Journal of Lie Theory
Volume **31** (2021) 287–300

Riesz Potentials for the κ-Generalized Fourier Transform

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Communicated by B. Ørsted

Abstract. We investigate the $L^p - L^q$ boundedness properties of the Riesz potentials I_κ^β and the fractional maximal function $M_{\kappa,\beta}$ associated with the κ-generalized Fourier transform. As application, we establish a Welland inequality.

Mathematics Subject Classification: Primary 42B25; Secondary 42B10.

Key Words: Generalized Fourier transform, Riesz potentials, fractional maximal function, Welland inequality.

1. Introduction

Dunkl theory is a far reaching generalization of Fourier analysis and special function theory related to root systems and where the Lebesgue measure is replaced by a weighted measure invariant under the refection group and parameterized by a multiplicity function κ.

Recently, Ben Said, Kobayashi and Ørsted have introduced in [4] a deformation of the Dunkl theory by a parameter $a > 0$, which arises from the “interpolation” of the two $sl(2;\mathbb{R})$ actions on the Weil representation of the metaplectic group $M_p(n;\mathbb{R})$ and the minimal unitary representation of the conformal group $O(n+1;2)$. They have studied the so-called $\mathcal{F}_{\kappa,a}$ transform which includes the Fourier transform ($\kappa = 0$ and $a = 2$), the Dunkl transform ([7],[8]) ($\kappa > 0$ and $a = 2$) and a new unitary operator $\mathcal{F}_\kappa = \mathcal{F}_{\kappa,1}$ having a rich structure, as much as the Dunkl transform, which we call the κ-generalized Fourier transform.

In this paper we are concerned with Riesz potentials and fractional maximal function associated with the κ-generalized Fourier transform $\mathcal{F}_\kappa$.
We consider the weight function υ_κ, which is defined by

$$\upsilon_\kappa(x) = \|x\|^{-1} \prod_{\alpha \in R_+} |< x, \alpha >|^{2\kappa(\alpha)}, \quad x \in \mathbb{R}^N.$$

Here R_+ is a fixed positive root system.

This is a G-invariant positive homogeneous of degree $2\gamma_\kappa - 1$, where $\gamma_\kappa = \sum_{\alpha \in R_+} \kappa_\alpha$.

Consider also the weighted Lebesgue measure ν_κ given by $d\nu_\kappa(x) = \upsilon_\kappa(x)dx$. Denote by $\aleph = 2\gamma_\kappa + N - 1$ the homogeneous dimension of the system.

ISSN 0949–5932 / \$2.50

For $f \in L^1(\mathbb{R}^N, \nu_\kappa)$, the κ-generalized Fourier transform is defined by

$$\mathcal{F}_\kappa f(y) = c_\kappa \int_{\mathbb{R}^N} f(x) B_\kappa(x,y) d\nu_\kappa(x), \quad y \in \mathbb{R}^N, \tag{1}$$

where c_κ is the constant given by $c_\kappa^{-1} = \displaystyle\int_{\mathbb{R}^N} e^{-\|x\|} d\nu_\kappa(x)$ and $B_\kappa(x,y)$ denotes the kernel given by

$$B_\kappa(x,y) = \Gamma(\frac{\aleph}{2}) V_\kappa \left[\widetilde{J}_{\kappa-1}(\sqrt{2\|x\|\|y\|(1+ < \frac{x}{\|x\|}, . >)} \right] \left(\frac{y}{\|y\|} \right). \tag{2}$$

Here V_κ denotes the Dunkl intertwining operator (see (6)) and $\widetilde{J}_\nu(z) = (\frac{z}{2})^\nu J_\nu(z)$, J_ν (see [15]), being the normalized Bessel function.

The generalized translation operator $f \to \tau_y f$, $y \in \mathbb{R}^N$ is defined on $L^2(\mathbb{R}^N, \nu_\kappa)$ by

$$\mathcal{F}_\kappa(\tau_y f)(\xi) = B_\kappa(y, \xi) \mathcal{F}_\kappa(f)(\xi), \quad \xi \in \mathbb{R}^N. \tag{3}$$

It plays the role of the ordinary translation $\tau_y f(.) = f(. - y)$ in $\mathbb{R}^N$, since the Euclidean Fourier transform satisfies $\widehat{\tau_y f}(x) = e^{-i<x,y>} \widehat{f}(x)$.

For $0 < \beta < \aleph$, the Riesz potentials I_β^κ is defined on $\mathcal{S}(\mathbb{R}^N)$ (the class of Schwartz functions) by

$$I_\beta^\kappa f(x) = \frac{\Gamma(\aleph - \beta)}{\Gamma(\beta)} \int_{\mathbb{R}^N} \frac{\tau_y f(x)}{\|y\|^{\aleph - \beta}} d\nu_\kappa(y), \quad x \in \mathbb{R}^N. \tag{4}$$

Up to this stage we have considered the Riesz potentials only for very smooth functions. But since the Riesz potentials are integral operators, it is natural to inquire their actions on the spaces $L^p(\mathbb{R}^N, \nu_\kappa)$.

For this purpose we formulate the following problem. Given $\beta \in]0, \aleph[$ for what pair (p,q) is it possible to extend I_β^κ to bounded operator from $L^p(\mathbb{R}^N, \nu_\kappa)$ to $L^q(\mathbb{R}^N, \nu_\kappa)$? That is, when do we have the inequality

$$\|I_\beta^\kappa f\|_{\kappa,q} \leq C \|f\|_{\kappa,p},$$

where $\|.\|_{\kappa,p}$ stands as the L^p-norm with respect to the measure ν_κ.

Our aim in this paper is to show that the last inequality holds if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\aleph}.$$

The boundedness of Riesz potentials will be used to establish the boundedness properties of the fractional maximal operator associated with the κ-generalized Fourier transform.

