_{[\gamma]}, \mathbf{1}) &= \pi([\tilde{\lambda}_1, \dots, \tilde{\lambda}_{t+1}], [\tilde{\mu}_1, \dots, \tilde{\mu}_t]), \end{aligned}$$

then there exists a set of indexes $S_{[\gamma]} = \{i_1, i_2, \dots, i_m\}$ so that for $i_k \in S_{[\gamma]}$

$$\begin{aligned} \tilde{\mu}_{i_k} &= \lambda_{i_k+1} = d_{2i_k+1}/2 = \tilde{\lambda}_{i_k+1} - 1 \\ \tilde{\lambda}_{i_k+1} &= \mu_{i_k} = d_{2i_k}/2 \\ d_{2i_k+1} &= d_{2i_k} - 2 \\ \gamma_{2i_k+1} &= \gamma_{2i_k} = d_{2i_k} - 1. \end{aligned} \tag{28}$$

When $j \notin S_{[\gamma]}$ we have $\mu_j = \tilde{\mu}_j$, $\lambda_j = \tilde{\lambda}_j$, $d_{2j} = \gamma_{2j}$, $d_{2j+1} = \gamma_{2j+1}$.

Proof. By (3) of Proposition A.3, there exists a set of indexes $S_{[\gamma]}$ so that $\tilde{\mu}_{i_k} = \lambda_{i_k}$, and $\tilde{\lambda}_{i_k+1} = \mu_{i_k}$ when $i_k \in S_{[\gamma]}$. The computation of the symbol of $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1})$ in Lemma A.2 gives $2\lambda_{i_k+1} = d_{2i_k+1}$; $\mu_{i_k} = 2d_{2i_k}$. Part (4) of Proposition A.3 implies that $\tilde{\mu}_{i_k} = \tilde{\lambda}_{i_k+1} - 1$. Hence, $d_{2i_k+1} = d_{2i_k} - 2$. Now, the identity $\gamma_{2i_k+1} = \gamma_{2i_k} = d_{2i_k} - 1$ follows from Lemma A.1. ■

The following technical lemma is used in the proof of Proposition 4.1 in the main body of the paper.

Lemma A.5. *Let $\sigma(\mathcal{O}_{[\mathbf{d}]}, \mathbf{1}) = \pi([\lambda_1, \lambda_2, \dots, \lambda_{t+1}], [\mu_1, \dots, \mu_t])$ be a special representation. Assume that $[\mathbf{d}]$ has all its parts even. For $k > t$ set $a_{k,t} = k - t$ and let $([\theta], [\psi]) = ([\theta_1, \dots, \theta_{k+1}], [\psi_1, \dots, \psi_k])$ be a pair of partitions so that*

$$\begin{aligned}\theta_j &= \lambda_j \text{ when } j = 1, \dots, t+1, \\ \psi_j &= \mu_j \text{ when } j = 1, \dots, t, \\ \psi_j &= 1 \text{ when } t+1 < j \leq k, \text{ and} \\ \theta_j &= 0 \text{ or } 1 \text{ when } j > t+1.\end{aligned}$$

Then, $\pi([\theta], [\psi])$ is a special representation. Moreover,

(1) *If t is odd and $\theta_j = 0$ when $j > t+1$, then the symbol of $\pi([\theta], [\psi])$ is*

$$\begin{pmatrix} 0 & 1 & 2 \cdots & \lambda_{t+1} + a_{k,t} & & \lambda_t + (k-t+1) \cdots \lambda_1 + k \\ 1 & 2 \cdots & a_{k,t} & & \mu_t + (k-t) & \cdots \mu_1 + (k-1) \end{pmatrix}.$$

(2) *If t is odd, $\theta_j = 1$ for $t+1 < j \leq t+m$, and $\theta_j = 0$ when $j \geq t+m+1$ then the first row in the symbol of $\pi([\theta], [\psi])$ is*

$$0 \ 1 \ 2 \cdots a_{k,t+m} - 1 \ a_{k,t+m} + 1 \cdots a_{k,t} \ \lambda_{t+1} + a_{k,t} \cdots \lambda_1 + k.$$

The second row is

$$1 \ 2 \cdots a_{k,t+m} - 1 \ a_{k,t+m} \cdots (a_{k,t} - 1)a_{k,t} \ \mu_t + a_{k,t} \cdots \mu_1 + (k-1),$$

where $a_{k,t+m} = k - t - m$.

(3) *Assume t is odd and $\theta_j = 1$ for all $j > t+1$. Then the first row of the symbol of $\pi([\theta], [\psi])$ is*

$$\begin{pmatrix} 1 & 2 & \dots & & a_{k,t} & \lambda_{t+1} + a_{k,t} & \lambda_t + (a_{k,t} + 1) & \lambda_1 + k \\ & 1 & 2 & \dots & a_{k,t} - 1 & a_{k,t} & \mu_t + a_{k,t} & \dots \mu_1 + (k-1) \end{pmatrix}.$$

(4) *If t is even and $\theta_j = 0$ when $j \geq t+1$, then the symbol of $\pi([\theta], [\psi])$ is*

$$\begin{pmatrix} 0 & 1 & 2 & \dots & a_{k,t} & \lambda_t + a_{k,t} + 1 & \lambda_{t-1} + a_{k,t} + 2 & \lambda_1 + k \\ & 1 & 2 & \dots & a_{k,t} & & \mu_t + a_{k,t} & \dots \mu_1 + (k-1) \end{pmatrix}.$$

(5) *If t is even, $\theta_j = 1$, $t+1 \leq j \leq t+m$, and $\theta_j = 0$ for $j > t+m$, then the first row in the symbol of $\pi([\theta], [\psi])$ is*

$$0 \ 1 \ 2 \cdots a_{t,m} [a_{k,t+m}] + 2 \cdots a_{k,t} + 1 \ \lambda_t + a_{k,t} + 1 \ \lambda_{t-1} + (a_{k,t} + 2) \cdots \lambda_1 + k.$$

The second row is

$$1 \ 2 \cdots a_{k,t+m} [a_{k,t+m} + 1] \cdots a_{k,t} \ \mu_t + a_{k,t} \cdots \mu_1 + (k-1).$$

(6) *Assume t is even and $\theta_j = 1$ for all $j \geq t+1$. Then the first row of the symbol of $\pi([\theta], [\psi])$ is*

$$0 \ 2 \ 3 \cdots a_{k,t} \ a_{k,t} + 1 \ \lambda_t + a_{k,t} + 1 \cdots \lambda_1 + k.$$

The second row is

$$1 \ 2 \cdots a_{k,t} - 1 \ a_{k,t} \ \mu_t + a_{k,t} \cdots \mu_1 + (k-1).$$

Proof. This is an easy application of part (2) in Theorem 2.18. ■

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