Journal of Lie Theory
Volume **31** (2021) 301–312


# On the Schur Multiplier and Covers of a Pair of Leibniz Algebras

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Communicated by M. Schlichenmaier

**Abstract.** We utilize the notion of relative central extension on pairs of Leibniz algebras to study their Schur multipliers and covers. We provide some necessary conditions for detecting a certain class of covers of a pair of Leibniz algebras. In addition, we prove that every perfect pair of Leibniz algebras admits at least one cover. Moreover, we show that under certain conditions this cover is unique up to isomorphism. Finally, we prove that any two covers of a finite dimensional Leibniz algebra induce isoclinic pairs of Leibniz algebras.

*Mathematics Subject Classification:* 17A32, 17B99.

*Key Words:* Pair of Leibniz algebras, Schur multiplier, cover.

## 1. Introduction

Leibniz algebras originated from works by Blokh [4, 5] as a noncommutative version of Lie algebras, and was rediscovered by Loday in [12, 13] in his study of periodicity in algebraic K-theory. Several questions arise from investigating extentions of properties of Lie algebras to Leibniz algebras. The aim of this paper is to extend certain properties of the Schur multiplier and covers of a pair of Lie algebras to a pair of Leibniz algebras. Recall from [17] that for a free presentation $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{q} \to 0$ of a Lie algebra $\mathfrak{q}$, and $\mathfrak{n}$ an ideal of $\mathfrak{q}$, the Schur multiplier of the pair $(\mathfrak{n}, \mathfrak{q})$ is defined to be the factor Lie algebra

$$\mathcal{M}(\mathfrak{n}, \mathfrak{q}) = \frac{\mathfrak{r} \cap [\mathfrak{s}, \mathfrak{f}]}{[\mathfrak{r}, \mathfrak{f}]},$$

where $\mathfrak{s}$ is an ideal in $\mathfrak{f}$ such that $\mathfrak{s}/\mathfrak{r} \cong \mathfrak{n}$.

The concept of Schur multiplier was introduced by Schur [21] in 1904 in his study of projective representations of a group. Since then, it has appeared in many studies related to mathematical concepts such as efficient presentations, homology and projective representations of various algebraic structures. Nearly a century later, the notion of Schur multiplier was extended by Ellis [8] to a pair of groups, and more recently to a pair of Lie algebras by Saeedi et al. [17]. Both of these studies rely on notions of relative central extensions presented by Loday in [11].

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ISSN 0949–5932 / $2.50 

In this paper, we consider this notion on a pair of Leibniz algebras and use it to study the Schur multiplier and covers on a pair of Leibniz algebras. It is worth noting that a study of Schur multiplier has been recently developed on the category of Leibniz algebras by Biyogmam and Casas [3]. Their study relies on Lie-central extensions, that is, central extensions relative to the *Liezation functor* $(-)_{\mathsf{Lie}} : \mathsf{Leib} \to \mathsf{Lie}$, which assigns to a Leibniz algebra $\mathfrak{q}$ the Lie algebra $\mathfrak{q}_{\mathsf{Lie}}$, where $\mathfrak{q}_{\mathsf{Lie}} = \mathfrak{q}/\mathfrak{q}^{\mathrm{ann}}$ with $\mathfrak{q}^{\mathrm{ann}}$ being the subspace of $\mathfrak{q}$ spanned by all elements of the form $[x,x]$, $x \in \mathfrak{q}$, and $\mathsf{Leib}$ denotes the category of Leibniz algebras and $\mathsf{Lie}$ denotes the category of Lie algebras. This approach applies essentially to non-Lie Leibniz algebras.

The approach in this paper provides results that apply to all Leibniz algebras, including Lie algebras. The rest of this article is organized in two sections. In section two, we provide preliminaries on Leibniz algebras and define the notions of relative central extension, multiplier and covers of a pair of Leibniz algebras. We prove that every perfect pair of Leibniz algebras admits at least one cover. Section three contains the main results of the paper. In particular, we provide some conditions under which all covers of a pair of finite dimensional Leibniz algebras are isomorphic. We also prove that any two covers of a finite dimensional Leibniz algebra yield isoclinic pairs of Leibniz algebras.

Our results are inspired by corresponding works in group theory [8, 14, 15] and Lie algebra [1, 18, 19].

## 2. The Schur multiplier and covers

Recall that a Leibniz algebra [12, 13] is a vector space $\mathfrak{q}$ equipped with a bilinear map $[-,-] : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$, usually called the Leibniz bracket of $\mathfrak{q}$, satisfying the Leibniz identity: $[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad x,y,z \in \mathfrak{q}$.
A subalgebra $\mathfrak{n}$ of a Leibniz algebra $\mathfrak{q}$ is said to be left ideal (resp. right ideal) of $\mathfrak{q}$ if $[h,g] \in \mathfrak{n}$ (resp. $[g,h] \in \mathfrak{n}$), for all $h \in \mathfrak{n},\ g \in \mathfrak{q}$. If $\mathfrak{n}$ is both left ideal and right ideal, then $\mathfrak{n}$ is called two-sided ideal of $\mathfrak{q}$. In this case $\mathfrak{q}/\mathfrak{n}$ naturally inherits a Leibniz algebra structure.
Let $\mathfrak{q}$ be a Leibniz algebra and $\mathfrak{n}$ be a two-sided ideal of $\mathfrak{q}$. Consider the pair $(\mathfrak{n},\mathfrak{q})$ of Leibniz algebras. Then

$$[\mathfrak{n},\mathfrak{q}] = \langle [n,q],[q,n] \mid n \in \mathfrak{n}, q \in \mathfrak{q} \rangle,$$

$$Z(\mathfrak{n},\mathfrak{q}) = \{n \in \mathfrak{n} \mid [n,q] = [q,n] = 0, \forall q \in \mathfrak{q}\}$$

are called commutator subalgebra and center of the pair $(\mathfrak{n},\mathfrak{q})$, respectively. Clearly, $[\mathfrak{q},\mathfrak{q}]$ and $Z(\mathfrak{q},\mathfrak{q})$ are the usual commutator subalgebra and center of $\mathfrak{q}$.
Let $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{q} \to 0$ be a free presentation of $\mathfrak{q}$. The Schur multiplier of the pair $(\mathfrak{n},\mathfrak{q})$ is defined to be the factor Leibniz algebra

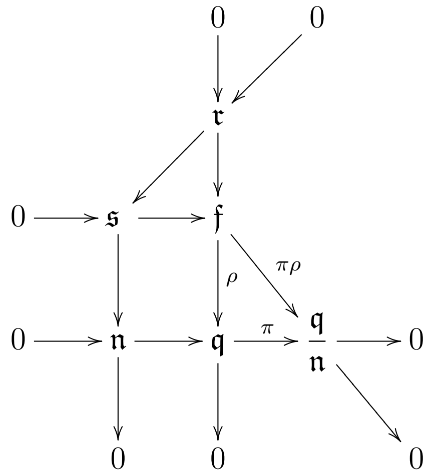
$$\mathcal{M}(\mathfrak{n},\mathfrak{q}) = \frac{\mathfrak{r} \cap [\mathfrak{s},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]},$$

where $\mathfrak{s}$ is a two-sided ideal in $\mathfrak{f}$ such that $\mathfrak{s}/\mathfrak{r} \cong \mathfrak{n}$. $\mathcal{M}(\mathfrak{q}) := \mathcal{M}(\mathfrak{q},\mathfrak{q})$ is referred to as the Schur multiplier of the Leibniz algebra $\mathfrak{q}$ (see also [9]). Clearly, this definition extends the definition of Schur multiplier of a pair of Lie algebras (resp. Schur multiplier of a Lie algebra) defined in [17] (resp. in [20]).