For $0 \leq \beta < \aleph$ and $f \in L^p(\mathbb{R}^N, \nu_\kappa)$, $1 \leq p < +\infty$, we define the fractional maximal function $M_{\kappa,\beta}(f)$ by

$$M_{\kappa,\beta}(f)(x) = \sup_{r>0} \frac{1}{(\nu_\kappa(B_r))^{\frac{\aleph-\beta}{\aleph}}} \left| \int_{\mathbb{R}^N} f(y) \tau_x \chi_{B_r}(y) d\nu_\kappa(y) \right|, \quad x \in \mathbb{R}^N, \tag{5}$$

χ_{B_r} being the characteristic function of the Euclidean ball of radius r centered at 0.

If $\beta = 0$, then $M_\kappa = M_{\kappa,0}$ is the Hardy-Littlewood-Paley function associated with the κ-generalized Fourier transform. In [2] and [3] the authors have proved that M_κ is $L^p \to L^p$ bounded if $p > 1$ and of a weak type $(1,1)$ on $\mathbb{R}$ and $\mathbb{R}^N$, respectively.

As application, we shall give pointwise estimates of I^κ_β. Especially, we will prove the Welland inequality for the κ-generalized Fourier transform, it says that:

For $0 < \beta < \aleph$ and $0 < \epsilon < \min(\beta, \aleph - \beta)$, there exists a constant $C > 0$ such that for any nonnegative locally integrable function $f : \mathbb{R}^N \to \mathbb{R}_+$ the following inequality holds:

$$I^\kappa_\beta f(x) \leq C\sqrt{M_{\kappa,\beta-\epsilon}(f)(x)M_{\kappa,\beta+\epsilon}(f)(x)}, \quad x \in \mathbb{R}^N.$$

The contents of this paper are as follows. In section 2, we collect some results about harmonic analysis associated with the Dunkl and the κ-generalized Fourier transform theories. Section 3, is devoted to the proof of the $L^p \to L^q$ boundedness of the Riesz potentials I^κ_β associated with the κ-generalized Fourier transform. Note that theses results have been established by Stein in [13] for the classical Fourier transform and Hassani, Mustapha and the second author for the Dunkl transform (see [9]). In section 4, we first study the $L^p \to L^q$ boundedness of the related fractional maximal function. Next, we study the pointwise estimates of Riesz potentials. Especially, we establish the Welland inequality for the κ-generalized Fourier transform. This inequality was proved for the classical Riesz potentials by Welland (see [16]).

2. Background

2.1. Dunkl theory

Let G be a finite reflection group on $\mathbb{R}^N$ with a fixed positive root system R_+, normalized such that $<\alpha,\alpha>= 2$ for all $\alpha \in R_+$, where $< .,. >$ denotes the usual Euclidean inner product. For a nonzero vector $\alpha \in \mathbb{R}^N$, let σ_α denote the reflection with respect to the hyperplane perpendicular to α, i.e.

$$\sigma_\alpha x = x - 2\frac{<x,\alpha>}{\|\alpha\|^2}\alpha, \quad x \in \mathbb{R}^N.$$

In consequence G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_\alpha, \alpha \in R_+\}$. Let κ be a nonnegative multiplicity function $\alpha \longrightarrow \kappa(\alpha)$ defined on R_+ with the property that $\kappa_u = \kappa_\alpha$ whenever σ_u is conjugate to σ_α in G, then $\alpha \longrightarrow \kappa(\alpha)$ is a G-invariant function.

Introduced by Dunkl in [6] and developed by many authors (see [5], [7], [8], [12] and [14]) the Dunkl operators T_j, $1 \leq j \leq d$ on $\mathbb{R}^N$ are the first-order differential-difference operators given by

$$T_j f(x) = \partial_j f(x) + \sum_{\alpha\in R_+} \kappa(\alpha)\frac{f(x) - f(\sigma_\alpha x)}{<x,\alpha>} <\alpha, e_j>, \qquad 1 \leq j \leq N,$$

where ∂_j denotes the usual partial derivatives and $e_1, ..., e_N$ the standard basis of $\mathbb{R}^N$. A fundamental property of these differential-difference operator is their commutativity, that is

$$T_k T_l = T_l T_k, \quad 1 \leq k, l \leq N.$$

The Dunkl Laplacian is $\Delta_\kappa = \sum_{j=1}^N T_j^2$.

Closely related to the Dunkl operators is the so-called intertwining operator V_κ, which is the unique linear isomorphism of $\bigoplus_{n\geq 0}\mathcal{P}_n$ determined by (see [8])

$$V_\kappa(\mathcal{P}_n)=\mathcal{P}_n,\quad V_\kappa(1)=1,\quad T_jV_\kappa=V_\kappa\partial_j,\quad 1\leq j\leq N,$$

with $\mathcal{P}_n$ the subspace of homogeneous polynomials of degree n in N variables. Even if the positivity of the intertwining operator has been established in ([10], [11]) by Rösler, an explicit formula is not known in general. However, the operator V_κ possesses the integral representation

$$V_\kappa f(y)=\int_{\mathbb{R}^N} f(y)d\mu_x(y), \tag{6}$$

where μ_x is a probability measure on $\mathbb{R}^N$ with support in the closed ball $B(0,\|x\|)$ of center 0 and radius $\|x\|$.

2.2. κ-Generalized Fourier transform

Recall that the kernel $B_\kappa(x,y)$, $x,y\in\mathbb{R}^N$ given by (2) satisfies

$$B_\kappa(0,y)=1,\qquad |B_\kappa(x,y)|\leq 1;\quad \|x\|\Delta_\kappa^x B_\kappa(x,y)=-\|y\|B_\kappa(x,y).$$

The kernel plays an important role in the development of the κ-generalized Dunkl transform given by (1). Note that when f is a suitable radial function (see [3]), i. e. $(f(x)=f_0(\|x\|))$ then $\mathcal{F}_\kappa(f)$ is also radial function and has the form

$$\mathcal{F}_\kappa(f)(\xi)=H_{\aleph-1}f_0(\|\xi\|),$$

where
$$H_\nu f_0(s)=\int_0^\infty f_0(u)\widetilde{J}_\nu(2\sqrt{us})u^\nu du.$$

Some properties of the κ-generalized Fourier transform are collected below ([3]).

