Spinorial Representations of Orthogonal Groups

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Abstract.   Let G be a real compact Lie group, such that $G = G^0 \rtimes C_2$, with G^0 simple. Here G^0 is the connected component of G containing the identity and C_2 is the cyclic group of order 2. We give criteria for whether an orthogonal representation $\pi \colon G \to \mathrm{O}(V)$ lifts to $\mathrm{Pin}(V)$ in terms of the highest weights of π and also in terms of character values. From these criteria we compute the first and second Stiefel-Whitney classes of the representations of the orthogonal groups.

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1. Introduction

Let G be a real compact Lie group such that $G = G^0 \rtimes C_2$, where G^0 is its connected component containing the identity and C_2 denotes the cyclic group of order 2. We take $C_2 = \{1, g_0\}$, where conjugation action of g_0 on G^0 gives a diagram automorphism of G^0. We in particular consider the groups G with G^0 simple of type A_n, D_n and E_6. These are of interest because other types do not admit a nontrivial diagram automorphism.

We call a real (resp. complex) representation (π, V) of G orthogonal if its image lies inside $\mathrm{O}(V)$, the real (resp. complex) orthogonal group. We know that $\mathrm{Pin}(V)$ is a topological double cover of $\mathrm{O}(V)$. Let $\rho : \mathrm{Pin}(V) \to \mathrm{O}(V)$ denote the covering map. An orthogonal representation π of G is spinorial if there exists a Lie group homomorphism $\widehat{\pi} : G \to \mathrm{Pin}(V)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
 & & \mathrm{Pin}(V) \\
 & \nearrow{\scriptstyle \widehat{\pi}} & \big\downarrow{\scriptstyle \rho} \\
G & \xrightarrow{\ \pi\ } & \mathrm{O}(V)
\end{array}
$$

i.e. $\rho \circ \widehat{\pi} = \pi$. We write $\mathrm{O}(n)$ (resp. $\mathrm{SO}(n)$) for $\mathrm{O}(n, \mathbb{R})$ (resp. $\mathrm{SO}(n, \mathbb{R})$). For real representations we take the quadratic form

$$
Q(x_1, x_2, \ldots, x_n) = -\sum x_i^2,
$$

and consider the corresponding real Pin group.

For any orthogonal complex representation π of a compact group G there exists a real representation π_0 such that $\pi \cong \pi_0 \otimes \mathbb{C}$. For details we refer the reader to [3, Chapter 2, Section 6]. Note that the representation π is spinorial if and only if π_0 is spinorial.

The irreducible representations of $G = G^0 \rtimes C_2$ arise in the following way. Take an irreducible representation (π^λ, V^λ) of G^0, parameterized by the highest weight λ. Denote the highest weight of the representation $\phi(x) = \pi^\lambda(g_0 x g_0^{-1})$ by $g_0 \cdot \lambda$. Consider the representation $\rho^\lambda = \mathrm{Ind}_{G^0}^G(\pi^\lambda)$. There are two possibilities.

Type I: The representation ρ^λ is irreducible. In this case we have $g_0 \cdot \lambda \neq \lambda$ and

$$\rho^\lambda \mid_{G^0} = \pi^\lambda \oplus \pi^{g_0 \cdot \lambda}.$$

Type II: The representation ρ^λ is reducible and $\rho^\lambda = \pi^{\lambda,+} \oplus \pi^{\lambda,-}$, such that $\dim \pi^{\lambda,+} = \dim \pi^{\lambda,-} = \dim \pi^\lambda$. In this case we have $g_0 \cdot \lambda = \lambda$ and

$$\pi^{\lambda,\pm} \mid_{G^0} = \pi^\lambda.$$

In fact every irreducible representation of G is either of Type I or Type II. From [9] we obtain a criterion for spinoriality of reductive connected algebraic groups over a field of characteristic zero. The criterion appears as the first condition in Theorem 1.1. We write $\mathfrak{g}$ for the Lie algebra of G. Let $\mathfrak{g}$ be simple. In the following theorem $p(\underline{\nu})$ is a certain constant related to group G^0 and $\chi_\lambda(C)$ is the trace of the Casimir element for the representation of G^0 with highest weight λ. For details we refer Sections 2 and 4 of this paper.

Theorem 1.1.   *An orthogonal representation of G of Type I is spinorial if and only if both the following conditions hold:*

(1) $p(\underline{\nu}) \cdot (\dim V^\lambda) \cdot \left(\dfrac{\chi_\lambda(C) + \chi_{g_0 \cdot \lambda}(C)}{\dim \mathfrak{g}} \right) \equiv 0 \pmod 2$,

(2) $\dim V^\lambda \equiv 0 \ or \ 3 \pmod 4$.

Theorem 1.2.   *An orthogonal representation $\pi^{\lambda,\pm}$ of G of Type II is spinorial if and only if both the following conditions hold:*

(1) $\dfrac{p(\underline{\nu}) \cdot \dim V^\lambda \cdot \chi_\lambda(C)}{\dim \mathfrak{g}} \equiv 0 \pmod 2$.

(2) $\dim V^\lambda - \chi_{\pi^{\lambda,\pm}}(g_0) \equiv 0 \ or \ 6 \pmod 8$.

We also provide a formula for $\chi_{\pi^{\lambda,\pm}}(g_0)$ in terms of the highest weight λ of the representation of G^0 (see Theorem 6.2).

We denote by w_1 and w_2 the first and second Stiefel-Whitney classes of π. Then an orthogonal representation π of G is spinorial if and only if $w_2(\pi) + w_1(\pi) \cup w_1(\pi) = 0$. Let m denote the multiplicity of -1 as an eigenvalue of $\pi^{\lambda,\pm}(g_0)$. Then for representations of Type II we have $m = (\dim V^\lambda - \chi_\pi(g_0))/2$. We use these results to compute the second Stiefel-Whitney classes for representations $\pi = \pi^{\lambda,\pm}$ of $\mathrm{O}(n)$ for $n \geq 4$:

$$w_2(\pi_0) = \frac{2(n-1)}{n(2n-1)} \cdot \dim V^\lambda(\lambda, \lambda + 2\delta) w_2(\gamma^n) + \frac{m(m-1)}{2} e_{\mathrm{cup}},$$

where e_{cup} is a certain cup product and γ_n is the n-plane vector bundle over the infinite Grassmannian G_n. For details see Section 9. Note that $G_n = BO(n)$ where $BO(n)$ denote the classifying space for $O(n)$. We also calculate $w_2(\pi_0)$ for the cases when $\pi = \rho^\lambda$ is an irreducible representation of $O(2n)$ and irreducible representations π of $O(2n+1)$.

We also have a character formula to detect spinoriality of representations of orthogonal groups. A representation π of $O(n)$ is spinorial if and only if both of the following conditions hold:

(1) $\chi_\pi(I) - \chi_\pi(d_1) \equiv 0$ or $6 \pmod 8$,

(2) $\chi_\pi(I) - \chi_\pi(d_2) \equiv 0$ or $6 \pmod 8$.

Here χ_π is the character of π and

$$d_1 = \mathrm{diag}(-1, 1, 1 \ldots, 1) \quad \text{and} \quad d_2 = \mathrm{diag}(-1, -1, 1, \ldots, 1),$$

where $\mathrm{diag}(a_1, \ldots, a_n)$ is the diagonal matrix with entries a_i. In fact the same formulae detect the spinoriality of orthogonal representations of $GL(n, \mathbb{R})$

The paper is arranged as follows. Section 2 reviews the basic definitions and notations. We give a criterion for spinoriality of semidirect product and establish a connection between spinoriality of a Lie group and its maximal compact subgroup in Section 3. We present brief reviews of the papers [9] and [15] highlighting the important results in Sections 4 and 5 respectively. In Section 6 we give criteria for spinoriality of an orthogonal, irreducible representation of compact real Lie groups G of the form $G^0 \rtimes C_2$ in terms of their highest weights. In Section 7 we solve the case of reducible representations. Section 8 deals with the particular case of orthogonal groups. We calculate first and second Stiefel-Whitney classes of real representations of orthogonal groups in Section 9 . Here we obtain expressions of first and second Stiefel-Whitney classes in terms of the highest weights of the representations. In Section 10 we provide criteria to detect spinorial representations of orthogonal groups in terms of character values. Finally in Section 11 we work out some examples like $O(2), O(4)$ and $O(8)$.

2. Notation and preliminaries

2.1. Compact Lie groups

Let G be a real compact Lie group such that $G = G^0 \rtimes C_2$ such that G^0 is simple. Let T be a maximal torus of G^0 with Lie algebra $\mathfrak{t}$. Let $X^*(T)$ (resp. $X_*(T)$) be the character (resp. co-character) lattice of T. Consider an irreducible orthogonal representation (π^λ, V^λ) of G^0, where λ denotes the highest weight. Let $\mathfrak{g}$ denote the Lie algebra of G^0. Consider a co-character $\nu \in X_*(T)$. Note that $\lambda \in X^*(T)$. We have $\chi_\lambda(C) = (\lambda, \lambda + 2\delta)$, where $(,)$ is the Killing form for $\mathfrak{g}$ and δ is half the sum of positive roots for $\mathfrak{g}$, and C is the Casimir element for $\mathfrak{g}$. Let $\pi_1(G) = X_*(T)/Q(T)$, where $Q(T)$ is the co-root lattice. (In Section 4 we review the usual pairing $\langle \alpha, \nu \rangle$, fix norms on $\mathfrak{t}$ and $\mathfrak{t}^*$ associated to the Killing form.)