**Proposition 2.1.** *Let* $(\mathfrak{n},\mathfrak{q})$ *be a pair of Leibniz algebras. Then the following sequence is exact:*

$$0 \to \mathcal{M}(\mathfrak{n},\mathfrak{q}) \to \mathcal{M}(\mathfrak{q}) \to \mathcal{M}(\mathfrak{q}/\mathfrak{n}) \to \frac{\mathfrak{n}\cap[\mathfrak{q},\mathfrak{q}]}{[\mathfrak{n},\mathfrak{q}]} \to 0.$$

**Proof.** We use an argument similar to the proof of [3, Proposition 4.1]. Let $0 \to \mathfrak{r} \to \mathfrak{f} \overset{\rho}{\to} \mathfrak{q} \to 0$ be a free presentation of the Leibniz algebra $\mathfrak{q}$, and $\mathfrak{s}$ a two-sided ideal of $\mathfrak{f}$ such that $\mathfrak{n} \cong \mathfrak{s}/\mathfrak{r}$. Then $0 \to \mathfrak{s} \to \mathfrak{f} \overset{\pi\rho}{\to} \mathfrak{q}/\mathfrak{n} \to 0$ is a free presentation of the Leibniz algebra $\mathfrak{q}/\mathfrak{n}$, where $\pi$ is the canonical projection, and we have the following diagram of free presentations:




Define $\Pi : \mathcal{M}(\mathfrak{q}) = \frac{\mathfrak{r}\cap[\mathfrak{f},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]} \to \frac{\mathfrak{s}\cap[\mathfrak{f},\mathfrak{f}]}{[\mathfrak{s},\mathfrak{f}]} = \mathcal{M}(\mathfrak{q}/\mathfrak{n})$ by $\Pi(r+[\mathfrak{r},\mathfrak{f}]) = r+[\mathfrak{s},\mathfrak{f}]$. Clearly $\ker(\Pi) = \frac{\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]} = \mathcal{M}(\mathfrak{n},\mathfrak{q})$ and $\mathsf{Coker}(\Pi) = \frac{\mathfrak{n}\cap[\mathfrak{q},\mathfrak{q}]}{[\mathfrak{n},\mathfrak{q}]}$ thanks to the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}] & \longrightarrow & \mathfrak{r}\cap[\mathfrak{f},\mathfrak{f}] & \longrightarrow & \dfrac{\mathfrak{r}\cap[\mathfrak{f},\mathfrak{f}]}{\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}]} \\
\downarrow & & \downarrow & & \downarrow \\
[\mathfrak{s},\mathfrak{f}] & \longrightarrow & \mathfrak{s}\cap[\mathfrak{f},\mathfrak{f}] & \longrightarrow & \dfrac{\mathfrak{s}\cap[\mathfrak{f},\mathfrak{f}]}{[\mathfrak{s},\mathfrak{f}]} \\
\downarrow & & \downarrow & & \downarrow \\
[\mathfrak{n},\mathfrak{q}] \cong \dfrac{[\mathfrak{s},\mathfrak{f}]}{\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}]} & \longrightarrow & \mathfrak{n}\cap[\mathfrak{q},\mathfrak{q}] & \longrightarrow & \dfrac{\mathfrak{n}\cap[\mathfrak{q},\mathfrak{q}]}{[\mathfrak{n},\mathfrak{q}]}
\end{array}$$

■

**Definition 2.2.** [6] Let $\mathfrak{m}$ and $\mathfrak{n}$ be Leibniz algebras. A *Leibniz action* of $\mathfrak{m}$ on $\mathfrak{n}$ is a couple of bilinear maps $\mathfrak{m}\times\mathfrak{n}\to\mathfrak{n}$ given by $(m,n)\mapsto {}^{m}n$, and $\mathfrak{n}\times\mathfrak{m}\to\mathfrak{n}$ by $(n,m)\mapsto n^{m}$, satisfying the following axioms:

$$\begin{aligned}
{}^{[m,m']}n &= {}^{m}({}^{m'}n) + ({}^{m}n)^{m'}, & {}^{m}[n,n'] &= [{}^{m}n,n'] - [{}^{m}n',n],\\
n^{[m,m']} &= (n^{m})^{m'} - (n^{m'})^{m}, & [n,n']^{m} &= [n^{m},n'] + [n,n'^{m}],\\
{}^{m}({}^{m'}n) + {}^{m}(n^{m'}) &= 0, & [n,{}^{m}n'] + [n,n'^{m}] &= 0,
\end{aligned}$$

for every $m,m'\in\mathfrak{m}$ and $n,n'\in\mathfrak{n}$.

**Definition 2.3.** Let $(\mathfrak{n},\mathfrak{q})$ be a pair of Leibniz algebras. A *relative central extension* of the pair $(\mathfrak{n},\mathfrak{q})$ is a homomorphism of Leibniz algebras $\varphi:\mathfrak{m}\to\mathfrak{q}$ together with a Leibniz action of $\mathfrak{q}$ on $\mathfrak{m}$ satisfying the following conditions:

(i) $\varphi(\mathfrak{m})=\mathfrak{n}$,

(ii) $\varphi({}^{q}m)=[q,\varphi(m)]$ and $\varphi(m^{q})=[\varphi(m),q]$, for all $q\in\mathfrak{q}$ and $m\in\mathfrak{m}$,

(iii) ${}^{\varphi(m_1)}m_2=[m_1,m_2]=m_1^{\varphi(m_2)}$, for all $m_1,m_2\in\mathfrak{m}$,

(iv) $\ker\varphi\subseteq Z(\mathfrak{m},\mathfrak{q})$, in which $Z(\mathfrak{m},\mathfrak{q})=\{m\in\mathfrak{m}\mid {}^{q}m=m^{q}=0,\forall q\in\mathfrak{q}\}$.

Moreover, the relative central extension $\varphi:\mathfrak{m}\to\mathfrak{q}$ is said to be a cover of $(\mathfrak{n},\mathfrak{q})$, if

$$\mathcal{M}(\mathfrak{n},\mathfrak{q})\cong\ker\varphi\subseteq[\mathfrak{m},\mathfrak{q}],$$

where $[\mathfrak{m},\mathfrak{q}]=\langle {}^{q}m,m^{q}\mid m\in\mathfrak{m},q\in\mathfrak{q}\rangle$. Now if $\mathfrak{n}=\mathfrak{q}$, then a cover $\varphi:\mathfrak{m}\to\mathfrak{q}$ of the pair $(\mathfrak{q},\mathfrak{q})$ together with the action ${}^{q}m={}^{\varphi(m_1)}m$ and $m^{q}=m^{\varphi(m_1)}$, where $q=\varphi(m_1)$ for some $m_1\in\mathfrak{m}$, is a cover of $\mathfrak{q}$.

Finally, a Leibniz algebra $\mathfrak{q}$ and a pair $(\mathfrak{n},\mathfrak{q})$ of Leibniz algebras are said to be *perfect*, if $\mathfrak{q}=\mathfrak{q}^2$ (in which $\mathfrak{q}^2$ denotes $[\mathfrak{q},\mathfrak{q}]$) and $[\mathfrak{n},\mathfrak{q}]=\mathfrak{n}$, respectively.

**Example 2.4.** (i) Let $\mathfrak{q}$ be a Leibniz algebra and $\mathfrak{n}$ be a two-sided ideal of $\mathfrak{q}$. Then the inclusion map $i:\mathfrak{n}\to\mathfrak{q}$ is a relative central extension of $(\mathfrak{n},\mathfrak{q})$, where the Leibniz action of $\mathfrak{q}$ on $\mathfrak{n}$ is given by the Leibniz bracket.