Proposition 2.1. (i) *The κ-generalized Fourier transform $\mathcal{F}_\kappa$ is a topological isomorphism of $\mathcal{S}(\mathbb{R}^N)$ into itself and its inverse is given by $\mathcal{F}_\kappa^{-1}=\mathcal{F}_\kappa$.*

(ii) *(Plancherel Theorem) The κ-generalized Fourier transform extends to an isometry of $L^2(\mathbb{R}^N,\nu_\kappa)$.*

The κ-generalized Fourier transform allows us to define the generalized translation operator on $L^2(\mathbb{R}^N,\nu_\kappa)$, by (3). In the analysis of this translation a particular role is played by the space

$$A_\kappa(\mathbb{R}^N)=\{f\in L^1(\mathbb{R}^N,\nu_\kappa)/\ \mathcal{F}_\kappa f\in L^1(\mathbb{R}^N,\nu_\kappa)\}.$$

Note that $A_\kappa(\mathbb{R}^N)$ is contained in the intersection of $L^1(\mathbb{R}^N,\nu_\kappa)$ and L^∞ and hence is a subspace of $L^2(\mathbb{R}^N,\nu_\kappa)$.

The operator τ_y satisfies the following properties.

Proposition 2.2. *Assume that* $f \in A_\kappa(\mathbb{R}^N)$ *and* $g \in L^1(\mathbb{R}^N, \nu_\kappa) \cap L^\infty$. *Then*

(i) *For every* $x, y \in \mathbb{R}^N$, *we have* $\tau_y f(x) = \tau_x f(y)$.

(ii) *For every* $y \in \mathbb{R}^N$, *the operator* τ_y *satisfies*

$$\int_{\mathbb{R}^N} \tau_y f(x) g(x) d\nu_\kappa(x) = \int_{\mathbb{R}^N} f(x) \tau_y g(x) d\nu_\kappa(x). \tag{7}$$

A formula of $\tau_y f$ is known, at the moment, only in two cases. One in the case of $G = \mathbb{Z}_2, \upsilon_\kappa(x) = |x|^{2\kappa-1}$ on $\mathbb{R}$ (see [1]). Another case where a formula of $\tau_y f$ is known when f is radial function in $A_\kappa(\mathbb{R}^N)$ $(f(x) = f_o(\|x\|))$, G being any reflection group(see [3])

$$\tau_y f(x) = \frac{\Gamma(\frac{\aleph}{2})}{\Gamma(\frac{\aleph}{2} - \frac{1}{2})} V_\kappa \left[\int_{-1}^{1} f_0(\ll x, y; u, . \gg) \left(1 - u^2\right)^{\frac{\aleph}{2} - \frac{1}{2}} du \right] (\frac{y}{\|y\|}), \tag{8}$$

where $\ll x, y; u, . \gg = \|x\| + \|y\| - \sqrt{2\|x\|\|y\|(1 + < \frac{x}{\|x\|}, . >)u}$.

From (8) and (6) it follows that $\tau_y f(x) \geq 0$ for all $y \in \mathbb{R}^N, f(x) = f_0(\|x\|) \geq 0$. Further properties of $\tau_y f$ (f being radial) follow from this formula. These properties are collected in the following proposition.

Proposition 2.3. (See [3])

(i) *For every* $f \in L^1_{rad}(\mathbb{R}^N, \nu_\kappa)$ *the subspace of radial functions in* $L^1(\mathbb{R}^N, \nu_\kappa)$, *we have:*

$$\int_{\mathbb{R}^N} \tau_y f(x) d\nu_\kappa(x) = \int_{\mathbb{R}^N} f(x) d\nu_\kappa(x). \tag{9}$$

(ii) *For* $1 \leq p \leq 2$, $\tau_y : L^p_{rad}(\mathbb{R}^N, \nu_\kappa) \longrightarrow L^p_{rad}(\mathbb{R}^N, \nu_\kappa)$ *is bounded operator.*

(iii) *For every* $s > 0$ *and* $y \in \mathbb{R}^N$, *we have*

$$\tau_y(e^{-s\|.\|})(x) \leq e^{-s\left(\sqrt{\|x\|} - \sqrt{\|y\|}\right)^2}, \quad x \in \mathbb{R}^N. \tag{10}$$

The generalized translation operator can be used to define a generalized convolution.

Definition 2.4. For $f, g \in L^2(\mathbb{R}^N, \nu_\kappa)$, we define the generalized convolution $\star_\kappa$, by

$$f \star_\kappa g(x) = c_\kappa \int_{\mathbb{R}^N} f(y) \tau_x g(y) d\nu_\kappa(y), \quad x \in \mathbb{R}^N.$$

Note that the generalized convolution $\star_\kappa$ is well defined since $\tau_x g \in L^2(\mathbb{R}^N, \nu_\kappa)$ and it may be rewrite

$$f \star_\kappa g(x) = c_\kappa \int_{\mathbb{R}^N} \mathcal{F}_\kappa f(\lambda) \mathcal{F}_\kappa g(\lambda) B_\kappa(x, \lambda) d\nu_\kappa(\lambda), \quad x \in \mathbb{R}^N.$$

This convolution was considered in [3]. It satisfies

$$f \star_\kappa g = g \star_\kappa f; \quad \mathcal{F}_\kappa(f \star_\kappa g) = \mathcal{F}_\kappa f \cdot \mathcal{F}_\kappa f.$$

The following theorem holds.

