

On Compact Abelian Lie Groups of Homeomorphisms of \mathbb{R}^m

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Abstract. Let G be a compact Lie group of homeomorphisms of \mathbb{R}^m . The Naive conjecture saying that G is conjugate to a subgroup of the orthogonal group $O(m)$ is known to be false for higher dimension. In this paper we give a partial answer by considering the action of the group $S = S(K_1) \times \dots \times S(K_q)$ on $\mathbb{R}^m = K_1 \oplus \dots \oplus K_q$; where $K_i = \mathbb{R}$ or \mathbb{C} and $S(K_i) = \{x \in K_i : |x| = 1\}$ for $1 \leq i \leq q$, and we show that G is contained in S if and only if every element of G centralizes S .

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1. Introduction

For $m \in \mathbb{N}$, let $\text{Homeo}(\mathbb{R}^m)$ be the group of all homeomorphisms of the Euclidean space \mathbb{R}^m , equipped with the compact-open topology. In all this paper, a compact group of homeomorphisms of \mathbb{R}^m means a topological subgroup of $\text{Homeo}(\mathbb{R}^m)$ which is compact. The orthogonal group of \mathbb{R}^m is denoted by $O(m)$. Two subgroups G_1 and G_2 of $\text{Homeo}(\mathbb{R}^m)$ are said to be *conjugate* if there exists $\alpha \in \text{Homeo}(\mathbb{R}^m)$ such that

$$G_1 = \alpha G_2 \alpha^{-1}.$$

Naive conjecture ([6]): Let G be a compact Lie group of homeomorphisms of \mathbb{R}^m , then G is conjugate to a subgroup of $O(m)$.

It is well known that the conjecture is true for $m = 2$, and it is false for $m = 3$ (Bing's example, [2]). However, in dimension 3, there are partial results; the Naive conjecture is true if G is connected ([5]) or if G is a group of diffeomorphisms ([6]). For $m \geq 4$, the conjecture is known to be false in either category ([4]). In this paper, we continue to give some partial results on the Naive conjecture. Some notation introduced in this section will be used in Section 2.

The *centralizer* $C(H)$ of a subset H of $\text{Homeo}(\mathbb{R}^m)$ is defined by:

$$C(H) = \{g \in \text{Homeo}(\mathbb{R}^m) : gh = hg, \forall h \in H\}.$$

So, an element g centralizes H if $g \in C(H)$.

For a homeomorphism h of a metric space E , the set of fixed points of h is denoted by $Fix(h)$, and for a subgroup $G \subset Homeo(E)$ the set of fixed points of G is

$$Fix(G) = \{x \in E : g(x) = x, \forall g \in G\}.$$

A subset $B \subset E$ is said to be *invariant* by h (resp. by G) if $h(B) = B$ (resp. if $g(B) = B$ for every $g \in G$). The identity map of E is denoted by Id_E . The boundary of a manifold M is denoted by ∂M , and the notation \mathbb{R}_+ is defined by

$$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}.$$

For $K = \mathbb{R}$ or $\mathbb{C} \cong \mathbb{R}^2$, let $S(K) = \{x \in K : |x| = 1\}$. Let m be any integer ≥ 1 , and let q be an integer ≥ 1 such that the Euclidean space \mathbb{R}^m is identified to the vector space $V = K_1 \oplus K_2 \oplus \dots \oplus K_q$, where $K_i = \mathbb{R}$ or \mathbb{C} for $1 \leq i \leq q$. The group

$$S = S(K_1) \times S(K_2) \times \dots \times S(K_q)$$

acts on V in the natural way. Clearly S is a compact abelian subgroup of the orthogonal group $O(m)$.