(ii) Consider the 4-dimensional Leibniz algebra $\mathfrak{m}=\langle x_1,x_2,x_3,x_4\rangle$ over a field $\mathbb{K}$, with non-zero multiplication $[x_1,x_1]=x_2$, and also the 2-dimensional abelian Leibniz algebra $\mathfrak{q}=\langle a,b\rangle$ and the two-sided ideal $\mathfrak{n}=\langle a\rangle$ of $\mathfrak{q}$. Since $[\mathfrak{n},\mathfrak{q}]=0$, we have $\mathcal{M}(\mathfrak{n},\mathfrak{q})=[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]$ where $0\to\mathfrak{r}\to\mathfrak{f}\xrightarrow{\pi}\mathfrak{q}\to 0$ is a free presentation of $\mathfrak{q}$ such that $\mathfrak{n}\cong\mathfrak{s}/\mathfrak{r}$. Thus $\mathfrak{s}/\mathfrak{r}=\langle s+\mathfrak{r}\rangle$ and $\mathfrak{f}/\mathfrak{r}=\langle s+\mathfrak{r},f+\mathfrak{r}\rangle$, for some $s\in\mathfrak{s}$ and $f\in\mathfrak{f}$ such that $\pi(s)=a$ and $\pi(f)=b$, and hence

$$\mathcal{M}(\mathfrak{n},\mathfrak{q})=\langle[s,s]+[\mathfrak{r},\mathfrak{f}],[s,f]+[\mathfrak{r},\mathfrak{f}],[f,s]+[\mathfrak{r},\mathfrak{f}]\rangle.$$

Moreover, consider the (non-zero) action of $\mathfrak{q}$ on $\mathfrak{m}$ by ${}^{a}x_1=x_1^{a}=x_2$, ${}^{b}x_1=x_3$ and $x_1^{b}=x_4$. Now define $\varphi:\mathfrak{m}\to\mathfrak{q}$ by $\varphi(x_1)=a$ and $\varphi(x_i)=0$, for $2\le i\le 4$. One can easily check that $\mathcal{M}(\mathfrak{n},\mathfrak{q})\cong\ker\varphi=[\mathfrak{m},\mathfrak{q}]=Z(\mathfrak{m},\mathfrak{q})=\langle x_2,x_3,x_4\rangle$ and $\varphi$ is a cover of the pair $(\mathfrak{n},\mathfrak{q})$.

Next, we prove that every perfect pair of Leibniz algebras admits at least one cover.

**Proposition 2.5.** *Let $(\mathfrak{n},\mathfrak{q})$ be a perfect pair of Leibniz algebras with a free presentation $0\to\mathfrak{r}\to\mathfrak{f}\xrightarrow{\pi}\mathfrak{q}\to 0$ such that $\mathfrak{n}\cong\mathfrak{s}/\mathfrak{r}$, for some two-sided ideal $\mathfrak{s}$ in $\mathfrak{f}$. Then $\rho:[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]\longrightarrow\mathfrak{q}$ given by $\rho(x+[\mathfrak{r},\mathfrak{f}])=x+\mathfrak{r}$, is a cover of $(\mathfrak{n},\mathfrak{q})$.*

**Proof.** Consider the action of $\mathfrak{q}$ on $[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]$ given by the following bilinear maps:

$$\bar{\delta}:\frac{[\mathfrak{s},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]}\times\mathfrak{q}\longrightarrow\frac{[\mathfrak{s},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]}\qquad\qquad \bar{\delta}':\mathfrak{q}\times\frac{[\mathfrak{s},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]}\longrightarrow\frac{[\mathfrak{s},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]}$$

$$([s,f]+[\mathfrak{r},\mathfrak{f}],q)\longmapsto[[s,f],t]+[\mathfrak{r},\mathfrak{f}],\qquad (q,[s,f]+[\mathfrak{r},\mathfrak{f}])\longmapsto[t,[s,f]]+[\mathfrak{r},\mathfrak{f}],$$

$$([f,s]+[\mathfrak{r},\mathfrak{f}],q)\longmapsto[[f,s],t]+[\mathfrak{r},\mathfrak{f}],\qquad (q,[f,s]+[\mathfrak{r},\mathfrak{f}])\longmapsto[t,[f,s]]+[\mathfrak{r},\mathfrak{f}],$$

where $\pi(t)=q$.

As $(\mathfrak{n},\mathfrak{q})$ is perfect, $([\mathfrak{s},\mathfrak{f}]+\mathfrak{r})/\mathfrak{r}=\mathfrak{s}/\mathfrak{r}$. Clearly, $\rho$ is a cover of the pair $(\mathfrak{n},\mathfrak{q})$, since $\ker\rho=(\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}])/[\mathfrak{r},\mathfrak{f}]\cong\mathcal{M}(\mathfrak{n},\mathfrak{q})$, and thus

$$\frac{[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]}{\ker\rho}\cong\frac{[\mathfrak{s},\mathfrak{f}]}{\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}]}\cong\frac{[\mathfrak{s},\mathfrak{f}]+\mathfrak{r}}{\mathfrak{r}}=\frac{\mathfrak{s}}{\mathfrak{r}}\cong\mathfrak{n}.$$

It remains to show that $\ker\rho\leq[[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}]$. It is enough to prove that

$$[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]\leq[[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}]$$

since $\ker\rho\leq[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]$. Indeed, recall that

$$[[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}]=\langle[x,q],[q,x]\mid x\in[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}],q\in\mathfrak{q}\rangle,$$

such that
$$[x,q]=[[s_0,f_0]+[\mathfrak{r},\mathfrak{f}],q]=[[s_0,f_0],t]+[\mathfrak{r},\mathfrak{f}],$$

for some $s_0\in\mathfrak{s}$ and $f_0,t\in\mathfrak{f}$, where $\pi(t)=q$. Now, let $s\in\mathfrak{s}$ and $f\in\mathfrak{f}$. Since $[\mathfrak{s},\mathfrak{f}]+\mathfrak{r}=\mathfrak{s}$, there are $s'\in\mathfrak{s}$, $f'\in\mathfrak{f}$ and $r\in\mathfrak{r}$ such that

$$[s,f]+[\mathfrak{r},\mathfrak{f}]=[[s',f']+r,f]+[\mathfrak{r},\mathfrak{f}]=[[s',f'],f]+[\mathfrak{r},\mathfrak{f}]\in[[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}].$$

Similarly, one may show that $[f,s]+[\mathfrak{r},\mathfrak{f}]\in[[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}]$. Therefore we have reached $[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]]\leq[[\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}]$. ■

A relative central extension $\sigma:\mathfrak{m}\to\mathfrak{q}$ of a pair $(\mathfrak{n},\mathfrak{q})$ of Leibniz algebras is said to be *universal*, if for every relative central extension $\theta:\mathfrak{m}'\to\mathfrak{q}$, there exists a unique homomorphism $\varphi:\mathfrak{m}\to\mathfrak{m}'$ such that $\theta\varphi=\sigma$. A unified framework for the study of universal central extensions using techniques from categorical Galois theory can be found in [7].

**Proposition 2.6.** *If a pair $(\mathfrak{n},\mathfrak{q})$ of Leibniz algebras has a universal relative central extension, then it is perfect.*

**Proof.** Let $\sigma:\mathfrak{m}\to\mathfrak{q}$ be a universal relative central extension of $(\mathfrak{n},\mathfrak{q})$. Consider the following exact sequence:

$$0\to\ker\sigma\oplus\frac{\mathfrak{n}}{[\mathfrak{n},\mathfrak{q}]}\to\mathfrak{m}\oplus\frac{\mathfrak{n}}{[\mathfrak{n},\mathfrak{q}]}\xrightarrow{\theta}\mathfrak{q}\to 0,$$

where $\theta(m,n+[\mathfrak{n},\mathfrak{q}])=\sigma(m)$, for all $m\in\mathfrak{m}$, $n\in\mathfrak{n}$. One may easily check that $\ker\theta\leq Z(\mathfrak{m}\oplus\mathfrak{n}/[\mathfrak{n},\mathfrak{q}],\mathfrak{q})$. Therefore, $\theta$ is a relative central extension of the pair $(\mathfrak{n},\mathfrak{q})$. Now, define homomorphisms

$$\varphi_i:\mathfrak{m}\to\mathfrak{m}\oplus\frac{\mathfrak{n}}{[\mathfrak{n},\mathfrak{q}]}\quad(i=1,2),$$

by $\varphi_1(m)=(m,0)$ and $\varphi_2(m)=(m,\sigma(m)+[\mathfrak{n},\mathfrak{q}])$. Since $\theta\varphi_i=\sigma$ $(i=1,2)$, by the universal property of $\sigma$, we get $\varphi_1=\varphi_2$, which implies that $[\mathfrak{n},\mathfrak{q}]=\mathfrak{n}$. ■

## 3. Main results

Throughout this section, we assume that the relative central extension (or cover) $\mathfrak{m}$ contains at least one maximal subalgebra, which means that the Frattini subalgebra $\Phi(\mathfrak{m})\neq\mathfrak{m}$. It is well-known that every finite dimensional Leibniz algebra contains at least one maximal subalgebra.