Theorem 2.5. *Let g be a bounded radial function in $L^1(\mathbb{R}^N,\nu_\kappa)$. Then $f\star_\kappa g$ initially defined on the intersection of $L^1(\mathbb{R}^N,\nu_\kappa)$ and $L^2(\mathbb{R}^N,\nu_\kappa)$ extends to all $L^p(\mathbb{R}^N,\nu_\kappa)$, $1\le p\le\infty$ as a bounded operator. In particular we have*

$$\|f\star_\kappa g\|_{\kappa,p}\le\|g\|_{\kappa,1}\|f\|_{\kappa,p}. \tag{11}$$

3. Riesz potentials

We consider the boundedness of I_β^κ (given by (4)) as an operator from $L^p(\mathbb{R}^N,\nu_\kappa)$ to $L^q(\mathbb{R}^N,\nu_\kappa)$. The following necessary condition holds:

Proposition 3.1. *Let $1\le p<q<+\infty$ and $\beta\in]0,\aleph[$. If*

$$\|I_\beta^\kappa f\|_{\kappa,q}\le C\|f\|_{\kappa,p},\quad f\in\mathcal{S}(\mathbb{R}^N),$$

then it is necessary that

$$\frac{1}{p}-\frac{1}{q}=\frac{\beta}{\aleph}. \tag{12}$$

Proof. For $s>0$, let $\psi_s(x)=e^{-s\|x\|}$, $x\in\mathbb{R}^N$, then using (11), a change of variable shows that $I_\beta^\kappa\left(e^{-s\|\cdot\|}\right)(x)=s^{-\beta}I_\beta^\kappa\left(e^{-\|\cdot\|}\right)(sx)$. Consequently, setting $\psi(x)=e^{-\|x\|}$, a change of variables gives

$$\|I_\beta^\kappa\psi_s\|_{\kappa,q}=s^{-\beta-\frac{\aleph}{q}}\|I_\beta^\kappa\psi\|_{\kappa,q};\quad \|\psi_s\|_{\kappa,p}=s^{-\frac{2\gamma_\kappa+d-1}{p}}\|\psi\|_{\kappa,p}. \tag{13}$$

Letting $s\to0$ and $s\to+\infty$, we show that if $\|I_\beta^\kappa f\|_{\kappa,q}\le C\|f\|_{\kappa,p}$, we must have $\beta+\dfrac{\aleph}{q}-\dfrac{\aleph}{p}=0$, which gives (12). ∎

We shall see below that the condition (12) is also sufficient, save for the case $p=1$, (then $q=\frac{\aleph}{\aleph-\beta}$), in this case we have only a weak type $(1,q)$ mapping.

Theorem 3.2. *Let β be a real number such that $0<\beta<\aleph$ and let (p,q) be a pair of reals numbers such that $1\le p<q<\infty$, and $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{\aleph}$. Then*

(i) *If $p>1$, then the mapping $f\longrightarrow I_\beta^\kappa f$ can be extended to a bounded operator from $L^p(\mathbb{R}^N,\nu_\kappa)$ to $L^q(\mathbb{R}^N,\nu_\kappa)$ and*

$$\|I_\beta^\kappa f\|_{\kappa,q}\le A_{p,\beta}\|f\|_{\kappa,p},\quad f\in L^p(\mathbb{R}^N,\nu_\kappa)$$

where $A_{p,\beta}>0$ depends only on p and β .

(ii) *If $p=1$, $f\longrightarrow I_\beta^\kappa f$ can be extended to a mapping of weak type $(1,q)$, meaning that for all $\lambda>0$*

$$\int_{\{x:|I_\beta^\kappa f(x)|>\lambda\}}d\nu_\kappa(x)\le A_\beta\left(\frac{\|f\|_{\kappa,1}}{\lambda}\right)^q, f\in L^1(\mathbb{R}^N,\nu_\kappa),$$

where $A_\beta>0$ depends only on β.

Proof. Recall the simple formula $\|y\|^{-a}=\dfrac{1}{\Gamma(a)}\displaystyle\int_0^\infty s^a e^{-a\|y\|}\frac{ds}{s}$.

Applying this formula with $a = \aleph - \beta$ and using Fubini's theorem, we obtain

$$
\begin{aligned}
I_\beta^\kappa f(x) &= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\aleph-\beta} \left(\int_{\mathbb{R}^N} e^{-s\|\cdot\|} \tau_y f(x) d\nu_\kappa(y) \right) \frac{ds}{s} \\
&= \frac{1}{c_\kappa \Gamma(\beta)} \int_0^\infty f \star_\kappa \psi_s(x) \frac{ds}{s}, \quad f \in S(\mathbb{R}^N),
\end{aligned}
$$

here $\psi_s(y) = e^{-s\|y\|}$. Thus

$$
I_\beta^\kappa f(x) = \frac{1}{c_\kappa \Gamma(\beta)} \int_0^\infty f \star_\kappa \psi_s(x) \frac{ds}{s}, \quad f \in S(\mathbb{R}^N). \tag{14}
$$

By (10), we obtain

$$
\tau_x(e^{-s\|\cdot\|})(y) \leq 1. \tag{15}
$$

Thus the mapping $f \longrightarrow I_\beta^\kappa f$ can be extended to all functions $f \in L^p(\mathbb{R}^N, \nu_\kappa), p \geq 1$. Let (p, q) be a pair of real numbers satisfying (12) and let $f \in L^p(\mathbb{R}^N, \nu_\kappa)$ normalized such that $\|f\|_{\kappa,p} = 1$.

We shall prove that the integral $\dfrac{1}{c_\kappa \Gamma(\beta)} \displaystyle\int_0^\infty s^{\aleph-\beta} f \star_\kappa \psi_s(x) \frac{ds}{s}$ converges absolutely for almost every x. We decompose $I_\beta^\kappa f(x) = G_a f(x) + G^a f(x)$

where

$$
G_a f(x) = \frac{1}{c_\kappa \Gamma(\beta)} \int_0^a s^{\aleph-\beta} f \star_\kappa \psi_s(x) \frac{ds}{s}
$$

and

$$
G^a f(x) = \frac{1}{c_\kappa \Gamma(\beta)} \int_a^{+\infty} s^{\aleph-\beta} f \star_\kappa \psi_s(x) \frac{ds}{s},
$$

where $a > 0$ is a nonnegative constant which will be fixed later. Let $x \in \mathbb{R}^N$. Then, by Hölder's inequality, (9), (13) and (15), we obtain

$$
\begin{aligned}
&\sup_x |f \star_\kappa \psi_s(x)| \leq c_\kappa \int_{\mathbb{R}^N} \tau_x \psi_s(y) |f(y)| d\nu_\kappa(y) \\
&\leq c_\kappa \left(\int_{\mathbb{R}^N} \tau_x \psi_s(y) d\nu_\kappa(y) \right)^{1-\frac{1}{p}} \left(\int_{\mathbb{R}^N} \tau_x \psi_s(y) |f(y)|^p d\nu_\kappa(y) \right)^{\frac{1}{p}} \leq \frac{(c_\kappa)^{\frac{1}{p}}}{s^{\aleph(1-\frac{1}{p})}}.
\end{aligned}
$$