The main result of the paper (Theorem 2.2) states that: a compact Lie group G of homeomorphisms of $\mathbb{R}^m = V$ is contained in S if and only if every element of G centralizes S . The main ingredient of the proof is the following theorem:

Theorem 1.1. (*Newman's theorem* [3]) *Let G be a compact Lie group acting effectively on a connected topological manifold M , then the subset $Fix(G)$ is nowhere dense.*

As a consequence of Theorem 1.1 one can deduce that if g is a homeomorphism of a connected manifold M with non-empty boundary ∂M such that g is of finite order and coincides with the identity on the boundary, then g is the identity. This is seen by glueing a second copy M' of M to M along the boundary ∂M and extending g by the identity on M' .

2. Characterization of some compact Lie groups of homeomorphisms of \mathbb{R}^m

This section contains a proof of the main result. We consider the special subgroups S of the orthogonal group $O(m)$ defined in Section 1. We use the same notation as in Section 1. We begin by proving the following proposition.

Proposition 2.1. *Let h be a homeomorphism of \mathbb{R}^m of finite order. If h centralizes S , then h is contained in S .*

Proof. The proof contains two steps. In the first step we prove that h induces the identity map on the orbit space \mathbb{R}^m/S , and in the second step we prove that such a homeomorphism must be an element of S .

Step 1 is by finite induction on q . For $q = 1$, we have $V = K_1 = \mathbb{R}$ or \mathbb{C} . If $K_1 = \mathbb{R}$, then $S(K_1) = \{\pm 1\}$ and $h(0) = 0$ since h centralizes S . The map \bar{h} induced by h on the orbit space $\mathbb{R}/S = \mathbb{R}_+$ is of finite order and fixes the boundary of \mathbb{R}_+ hence it is the identity by Theorem 1.1. If $K_1 = \mathbb{C}$, then $S(K_1)$ is the group

of rotations of the plane and h fixes the origin which is the boundary of the orbit space $\mathbb{C}/S = \mathbb{R}_+$. Then by the same argument as in the preceding case h induces the identity map on the orbit space \mathbb{C}/S . So Step 1 is true for $q = 1$. Assume that the result of Step 1 is true up to $q - 1$; we will show that it is also true for q . Put $V_1 = K_2 \oplus \dots \oplus K_q$, $V_q = K_1 \oplus \dots \oplus K_{q-1}$, and for every $2 \leq i \leq q - 1$, $V_i = K_1 \oplus \dots \oplus K_{i-1} \oplus K_{i+1} \oplus \dots \oplus K_q$. For every $1 \leq i \leq q$ there exists $s_i \in S$ such that $Fix(s_i) = V_i$. Then $h(V_i) = V_i$ since h centralizes S . On the other hand, V_i is invariant by every element of S . Clearly the restriction $h|_{V_i}$ centralizes the restriction $S|_{V_i}$. Then by induction the homeomorphism \bar{h} of V/S induced by h coincides with the identity on every subset V_i/S ($1 \leq i \leq q$) and it is of finite order since h is. Obviously, $V/S = \mathbb{R}_+^q$, $V_1/S = \{0\} \times \mathbb{R}_+^{q-1}$, $V_q/S = \mathbb{R}_+^{q-1} \times \{0\}$, and for every $2 \leq i \leq q - 1$, $V_i/S = \mathbb{R}_+^{i-1} \times \{0\} \times \mathbb{R}_+^{q-i}$. Then $\partial(V/S) = \cup_{i=1}^q V_i/S$. Hence \bar{h} is the identity by Theorem 1.1.

Now, we prove Step 2. We know that h induces the identity map on the orbit space \mathbb{R}^m/S by Step 1. So h maps every orbit O under S to itself and since h commutes with the action of S , h acts on each orbit $O \in \mathbb{R}^m/S$ by an element $s(O) \in S$ since h commutes with the action of S . For every O in the interior A of the orbit space \mathbb{R}^m/S the element $s(O)$ is unique and of finite order dividing the order of h . So, we have a map of A to a finite subgroup of S . This map is continuous since h is, hence constant since A is connected. The latter means that there is a unique element $s \in S$ such that h acts like s on an open dense subset of \mathbb{R}^m . Thus $h = s$ on all of \mathbb{R}^m . ■