Let $(\mathfrak{n},\mathfrak{q})$ be a pair of Leibniz algebras with a free presentation $0\to\mathfrak{r}\to\mathfrak{f}\xrightarrow{\pi}\mathfrak{q}\to 0$ such that $\mathfrak{n}\cong\mathfrak{s}/\mathfrak{r}$ for a two-sided ideal $\mathfrak{s}$ in $\mathfrak{f}$. Consider the homomorphism $\delta:\mathfrak{s}/[\mathfrak{r},\mathfrak{f}]\longrightarrow\mathfrak{q}$ given by $s+[\mathfrak{r},\mathfrak{f}]\longmapsto\pi(s)$. It is easy to see that $\delta$ is a relative central extension by the following action

$$\begin{array}{rclcrcl} \bar{\delta}:\dfrac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]}\times\dfrac{\mathfrak{f}}{\mathfrak{r}} & \longrightarrow & \dfrac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]} & \qquad & \bar{\delta}':\dfrac{\mathfrak{f}}{\mathfrak{r}}\times\dfrac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]} & \longrightarrow & \dfrac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]}\\ (s+[\mathfrak{r},\mathfrak{f}],f+\mathfrak{r}) & \longmapsto & [s,f]+[\mathfrak{r},\mathfrak{f}], & & (f+\mathfrak{r},s+[\mathfrak{r},\mathfrak{f}]) & \longmapsto & [f,s]+[\mathfrak{r},\mathfrak{f}]. \end{array}$$

In the next result, we show that under certain conditions, any relative central extensions of an arbitrary pair $(\mathfrak{n},\mathfrak{q})$ of Leibniz algebras is a homomorphic image of the relative central extension $\delta$.

**Lemma 3.1.** *Let $\mathfrak{q}\cong\mathfrak{f}/\mathfrak{r}$, where $\mathfrak{f}$ is a free Leibniz algebra, and let $\mathfrak{n}$ be a two-sided ideal of $\mathfrak{q}$ with $\mathfrak{n}\cong\mathfrak{s}/\mathfrak{r}$ for some two-sided ideal $\mathfrak{s}$ in $\mathfrak{f}$. If $\sigma:\mathfrak{m}\to\mathfrak{q}$ is a relative central extension of a pair $(\mathfrak{n},\mathfrak{q})$, then there exists a homomorphism $\beta:\mathfrak{s}/[\mathfrak{r},\mathfrak{f}]\longrightarrow\mathfrak{m}$ such that the following diagram is commutative:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{\mathfrak{r}}{[\mathfrak{r},\mathfrak{f}]} & \longrightarrow & \dfrac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]} & \xrightarrow{\delta} & \mathfrak{n} & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle\beta|} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle 1_{\mathfrak{n}}} & & \\ 0 & \longrightarrow & \ker\sigma & \longrightarrow & \mathfrak{m} & \xrightarrow{\sigma} & \mathfrak{n} & \longrightarrow & 0, \end{array}$$

*where $\delta$ is the relative central extension defined above. In particular, if $\mathfrak{m}$ is a perfect Leibniz algebra, then $\beta$ is an epimorphism.*

**Proof.** Consider the map $\kappa:\mathfrak{q}\to\mathfrak{n}$ by $\kappa(q)=q$ if $q\in\mathfrak{n}$, otherwise $\kappa(q)=0$. Clearly $\kappa$ induces a homomorphism $\theta:\mathfrak{f}\to\mathfrak{n}$, and since $\mathfrak{f}$ is a free Leibniz algebra, there is a homomorphism $\alpha:\mathfrak{f}\to\mathfrak{m}$ such that $\sigma\alpha=\theta$. For $x\in\mathfrak{f}$ and $r\in\mathfrak{r}$, $\alpha([r,x])=[\alpha(r),\alpha(x)]=0$, since $\alpha(r)\in\ker\sigma\leq Z(\mathfrak{m},\mathfrak{q})\leq Z(\mathfrak{m})$. Therefore $[\mathfrak{r},\mathfrak{f}]\leq\ker\alpha|_{\mathfrak{s}}$, which gives $\beta:\mathfrak{s}/[\mathfrak{r},\mathfrak{f}]\to\mathfrak{m}$. Clearly $\beta(\mathfrak{r}/[\mathfrak{r},\mathfrak{f}])\leq\ker\sigma$, which induces the restriction of $\beta$ to $\mathfrak{r}/[\mathfrak{r},\mathfrak{f}]$. Now, let $\ker\sigma\leq Z(\mathfrak{m},\mathfrak{q})$ and $\Phi(\mathfrak{m})\neq\mathfrak{m}$. For every $m\in\mathfrak{m}$, there exists $\bar{s}\in\mathfrak{s}/[\mathfrak{r},\mathfrak{f}]$ such that $\sigma(m)=\delta(\bar{s})=\sigma\beta(\bar{s})$. Hence $m=\beta(\bar{s})+a$, for some $a\in\ker\sigma$. So $\mathfrak{m}=\beta(\mathfrak{s}/[\mathfrak{r},\mathfrak{f}])+\ker\sigma$. Since $\mathfrak{m}=\mathfrak{m}^2$, thus

$$\ker\sigma\leq Z(\mathfrak{m},\mathfrak{q})\cap\mathfrak{m}^2\leq Z(\mathfrak{m})\cap\mathfrak{m}^2\leq\Phi(\mathfrak{m}).$$

Therefore, $\mathfrak{m}=\beta(\mathfrak{s}/[\mathfrak{r},\mathfrak{f}])$ and hence $\beta$ is an epimorphism. ■

The following example shows that the perfectness condition of the above lemma is necessary.

**Example 3.2.** Let $\mathfrak{q}$ be an abelian Leibniz algebra with the free presentation $\mathfrak{q}\cong\mathfrak{f}/\mathfrak{f}^2$ and also let $\mathfrak{n}=\mathfrak{q}$ and $\mathfrak{m}=\mathfrak{q}\oplus\mathfrak{q}$ (which is not perfect). Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{\mathfrak{f}^2}{\mathfrak{f}^3} & \xrightarrow{\subseteq} & \dfrac{\mathfrak{f}}{\mathfrak{f}^3} & \xrightarrow{\delta} & \mathfrak{q}=\mathfrak{n} & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle\beta|} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle 1_{\mathfrak{q}}} & & \\ 0 & \longrightarrow & \mathfrak{q} & \xrightarrow{\alpha} & \mathfrak{m}=\mathfrak{q}\oplus\mathfrak{q} & \xrightarrow{\sigma} & \mathfrak{q} & \longrightarrow & 0, \end{array}$$

where $\alpha(x) = (x, 0)$ and $\sigma(x, y) = y$, for every $x, y \in \mathfrak{q}$. Therefore $\beta(\mathfrak{f}^2/\mathfrak{f}^3) = [\beta(\mathfrak{f}/\mathfrak{f}^3), \beta(\mathfrak{f}/\mathfrak{f}^3)] \leq \mathfrak{m}^2 = 0$. Thus $\beta$ is not an epimorphism.

The following is an immediate consequence of the above lemma.

**Corollary 3.3.** *Let $\sigma : \mathfrak{m} \to \mathfrak{q}$ be a cover of $(\mathfrak{n}, \mathfrak{q})$ such that $\mathfrak{m}$ is perfect. Then $\mathfrak{m}$ is a homomorphic image of $\mathfrak{s}/[\mathfrak{r}, \mathfrak{f}]$.*

Now, we prove the first main theorem.