2.2. Root systems

Let R be the root system with respect to T. Let $\tau \in \Gamma$, where Γ denotes the group of diagram automorphisms of the Dynkin diagram of G^0. In fact τ induces an outer automorphism of G^0. We in particular consider the groups $G = G^0 \rtimes \langle g_0 \rangle$ such that the conjugation action of g_0 on G^0 gives an outer automorphism of G^0. If G^0 is not simply connected we have $G^0 = \hat{G}^0/Z_1$, where $\hat{G}^0$ denotes the universal covering group of G^0, and Z_1 is some subgroup of the center of $\hat{G}^0$. Let Γ_{Z_1} be the subgroup of Γ which leaves Z_1 fixed. Define $\widetilde{G^0} = G^0 \rtimes_\phi \Gamma_{Z_1}$, where $\phi : \Gamma \to \operatorname{Aut}(G^0)$ is a homomorphism (see [15, Section 2.1] for details) . Take $\bar{G}^0 \subset \widetilde{G^0}$ to be any sub extension of G^0. Write $S = (T^\tau)^0$ for the connected component of group of fixed points of τ inside T with Lie algebra $\mathfrak{s}$. Define $S_0 = T^\tau$, the sub torus fixed by τ.

Define
$$R^\tau = \{\alpha \mid_{\mathfrak{s}_0} : \alpha \in R\},$$

where $\mathfrak{s}_0$ denotes the Lie algebra of S_0 and $R^1 = R^{\tau\vee}$. Note that R^τ is also a root system. Here $R^{\tau\vee} = \{\frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in R^\tau\}$, where the bilinear form $(\, , \,)$ is a suitable multiple of Killing form such that $(\alpha,\alpha) = 2$ for a long root α. Let $e : \mathbb{C} \to \mathbb{C}$ denote the exponential map. For $\mu \in \mathfrak{t}^*$ we have the map $e(\mu) : \mathfrak{t} \to \mathbb{C}$ given by $e(\mu)(h) \mapsto e^{\mu(h)}$ for $h \in \mathfrak{t}$. Write $\rho^\tau = \frac{1}{2} \sum_{\beta \in R^{1+}} \beta$.

3. Lifting criteria for semi-direct products

One can detect the spinoriality of a representation of a Lie group from the spinoriality of its restrictions to certain subgroups. We state the result as the following lemma. For a Lie group G, let G^0 denote its connected component containing identity.

Lemma 3.1. *Let G be a Lie group and H be a subgroup of it such that $G = G^0 \cdot H$. Then any orthogonal representation π of G is spinorial if and only if $\pi_1 = \pi \mid_{G^0}$ and $\pi_2 = \pi \mid_H$ are spinorial and the lifts of π_1 and π_2 agree on $G^0 \cap H$.*

Proof. If π is spinorial then π_1 and π_2 are spinorial. For the converse let the representations π_1 and π_2 be spinorial. We write $\hat{\pi}_i$ for the lift of π_i. We have

$$G = \bigsqcup_{\alpha \in I} G^0 h_\alpha,$$

where h_α's are the representatives of the cosets of G^0 in G and I is an indexing set. Now any element of $g \in G$ can be written as $g = a \cdot h_\alpha$, where $a \in G^0$. We define the lift of π as

$$\hat{\pi}(g) = \hat{\pi}(a \cdot h_\alpha) = \hat{\pi}_1(a)\hat{\pi}_2(h_\alpha).$$

For a different coset representation if we have $g = a' \cdot h_\beta$, then $a'^{-1}a = h_\beta h_\alpha^{-1} \in G^0 \cap H$. Since $\hat{\pi}_1$ and $\hat{\pi}_2$ agree on $G^0 \cap H$, we obtain $\hat{\pi}_1(a'^{-1}a) = \hat{\pi}_2(h_\beta h_\alpha^{-1})$. This gives $\hat{\pi}_1(a)\hat{\pi}_2(h_\alpha) = \hat{\pi}_1(a')\hat{\pi}_2(h_\beta)$. Therefore the lift $\hat{\pi}$ is well-defined.

To prove that $\hat{\pi}$ is a lift it suffices to show that $\hat{\pi}$ is a homomorphism, i.e. for two elements $g_1, g_2 \in G$, we require

$$\hat{\pi}(g_1 g_2) = \hat{\pi}(g_1)\hat{\pi}(g_2). \tag{1}$$

Let $g_1 = a_1 h_\beta$, $g_2 = a_2 h_\gamma$ and $h_\beta \cdot h_\gamma \in G^0 h_\alpha$. Therefore $h_\beta \cdot h_\gamma \cdot h_\alpha^{-1} \in G^0$.

We write $\quad \hat{\pi}(g_1 g_2) = \hat{\pi}(a_1 h_\beta a_2 h_\gamma h_\alpha^{-1} h_\alpha) = \hat{\pi}((a_1 h_\beta a_2 h_\beta^{-1})(h_\beta h_\gamma h_\alpha^{-1}) h_\alpha)$

$$= \hat{\pi}_1((a_1 h_\beta a_2 h_\beta^{-1})(h_\beta h_\gamma h_\alpha^{-1})) \hat{\pi}_2(h_\alpha) = \hat{\pi}_1(a_1 h_\beta a_2 h_\beta^{-1}) \hat{\pi}_1(h_\beta h_\gamma h_\alpha^{-1}) \hat{\pi}_2(h_\alpha).$$

Note that since G^0 is normal in G, we have $a_1 h_\beta a_2 h_\beta^{-1} \in G^0$. We can rewrite the requirement mentioned in (1) as

$$\hat{\pi}_1(a_1 h_\beta a_2 h_\beta^{-1}) \hat{\pi}_1(h_\beta h_\gamma h_\alpha^{-1}) \hat{\pi}_2(h_\alpha) = \hat{\pi}_1(a_1) \hat{\pi}_2(h_\beta) \hat{\pi}_1(a_2) \hat{\pi}_2(h_\gamma).$$

Consider the element

$$x = (\hat{\pi}_1(a_1 h_\beta a_2 h_\beta^{-1}) \hat{\pi}_1(h_\beta h_\gamma h_\alpha^{-1}) \hat{\pi}_2(h_\alpha))^{-1} \hat{\pi}_1(a_1) \hat{\pi}_2(h_\beta) \hat{\pi}_1(a_2) \hat{\pi}_2(h_\gamma).$$

Taking the image of x under the covering map ρ we obtain

$$\rho(x) = (\pi_1(a_1 h_\beta a_2 h_\beta^{-1}) \pi_1(h_\beta h_\gamma h_\alpha^{-1}) \pi_2(h_\alpha))^{-1} \pi_1(a_1) \pi_2(h_\beta) \pi_1(a_2) \pi_2(h_\gamma).$$

Since both π_1 and π_2 are restrictions of the same representation we replace them by π for the convenience of the computation. Thus we obtain

$$\rho(x) = (\pi(a_1 h_\beta a_2 h_\beta^{-1}) \pi(h_\beta h_\gamma h_\alpha^{-1}) \pi(h_\alpha))^{-1} \pi(a_1) \pi_2(h_\beta) \pi(a_2) \pi(h_\gamma)$$

$$= (\pi(a_1 h_\beta a_2 h_\gamma))^{-1} \pi(a_1 h_\beta a_2 h_\gamma) = 1.$$

Therefore we should have $x = \pm 1$. If we fix h_β and h_γ then x becomes a continuous function on $G^0 \times G^0$. Note that h_α depends only on h_β and h_γ. Since $G^0 \times G^0$ is connected x takes a constant value on this domain. Taking $a_1 = a_2 = 1$ we obtain

$$x = (\hat{\pi}_1(h_\beta h_\beta^{-1}) \hat{\pi}_1(h_\beta h_\gamma h_\alpha^{-1}) \hat{\pi}_2(h_\alpha))^{-1} \hat{\pi}_1(1) \hat{\pi}_2(h_\beta) \hat{\pi}_1(1) \hat{\pi}_2(h_\gamma)$$

$$= (\hat{\pi}_2(h_\beta h_\gamma h_\alpha^{-1} h_\alpha))^{-1} \hat{\pi}_2(h_\beta h_\gamma) \quad \text{as} \quad h_\beta h_\gamma h_\alpha^{-1} \in G^0 \cap H = 1$$

So x takes the value 1 at 1×1, so x takes the value 1 on $G^0 \times G^0$. Since this is true for all h_β, h_γ, we conclude that $x = 1$. In other words the map $\hat{\pi}$ is a homomorphism.

Next we claim that the map $\hat{\pi}$ is continuous. Since G^0 is open in G, so is $G^0 h_\alpha$.

We have $\quad \hat{\pi}\mid_{G^0 h_\alpha}(a h_\alpha) = \hat{\pi}_1(a) \hat{\pi}_2(h_\alpha), \quad$ for $a h_\alpha \in G^0 h_\alpha$.

Note that the set $\{G^0 h_\alpha \mid \alpha \in I\}$ forms an open cover of G and $\hat{\pi}\mid_{G^0 h_\alpha}$ is continuous for all $\alpha \in I$. Therefore $\hat{\pi}$ is continuous. ∎

Consider a group G with the following conditions:

(1) $G = G_1 \rtimes_\phi G_2$,

(2) G_1 is a connected Lie group,

(3) G_2 is a discrete group.

We prove the following theorem.