Hence

$$
|G_a f(x)| \leq \frac{(c_\kappa)^{\frac{1}{p}}}{\Gamma(\beta)} \int_0^a s^{\frac{\aleph}{p}-\beta} \frac{ds}{s} = A a^{\frac{\aleph}{p}-\beta}, \quad \text{where } A = \frac{1}{\Gamma(\beta)(c_\kappa)^{1-\frac{1}{p}}}.
$$

This gives

$$
\|G_a f\|_{\kappa,\infty} \leq A a^{\frac{\aleph}{p}-\beta}. \tag{16}
$$

Using (10), we obtain

$$
\|G^a f\|_{\kappa,p} \leq \frac{a^{-\beta}}{\Gamma(\beta) c_\kappa}.
$$

This last inequality combined with (16) implies that the integral representing $I_\beta^\kappa f$ given by (14) converges absolutely for almost every $x \in \mathbb{R}^N$.

We shall show next that if $1 \leq p < q < \infty$ satisfying (12), then the mapping $f \to I_\beta^\kappa f$ is of weak type (p, q), in the sense that

$$
\nu_\kappa\left(\{x : \ |I_\beta^\kappa f(x)| > \lambda\}\right) \leq \left(C_{p,q} \frac{\|f\|_{\kappa,p}}{\lambda} \right)^q, \quad f \in L^p(\mathbb{R}^d, \nu_\kappa), \quad \lambda > 0. \tag{17}
$$

Let $\lambda > 0$, then

$$\nu_\kappa\left(\{x : |I^\kappa_\beta f(x)| > \lambda\}\right) \le \nu_\kappa\left(\{x : |G_a f(x)| > \frac{\lambda}{2}\}\right) + \nu_\kappa\left(\{x : |G^a f(x)| > \frac{\lambda}{2}\}\right).$$

Choose therefore a satisfying $$\frac{\lambda}{2} = A a^{\frac{\aleph}{p} - \beta}, \tag{18}$$

we obtain thanks to (16) that $\nu_\kappa\left(\{x : |G_a f(x)| > \frac{\lambda}{2}\}\right) = 0$. Thus

$$\nu_\kappa\left(\{x : |I^\beta_k f(x)| > \lambda\}\right) \le \nu_\kappa\left(\{x : |G^a f(x)| > \frac{\lambda}{2}\}\right) = \left(\frac{2}{\lambda}\right)^p \|G^a f\|_{\kappa,p},$$

and then $$\nu_\kappa\left(\{x : |I^\kappa_\beta f(x)| > \lambda\}\right) \le \left(\frac{2B}{\lambda} a^{-\beta}\right)^p.$$

Using (18) we deduce that

$$\nu_\kappa\left(\{x : |I^\kappa_\beta f(x)| > \lambda\}\right) \le \frac{2^p B^p (2C)^{\frac{\beta p^2}{\aleph - \beta p}}}{\lambda^{\frac{p\aleph}{\aleph - \beta p}}} = C_{p,\beta}\left(\frac{\|f\|_{\kappa,p}}{\lambda}\right)^q$$

since $q = p\aleph/(\aleph - \beta p)$ and $\|f\|_{\kappa,p} = 1$.

This is (17), and so the mapping $f \longrightarrow I^\kappa_\beta f$ is of weak type (p,q).

The special case for $p = 1$ gives then part (ii) of Theorem 3.2. Part (i) of Theorem 3.2 follows by an obvious use of a real interpolation of the spaces $L^p(\mathbb{R}^N, \nu_\kappa)$ and the Marcinkiewicz interpolation theorem. ■

4. Fractional maximal function

We begin this section by proving that the condition (12) is necessary for the boundedness of the operator $M_{\kappa,\beta}$ given by (5).

Proposition 4.1. (i) *Let* $1 < p < q < +\infty$ *and* $\beta \in]0, \aleph[$. *If*

$$\|M_{\kappa,\beta} f\|_{\kappa,q} \le C\|f\|_{\kappa,p}, \quad f \in L^p(\mathbb{R}^N, \nu_\kappa),$$

then it is necessary that (p,q) *satisfies* (12).

(ii) *Let* $q \in]1, +\infty[$. *If*

$$\nu_\kappa(\{x : M_{\kappa,\beta} f(x) > \lambda\}) \le C\left(\frac{\|f\|_{\kappa,1}}{\lambda}\right)^q, \quad \lambda > 0,$$

then it is necessary that $q = \dfrac{\aleph}{\aleph - \beta}$.

Proof. (i) We consider the dilation operator $\delta_r f(x) = f(rx)$, $r > 0$, then

$$\|\delta_r f\|_{\kappa,p} = r^{-\frac{\aleph}{p}} \|f\|_{\kappa,p}; \quad \delta_{r^{-1}} M_{\kappa,\beta} \delta_r = r^{-\beta} M_{\kappa,\beta}.$$

If $p > 1$, the estimate $\|M_{\kappa,\beta} f\|_{\kappa,q} \le C\|f\|_{\kappa,p}$ implies that

$$\|M_{\kappa,\beta} f\|_{\kappa,q} \le C r^{\beta + \frac{\aleph}{q} - \frac{\aleph}{p}} \|f\|_{\kappa,p}, \qquad r > 0.$$

Letting $r \longrightarrow \infty$ and $r \longrightarrow 0$ we get $\dfrac{1}{q} - \dfrac{1}{p} = \dfrac{\beta}{\aleph}$.

(ii) The same argument proves the result in the weak case. ∎

Now, we shall prove the following fractional maximal theorem.