Theorem 2.2. *Let G be a compact Lie group of homeomorphisms of \mathbb{R}^m . Then G is contained in S if and only if every element of G centralizes S .*

Proof. Necessity follows from the fact that S is an abelian group. To prove sufficiency, assume that every element of G centralizes S . Since G is a compact Lie group, it contains a dense subset of elements of finite order. So, the result is true by Proposition 2.1. ■

As a consequence of Theorem 2.2, we obtain in the following corollary a condition under which a compact Lie group of homeomorphisms of \mathbb{R}^m is conjugate to some special subgroup of the orthogonal group $O(m)$; these are the subgroups of the groups S whose form is given in Section 1.

Corollary 2.3. *Let G be a compact Lie group of homeomorphisms of \mathbb{R}^m . Then G is conjugate to a subgroup of S if and only if for some $\alpha \in Homeo(\mathbb{R}^m)$, every element of $\alpha G \alpha^{-1}$ centralizes S .*

3. Examples

In this section, we give two examples showing the importance of the condition “centralizes S ” in Theorem 2.2.

Example 3.1. Here we give a compact abelian Lie subgroup G of $Homeo(\mathbb{R}^7)$ such that every element of G centralizes the group $L = \{Id_{\mathbb{R}^4}\} \times S(\mathbb{C}) \times \{Id_{\mathbb{R}}\}$, but G cannot be conjugate to a subgroup of the orthogonal group $O(7)$. Indeed, the group L does not have the same form as the group S defined in Section 1.

Let W be the Whitehead manifold. Since $W \times \mathbb{R}$ is homeomorphic to \mathbb{R}^4 , there is a homeomorphism

$$\begin{aligned} \varphi : \quad \mathbb{R}^4 \times \mathbb{R}^3 &\longrightarrow W \times \mathbb{R} \times \mathbb{R}^3 \\ (x, y_1, y_2, y_3) &\longmapsto (\varphi_1(x), \varphi_2(x), y_3, y_1, y_2); \end{aligned}$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, and $(y_1, y_2, y_3) \in \mathbb{R}^3$.

Consider the group $G = \varphi^{-1}(\{Id_W\} \times S(\mathbb{C}) \times S(\mathbb{C}))\varphi$. Then G is a compact abelian Lie subgroup of $\text{Homeo}(\mathbb{R}^7)$, moreover it is easy to prove that every element of G centralizes the group $L = Id_{\mathbb{R}^4} \times S(\mathbb{C}) \times Id_{\mathbb{R}}$. However, G cannot be conjugate to a subgroup of $O(7)$ since $\text{Fix}(G) = \varphi^{-1}(W \times \{0_{\mathbb{R}^4}\})$ which is homeomorphic to W and not homeomorphic to \mathbb{R}^3 .

Example 3.2. In this example we give a compact abelian Lie subgroup G of $\text{Homeo}(\mathbb{R}^7)$ such that every element of G centralizes the group $\{Id_{\mathbb{R}^4}\} \times S(\mathbb{C}) \times S(\mathbb{R})$, but G cannot be conjugate to a subgroup of $O(7)$.

With the notation of Example 3.1, define the group

$$G = \varphi^{-1}(\{Id_W\} \times S(\mathbb{R}) \times S(\mathbb{R}) \times S(\mathbb{C}))\varphi.$$

Then G is a compact abelian Lie subgroup of $\text{Homeo}(\mathbb{R}^7)$. Moreover, it is easy to see that every element of G centralizes the group $\{Id_{\mathbb{R}^4}\} \times S(\mathbb{C}) \times S(\mathbb{R})$. However, $\text{Fix}(G) = \varphi^{-1}(W \times \{0_{\mathbb{R}^4}\})$, which means that G cannot be conjugate to a subgroup of $O(7)$.

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