Theorem 3.2. *A representation ϕ of G is spinorial if and only if $\phi\mid_{G_1}$ and $\phi\mid_{G_2}$ are spinorial.*

Proof. Since $G = G_1 \rtimes_\phi G_2$ we obtain (1) $G^1 = G^0$ and $G = G_1 \cdot G_2$, and (2) $G_1 \cap G_2 = \{e\}$. Therefore the theorem follows from Lemma 3.1. $\blacksquare$

Lemma 3.3. *Let G, G', H be connected real Lie groups and $\phi : H \to G$ be a homomorphism. Let $\alpha : G' \to G$ be a cover. Then ϕ can be lifted to G' if and only if the image of ϕ_* in $\pi_1(G)$ is contained in the image of α_*.*

Proof. It follows from the lifting theorem in algebraic topology, that there is a unique continuous topological lift ψ, which takes identity of H to identity of G'. We will prove that ψ is a group homomorphism.

$$
\begin{array}{ccc}
 & & G' \\
 & \nearrow^{\widehat{\psi}} & \downarrow{\alpha} \\
H & \xrightarrow{\phi} & G
\end{array}
$$
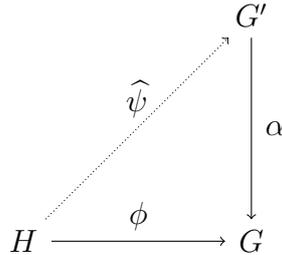
Let $*$ denote the multiplication in any group. We have $\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$. So we get $\alpha \circ \psi(g_1 * g_2) = \alpha\psi(g_1) * \alpha\psi(g_2)$. Hence $\alpha(\psi(g_1 * g_2)\psi(g_1)^{-1}\psi(g_2)^{-1}) = 1$. The image of the map $p : H \times H \to G'$ given by $(g_1, g_2) \to \psi(g_1 * g_2)\psi(g_1)^{-1}\psi(g_2)^{-1}$ is connected, since H is connected. The kernel of α is discrete, as it is a covering map. Thus, we get $(\psi(g_1 * g_2)\psi(g_1)^{-1}\psi(g_2)^{-1}) = 1$. Hence $\psi(g_1 * g_2) = \psi(g_1)\psi(g_2)$. Thus ψ is, in fact, a group homomorphism. $\blacksquare$

In fact for a real, reductive Lie group G, the spinoriality of a representation of it can be detected by the spinoriality of its restriction to its maximal compact subgroup.

Theorem 3.4. *Let G be a reductive real Lie group such that G^0 has finite index in G. Let K be a maximal compact subgroup of G. Then an orthogonal representation ϕ of G is spinorial if and only if $\phi \mid_K$ is spinorial.*

Proof. One direction is obvious. Assume that $\phi \mid_K$ is spinorial. We denote the lift of $\phi \mid_K$ by $\hat{\phi}_K$. Note that [8, page 257 Theorem 2.2 part 3] the inclusion map $i : K \to G$ is a homotopy equivalence, which means

(1) We have $G = G^\circ \cdot K$, and (2) the map $i_* : \pi_1(K^\circ) \to \pi_1(G^\circ)$ is an isomorphism.

Since $\pi \mid_K$ lifts, from Lemma 3.3 we have $\phi_*(\pi_1(K^0)) \subset \rho_*(\pi_1(\mathrm{Spin}(V)))$. Thus we have $\phi_*(\pi_1(G^0)) \subset \rho_*(\pi_1(\mathrm{Spin}(V)))$ which says that $\phi \mid_{G^\circ}$ is spinorial. We denote the lift by $\hat{\phi}_0$. Since the lift $\hat{\phi}_0$ is unique we have $\hat{\phi}_0 \mid_{K^0} = \hat{\phi}_K \mid_{K^0}$. Note that $K^0 = G^0 \cap K$. Now the theorem follows from Lemma 3.1 by taking $H = K$. $\blacksquare$

4. Review of Joshi-Spallone

The paper [9] gives criteria for spinoriality of the representations of connected, reductive algebraic groups over fields of characteristic zero. We restrict our discussion to real, compact Lie groups. We continue with the notations as in Section 2.

Write $\langle \ , \ \rangle_T : X^*(T) \times X_*(T) \to \mathbb{Z}$ for the pairing

$$\langle \mu, \nu \rangle_T = n \Leftrightarrow \mu(\nu(t)) = t^n,$$

for $t \in \mathbb{R}^\times$, and $\langle \ , \ \rangle_\mathfrak{t} : \mathfrak{t}^* \times \mathfrak{t}$ for the natural pairing. Note that for $\mu \in X^*(T)$ and $\nu \in X_*(T)$, we have

$$\langle d\mu, d\nu(1) \rangle_\mathfrak{t} = \langle \mu, \nu \rangle_T.$$

So we may drop the subscripts and simply write '$\langle \mu, \nu \rangle$'. Write $(\ , \ )$ for the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$; it may be computed by

$$(x, y) = \sum_{\alpha \in R} \alpha(x)\alpha(y),$$

for $x, y \in \mathfrak{t}$. Also set $|x|^2 = (x, x)$. In particular, for $\nu \in X_*(T)$ we have

$$|\nu|^2 = \sum_{\alpha \in R} \langle \alpha, \nu \rangle^2 \in 2\mathbb{Z}.$$

The Killing form restricted further to $\mathfrak{t}$ induces an isomorphism $\sigma : \mathfrak{t}^* \cong \mathfrak{t}$. We use the same notation $(\ , \ )$ to denote the inverse form on $\mathfrak{t}^*$ defined for $\mu_1, \mu_2 \in \mathfrak{t}$ by $(\mu_1, \mu_2) = (\sigma(\mu_1), \sigma(\mu_2))$.

Now we pick co-characters $\nu_1, \ldots, \nu_r$ whose images generate $\pi_1(G)$, where $\pi_1(G) = X_*(T)/Q(T)$ i.e., co-character lattice modulo co-root lattice. Consider the integer

$$p(\underline{\nu}) = p(\nu_1, \ldots, \nu_r) = \frac{1}{2} \gcd \left(|\nu_1|^2, \ldots, |\nu_r|^2 \right).$$

The following theorem is from [9, Theorem 1]

Theorem 4.1. *The irreducible representation π_λ of a connected reductive Lie group G is spinorial if and only if*

$$\frac{p(\underline{\nu}) \dim V^\lambda \chi_\lambda(C)}{\dim \mathfrak{g}} \equiv 0 \pmod 2,$$

5. Review of Wendt

This paper [15] gives Weyl character formula for character values of representations of real compact Lie groups G with two connected components with G^0 of type A_n, D_n and E_6. We continue with the notations as in Section 2. We write

$$\rho^\tau = \frac{1}{2} \sum_{\bar{\beta} \in R_+^1} \bar{\beta}.$$

We have $\delta^\tau : \mathfrak{s}_0 \to \mathbb{C}$, defined as $\delta^\tau(x) = e(\rho^\tau(x)) \prod_{\bar{\beta} \in R_+^1} (1 - e(\bar{\beta}(x)))$ for $x \in \mathfrak{s}_0$. For $\mu \in \mathfrak{s}_0^*$, we have

$$A^\tau(\mu) = \sum_{w \in W^\tau} \epsilon(w) \cdot e(w\mu),$$

where W^τ is the Weyl group of the root system R^τ and ϵ is the sign character of the Weyl group W^τ. Since S_0 is regular in G, we can choose a Weyl chamber $K \subset \mathfrak{t}^*$ such that $K \cap \mathfrak{s}_0^*$ is non-empty. Write $\bar{K}$ to denote the closure of K. Let I denote

the lattice $I = \ker(\exp) \cap \mathfrak{t}$ and $I^* \subset \mathfrak{t}^*$ be its dual. For this paper we consider $G = \bar{G}^0 = G^0 \rtimes C_2$, where $C_2 = \langle g_0 \rangle$. Let π^λ denote the irreducible representation of G^0 with highest weight λ. Write $\tau(\lambda)$ to denote the highest weight of the representation $\pi^\lambda(\tau \cdot g)$. Following Section 1 the representation $\rho^\lambda = \mathrm{Ind}_{G_0}^{G} \pi^\lambda$ is either of Type I or Type II. Let χ_λ denote the character of π_λ and $\tilde{\chi}_\lambda$ denote the irreducible character of G. We in particular take $\tau = C_{g_0}$, the conjugation action of g_0 on G^0. We define a function $\tilde{\chi}_\lambda^\tau : \mathfrak{s}_0 \to \mathbb{C}$ as $\tilde{\chi}_\lambda^\tau(h) = \tilde{\chi}_\lambda(g_0 \cdot \exp(h))$. From [15, Theorem 2.6] and [15, Corollary 2.7] we obtain the following result.

Theorem 5.1. *There exists an irreducible character $\tilde{\chi}_\lambda$ of $\bar{G}^0$ for each $\lambda \in I^* \cap \bar{K}$. If $\lambda \notin \mathfrak{s}_0^*$ then ρ^λ is irreducible. In this case we have*

$$\tilde{\chi}_\lambda \mid_{G^0} = \chi_\lambda + \chi_{\tau(\lambda)}, \quad and \quad \tilde{\chi}_\lambda \mid_{g_0 G^0} = 0,$$

For each $\lambda \in \mathfrak{s}_0^*$, ρ^λ splits into two irreducibles $\pi_\lambda^\pm$. In this case we have

$$\tilde{\chi}_\lambda \mid_{G^0} = \chi_\lambda, \quad and$$

$$\tilde{\chi}_\lambda^\tau(g_0 \cdot \exp(h)) = \tilde{\chi}_\lambda^\tau(h) = \pm A^\tau(\lambda + \rho^\tau)(h)/A^\tau(\rho^\tau)(h), \quad where \; h \in \mathfrak{s}_0.$$

From [15, page 36] we have the following table showing relation between type of R and type of R^τ.