Theorem 4.2. *Let β be a real number such that $0 < \beta < \aleph$ and let (p,q) be a pair of real numbers such that $1 \leq p < q < \infty$ and satisfying (12). Then*

(i) *The maximal fractional operator $M_{\kappa,\beta}$ is bounded from $L^p(\mathbb{R}^N, \nu_\kappa)$ to $L^q(\mathbb{R}^N, \nu_\kappa)$ for $p > 1$.*

(ii) *The maximal fractional operator $M_{\kappa,\beta}$ is of weak type $(1,q)$, that is,*

$$\nu_\kappa\left(\{x : M_{\kappa,\beta} f(x) > \lambda\}\right) \leq C_\beta \left(\frac{\|f\|_{\kappa,1}}{\lambda}\right)^q, \quad \text{for } f \in L^1(\mathbb{R}^N, \nu_\kappa),\ \lambda > 0,$$

where $C_\beta > 0$ depends only on β.

Proof. Based on a passage to polar coordinates, we have for $r > 0$

$$\nu_\kappa(B_r) = \frac{c_\kappa r^{\aleph}}{\Gamma(\aleph + 1)}.$$

So we deduce from (7) that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} M_{\kappa,\beta}(f)(x) &= \sup_{r>0} \frac{m_\kappa}{r^{\aleph-\beta}} \left| \int_{B_r} \tau_x f(y) d\nu_\kappa(y) \right| \\ &= \sup_{r>0} \frac{m_\kappa}{r^{\aleph-\beta}} \left| \int_{\mathbb{R}^N} \tau_{\chi_{B_r}}(y) f(y) d\nu_\kappa(y) \right|, \quad x \in \mathbb{R}^N, \end{aligned} \tag{19}$$

where $m_\kappa = \left(\dfrac{\Gamma(\aleph+1)}{c_\kappa}\right)^{1-\frac{\beta}{\aleph}}$. Invoking (19) we obtain

$$M_{\kappa,\beta} f(x) \leq C_{\kappa,N,\beta} I^\kappa_\beta(|f|)(x), \tag{20}$$

where $C_{\kappa,N,\beta}$ is the positive constant given by $C_{\kappa,N,\beta} = \dfrac{\Gamma(\beta)}{m_\kappa \Gamma(\aleph)}$.

Thus the assertions (i) and (ii) follow from (20) and Theorem 3.2. ∎

Now, we will prove some estimates of the Riesz potentials associated with the κ-generalized Fourier transform.

Notation. We denote by $L^1_{loc}(\mathbb{R}^N, \nu_\kappa)$ the space of locally integrable functions with respect to the measure ν_κ.

Theorem 4.3. *Let $0 < \beta < \aleph$.*

(i) *(Welland inequality) For $0 < \epsilon < \min(\beta, \aleph - \beta)$, there exists a constant $C_\epsilon > 0$ such that for every $f \in L^1_{loc}(\mathbb{R}^N, \nu_\kappa)$ and for every $x \in \mathbb{R}^N$ the following inequality holds:*

$$\left| I^\kappa_\beta(f)(x) \right| \leq C_{\epsilon,\beta} \left(M_{\kappa,\beta-\epsilon} f(x)\right)^{\frac{1}{2}} \left(M_{\kappa,\beta+\epsilon} f(x)\right)^{\frac{1}{2}}.$$

(ii) *For* $1 \leq p < \frac{\lambda}{p}$, *there exists a constant* $C > 0$ *such that for every* $f \in L^1_{loc}(\mathbb{R}^N, \nu_\kappa)$ *and for every* $x \in \mathbb{R}^N$ *the following inequality holds:*

$$\left|I^\kappa_\beta(f)(x)\right| \leq C_{p,\lambda}\left(M_{\kappa,\frac{\lambda}{p}}f(x)\right)^{\frac{\beta p}{\lambda}}(M_\kappa f(x))^{1-\frac{\beta p}{\lambda}}.$$

Proof. Let f be a function in $L^1_{loc}(\mathbb{R}^N, \nu_\kappa)$. For any $r > 0$, we decompose the integral I^κ_β as the sum of two integrals:

$$I^\kappa_\beta f(x) = \frac{\Gamma(\aleph-\beta)}{\Gamma(\beta)}\left(Q_r f(x) + Q^r f(x)\right), \tag{21}$$

where $Q_r f(x) = \int_{B_r} \frac{\tau_y f(x)}{\|y\|^{\aleph-\beta}} d\nu_\kappa(y)$ and $Q^r f(x) = \int_{\mathbb{R}^N \setminus B_r} \frac{\tau_y f(x)}{\|y\|^{\aleph-\beta}} d\nu_\kappa(y)$.

(i) Note that for every $\rho, \varsigma > 0$, we have

$$\left|\int_{B_\rho} \tau_y f(x) d\nu_\kappa(y)\right| \leq \frac{\rho^{\aleph-\varsigma}}{m_\kappa} M_{\kappa,\varsigma} f(x). \tag{22}$$

Let $0 < \epsilon < \beta$; then we have

$$\begin{aligned}|Q_r f(x)| &\leq \sum_{n=0}^{+\infty}\left|\int_{B_{2^{-n}r}\setminus B_{2^{-n-1}r}} \frac{\tau_y f(x)}{\|y\|^{\aleph-\beta}} d\nu_\kappa(y)\right| \\ &\leq \sum_{n=0}^{+\infty} \frac{1}{(2^{-n-1}r)^{\aleph-\beta}}\left|\int_{B_{2^{-n}r}} \tau_y f(x) d\nu_\kappa(y)\right|.\end{aligned}$$

Applying (22), with $\rho = 2^{-n}r$ and $\varsigma = \beta - \epsilon$, we obtain

$$|Q_r f(x)| \leq \frac{2^{\aleph-\beta}}{m_\kappa} r^\epsilon M_{\kappa,\beta-\epsilon} f(x) \sum_{n=0}^{+\infty} 2^{-n\epsilon}.$$

Thus $|Q_r f(x)| \leq C_1 r^\epsilon M_{\kappa,\beta-\epsilon} f(x)$, where $C_1 = \frac{2^{\aleph-\beta} m_\kappa}{1-2^{-\epsilon}}$. (23)