**Theorem 3.4.** *Let $(\mathfrak{n}, \mathfrak{q})$ be a pair of Leibniz algebras with finite dimensional Schur multiplier, $\mathfrak{q} \cong \mathfrak{f}/\mathfrak{r}$ and $\mathfrak{n} \cong \mathfrak{s}/\mathfrak{r}$, where $\mathfrak{f}$ is a free Leibniz algebra and $\mathfrak{s}$ is a two-sided ideal of $\mathfrak{f}$. Then for every cover $\sigma : \mathfrak{m} \to \mathfrak{q}$ of $(\mathfrak{n}, \mathfrak{q})$ in which $\mathfrak{m}$ is perfect, there exists a two-sided ideal $\mathfrak{t}$ of $\mathfrak{f}$ such that*

(i) $\mathfrak{m} \cong \mathfrak{s}/\mathfrak{t}$ *and* $\ker\sigma \cong \mathfrak{r}/\mathfrak{t}$,

(ii) $\dfrac{\mathfrak{r}}{[\mathfrak{r}, \mathfrak{f}]} = \mathcal{M}(\mathfrak{n}, \mathfrak{q}) \oplus \dfrac{\mathfrak{t}}{[\mathfrak{r}, \mathfrak{f}]}$.

**Proof.** Since $0 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{\pi} \mathfrak{q} \to 0$ is a free presentation of $\mathfrak{q}$, it follows by the proof of Lemma 3.1 that there exists a homomorphism $\psi : \mathfrak{s} \to \mathfrak{m}$ which induces the epimorphism $\beta : \mathfrak{s}/[\mathfrak{r}, \mathfrak{f}] \to \mathfrak{m}$, and so $\psi$ is onto. Put $\mathfrak{t} = \ker\psi$, then $\mathfrak{s}/\mathfrak{t} \cong \mathfrak{m}$. Clearly $\psi(\mathfrak{r}) = \ker\sigma$ and by Lemma 3.1, $\psi([\mathfrak{r}, \mathfrak{f}]) = 0$. Hence $[\mathfrak{r}, \mathfrak{f}] \leq \mathfrak{t}$, which induces the epimorphism $\bar{\psi} : \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}] \to \ker\sigma$. We claim that $\bar{\psi}| : \mathcal{M}(\mathfrak{n}, \mathfrak{q}) \longrightarrow \ker\sigma$ is an isomorphism. Indeed, suppose that $a \in \ker\sigma$. Since $\psi(\mathfrak{r}) = \ker\sigma$, there exists $x \in \mathfrak{r}$ such that $\psi(x) = a$. On the other hand, by the proof of Lemma 3.1, we have

$$\mathfrak{m} = \langle \beta(\bar{s}) \mid \bar{s} \in \frac{\mathfrak{s}}{[\mathfrak{r}, \mathfrak{f}]} - \frac{\mathfrak{r}}{[\mathfrak{r}, \mathfrak{f}]} \rangle.$$

Now since $\mathfrak{m}$ is perfect, we have $\mathfrak{n} = \sigma(\mathfrak{m}) = \sigma([\mathfrak{m}, \mathfrak{m}]) = [\mathfrak{n}, \mathfrak{n}] \leq [\mathfrak{n}, \mathfrak{q}] \leq \mathfrak{n}$. Thus the pair $(\mathfrak{n}, \mathfrak{q})$ is also perfect. Therefore $[\mathfrak{s}, \mathfrak{f}] + \mathfrak{r} = \mathfrak{s}$. Moreover, $\ker\sigma = \psi(\mathfrak{r}) \leq \Phi(\mathfrak{m})$ and hence $[\mathfrak{m}, \mathfrak{q}] \leq \psi([\mathfrak{s}, \mathfrak{f}])$. It follows that $a = \psi(y)$ for some $y \in [\mathfrak{s}, \mathfrak{f}]$, since $\ker\sigma \leq [\mathfrak{m}, \mathfrak{q}]$. Thus $\psi(x) = a = \psi(y)$. Hence $y - x \in \ker\psi \leq \mathfrak{r}$ and so $y \in \mathfrak{r} \cap [\mathfrak{s}, \mathfrak{f}]$. Since $\ker\sigma \cong \mathcal{M}(\mathfrak{n}, \mathfrak{q}) \cong \dfrac{\mathfrak{r} \cap [\mathfrak{s}, \mathfrak{f}]}{[\mathfrak{r}, \mathfrak{f}]}$, and $\mathcal{M}(\mathfrak{n}, \mathfrak{q})$ is finite dimensional, $\bar{\psi}|$ is an isomorphism. As $\bar{\psi}$ is surjective, for every $\bar{r} \in \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}]$, there exists $\bar{x} \in \mathcal{M}(\mathfrak{n}, \mathfrak{q})$ such that $\bar{\psi}(\bar{x}) = \bar{\psi}(\bar{r})$. So $\bar{r} - \bar{x} \in \ker\psi$. Consequently, $\mathfrak{r}/[\mathfrak{r}, \mathfrak{f}] = \ker\bar{\psi} + \mathcal{M}(\mathfrak{n}, \mathfrak{q})$. Let $\bar{x} \in \ker\bar{\psi} \cap \mathcal{M}(\mathfrak{n}, \mathfrak{q})$. Then $\bar{x} \in \ker\bar{\psi}| = 0$. Therefore,

$$\frac{\mathfrak{r}}{[\mathfrak{r}, \mathfrak{f}]} \cong \mathcal{M}(\mathfrak{n}, \mathfrak{q}) \oplus \ker\bar{\psi} = \mathcal{M}(\mathfrak{n}, \mathfrak{q}) \oplus \frac{\mathfrak{t}}{[\mathfrak{r}, \mathfrak{f}]},$$

which completes the proof. ∎

**Corollary 3.5.** *Let $\sigma : \mathfrak{m} \to \mathfrak{q}$ be a relative central extension of a pair $(\mathfrak{n}, \mathfrak{q})$ of finite dimensional Leibniz algebras such that $\mathfrak{m}$ is perfect. Then $\mathfrak{m}$ is a homomorphic image of a cover of $(\mathfrak{n}, \mathfrak{q})$.*

**Proof.** Let $\mathfrak{q} \cong \mathfrak{f}/\mathfrak{r}$ and $\mathfrak{n} \cong \mathfrak{s}/\mathfrak{r}$. By Lemma 3.1, there is an epimorphism $\beta : \mathfrak{s}/[\mathfrak{r}, \mathfrak{f}] \to \mathfrak{m}$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathfrak{r}}{[\mathfrak{r},\mathfrak{f}]} & \longrightarrow & \dfrac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]} & \xrightarrow{\ \delta\ } & \mathfrak{n} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta|} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle 1_{\mathfrak{n}}} & & \\
0 & \longrightarrow & \ker\sigma & \longrightarrow & \mathfrak{m} & \xrightarrow{\ \sigma\ } & \mathfrak{n} & \longrightarrow & 0,
\end{array}$$

where $\delta$ is the relative central extension defined in Lemma 3.1. Now put $\ker\beta| = \ker\beta = \mathfrak{u}/[\mathfrak{r}, \mathfrak{f}]$ for some two-sided ideal $\mathfrak{u}$ in $\mathfrak{r}$. By the proof of Theorem 3.4, we have

$$\frac{\mathfrak{r}}{\mathfrak{u}} \cong \frac{\mathfrak{r}/[\mathfrak{r},\mathfrak{f}]}{\mathfrak{u}/[\mathfrak{r},\mathfrak{f}]} \cong \ker\sigma \cong \frac{(\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}])/[\mathfrak{r},\mathfrak{f}]}{(\mathfrak{u}\cap[\mathfrak{s},\mathfrak{f}])/[\mathfrak{r},\mathfrak{f}]} \cong \frac{(\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}])+\mathfrak{u}}{\mathfrak{u}},$$

and since $\ker\sigma$ is finite dimensional, we have $(\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}])+\mathfrak{u} = \mathfrak{r}$. Now, let $\mathfrak{t}$ be a two-sided ideal of $\mathfrak{f}$ such that $\mathfrak{t}/[\mathfrak{r},\mathfrak{f}]$ is the complement of $(\mathfrak{u}\cap[\mathfrak{s},\mathfrak{f}])/[\mathfrak{r},\mathfrak{f}]$ in $\mathfrak{u}/[\mathfrak{r},\mathfrak{f}]$. Then, $\mathfrak{t}\cap(\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}]) = [\mathfrak{r},\mathfrak{f}]$ and $\mathfrak{t}+(\mathfrak{r}\cap[\mathfrak{s},\mathfrak{f}]) = \mathfrak{r}$, which implies that

$$\frac{\mathfrak{r}}{[\mathfrak{r},\mathfrak{f}]} = \mathcal{M}(\mathfrak{n},\mathfrak{q}) \oplus \frac{\mathfrak{t}}{[\mathfrak{r},\mathfrak{f}]}.$$