Table 1: Table showing relation between type of R and type of R^τ.

R	A_{2n-1}	A_{2n}	$D_n (n \geq 4)$	D_4	E_6
$\mathrm{ord}(\tau)$	2	2	2	3	2
R^τ	C_n	BC_n	B_{n-1}	G_2	F_4

For the even orthogonal groups $\mathrm{O}(2n)$, we take τ to be the automorphism of the Dynkin diagram of type D_n switching two of the extremal nodes. It corresponds to the conjugation map by

$$g_0 = \left(\sum_{i=1}^{2n-2} e_{i,i} + e_{2n-1,2n} + e_{2n,2n-1} \right),$$

where $e_{i,j}$ denotes the elementary $2n \times 2n$ matrix with 1 at the (i,j)-th position and 0 everywhere else. Let $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Let T be the maximal torus $T = \mathrm{diag}(R_{\theta_1}, \ldots, R_{\theta_n})$. Thus

$$S_0 = \mathrm{diag}(R_{\theta_1}, \ldots, R_{\theta_{n-1}}, I_2).$$

6. Main Theorem

In this section we give the criteria for the spinoriality of the irreducible orthogonal representations of $G = G^0 \rtimes C_2$. Let 'sgn' denote the non-trivial character of C_2.

Lemma 6.1.   *Let (π, V) be a representation of $C_2 = \{\pm 1\}$, such that*

$$\pi = \underbrace{\mathrm{sgn} \oplus \mathrm{sgn} \oplus \cdots \oplus \mathrm{sgn}}_{m \ times} \oplus \mathbb{1} \oplus \cdots \oplus \mathbb{1}.$$

Then π is spinorial if and only if $m \equiv 0$ or 3 (mod 4), where $m = \frac{\chi_\pi(1) - \chi_\pi(-1)}{2}$.

The proof of Lemma 6.1 follows from [6, Section 3.1].

Theorem 6.2.   *For the irreducible representation $(\pi^{\lambda,\pm}, V)$ of G we have*

$$\chi_{\pi^{\lambda,\pm}}(g_0) = \pm \frac{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \rho^\tau \rangle}.$$

Proof.   From Theorem 5.1 we obtain

$$\chi_\lambda^\tau(h) = \pm \frac{A^\tau(\lambda + \rho^\tau)}{\delta^\tau}(h), \tag{2}$$

where $A^\tau(\mu) = \sum_{w \in W^\tau} \epsilon(w) e(w(\mu))$ and by [15, Corollary 2.7] we have $\delta^\tau = A^\tau(\rho^\tau)$.

Note that $\chi_\lambda^\tau(h) = \widetilde{\chi}_\lambda(g_0 \exp(h)) = \chi_{\pi^{\lambda,\pm}}(g_0 \exp(h))$, for $h \in \mathfrak{s}_0$, \tag{3}

For our case τ denotes the conjugation action by g_0. We have $\chi_{\pi^{\lambda,\pm}}(g_0) = \chi_\lambda^\tau(0)$. Let ν be a co-character of $\mathfrak{s}_0$ such that $\nu(1) = h$. From Section 4 we obtain

$$\langle w(\mu), xh \rangle = x \langle w(\mu), \nu(1) \rangle = x \langle w(\mu), \nu \rangle,$$

where $x \in \mathbb{R}$. We write $\langle w(\mu), \nu \rangle$ to denote $\langle w(\mu), \nu(1) \rangle$. Therefore

$$A^\tau(\mu)(xh) = \sum_{w \in W^\tau} \epsilon(w) e(x \langle w(\mu), \nu \rangle). \tag{4}$$

The character value $\widetilde{\chi}_\lambda(g_0 \exp(h))$ is a continuous function of h. Therefore the limit of the function exists at $h = 0$. We take the limit along the line $xh \in \mathfrak{s}_0$. This allows us to use Equation (3) to calculate

$$\lim_{x \to 0} \frac{A^\tau(\lambda + \rho^\tau)(xh)}{A^\tau(\rho^\tau)(xh)}.$$

Note that $A^\tau(\lambda + \rho^\tau)(0) = A^\tau(\rho^\tau)(0) = 0$. From L'Hospital's rule we have

$$\lim_{x \to 0} \frac{A^\tau(\lambda + \rho^\tau)(xh)}{A^\tau(\rho^\tau)(xh)} = \lim_{x \to 0} \frac{\frac{d^i}{dx^i}\left(A^\tau(\lambda + \rho^\tau)(xh)\right)}{\frac{d^i}{dx^i}\left(A^\tau(\rho^\tau)(xh)\right)},$$

where i is the least natural number so that the numerator and denominator are non-zero. We calculate

$$\frac{d^i}{dx^i} A^\tau(\lambda + \rho^\tau)(xh) \mid_{x=0} = \sum_{w \in W^\tau} \epsilon(w) \langle w(\lambda + \rho^\tau), \nu \rangle^i. \tag{5}$$

From [9, Proposition 6] we obtain

$$\sum_{w \in W} \epsilon(w)\langle w(\mu), \nu \rangle^i = \begin{cases} 0, & \text{if } 1 \le i \le N-1, \\ N! \dfrac{d_\mu d_\nu}{d_\delta}, & \text{if } i = N, \end{cases}$$

where N denotes the number of positive roots of R^τ and $d_\nu = \prod_{\alpha \in R^{\tau+}} \langle \alpha, \nu \rangle$ and $d_\mu = \prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \mu \rangle$. We calculate

$$\lim_{x \to 0} \frac{A^\tau(\lambda + \rho^\tau)(xh)}{A^\tau(\rho^\tau)(xh)} = \frac{N! \frac{d_{\lambda+\rho^\tau} d_\nu}{d_\delta}}{N! \frac{d_{\rho^\tau} d_\nu}{d_\delta}} = \frac{d_{\lambda+\rho^\tau}}{d_{\rho^\tau}} = \frac{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \rho^\tau \rangle}.$$

Note that the expression $\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \lambda + \rho^\tau \rangle$ is a polynomial in λ. Therefore it is non-zero for some λ. This guarantees that the denominator should be non-zero. ∎

Remark 6.3. Since the results in [15] hold true for the groups G with G^0 of type D_n, Theorem 6.2 also holds for the groups $O(2)$ and $O(4)$.

Proposition 6.4. *Let π be an irreducible representation of G. Let m denote the multiplicity of -1 as an eigenvalue of $\pi(g_0)$. Then*

$$m = \begin{cases} \dim V^\lambda, & \text{when } \pi \text{ is of Type I}, \\ (\dim V^\lambda - \chi_\pi(g_0))/2, & \text{when } \pi \text{ is of Type II}. \end{cases}$$

Proof. As g_0 is an involution we have

$$\rho(g_0) \sim \begin{pmatrix} -I_m & 0 \\ 0 & I_l \end{pmatrix}. \tag{6}$$

If ρ^λ is irreducible then $l + m = 2 \dim V^\lambda$. From [15, Theorem 2.6] it follows that $\chi_\lambda(g_0) = 0$. This means $l - m = 0$. So we deduce that $l = m = \dim V^\lambda$.
For $\pi = \pi^{\lambda,\pm}$, $\chi_{\pi^{\lambda,\pm}}(g_0) = l - m$ and $l + m = \dim V^\lambda$. This gives the result. ∎

Proof of Theorem 1.1. Note that $\rho^\lambda \mid_{G^0} = \pi^\lambda \oplus \pi^{g_0 \cdot \lambda}$, (see 1 for reference). From Theorem 3.2 and Theorem [9, Theorem 5], we conclude that $\rho^\lambda \mid_{G^0}$ is spinorial if and only if

$$\frac{p(\nu) \cdot (\dim V^\lambda) \cdot (\chi_\lambda(C) + \chi_{g_0 \cdot \lambda}(C))}{\dim \mathfrak{g}} \equiv 0 \pmod 2,$$

Hence the first condition follows. To obtain the other condition use Lemma 6.1 and Proposition 6.4. ∎

Proof of Theorem 1.2. The proof is similar to the previous one. ∎

6.1. Case of $G^0 \times C_2$

Let π_λ denote the irreducible representation of G^0 and ρ be an irreducible representation of C_2. We write sgn to denote the sign representation of C_2.

Theorem 6.5. *An irreducible, orthogonal representation $\pi = \pi_\lambda \otimes \rho$ of G, for $n \in \mathbb{N}$, is spinorial if and only if the following two conditions hold:*

(1) $\dfrac{p(\underline{\nu}) \cdot (\dim V^\lambda) \cdot \chi_\lambda(C)}{\dim \mathfrak{g}} \equiv 0 \pmod{2}$, *for all ν in $\pi_1(G^0)$,*

(2) $\dim V^\lambda \equiv 0$ *or $3 \pmod 4$ if $\rho = \mathrm{sgn}$. Otherwise the first condition is sufficient.*

Proof. We know that $G = G^0 \times C_2$. Here we identify C_2 with $\{\pm 1\} \subset G$. By Theorem 3.2, π is spinorial if and only if $\pi \mid_{G^0}$ and $\pi \mid_{C_2}$ are spinorial. We obtain the first condition from Section 4.