On the other hand, for $0 < \epsilon < \aleph - \beta$, we have

$$\begin{aligned}|Q^r f(x)| &\leq \sum_{n=0}^{+\infty}\left|\int_{B_{2^{n+1}r}\setminus B_{2^n r}} \frac{\tau_y f(x)}{\|y\|^{\aleph-\beta}} d\nu_\kappa(y)\right| \\ &\leq \sum_{n=0}^{+\infty} \frac{1}{(2^n r)^{\aleph-\beta}}\left|\int_{B_{2^{n+1}r}} \tau_y f(x) d\nu_\kappa(y)\right|.\end{aligned}$$

Applying again (22), with $\rho = 2^n r$ and $\varsigma = \beta + \epsilon$, we obtain

$$|Q^r f(x)| \leq \frac{2^{\aleph-\beta+\epsilon}}{m_\kappa} r^{-\epsilon} M_{\kappa,\beta+\epsilon} f(x) \sum_{n=0}^{+\infty} 2^{-n\epsilon}.$$

Thus
$$|Q^r f(x)| \leq C_2 r^{-\epsilon} M_{\kappa,\beta+\epsilon} f(x), \tag{24}$$

where $C_2 = 2^{\aleph-\beta} m_\kappa/(2^\epsilon - 1)$.

Hence, according to (21), (23) and (24), we obtain for any $0<\epsilon<\min(\beta,\aleph-\beta)$

$$\left|I_{\beta}^{\kappa}(f)(x)\right| \leq \frac{\Gamma(\beta)}{\Gamma(\aleph-\beta)}\left(C_1 r^{\epsilon} M_{\kappa,\beta-\epsilon}f(x)+C_2 r^{-\epsilon} M_{\kappa,\beta+\epsilon}f(x)\right). \tag{25}$$

Taking $r^{\epsilon}=\left(\frac{C_2 M_{\kappa,\beta+\epsilon}f(x)}{C_1 M_{\kappa,\beta-\epsilon}f(x)}\right)^{\frac{1}{2}}$ in (25) we get assertion (i) with

$$C_{\epsilon,\beta}=\frac{\Gamma(\beta)}{\Gamma(\aleph-\beta)}\frac{2^{\aleph-\beta+1}m_{\kappa}}{2^{\frac{\epsilon}{2}}-2^{-\frac{\epsilon}{2}}}.$$

(ii) Applying the same method as in (i) we obtain

$$\begin{aligned}
|Q_r f(x)| &\leq \sum_{n=0}^{+\infty}\left|\int_{B_{2^{-n}r}\setminus B_{2^{-n-1}r}} \frac{\tau_y f(x)}{\|y\|^{\aleph-\beta}}d\nu_{\kappa}(y)\right| \\
&\leq \sum_{n=0}^{+\infty}\frac{1}{(2^{-n-1}r)^{\aleph-\beta}}\left|\int_{B_{2^{-n}r}} \tau_y f(x)d\nu_{\kappa}(y)\right| \\
&\leq C_3 r^{\beta} M_{\kappa}f(x),
\end{aligned} \tag{26}$$

where $C_3=2^{\aleph}m_{\kappa}/(2^{\beta}-1)$. Similarly, we obtain

$$\begin{aligned}
|Q^r f(x)| &\leq \sum_{n=0}^{+\infty}\left|\int_{B_{2^{n+1}r}\setminus B_{2^{n}r}} \frac{\tau_y f(x)}{\|y\|^{\aleph-\beta}}d\nu_{\kappa}(y)\right| \\
&\leq \sum_{n=0}^{+\infty}\frac{1}{(2^{n}r)^{\aleph-\beta}}\left|\int_{B_{2^{n+1}r}} \tau_y f(x)d\nu_{\kappa}(y)\right| \\
&\leq C_4 r^{\beta-\frac{\lambda}{p}} M_{\kappa,\frac{\lambda}{p}}f(x),
\end{aligned} \tag{27}$$

where $C_4=2^{\aleph-\frac{\lambda}{p}}m_{\kappa}/(1-2^{\beta-\frac{\lambda}{p}})$. According to (21), (26) and (27), we obtain

$$\left|I_{\beta}^{\kappa}(f)(x)\right| \leq \frac{\Gamma(\beta)}{\Gamma(2\kappa+d-\alpha-1)}\left(C_3 r^{\beta} M_{\kappa}f(x)+C_4 r^{\beta-\frac{\lambda}{p}} M_{\kappa,\frac{\lambda}{p}}f(x)\right). \tag{28}$$

Taking $r=\left(\frac{C_4}{C_3}\frac{M_{\kappa,\frac{\lambda}{p}}f(x)}{M_{\kappa}f(x)}\right)^{\frac{p}{\lambda}}$ in (28) we get assertion (ii) with

$$C_{p,\beta,\lambda}=\frac{2\Gamma(\beta)}{\Gamma(\aleph-\beta)}(C_4)^{\frac{p\beta}{\lambda}}(C_3)^{1-\frac{p\beta}{\lambda}}.$$ ∎

Corollary 4.4. *Let $0<\beta<\aleph$ and let (p,q) be a pair of real numbers such that $1<p<q<\infty$ satisfying (12). Then the following properties are equivalent.*

(1) *The maximal fractional operator $M_{\kappa,\beta}$ is bounded from $L^p(\mathbb{R}^N,\nu_{\kappa})$ to $L^q(\mathbb{R}^N,\nu_{\kappa})$.*

(ii) *The Riesz potentials operator I_{β}^{κ} is bounded from $L^p(\mathbb{R}^N,\nu_{\kappa})$ to $L^q(\mathbb{R}^N,\nu_{\kappa})$.*

