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PBW Degenerations of Lie Superalgebras and their Typical Representations

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Abstract. We introduce the PBW degeneration for basic classical Lie superalgebras and construct for all type I, $\mathfrak{osp}(1,2n)$ and exceptional Lie superalgebras new monomial bases. These bases are parametrized by lattice points in convex lattice polytopes, sharing useful properties such as the integer decomposition property. This paper is the first step towards extending the framework of PBW degenerations to the Lie superalgebra setting.

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1. Introduction

The framework on PBW degenerations in Lie theory started around ten years ago with several works of Evgeny Feigin ([7, 10, 11]). Roughly speaking, the main idea is to *degenerate* a Lie algebra into an abelian Lie algebra. For example, on the level of universal enveloping algebras the PBW degree of monomials can be used to define a filtration such that the corresponding associated graded algebra is isomorphic to an ordinary polynomial ring. There is an induced filtration on any cyclic module and study of the induced associated graded space is in the focus for about a decade. This has been developed for finite-dimensional, simple complex Lie algebras and their finite-dimensional simple modules. Here, the associated graded space is a quotient of the polynomial ring, the commutative analogue of the well-known description of a finite-dimensional simple module as a quotient of the universal enveloping algebra.

This description raises a couple of interesting questions. Can we determine generators for the defining ideal? Can we provide a monomial basis for the associated graded space? In the classical, non-degenerate setup, these answers are known for a long time. In the degenerate setup, there are positive answers in type A [10] (and partial answers for Demazure modules [14]), type C [11] and type G [15]. Beyond these cases little is known, there hasn't been any new result on infinite series for the last seven years.

In the present paper, we will extend the framework of PBW degenerations to basic classical Lie superalgebras. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional, basic classical Lie superalgebra. The notion of PBW degree of monomials in the universal enveloping

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algebra is natural in this context, the elements of a fixed basis of $\mathfrak{g}$ will have degree one. The associated graded algebra is then a product of a symmetric algebra with an exterior algebra. Again, one has an induced filtration on cyclic modules and the natural questions of finding generators for the defining ideal and monomial bases show up in this context too.

We restrict ourselves to finite-dimensional typical representations $V(\lambda)$, for the necessary choices of positive roots and Borel subalgebras we refer the reader to Section 2. For the moment we assume an appropriate triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and set $\mathfrak{n}^\pm_{\bar{i}} = \mathfrak{g}_{\bar{i}} \cap \mathfrak{n}^\pm$. We consider the PBW degenerate Lie superalgebra $\mathfrak{n}^{-,a}$ and the induced PBW degenerate module $V^a(\lambda)$. Recall that $\mathfrak{n}^{-,a}$ is isomorphic to $\mathfrak{n}^-$ as a vector space with trivial Lie superbracket and hence the universal enveloping algebra $\mathbf{U}(\mathfrak{n}^{-,a})$ can be identified with $S(\mathfrak{n}^-_{\bar{0}}) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})$. The PBW filtration is compatible with the triangular decomposition in the sense that $\mathbf{U}(\mathfrak{n}^{-,a})$ and $V^a(\lambda)$ are natural $\mathbf{U}(\mathfrak{n}^+)$-modules.

We first consider the type I case, for technical reason we do not consider $A(n,n)$ here but our results can be extended without any difficulties for the central extension of $A(n,n)$. In this setup, the simple ideals in the underlying reductive Lie algebra $\mathfrak{g}_{\bar{0}}$ are of type A or C and hence a monomial basis for the PBW degenerate module of $\mathfrak{g}_{\bar{0}}$ are described using the so-called FFLV-polytopes, introduced in [10, 11]. For fixed λ, we denote this polytope by $P_{\mathfrak{g}_{\bar{0}}}(\lambda)$ and its lattice points by $S_{\mathfrak{g}_{\bar{0}}}(\lambda)$. We extend the results from classical Lie algebras to provide a complete answer for Lie superalgebras of type I.