Note that $\pi(-1) = \pm I$ by Schur's Lemma. If it is I, then $\pi \mid_{C_2}$ lifts trivially. Otherwise the second condition comes due to Lemma 6.1. ∎

7. Reducible representations

As before we take $C_2 = \langle g_0 \rangle$. Any orthogonal representation (π, V) of a real compact group G can be written as

$$\pi = \oplus_i \rho_i \oplus_j (\phi_j \oplus \phi_j^\vee),$$

where ρ_i is irreducible and orthogonal and ϕ_j is irreducible but not orthogonal. We have

$$\phi_j = \begin{cases} \mathrm{Ind}_{G^0}^G \phi_{j0}, & \text{when } \phi_j \text{ is of Type I,} \\ \psi_j^\pm, & \text{where } \mathrm{Ind}_{G^0}^G \phi_{j0} = \psi_j^+ \oplus \psi_j^-, \text{when } \phi_j \text{ is of Type II,} \end{cases}$$

where ϕ_{j0} is an irreducible representation of G^0. For a representation π of G, let m_π denote the multiplicity of -1 as an eigenvalue of $\pi(g_0)$. Note that

$$m_\pi = m_{\pi^\vee}.$$

Then we have $\quad m_{\phi_j \oplus \phi_j^\vee} = \begin{cases} 2 \dim \phi_{j0}, & \text{when } \phi_j \text{ is of Type I,} \\ \dim \phi_{j0} - \chi_{\phi_j}(g_0), & \text{when } \phi_j \text{ is of Type II.} \end{cases}$

Theorem 7.1. *Consider an orthogonal representation π of G of the form*

$$\pi = \oplus_i \rho_i \oplus_j (\phi_j \oplus \phi_j^\vee).$$

Let λ_i (resp. γ_j) denote the highest weight of ρ_i (resp. ϕ_j). Then π is spinorial if and only if both the conditions hold:

(1) $q_\pi(\nu) = p(\underline{\nu}) \sum\limits_i \dfrac{\dim \rho_i \cdot \chi_{\lambda_i}(C)}{\dim \mathfrak{g}} \equiv 0 \pmod 2,$

(2) $m_\pi = \sum_i m_{\rho_i} + \sum_j m_{(\phi_j \oplus \phi^\vee)} \equiv 0$ *or $3 \pmod 4$.*

8. Orthogonal groups

8.1. General representations

From [3, Corollary 7.8, page 297] we obtain that all the representations of the orthogonal group are orthogonal.

We have
$$O(l) = \begin{cases} SO(l) \times C_2, & \text{when } l = 2n + 1, \\ SO(l) \rtimes C_2, & \text{when } l = 2n. \end{cases}$$

We call them odd and even orthogonal groups, respectively. Take $C_2 = \langle g_0 \rangle$, where

$$g_0 = \begin{cases} -I_{2n+1}, & \text{for } O(2n+1), \\ \left(\sum_{i=1}^{2n-2} e_{i,i} + e_{2n-1,2n} + e_{2n,2n-1} \right), & \text{for } O(2n). \end{cases} \tag{7}$$

Here $e_{i,j}$ denotes the elementary matrices.

Corollary 8.1. *An irreducible representation $\pi_\lambda \otimes \rho$ of $O(2n+1)$, for $n \in \mathbb{N}$, is spinorial if and only if the following two conditions hold:*

(1) $\dfrac{(2n-1)}{n(2n+1)} \dim V^\lambda (\lambda, \lambda + 2\delta) \equiv 0 \pmod 2$.

(2) $\dim V^\lambda \equiv 0 \text{ or } 3 \pmod 4$ *if $\rho = \mathrm{sgn}$. Otherwise the first condition is sufficient.*

Proof. From [9, Remark 5] which says $p(\underline{\nu}) = 2n - 1$. The result then follows from Theorem 6.5. ∎

Again $O(2n) = SO(2n) \rtimes C_2$. Note that $D_2 \cong A_1 \oplus A_1$ and $D_3 \cong A_3$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the highest weight of π^λ, then π^λ is of type I when $\lambda_n \neq 0$. Otherwise it is of type II. For details we refer [3, Section 7.5, page 296].
The following theorem gives the criteria for spinoriality for the representations of Type I.

Corollary 8.2. *The irreducible representation ρ^λ of $O(2n), n \geq 3$, is spinorial if and only if both the following conditions hold:*

(1) $\dfrac{2 \cdot (n-1)}{n \cdot (2n-1)} \cdot \dim V^\lambda \left(\chi_\lambda(C) + \chi_{g_0 \cdot \lambda}(C) \right) \equiv 0 \pmod 2$,

(2) $\dim V^\lambda \equiv 0 \text{ or } 3 \pmod 4$.

Proof. The result follows from Theorem 1.1 and [9, Table 1, Section 11] which says $p(\underline{\nu}) = 2n - 2$. ∎

The next theorem gives lifting criteria for representations of Type II.

Corollary 8.3. *The representation $\pi_\lambda^\pm$ of $O(2n)$, for $n \geq 3$, is spinorial if and only if both the following, conditions hold:*

(1) $\dfrac{2 \cdot (n-1)}{n \cdot (2n-1)} \cdot \left(\dim V^\lambda \cdot \chi_\lambda(C) \right) \equiv 0 \pmod 2$,

(2) $\dim V^0 - \chi_\pi(g_0) \equiv 0 \text{ or } 6 \pmod 8$, *where $\pi = \pi^{\lambda, \pm}$.*

Proof. This follows from Theorem 1.2 and [9, Table 1 , Section 11] which says $p(\underline{\nu}) = 2n - 2$. ∎

Note that for $n \in \{1, 2\}$ the group $SO(n)$ is not simple. We work out those cases in Section 11.

8.2. Adjoint representation

The adjoint action of the orthogonal group $O(n)$ on its Lie algebra $\mathfrak{so}(n)$ preserves the Killing form $K(X, Y) = (n-2)Tr(XY)$ where $X, Y \in \mathfrak{so}(n)$.

Theorem 8.4. *The adjoint representation of the orthogonal group $O(l)$ is spinorial if and only if $l \equiv 0 \pmod 4$.*

Proof. We have [9, Corollary 4] which says that $\mathrm{Ad}\,|_{SO(l)}$ is spinorial if and only if $\delta \in X^*(T)$, where δ denotes half the sum of positive roots. For $l = 2n+1$, the group $SO(2n+1)$ is of type B_n. From [3, Chapter 5, Proposition 6.5], we have

$$\delta = \frac{1}{2} \sum_{i=1}^{n} (2n - 2i + 1)e_i \notin X^*(T).$$

Therefore the adjoint representations of the odd orthogonal groups $O(2n+1)$ are aspinorial.

For $l = 2n$, the group $SO(2n)$ is of type D_n. In this case from [3, Chapter 5, Proposition 6.4], we have

$$\delta = \sum_{i=1}^{n} (n-i)e_i \in X^*(T).$$

Therefore $\mathrm{Ad}\,|_{SO(2n)}$ is spinorial. Following Theorem 3.2, it remains to verify whether $\mathrm{Ad}\,|_{C_2}$ is spinorial. For that we calculate

$$\chi_{\mathrm{Ad}}(g_0) = \frac{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \rho^\tau \rangle}.$$

From Table 1 we obtain R^τ is of type B_{n-1}. Therefore one calculates

$$\rho^\tau = \frac{1}{2} \sum_{\alpha \in R^\tau} \frac{2\alpha}{(\alpha, \alpha)} = \frac{1}{2} \sum_{i<j} \{(e_i + e_j) + (e_i - e_j)\} + \frac{1}{2} \sum_{j=1}^{n-1} 2e_j = \sum_{i=1}^{n-1} (n-i)e_i.$$

From the given root system for the groups of type D_n in [3, Chapter 5, Proposition 6.4], we obtain the highest weight as $\lambda = e_1 + e_2$. Therefore

$$\lambda + \rho^\tau = ne_1 + (n-1)e_2 + \sum_{i=3}^{n-1} (n-i)e_i.$$

One calculates

$$\chi_{\mathrm{Ad}}(g_0) = \frac{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in R^{\tau+}} \langle \alpha^\vee, \rho^\tau \rangle} = \frac{\prod_{\alpha \in R^{\tau+}} \langle \frac{2\alpha}{(\alpha,\alpha)}, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in R^{\tau+}} \langle \frac{2\alpha}{(\alpha,\alpha)}, \rho^\tau \rangle} = \frac{\prod_{\alpha \in R^{\tau+}} \langle \alpha, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in R^{\tau+}} \langle \alpha, \rho^\tau \rangle}.$$

Note that $\lambda + \rho^\tau$ and ρ^τ differ only by first two terms. Therefore the terms in both the numerator and denominator containing the elements e_1 and e_2 will survive. The positive roots for groups of type B_{n-1} are $e_i \pm e_j$ for $i < j$, and e_j for $1 \le j \le n-1$. Consider the set

$$S = \{e_1 \pm e_j \mid j > 1\} \cup \{e_2 \pm e_j \mid j > 2\} \cup \{e_1, e_2\}.$$

We have

$$\chi_{\mathrm{Ad}}(g_0) = \frac{\prod_{\alpha \in S} \langle \alpha, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in S} \langle \alpha, \rho^\tau \rangle} = \frac{(2n-4)!/2 \cdot (2n-3)! \cdot (2n-1)}{(2n-4)! \cdot (2n-5)! \cdot (2n-3)} = 2n^2 - 5n + 2.$$

Following Proposition 6.4 we obtain the multiplicity of -1 as an eigenvalue of $\mathrm{Ad}(g_0)$:

$$\frac{1}{2} \cdot (\dim(\mathrm{Ad}) - \chi_{\mathrm{Ad}}(g_0)) = \frac{1}{2} \cdot \left(\frac{(2n)^2 - 2n}{2} - (2n^2 - 5n + 2) \right)$$
$$= 2n - 1.$$

Since $2n - 1$ is odd, from Lemma 6.1 we conclude that $\mathrm{Ad}\,|_{C_2}$ is spinorial if and only if $2n - 1 \equiv 3 \pmod 4$. Equivalently we require $2n \equiv 0 \pmod 4$. ∎

Remark 8.5. In general the adjoint representation of $G = G^0 \rtimes C_2$ is of type II. This is because $\mathrm{Ad}\,|_{G^0}$ remains irreducible.