Proof. The proof of the implication (i)$\Longrightarrow$(ii) goes as follows:
Let $0<\epsilon<\min(\beta,\aleph-\beta), 1<p<q<\infty$ satisfying (12). If we take

$$\frac{1}{q_\epsilon}=\frac{1}{q}+\frac{\epsilon}{\aleph}, \quad \frac{1}{\overline{q}_\epsilon}=\frac{1}{q}-\frac{\epsilon}{\aleph}, \quad p_1=\frac{2}{q}\,q_\epsilon, \quad p_2=\frac{2}{q}\,\overline{q}_\epsilon,$$

we have $\frac{1}{p_1}+\frac{1}{p_2}=1$. From Theorem 4.3 and Hölder's inequality we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left|I_\beta^\kappa f(x)\right|^q d\nu_\kappa(x)\\
&\leq C_\epsilon\int_{\mathbb{R}^N}\left(M_{\kappa,\beta-\epsilon}f(x)M_{\kappa,\beta+\epsilon}f(x)\right)^{\frac{q}{2}}d\nu_\kappa(x)\\
&\leq C_\epsilon\left(\int_{\mathbb{R}^N}\left(M_{\kappa,\beta-\epsilon}f(x)\right)^{\frac{q}{2}p_1}d\nu_\kappa(x)\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{R}^N}\left(M_{\kappa,\beta+\epsilon}f(x)\right)^{\frac{q}{2}p_2}d\nu_\kappa(x)\right)^{\frac{1}{p_2}}\\
&=C_\epsilon\left(\int_{\mathbb{R}^N}\left(M_{\kappa,\beta-\epsilon}f(x)\right)^{q_\epsilon}d\mu_k(x)\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{R}^N}\left(M_{\kappa,\beta+\epsilon}f(x)\right)^{\overline{q}_\epsilon}d\nu_\kappa(x)\right)^{\frac{1}{p_2}}\\
&=C_\epsilon\|M_{\kappa,\beta-\epsilon}f\|_{q_\epsilon}^{\frac{q}{2}}\|M_{\kappa,\beta+\epsilon}f\|_{\overline{q}_\epsilon}^{\frac{q}{2}}.
\end{aligned}$$

Finally, using Theorem 4.3 we conclude that

$$\|I_\beta^\kappa f\|_{\kappa,q}\leq C_\epsilon\|M_{\kappa,\beta-\epsilon}f\|_{\kappa,q_\epsilon}^{\frac{1}{2}}\|M_{\kappa,\beta+\epsilon}f\|_{\kappa,\overline{q}_\epsilon}^{\frac{1}{2}}\leq C'_\epsilon\|f\|_{\kappa,p}.$$

On the other hand, the implication $(ii)\Longrightarrow(i)$ follows from (20). Thus the proof is finished. ∎

Theorem 4.5. *Let $0<\beta<\aleph$. For $0<\theta<1$, there exists a constant C such that for any $f\in L^1_{loc}(\mathbb{R}^d,\nu_\kappa)$ and for any $x\in\mathbb{R}^N$ the following inequalities holds:*

$$\left|I_{\theta\beta}^\kappa f(x)\right| \leq C\left(\left|I_\beta^\kappa f(x)\right|\right)^\theta\left(M_{\kappa,\beta}f(x)\right)^{1-\theta}, \tag{29}$$

$$\left|I_{\theta\beta}^\kappa f(x)\right| \leq C\left(M_\kappa f(x)\right)^\theta\left(M_{\kappa,\beta}f(x)\right)^{1-\theta}. \tag{30}$$

Proof. We decompose $I_{\theta\beta}^\kappa f$ as in (21). Since for $r>0$,

$$\|y\|^{\beta\theta-\beta}\leq r^{\beta\theta-\beta}, \quad y\in\mathbb{R}^d\backslash B_r,$$

we deduce that

$$|Q^r f(x)|\leq r^{\beta\theta-\beta}\left|\int_{\mathbb{R}^d\backslash B_r}\frac{\tau_x f(y)}{\|y\|^{\aleph-\beta\theta}}d\nu_\kappa(y)\right|\leq\frac{\Gamma(\aleph)}{\Gamma(\beta)}\,r^{\beta\theta-\beta}\left|I_\beta^\kappa f(x)\right|. \tag{31}$$

According to (21), (26) and (31), we obtain

$$\left|I_{\theta\beta}^\kappa f(x)\right|\leq C_3\,r^\beta M_\kappa(f)(x)+\frac{\Gamma(\aleph)}{\Gamma(\beta)}\,r^{\beta\theta-\beta}\left|I_\beta^\kappa f(x)\right|. \tag{32}$$

Taking $r^\beta = \dfrac{\Gamma(\aleph)}{C_3\Gamma(\beta)} \dfrac{\left|I^\kappa_\beta f(x)\right|}{M_\kappa(|f|)(x)}$ in (32), we obtain the inequality (29).

To prove (30), we use the same method as in the proof of Theorem 4.3 to obtain

$$\begin{aligned} |Q_r f(x)| &\leq \sum_{n=0}^{+\infty} \left| \int_{B_{2^{n+1}r}\setminus B_{2^n r}} \frac{\tau_x f(y)}{\|y\|^{\aleph-\beta\theta}} d\nu_\kappa(y) \right| \\ &\leq \sum_{n=0}^{+\infty} \frac{1}{(2^n r)^{\aleph-\beta\theta}} \left| \int_{B_{2^{n+1}r}} \tau_x f(y) d\nu_\kappa(y) \right| \\ &\leq C_5 r^{\beta\theta-\beta} M_{\kappa,\beta} f(x), \end{aligned} \tag{33}$$

where $C_5 = 2^{\aleph-\beta} m_\kappa/(1-2^{\beta\theta-\beta})$. Taking into account (21), (26) and (33), we deduce

$$\left|I^\kappa_{\theta\beta} f(x)\right| \leq \frac{\Gamma(\beta)}{\Gamma(\aleph)} \left(C_3 r^{\beta\theta} M_\kappa(f)(x) + C_5 r^{\beta\theta-\beta} M_{\kappa,\beta}(f)(x)\right). \tag{34}$$

Taking $r^\beta = \dfrac{C_5 M_{\kappa,\beta}(f)(x)}{C_3 M_\kappa(f)(x)}$ in (34), we obtain the inequality (30) with

$$C = \frac{\Gamma(\beta)}{\Gamma(\aleph)} C_3^{1-\theta} C_5^\theta. \qquad \blacksquare$$

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Received March 30, 2020
and in final form September 8, 2020