It is easy to see that $\theta : \mathfrak{s}/\mathfrak{t} \to \mathfrak{f}/\mathfrak{r}$ given by $\theta(s+\mathfrak{t}) = s+\mathfrak{r}$, together with the action

$$\begin{array}{rcl}
\bar{\theta} : \dfrac{\mathfrak{s}}{\mathfrak{t}} \times \dfrac{\mathfrak{f}}{\mathfrak{r}} & \longrightarrow & \dfrac{\mathfrak{s}}{\mathfrak{t}} \\[2ex]
(s+\mathfrak{t}, f+\mathfrak{r}) & \longmapsto & [s,f]+\mathfrak{t},
\end{array}
\qquad
\begin{array}{rcl}
\bar{\theta}' : \dfrac{\mathfrak{f}}{\mathfrak{r}} \times \dfrac{\mathfrak{s}}{\mathfrak{t}} & \longrightarrow & \dfrac{\mathfrak{s}}{\mathfrak{t}} \\[2ex]
(f+\mathfrak{r}, s+\mathfrak{t}) & \longmapsto & [f,s]+\mathfrak{t},
\end{array}$$

is a cover of the pair $(\mathfrak{n},\mathfrak{q})$ such that $\dfrac{\mathfrak{s}/\mathfrak{t}}{\mathfrak{u}/\mathfrak{t}} \cong \mathfrak{m}$. ■

We prove next that under some conditions all covers of a pair of finite dimensional Leibniz algebras are isomorphic. This shows that such a cover is unique up to isomorphism.

**Theorem 3.6.** *Let $\sigma_i : \mathfrak{m}_i \to \mathfrak{q}$ $(i = 1, 2)$ be two covers of a pair $(\mathfrak{n},\mathfrak{q})$ of finite dimensional Leibniz algebras such that $\mathfrak{m}_i^2 = \mathfrak{m}_i$. Then*

(i) $\mathfrak{m}_1 \cong \mathfrak{m}_2$,

(ii) $\mathfrak{m}_1/Z(\mathfrak{m}_1,\mathfrak{q}) \cong \mathfrak{m}_2/Z(\mathfrak{m}_2,\mathfrak{q})$, *and*

(iii) $Z(\mathfrak{m}_1,\mathfrak{q})/\ker\sigma_1 \cong Z(\mathfrak{m}_2,\mathfrak{q})/\ker\sigma_2$.

**Proof.** (i) Let $\sigma : \mathfrak{m} \to \mathfrak{q}$ be a cover satisfying the assumptions. As $\mathfrak{m}$ is perfect, the pair $(\mathfrak{n},\mathfrak{q})$ is also perfect, i.e. $[\mathfrak{n},\mathfrak{q}] = \mathfrak{n}$. Also, since $\sigma : \mathfrak{m} \to \mathfrak{q}$ is a cover, we have $\mathfrak{m}/\ker\sigma \cong \mathfrak{n}$ and $\ker\sigma \leq [\mathfrak{m},\mathfrak{q}]$. Hence

$$\frac{\mathfrak{m}/\ker\sigma}{[\mathfrak{m},\mathfrak{q}]/\ker\sigma} \cong \frac{\mathfrak{n}}{[\mathfrak{n},\mathfrak{q}]},$$

and so $[\mathfrak{m},\mathfrak{q}] = \mathfrak{m}$. Therefore $\sigma : [\mathfrak{m},\mathfrak{q}] \longrightarrow [\mathfrak{n},\mathfrak{q}]$ is an epimorphism.

Thus $\quad \dim[\mathfrak{m},\mathfrak{q}] = \dim\ker\sigma + \dim[\mathfrak{n},\mathfrak{q}] = \dim\mathcal{M}(\mathfrak{n},\mathfrak{q}) + \dim[\mathfrak{n},\mathfrak{q}]$.

Now assume that $\mathfrak{q} \cong \mathfrak{f}/\mathfrak{r}$ such that $\mathfrak{f}$ is a free Leibniz algebra and $\mathfrak{n} \cong \mathfrak{s}/\mathfrak{r}$. By Lemma 3.1, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathfrak{r}}{[\mathfrak{r},\mathfrak{f}]} & \longrightarrow & \dfrac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]} & \xrightarrow{\delta} & \mathfrak{n} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta|} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle 1_{\mathfrak{n}}} & & \\
0 & \longrightarrow & \ker\sigma & \longrightarrow & \mathfrak{m} & \xrightarrow{\sigma} & \mathfrak{n} & \longrightarrow & 0.
\end{array}$$

Since $[\mathfrak{s}/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}] = [\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]$ and $\delta| : [\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}] \longrightarrow [\mathfrak{n},\mathfrak{q}]$ is an epimorphism, thus

$$\dim([\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]) = \dim\mathcal{M}(\mathfrak{n},\mathfrak{q}) + \dim[\mathfrak{n},\mathfrak{q}].$$

Thus $\dim\mathfrak{m} = \dim[\mathfrak{m},\mathfrak{q}] = \dim([\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}])$. Clearly $\beta([\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]) \leq [\mathfrak{m},\mathfrak{q}]$. On the other hand, $\mathfrak{m} = [\mathfrak{m},\mathfrak{q}] \leq \beta([\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}])$ and so $\mathfrak{m} \cong [\mathfrak{s},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}]$.

(ii) By Theorem 3.4, there exists a two-sided ideal $\mathfrak{t}$ in $\mathfrak{f}$ such that $\mathfrak{m} \cong \mathfrak{s}/\mathfrak{t}$. This leads to an action of $\mathfrak{q}$ on $\mathfrak{s}/\mathfrak{t}$, which implies that $Z(\mathfrak{s}/\mathfrak{t},\mathfrak{q}) = Z(\mathfrak{m},\mathfrak{q})$.

Put $\mathfrak{u}/\mathfrak{t} = Z(\mathfrak{s}/\mathfrak{t},\mathfrak{q})$ and let $x \in Z(\mathfrak{s}/[\mathfrak{r},\mathfrak{f}],\mathfrak{q})$. Then $[x,q] \in [\mathfrak{r},\mathfrak{f}] \leq \mathfrak{t}$ and so $Z(\mathfrak{s}/[\mathfrak{r},\mathfrak{f}],\mathfrak{q}) \leq \mathfrak{u}/[\mathfrak{r},\mathfrak{f}]$. Now suppose that $x + [\mathfrak{r},\mathfrak{f}] \in \mathfrak{u}/[\mathfrak{r},\mathfrak{f}]$. Then

$$[x,q] \in \mathfrak{t} \leq \mathfrak{r} \cap [\mathfrak{s},\mathfrak{f}] \cap \mathfrak{t} = [\mathfrak{r},\mathfrak{f}].$$

Hence $\mathfrak{u}/[\mathfrak{r},\mathfrak{f}] = Z(\mathfrak{s}/[\mathfrak{r},\mathfrak{f}],\mathfrak{q})$. Therefore, $\dfrac{\mathfrak{m}}{Z(\mathfrak{m},\mathfrak{q})} \cong \dfrac{\mathfrak{s}/\mathfrak{t}}{\mathfrak{u}/\mathfrak{t}} \cong \dfrac{\mathfrak{s}}{\mathfrak{u}}$.