9. Stiefel-Whitney classes for representations of orthogonal groups

For an orthogonal representation π of a Lie group G one can define Stiefel-Whitney classes $w_i(\pi) \in H^i(BG, \mathbb{Z}/2\mathbb{Z})$, for $0 \le i \le \dim \pi$. Here BG denotes a classifying space of G and $H^i(BG, \mathbb{Z}/2\mathbb{Z})$ denotes the singular cohomology of BG with coefficients in $\mathbb{Z}/2\mathbb{Z}$. For details we refer the reader to [2, Section 2.6, page 50]. The well known fact that

$$H^i_{\mathrm{cts}}(G, \mathbb{Z}/2\mathbb{Z}) \cong H^i(BG, \mathbb{Z}/2\mathbb{Z}),$$

allows us to work with the continuous group cohomology $H^i_{\mathrm{cts}}(G, \mathbb{Z}/2\mathbb{Z})$. One can see [4, Proposition 5.2] for a reference of the fact. We simply write $H^i(G, \mathbb{Z}/2\mathbb{Z})$ for $H^i_{\mathrm{cts}}(G, \mathbb{Z}/2\mathbb{Z})$.

We first calculate the second Stiefel-Whitney class for a representation of C_2.

Lemma 9.1. *Let (π, V) be a representation of C_2 and*

$$\pi = \underbrace{\mathrm{sgn} \oplus \mathrm{sgn} \oplus \cdots \oplus \mathrm{sgn}}_{m \ times} \oplus \mathbb{1} \oplus \cdots \oplus \mathbb{1},$$

then $$w_2(\pi) = \frac{m(m-1)}{2} \cdot w_1(\mathrm{sgn}) \cup w_1(\mathrm{sgn}).$$

Proof. We have $w(\pi) = (1 + w_1(\mathrm{sgn}))^m$.

Therefore $$w_2(\pi) = \binom{m}{2} \cdot w_1(\mathrm{sgn}) \cup w_1(\mathrm{sgn}).$$ ∎

For $G = \mathrm{O}(n)$ we have $BG = G_n$, where G_n denotes the infinite Grassmannian. Let γ^n denote the real n-plane vector bundle over G_n and ϕ_n denote the standard representation of $\mathrm{O}(n)$ on $\mathbb{R}^n$. In fact the vector bundle γ^n is isomorphic to the vector bundle associated to ϕ_n over G_n. From [10, Theorem B.7, page 381] we obtain

$$H^*(\mathrm{O}(n), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, \ldots, w_n], \tag{8}$$

where w_i denotes the i-th Stiefel-Whitney class of the vector bundle γ^n over G_n. In other words $w_i = w_i(\gamma^n) = w_i(\phi_n)$. From the same reference we obtain $H^*(\mathrm{SO}(n), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_2', \ldots, w_n']$, where

$$w_j' \in H^j(\mathrm{SO}(n), \mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad w_j' = w_j(i^*(\gamma^n)).$$

Here $i : \mathrm{B\,SO}(n) \to \mathrm{B\,O}(n)$ denotes the induced map from the inclusion of $\mathrm{SO}(n)$ into $\mathrm{O}(n)$.

Let 'det' denote the determinant of ϕ_n. Note that $w_1(\phi_n) = w_1(\det)$ is the only non-zero element of $H^1(G, \mathbb{Z}/2\mathbb{Z})$. We write $e_{\mathrm{cup}} = w_1(\det) \cup w_1(\det)$. From Equation (8) it follows that

$$H^2(\mathrm{O}(n), \mathbb{Z}/2\mathbb{Z}) = \langle w_2(\gamma^n), e_{\mathrm{cup}} \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

For any representation π of $\mathrm{O}(n)$ we have

$$w_2(\pi) = aw_2(\gamma^n) + be_{\mathrm{cup}}, \quad \text{where } a, b \in \mathbb{Z}/2\mathbb{Z}.$$

For the subgroups C_2 and $\mathrm{SO}(n)$ of $\mathrm{O}(n)$ we obtain the two restriction maps $i_1^* : H^2(\mathrm{O}(n), \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathrm{SO}(n), \mathbb{Z}/2\mathbb{Z})$ and $i_2^* : H^2(\mathrm{O}(n), \mathbb{Z}/2\mathbb{Z}) \to H^2(C_2, \mathbb{Z}/2\mathbb{Z})$.

Lemma 9.2. *The map*

$$i^* : H^2(\mathrm{O}(n), \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathrm{SO}(n), \mathbb{Z}/2\mathbb{Z}) \oplus H^2(C_2, \mathbb{Z}/2\mathbb{Z}),$$

given by $i^*(\alpha) = i_1^*(\alpha) \oplus i_2^*(\alpha)$, *for* $\alpha \in H^2(\mathrm{O}(n), \mathbb{Z}/2\mathbb{Z})$, *is an isomorphism.*

Proof. Since i^* is a linear map between 2-dimensional $\mathbb{Z}/2\mathbb{Z}$ vector spaces, it suffices to show that its rank is 2. From [7, p. 328] we obtain that

$$w_2 + w_1 \cup w_1 \in H^2(\mathrm{O}(n), \mathbb{Z}/2\mathbb{Z})$$

corresponds to the group extension $\mathrm{Pin}(n)$ of $\mathrm{O}(n)$, where w_i denotes the Stiefel-Whitney classes of γ^n over G_n. The restriction map $\phi_n \mid_{\mathrm{SO}(n)}: \mathrm{SO}(n) \to \mathrm{SO}(n)$ is in fact the identity map on $\mathrm{SO}(n)$. If there exists a lift $l : \mathrm{SO}(n) \to \mathrm{Spin}(n)$, then it becomes a section of the covering map $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, as $\rho \circ l$ is the identity map on $\mathrm{SO}(n)$. This violates the fact that $\mathrm{Spin}(n)$ is a non-trivial double cover of $\mathrm{SO}(n)$. Therefore $\phi_n \mid_{\mathrm{SO}(n)}$ is aspinorial and hence $i_1^*(w_2(\phi_n)) = w_2(\phi_n \mid_{\mathrm{SO}(n)}) \neq 0$. Note that $H^*(C_2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]$. Also observe that $\phi_n \mid_{C_2}$ is the identity map on C_2, where

$$C_2 = \begin{cases} \{\pm I\}, & \text{for } n \text{ is odd}, \\ \langle g_0 \rangle, & \text{for } n \text{ is even}. \end{cases}$$

In any case we have $w_1(\phi_n \mid_{C_2}) = \det \circ \phi_n \mid_{C_2} = x$.

The representation $\phi_n \mid_{C_2}$ is spinorial if and only if $n \equiv 3 \pmod 4$. Thus in this case $i_2^*(w_2(\phi_n \mid_{C_2})) + x^2 = 0$ which implies $i_2^*(w_2) = x^2$, and 0 otherwise. Therefore we obtain

$$i_2^*(w_2(\phi_n)) = \begin{cases} 1, & \text{when } n \equiv 3 \pmod 4 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $i^*(w_2(\phi_n)) = (1,1)$ or $(1,0)$.

Consider the representation $\psi = \det \oplus \det$. As $\det |_{SO(n)} = 1$, we obtain $i_1^*(w_2(\psi)) = 0$.

Note that
$$\psi(g_0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So $w_1(\psi |_{C_2}) = \det(\psi |_{C_2}) = 1$ and $\psi |_{C_2}$ is aspinorial. This gives $i^*(w_2(\psi)) = (0,1)$. Thus i^* has rank 2, as required. ∎

Theorem 9.3. *Let $\pi_\lambda \otimes \rho$ be an irreducible representation of $O(2n+1)$, where ρ is an irreducible representation of C_2 and π_λ be the irreducible representation of $SO(2n+1)$ with highest weight λ. Then*

$$w_2((\pi_\lambda \otimes \rho)_0) = \frac{(2n-1)}{n(2n+1)} \dim V^\lambda \cdot \chi_\lambda(C) w_2(\gamma^n) + \frac{m(m-1)}{2} e_{cup},$$

where
$$m = \begin{cases} \dim \pi_\lambda, & \text{for } \rho = \mathrm{sgn}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. From Lemma 9.2 we obtain $w_2(\pi_0) = aw_2(\gamma^n) + be_{cup}$. We choose $a = \frac{(2n-1)}{n(2n+1)} \dim V^\lambda \cdot \chi_\lambda(C)$ by [9, Remark 5] which says $p(\underline{\nu}) = 2n - 1$ and $b = \frac{m(m-1)}{2}$. Note that $\pi_\lambda \otimes \rho |_{SO_{2n+1}}$ is spinorial if and only if $a \equiv 0 \pmod 2$. We use Lemma 9.1 for b. Hence the result follows. ∎

Theorem 9.4. *Let (π, W) be an irreducible representation of $O(2n)$, where $n \geq 4$. Let R^+ be the root system C_{n-1}. Then we have*

$$w_2(\pi_0) = \frac{2 \cdot (n-1)}{n \cdot (2n-1)} \dim V^\lambda \left(\chi_\lambda(C) + \chi_{g_0 \cdot \lambda}(C) \right) w_2(\gamma^n) + \frac{m(m-1)}{2} e_{cup},$$

when $\pi = \rho^\lambda$ is irreducible. For $\pi = \pi^{\lambda,\pm}$, we have

$$w_2(\pi_0) = \frac{2 \cdot (n-1)}{n \cdot (2n-1)} \dim V^\lambda \cdot \chi_\lambda(C) w_2(\gamma^n) + \frac{m(m-1)}{2} e_{cup}.$$

Here m is as mentioned in Proposition 6.4 and $\chi_{\pi^{\lambda,\pm}}(g_0)$ is as in Theorem 6.2.

Proof. The proof follows by a similar argument as in Theorem 9.3. ∎

Remark 9.5. We have $H^2(SO(l), \mathbb{Z}/2\mathbb{Z}) = \langle w_2' \rangle$. Therefore for an orthogonal representation π^λ of $SO(n)$ by [9, Remark 5] and [9, Table 1, Section 11] which says $p(\underline{\nu}) = l - 2$ we obtain

$$w_2((\pi^\lambda)_0) = \begin{cases} \dfrac{(2n-1)}{n(2n+1)} \dim V^\lambda \cdot \chi_\lambda(C) w_2', & \text{when } l = 2n+1, \\ \dfrac{2 \cdot (n-1)}{n \cdot (2n-1)} \dim V^\lambda \cdot \chi_\lambda(C) w_2', & \text{when } l = 2n. \end{cases}$$

9.1. Calculation of w_1

Let (π, W) be an irreducible representation of $O(n)$. From [12] we obtain immediately $w_1(\pi_0) = \det(\pi_0)$.

Theorem 9.6. *For* $O(2n + 1)$*, if* $\pi = \pi^\lambda \otimes \rho$ *as in Corollary 9.3, then*

$$w_1(\pi_0) = \begin{cases} \dim V^\lambda \cdot w_1(\gamma^n), & \text{when } \rho = \text{sgn} \\ 0, & \text{when } \rho = \mathbb{1}. \end{cases}$$

For $O(2n)$ *we have* $w_1(\pi_0) = m \cdot w_1(\gamma^n)$*, where* m *is as in Proposition 6.4.*

Proof. Note that $\pi(SO(n)) \subset SL(n)$. Therefore it is enough to determine $\det(\pi(g_0))$ (resp. $\det(\pi(-I))$) for $O(2n)$ (resp. $O(2n+1)$). ∎

10. A character formula

We begin this section with a detection result.

Proposition 10.1. *Let* G_1 *and* G_2 *be two groups and* $f : G_1 \to G_2$ *be a morphism. If the map* $f^* : H^i(G_2, \mathbb{Z}/2\mathbb{Z}) \to H^i(G_1, \mathbb{Z}/2\mathbb{Z})$ *is injective for* $i \in \{1, 2\}$*, then any representation* ϕ *of* G_2 *is spinorial if and only if* $\phi \circ f$ *is spinorial.*

Proof. We have $w_i(\phi \circ f) = f^*(w_i(\phi))$, for $i \in \{1, 2\}$. This gives

$$w_2(\phi \circ f) + w_1(\phi \circ f) \cup w_1(\phi \circ f) = f^*(w_2(\phi) + w_1(\phi) \cup w_1(\phi)). \tag{9}$$

From [7, p. 238] we obtain ϕ is spinorial if and only if $w_2(\phi)) + w_1(\phi) \cup w_1(\phi) = 0$. If ϕ is spinorial then following equation (9) we conclude that $\phi \circ f$ is spinorial. On the other hand if $\phi \circ f$ is spinorial then we obtain $f^*(w_2(\phi) + w_1(\phi) \cup w_1(\phi)) = 0$. Since f^* is injective we have the result. ∎

Consider the subgroup $D_i = \text{diag}\{\underbrace{\pm 1, \pm 1, \ldots, \pm 1}_{i \text{ times}}, 1, \ldots 1 \ldots, 1\}$ of $O(n)$ consisting of the diagonal matrices for $1 \le i \le n$. We write

$$d_i = \text{diag}(\underbrace{-1, -1, \ldots, -1}_{i \text{ times}}, 1, \ldots 1 \ldots, 1).$$

The next result enables us to detect spinorial representations of $O(n)$ from their character values.

Theorem 10.2. *A representation* π *of* $O(n)$ *is spinorial if and only if the following two conditions hold:*

(1) $\chi_\pi(1) - \chi_\pi(d_1) \equiv 0 \text{ or } 6 \pmod{8}$,

(2) $\chi_\pi(1) - \chi_\pi(d_2) \equiv 0 \text{ or } 6 \pmod{8}$.

Proof. Let W_i denote the permutation group of D_i. Note that $W_i = S_i$. From [14, Theorem 2.2, page 82] we obtain an isomorphism

$$\alpha : H^*(O(n), \mathbb{Z}/2\mathbb{Z}) \to H^*(D_n, \mathbb{Z}/2\mathbb{Z})^{W_n}.$$

From [1, Theorem 4.4, p. 69] we have $H^*(D_i, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x_1, x_2, \ldots, x_i]$, the invariant subgroup $H^*(D_i, \mathbb{Z}/2\mathbb{Z})^{W_i} = \mathbb{Z}/2\mathbb{Z}[x_1, x_2, \ldots, x_i]^{S_i}$ is the ring of symmetric polynomials in i variables.

We obtain an injection $f_i : H^i(D_n, \mathbb{Z}/2\mathbb{Z})^{W_n} \to H^i(D_i, \mathbb{Z}/2\mathbb{Z})^{W_i}$ by putting $x_j = 0$ for $i + 1 \leq j \leq n$. Therefore we obtain an injection

$$f_i \circ \alpha : H^i(O(n), \mathbb{Z}/2\mathbb{Z}) \to H^i(D_i, \mathbb{Z}/2\mathbb{Z})^{W_i}.$$

In particular $f_2 \circ \alpha : H^2(O(n), \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^2(D_2, \mathbb{Z}/2\mathbb{Z})$ is an injection. From Proposition 10.1 we conclude that π is spinorial if and only if $\pi\mid_{D_2}$ is spinorial. The non-trivial elements of the group D_2 are the involutions d_1, d_2 and $d_1 d_2$. Therefore $D_2 = \langle d_1, d_2 \rangle \cong C_2 \times C_2$. From Lemma 6.1 we obtain that $\pi\mid_{C_2 = \langle d_1 \rangle}$ is spinorial if and only if the first condition mentioned in the theorem holds. Since d_1 and $d_1 d_2$ are conjugate in $O(n)$, the same condition works for $\pi\mid_{C_2 = \langle d_1 d_2 \rangle}$. Similarly the representation $\pi\mid_{C_2 = \langle d_2 \rangle}$ is spinorial if and only if the second condition holds. ∎

Remark 10.3.   This method was suggested by Steven Spallone.

Using Theorem 3.4 one obtains similar result for orthogonal representations of $\mathrm{GL}_n(\mathbb{R})$. We state it as the following corollary.

Corollary 10.4.   *An orthogonal representation π of $\mathrm{GL}_n(\mathbb{R})$ is spinorial if and only if both the following conditions hold:*

(1) $\chi_\pi(1) - \chi_\pi(d_1) \equiv 0$ *or* $6 \pmod 8$,

(2) $\chi_\pi(1) - \chi_\pi(d_2) \equiv 0$ *or* $6 \pmod 8$.

Remark 10.5.   One can use the injective map

$$f_2 \circ \alpha : H^2(O(n), \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^2(D_2, \mathbb{Z}/2\mathbb{Z})$$

to calculate $w_2(\pi_0)$ for a representation π of $O(n)$. Consider the two 1-dimensional representations of $C_2 \times C_2$, namely $\phi_2 : (d_1, d_1 d_2) \to (-1, 1)$ and $\phi_3 : (d_1, d_1 d_2) \to (1, -1)$. For $i \in \{1, 2\}$ we write

$$g_{d_i} = \frac{\chi_\pi(1) - \chi_\pi(d_i)}{2} \quad \text{and} \quad g_{d_1 d_2} = \frac{\chi_\pi(1) - \chi_\pi(d_1 d_2)}{2}.$$

For a representation (π, V) of $O(n)$ we have

$$w_2(\pi_0) = \left[\frac{g_{d_1}}{2}\right] \alpha^2 + \left[\frac{g_{d_1 d_2}}{2}\right] \beta^2 + \left(\left[\frac{g_{d_1 d_2}}{2}\right] + \left[\frac{g_{d_1}}{2}\right] + \left[\frac{g_{d_2}}{2}\right]\right) \alpha\beta,$$

where $\alpha = w_1(\phi_2)$, $\beta = w_1(\phi_3)$ and $[\cdot]$ denotes the greatest integer function. For details we refer to [13].

11.   Examples

We work out the cases for the irreducible representations of $O(2)$ and $O(4)$ and $O(8)$. Note that we cannot apply Theorem 1.1 to determine the irreducible spinorial representations of the first two groups as $SO(2)$ and $SO(4)$ are not simple.