(iii) By Lemma 3.1 we have $\dfrac{Z(\mathfrak{m},\mathfrak{q})}{\ker\sigma} \cong \dfrac{\mathfrak{u}/\mathfrak{t}}{\mathfrak{r}/\mathfrak{t}} \cong \dfrac{\mathfrak{u}}{\mathfrak{r}}$, which completes the proof. ■

**Theorem 3.7.** *Let $(\mathfrak{n},\mathfrak{q})$ be a pair of Leibniz algebras with finite dimensional Schur multiplier, and $\sigma_i : \mathfrak{m}_i \to \mathfrak{q}$, $(i = 1,2)$ be two covers of the pair $(\mathfrak{n},\mathfrak{q})$ such that $\mathfrak{m}_i^2 = \mathfrak{m}_i$. If $\alpha : \mathfrak{m}_1 \to \mathfrak{m}_2$ is an epimorphism such that $\alpha(\ker\sigma_1) = \ker\sigma_2$, then $\alpha$ is an isomorphism.*

**Proof.** Let $\mathfrak{q} \cong \mathfrak{f}/\mathfrak{r}$ be a free presentation of $\mathfrak{q}$ and $\mathfrak{n} \cong \mathfrak{s}/\mathfrak{r}$, for some two-sided ideal $\mathfrak{s}$ in $\mathfrak{f}$. Using Theorem 3.4, there exist two-sided ideals $\mathfrak{t}_i$ $(i = 1,2)$ in $\mathfrak{f}$ such that $\mathfrak{m}_i \cong \mathfrak{s}/\mathfrak{t}_i$, $\ker(\sigma_i) \cong \mathfrak{r}/\mathfrak{t}_i$ and $\mathfrak{r}/[\mathfrak{r},\mathfrak{f}] = \mathcal{M}(\mathfrak{n},\mathfrak{q}) \oplus \mathfrak{t}_i/[\mathfrak{r},\mathfrak{f}]$. Therefore, one may consider $\alpha$ as an epimorphism from $\mathfrak{s}/\mathfrak{t}_1$ onto $\mathfrak{s}/\mathfrak{t}_2$ with $\alpha(\mathfrak{r}/\mathfrak{t}_1) = \mathfrak{r}/\mathfrak{t}_2$. By Lemma 3.1, there exists an epimorphism $\beta : \mathfrak{s}/[\mathfrak{r},\mathfrak{f}] \to \mathfrak{s}/\mathfrak{t}_2$ such that $\ker\beta = \mathfrak{t}_2/[\mathfrak{r},\mathfrak{f}]$.

Thus, the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{r}/[\mathfrak{r},\mathfrak{f}] & \longrightarrow & \mathfrak{s}/[\mathfrak{r},\mathfrak{f}] & \longrightarrow & \mathfrak{n} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta_1} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \beta_2} & & \\
0 & \longrightarrow & \mathfrak{r}/\mathfrak{t}_2 & \longrightarrow & \mathfrak{s}/\mathfrak{t}_2 & \longrightarrow & \mathfrak{n} & \longrightarrow & 0,
\end{array}$$

where $\beta_1$ and $\beta_2$ are the restriction and the induced homomorphism of $\beta$, respectively. It is easy to see that $\beta_2$ is an isomorphism. So, we obtain a homomorphism $\varphi : \mathfrak{s} \to \mathfrak{s}/\mathfrak{t}_1$ with $\alpha\varphi = \beta\gamma$, in which $\gamma$ is the natural epimorphism from $\mathfrak{s}$ onto $\mathfrak{s}/[\mathfrak{r},\mathfrak{f}]$. Since $[\mathfrak{r},\mathfrak{f}] \subseteq \mathfrak{t}_1$, $\varphi$ induces a homomorphism $\widehat{\varphi} : \mathfrak{s}/[\mathfrak{r},\mathfrak{f}] \to \mathfrak{s}/\mathfrak{t}_1$ and hence the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{r}/[\mathfrak{r},\mathfrak{f}] & \longrightarrow & \mathfrak{s}/[\mathfrak{r},\mathfrak{f}] & \longrightarrow & \mathfrak{n} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \widehat{\varphi}} & & \downarrow{\scriptstyle \varphi_2} & & \\
0 & \longrightarrow & \mathfrak{r}/\mathfrak{t}_1 & \longrightarrow & \mathfrak{s}/\mathfrak{t}_1 & \longrightarrow & \mathfrak{n} & \longrightarrow & 0
\end{array}$$

and $\alpha\widehat{\varphi} = \beta$, where $\varphi_1$ is the restriction of $\widehat{\varphi}$, and $\varphi_2 = (\alpha')^{-1}\beta_2$ is an isomorphism, in which $\alpha' : \mathfrak{n} \to \mathfrak{n}$ is the induced isomorphism by $\alpha$. Thus, $\mathfrak{s}/\mathfrak{t}_1 = \mathfrak{r}/\mathfrak{t}_1 + \mathrm{Im}\widehat{\varphi}$ and since $\mathfrak{s}/\mathfrak{t}_1$ is perfect and $\mathfrak{r}/\mathfrak{t}_1$ is abelian, we have

$$\mathfrak{r}/\mathfrak{t}_1 \leq [\mathfrak{s}/\mathfrak{t}_1, \mathfrak{s}/\mathfrak{t}_1] = [\mathfrak{r}/\mathfrak{t}_1 + \mathrm{Im}\widehat{\varphi}, \mathfrak{r}/\mathfrak{t}_1 + \mathrm{Im}\widehat{\varphi}] \leq \mathrm{Im}\widehat{\varphi}.$$

Hence $\widehat{\varphi}$ is onto. Now, put $\ker\widehat{\varphi} = \mathfrak{u}/[\mathfrak{r},\mathfrak{f}]$, for some two-sided ideal $\mathfrak{u}$ in $\mathfrak{s}$. One can easily check that $\mathfrak{u} \subseteq \mathfrak{t}_2$ and $\mathfrak{u} + (\mathfrak{r} \cap [\mathfrak{s},\mathfrak{f}]) = \mathfrak{r}$. Therefore $\mathfrak{u} = \mathfrak{t}_2$ and so $\alpha$ is an isomorphism. ∎

In [10], it is proved that every two covering groups of a finite group are isoclinic. Also, Salemkar et al. [20] showed that if $L$ is a Lie algebra with finite dimensional Schur multiplier, then all covers of $L$ are isoclinic.

Let $\sigma_i : \mathfrak{m}_i \to \mathfrak{q}$ $(i = 1, 2)$ be two covers of a pair $(\mathfrak{n}, \mathfrak{q})$ of Leibniz algebras. In the following result, we show under certain assumptions that two pairs $(\ker\sigma_1, \mathfrak{m}_1)$ and $(\ker\sigma_2, \mathfrak{m}_2)$ are isoclinic. Note that a different notion of isoclinism, namely Lie-isoclinism, was already discussed on pairs of Leibniz algebras in [16], which generalizes the concept of Lie-isoclinism of Leibniz algebras given in [2].

**Definition 3.8.** Two pairs $(\mathfrak{n}, \mathfrak{q})$ and $(\mathfrak{n}', \mathfrak{q}')$ of Leibniz algebras are said to be *isoclinic*, and are denoted by $(\mathfrak{n}, \mathfrak{q}) \sim (\mathfrak{n}', \mathfrak{q}')$, if there exist two isomorphisms

$$\alpha : \mathfrak{q}/Z(\mathfrak{n},\mathfrak{q}) \longrightarrow \mathfrak{q}'/Z(\mathfrak{n}',\mathfrak{q}') \ \text{ with } \ \alpha(\mathfrak{n}/Z(\mathfrak{n},\mathfrak{q})) = \mathfrak{n}'/Z(\mathfrak{n}',\mathfrak{q}'), \ \text{ and}$$
$$\beta : [\mathfrak{n},\mathfrak{q}] \longrightarrow [\mathfrak{n}',\mathfrak{q}'] \ \text{ with } \ \beta([n,q]) = [n',q'] \ \text{ and } \ \beta([q,n]) = [q',n'],$$

whenever $\alpha(q + Z(\mathfrak{n},\mathfrak{q})) = q' + Z(\mathfrak{n}',\mathfrak{q}')$ and $\alpha(n + Z(\mathfrak{n},\mathfrak{q})) = n' + Z(\mathfrak{n}',\mathfrak{q}')$.