11.1. Case of $O(2)$

We have $O(2) = S^1 \rtimes C_2$. The irreducible representations of S^1 are π_n given by $\pi_n(z) = z^n$. We write $\rho_n = \mathrm{Ind}_{SO(2)}^{O(2)}(\pi_n)$. From [11, Section 5.1] we obtain that

for $n > 0$, the irreducible representations are given by ρ_n. On the other hand $\rho_0 = \mathbb{1} \oplus \det$. We write g_0 to denote the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. From Theorem 5.1 we obtain $\chi_{\rho_n}(g_0) = 0$. Therefore $\rho_n(g_0) \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence by Lemma 6.1, the representations ρ_n are aspinorial for $n > 0$. The det representation is aspinorial by the same lemma.

11.2. Case of $O(4)$

We know that $\mathrm{Spin}(4) \cong \mathrm{SU}_2 \times \mathrm{SU}_2$, whose irreducible representations are of the form $V_{a,b} = \mathrm{Sym}^a V_0 \boxtimes \mathrm{Sym}^b V_0$, where V_0 denotes the standard representation of SU_2. The representations which factor through $SO(4)$ have the property $a \equiv b$ (mod 2). We write $g_0 = \left(\sum_{i=1}^{2} e_{i,i} + e_{3,4} + e_{4,3} \right)$. From Proposition 6.4 we obtain

$$m = \dim(V_{a,b}) = (a+1)(b+1),$$

when $\rho^{(a,b)}$ is irreducible. So $\rho^{(a,b)}$ is spinorial if and only if the two following conditions hold:

(1) $(a+1)(b+1) \equiv 0$ or 3 (mod 4)

(2) $\dfrac{1}{4}\left((a+1)\dbinom{b+2}{3} + (b+1)\dbinom{a+2}{3} \right) \equiv 0$ (mod 2).

We have the second condition from [9, Example 3, page 21]. Now we consider the case when $\rho^{a,b}$ is reducible. The root system of $SO(4)$ is $D_2 \cong A_1 \times A_1$. Therefore $D_2^\tau \cong A_1^\tau \times A_1^\tau \cong A_1 \times A_1$. From [5, Exercise 14.36, page 210] we obtain the Killing form $(\ ,\ )$ for $\mathfrak{so}(2n)$ as

$$(\mu_1, \mu_2) = \frac{1}{2(2n-2)}\mu_1 \cdot \mu_2, \tag{10}$$

where $\mu_1 \cdot \mu_2$ denotes the normal inner product. The positive roots are obviously $S = \{e_1 - e_2, e_1 + e_2\}$. We calculate

$$(e_1 - e_2)^\vee = 4 \cdot \frac{2(e_1 - e_2)}{(e_1 - e_2) \cdot (e_1 - e_2)} = 4(e_1 - e_2).$$

Similarly we have $(e_1 + e_2)^\vee = 4(e_1 + e_2)$. The highest weight of $V_{a,b}$ is $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$. Now for calculating ρ^τ we take the normalized Killing form mentioned in Section 5.

$$\rho^\tau = \frac{1}{2} \sum_S \frac{2\alpha}{(\alpha, \alpha)} = \frac{1}{2}(e_1 + e_2 + e_1 - e_2) = e_1.$$

From Theorem 6.2 we obtain

$$\chi_{\pi^{\lambda,\pm}}(g_0) = \pm \frac{\prod_{\alpha \in S}\langle \alpha^\vee, \lambda + \rho^\tau \rangle}{\prod_{\alpha \in S}\langle \alpha^\vee, \rho^\tau \rangle}.$$

Here $\langle \alpha^\vee, x \rangle = \left(\frac{2\alpha}{(\alpha,\alpha)}, x\right)$ where $(\ ,\ )$ is the Killing form. We have

$$\lambda + \rho^\tau = \left(\frac{a+b}{2} + 1\right)e_1 + \left(\frac{b-a}{2}\right)e_2.$$

So we calculate

$$\chi_{\pi^{\lambda,\pm}}(g_0) = \pm \frac{(e_1 - e_2, (\frac{a+b}{2}+1)e_1 + (\frac{b-a}{2})e_2)(e_1 + e_2, (\frac{a+b}{2}+1)e_1 + (\frac{b-a}{2})e_2)}{(e_1 - e_2, e_1)(e_1 + e_2, e_1)}$$

$$= \pm \frac{(a+1)(b+1)}{1} = \pm(a+1)(b+1).$$

From Proposition 6.4 we obtain

$$m = \begin{cases} 0 & \text{when} \quad \pi = \pi^{\lambda,+} \\ (a+1)(b+1) & \text{when} \quad \pi = \pi^{\lambda,-}. \end{cases}$$

Therefore we conclude the representation $\pi^{\lambda,-}$ is spinorial if and only if the following two conditions hold:

(1) $(a+1)(b+1) \equiv 0$ or $3 \pmod 4$,

(2) $\frac{1}{4}\left((a+1)\binom{b+2}{3} + (b+1)\binom{a+2}{3}\right) \equiv 0 \pmod 2$.

On the other hand the representation $\pi^{\lambda,+}$ is spinorial if and only if

$$\frac{1}{4}\left((a+1)\binom{b+2}{3} + (b+1)\binom{a+2}{3}\right) \equiv 0 \pmod 2.$$

11.3. Case of $O(8)$

Consider the representations V^λ of SO(8) with highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, 0)$. In these cases $\mathrm{Ind}_{\mathrm{SO}(8)}^{\mathrm{O}(8)} V^\lambda$ is reducible with irreducible components $\pi^{\lambda,\pm}$ such that $\pi^{\lambda,\pm}\mid_{\mathrm{SO}(8)} = V^\lambda$. From [9, Table 1, Section 11] which says $p(\nu) = 2n-2$, we have the representation V^λ is spinorial if and only if the integer

$$\frac{2(n-1)}{n(2n-1)} \dim V^\lambda \cdot \chi_\lambda(C), \tag{11}$$

is even. From [3, Theorem 1.7, page 242] we obtain the Weyl dimension formula

$$\dim V^\lambda = \prod_{\alpha \in \phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}, \tag{12}$$

where (,) denotes the Killing form as mentioned in (10).

As an example we solve here for the case $\lambda = (1, 0, 0, 0)$.

The root system for SO(8) is D_4. The positive roots ϕ^+ are $e_i - e_j$, $e_i + e_j$ for $1 \le i < j \le 4$. The half sum of positive roots is $\rho = (3, 2, 1, 0)$, then using equation (12) we calculate

$$\dim V^\lambda = \frac{\prod_{1\le i<j\le 4}((4,2,1,0)\cdot(e_i - e_j))\prod_{1\le i<j\le 4}((4,2,1,0)\cdot(e_i + e_j))}{\prod_{1\le i<j\le 4}((3,2,1,0)\cdot(e_i - e_j))\prod_{1\le i<j\le 4}((3,2,1,0)\cdot(e_i + e_j))} = 8.$$

From [9, Section 2.3] we have $\chi_\lambda(C) = (\lambda, \lambda + 2\rho)$. This gives

$$\chi_\lambda(C) = \frac{1}{12}\left((1,0,0,0)\cdot((1,0,0,0) + 2(3,2,1,0))\right) = 7/12.$$

Putting $n = 4$ in Equation 11 we obtain

$$\frac{2(n-1)}{n(2n-1)} \dim V^{\lambda} \cdot \chi_{\lambda}(C) = \frac{6}{28} \cdot 8 \cdot \frac{7}{12} = 1. \tag{13}$$

Therefore $\pi^{(1,0,0,0),\pm}$ is aspinorial.

To find the Stiefel-Whitney class of this representation we need m as in Proposition 6.4. We take $\mu = \lambda \mid_{\mathfrak{s}_0^*} = (1,0,0)$. The root system of SO(8) is D_4. From Table 1 we have $D_4^{\tau} = B_3$. Let S be the set of positive roots of B_3. Then $S = \{e_i \pm e_j \mid 1 \le i < j \le 3\} \cup \{e_i \mid 1 \le i \le 3\}$. Therefore we calculate

$$\rho^{\tau} = \frac{1}{2} \sum_S \frac{2\alpha}{(\alpha,\alpha)} = \sum_S \frac{\alpha}{(\alpha,\alpha)} = (3,2,1).$$

From Theorem 6.2 we obtain

$$\chi_{\pi^{\lambda,\pm}}(g_0) = \pm \frac{\prod_{\alpha \in S} \langle \alpha^{\vee}, \lambda + \rho^{\tau} \rangle}{\prod_{\alpha \in S} \langle \alpha^{\vee}, \rho^{\tau} \rangle},$$

where $\alpha^{\vee} = \frac{2\alpha}{(\alpha,\alpha)}$. Putting the value of $\alpha^{\vee}$ we calculate

$$\chi_{\pi^{\lambda,\pm}}(g_0) = \pm \frac{\prod_{\alpha \in S}(\alpha, \lambda + \rho^{\tau})}{\prod_{\alpha \in S}(\alpha, \rho^{\tau})} = \pm \frac{\prod_{\alpha \in S}(\alpha, (4,2,1))}{\prod_{\alpha \in S}(\alpha, (3,2,1))} = \pm 6.$$

From Theorem 6.4 we have $m = 7$ for $\pi^{\lambda,+}$ and $m = 1$ for $\pi^{\lambda,-}$. Putting these values in Theorem 9.4 we obtain

$$w_2(\pi) = \begin{cases} w_2(\gamma), & \text{for } \pi = \pi^{\lambda,+} \\ w_2(\gamma) + e_{\text{cup}}, & \text{for } \pi = \pi^{\lambda,-}. \end{cases}$$

For the notations γ and e_{cup} see Section 9.

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