Note that this definition clearly implies the commutativity of the following diagram:

$$\begin{array}{ccccc}
\dfrac{\mathfrak{n}}{Z(\mathfrak{n},\mathfrak{q})} \times \dfrac{\mathfrak{q}}{Z(\mathfrak{n},\mathfrak{q})} & \xrightarrow{\delta_1} & [\mathfrak{n},\mathfrak{q}] & \xleftarrow{\delta_2} & \dfrac{\mathfrak{q}}{Z(\mathfrak{n},\mathfrak{q})} \times \dfrac{\mathfrak{n}}{Z(\mathfrak{n},\mathfrak{q})}\\
\downarrow{\scriptstyle \alpha^2} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha^2}\\
\dfrac{\mathfrak{n}'}{Z(\mathfrak{n}',\mathfrak{q}')} \times \dfrac{\mathfrak{q}'}{Z(\mathfrak{n}',\mathfrak{q}')} & \xrightarrow{\sigma_1} & [\mathfrak{n}',\mathfrak{q}'] & \xleftarrow{\sigma_2} & \dfrac{\mathfrak{q}'}{Z(\mathfrak{n}',\mathfrak{q}')} \times \dfrac{\mathfrak{n}'}{Z(\mathfrak{n}',\mathfrak{q}')}
\end{array}$$

where $\delta_i(\bar{x}_1, \bar{x}_2) = [x_1, x_2]$ and $\sigma_i(\bar{y}_1, \bar{y}_2) = [y_1, y_2]$.

**Proposition 3.9.** *Let $(\mathfrak{n}, \mathfrak{q})$ be a pair of Leibniz algebras with finite dimensional Schur multiplier and $\sigma_i : \mathfrak{m}_i \to \mathfrak{q}$ $(i = 1, 2)$ be two covers of $(\mathfrak{n}, \mathfrak{q})$ such that $\mathfrak{m}_i^2 = \mathfrak{m}_i$. Then*

$$(\ker\sigma_1, \mathfrak{m}_1) \sim (\ker\sigma_2, \mathfrak{m}_2).$$

**Proof.** Let $(\mathfrak{n},\mathfrak{q})$ and $(\mathfrak{n}',\mathfrak{q}')$ be two pairs of Leibniz algebras and $\varphi:\mathfrak{q}\to\mathfrak{q}'$ be an epimorphism such that $\varphi(\mathfrak{n})=\mathfrak{n}'$ and $\ker\varphi\cap\mathfrak{n}=0$. Now consider two isomorphisms $\alpha:\mathfrak{q}/Z(\mathfrak{n},\mathfrak{q})\longrightarrow\mathfrak{q}'/Z(\mathfrak{n}',\mathfrak{q}')$ given by $\alpha(q+Z(\mathfrak{n},\mathfrak{q}))=\varphi(q)+Z(\mathfrak{n}',\mathfrak{q}')$, and $\beta:[\mathfrak{n},\mathfrak{q}]\longrightarrow[\mathfrak{n}',\mathfrak{q}']$ by $\beta([n,q])=[\varphi(n),\varphi(q)]$ and $\beta([q,n])=[\varphi(q),\varphi(n)]$.

Therefore $(\mathfrak{n},\mathfrak{q})\sim(\mathfrak{n}',\mathfrak{q}')$. Now, by Theorem 3.4, if $\sigma:\mathfrak{m}\to\mathfrak{q}$ is a cover of the pair $(\mathfrak{n},\mathfrak{q})$, then there exists an epimorphism $\varphi:\mathfrak{s}/[\mathfrak{r},\mathfrak{f}]\longrightarrow\mathfrak{m}$ such that we have $\varphi(([\mathfrak{s},\mathfrak{f}]\cap\mathfrak{r})/[\mathfrak{r},\mathfrak{f}])=\ker\sigma$. Since

$$\ker\varphi\cap\Big[\frac{[\mathfrak{s},\mathfrak{f}]\cap\mathfrak{r}}{[\mathfrak{r},\mathfrak{f}]},\frac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]}\Big]=0,$$

we have
$$(\ker\sigma,\mathfrak{m})\sim\Big(\frac{[\mathfrak{s},\mathfrak{f}]\cap\mathfrak{r}}{[\mathfrak{r},\mathfrak{f}]},\frac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]}\Big),$$

which completes the proof. ■

**Acknowledgments.** The authors would like to thank the referee for his/her valuable comments which helped to improve the article.

## References

[1] H. Arabyani, H. Safa: *Some properties of c-covers of a pair of Lie algebras*, Quaest. Math. 42(1) (2019) 37–45.

[2] G. R. Biyogmam, J. M. Casas: *On Lie-isoclinic Leibniz algebras*, J. Algebra 499 (2018) 337–357.

[3] G. R. Biyogmam, J. M. Casas: *The c-nilpotent Schur Lie-multiplier of Leibniz algebras*, J. Geom. Phys. 138 (2019) 55–69.

[4] A. Blokh: *A generalization of the concept of a Lie algebra*, Dokl. Akad. Nauk SSSR 165(3) (1965) 471–473.

[5] A. Blokh: *Cartan-Eilenberg homology theory for a generalized class of Lie algebras*, Dokl. Akad. Nauk SSSR 175(2) (1967) 266–268.

[6] J. M. Casas, H. Ravanbod: *Some properties of the Schur multiplier and stem covers of Leibniz crossed modules*, Bull. Malaysian Math. Sci. Soc. 43 (2020) 3437-–3456.

[7] J. M. Casas, T. Van der Linden: *Universal central extensions in semi-abelian categories*, Appl. Categorical Structures 22 (2014) 253–268.

[8] G. Ellis: *Schur multiplier of a pair of groups*, Appl. Categorical Structures 6 (1998) 355–371.

[9] S. N. Hosseini, B. Edalatzadeh, A. R. Salemkar: *The second relative homology of Leibniz algebras*, J. Pure Appl. Algebra 224 (2020) 206–217.

[10] M. R. Jones: *Isoclinisms and covering groups*, Bull. Austral. Math. Soc. 11 (1974) 71–76.

[11] J.-L. Loday: *Cohomologie et group de Steinberg relatifs*, J. Algebra 54 (1978) 178–202.

[12] J.-L. Loday: *Cyclic Homology*, Grundlagen der Mathematischen Wissenschaften 301, Springer, Berlin (1992).

[13] J.-L. Loday: *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseignement Math. 39 (1993) 269–293.

[14] M. R. R. Moghaddam, A. R. Salemkar, K. Chiti: *Some properties on the Schur multiplier of a pair of groups*, J. Algebra 312 (2007) 1–8.

[15] A. Pourmirzaei, M. Hassanzadeh, B. Mashayekhy: *On relative central extensions and covering pairs*, J. Algebraic Systems 4(1) (2016) 1–13.

[16] Z. Riyahi, J. M. Casas: *Lie-isoclinism of pairs of Leibniz algebras,* Bull. Malaysian Math. Sci. Soc. 43(1) (2020) 283–296.

[17] F. Saeedi, A. R. Salemkar, B. Edalatzadeh: *The commutator subalgebra and Schur multiplier of a pair of nilpotent Lie algebras*, J. Lie Theory 2 (2011) 491–498.

[18] H. Safa, H. Arabyani: *On $c$-nilpotent multiplier and $c$-covers of a pair of Lie algebras*, Commun. Algebra 45(10) (2017) 4429–4434.

[19] H. Safa, H. Arabyani, M. Norouzi: *On the $c$-nilpotent multiplier of a pair of Lie algebras*, Publ. Math. 97 (2020) 253–264.

[20] A. R. Salemkar, V. Alamian, H. Mohammadzadeh: *Some properties of the Schur multiplier and covers of Lie algebras*, Commun. Algebra 36 (2008) 697–707.

[21] I. Schur: *Über die Darstellung der endlichen Gruppen durch gebrochen lineare Substitutionen*, J. Reine Angew. Math. 127 (1904) 20–50.

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Received March 16, 2020
and in final form May 6, 2020