Theorem 1.1. *Let $\lambda, \mu \in P^+$ be dominant integral typical weights and d the number of positive odd roots. We set $P(\lambda) := P_{\mathfrak{g}_{\bar{0}}}(\lambda) \times \{0,1\}^d$ and $S(\lambda)$ denotes the lattice points in $P(\lambda)$.*

1. *We have $P(\lambda+\mu) \subseteq P(\lambda) + P(\mu)$ and $S(\lambda+\mu) \subseteq S(\lambda) + S(\mu)$.*
2. *The lattice points $S(\lambda) \subseteq P(\lambda)$ parametrize a monomial basis for $V^a(\lambda)$.*
3. *We have $V^a(\lambda+\mu) \subseteq V^a(\lambda) \otimes V^a(\mu)$.*
4. *We have an isomorphism $V^a(\lambda) \cong \mathbf{U}(\mathfrak{n}^{-,a})/\mathbf{I}(\lambda)$, where $\mathbf{I}(\lambda)$ is generated by the set*
$$\left\{ \mathbf{U}(\mathfrak{n}^+_{\bar{0}}) \circ x_{-\alpha}^{2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}+1} : \alpha \in R^+_{\bar{0}} \right\} \subseteq \mathbf{U}(\mathfrak{n}^{-,a}).$$

For Lie superalgebras of type II, the answer is more complicated, as $V(\lambda)$ is a proper quotient of the induced module (see Remark 2.5). Our first result reduces the problem of finding a monomial basis for the associated graded space to the computation of monomial bases for $\operatorname{gr} V(\lambda)$, where the highest weight λ is supported only on a unique simple ideal of $\mathfrak{g}_{\bar{0}}$, see Proposition 2.8. This reduction is used for example in Section 5 to construct a bases in the exceptional cases. We explain our results in the case of $\mathfrak{osp}(1,2n)$, which will be the first new infinite series to be solved since [11]. Inspired by the definition of symplectic Dyck path, we introduce the notion of orthosymplectic Dyck paths (see Definition 4.1) and define for $\lambda \in P^+$ a convex lattice polytope $P_{\mathfrak{osp}}(\lambda)$ and denote by $S_{\mathfrak{osp}}(\lambda)$ the corresponding lattice points. In contrast to the type A and C simple Lie algebras, this is not a marked chain polytope (see [1]). Nevertheless it shares certain useful properties with those. Our main result is the following.

Theorem 1.2. *All the statements of the previous theorem are true for* $\mathfrak{osp}(1,2n)$ *using* $P_{\mathfrak{osp}}(\lambda)$ *and* $S_{\mathfrak{osp}}(\lambda)$. *For the precise description of the ideal* $\mathbf{I}(\lambda)$, *see Theorem 4.2.*

Recall that in the classical Lie algebra cases A and C, one has $S(\lambda+\mu) = S(\lambda)+S(\mu)$. In the super setting, this can't be an equality, as the power of positive odd root vectors is limited to 1. Define

$$\Sigma = \bigcup_{\lambda \in P^+} S(\lambda) \times \{\lambda\}, \ \ \Sigma_{\mathfrak{osp}} = \bigcup_{\lambda \in P^+} S_{\mathfrak{osp}}(\lambda) \times \{\lambda\},$$

then $P(\lambda)$ (resp. $P_{\mathfrak{osp}}(\lambda)$) is a slice of the polyhderal cone defined by the convex hull of Σ (resp. $\Sigma_{\mathfrak{osp}}$). We have shown in particular, that the semi group Σ (resp. $\Sigma_{\mathfrak{osp}}$) is finitely generated by the lattice points for fundamental weights $S(\varpi_i)\times\{\varpi_i\}$ (resp. $S_{\mathfrak{osp}}(\varpi_i) \times \{\varpi_i\}$).
In the remaining infinite Lie superalgebra series of type II, we have statements reducing the problem of finding monomial bases for $V^a(\lambda)$ to the computation of monomial bases for PBW degenerate modules for simple Lie algebras of type B. Since these monomial bases are not yet described in full generality (see [2] and [22] for partial results), we omit our reduction statements for now and refer to future publications.

Nevertheless, we are able to provide similar results for the exceptional types $F(4)$, $G(3)$ and $D(2,1;\alpha)$. Here the results are less uniform as the semi groups of typical highest weights are not finitely generated. We provide for all typical representations a convex lattice polytope whose lattice points parametrize a monomial bases of the PBW degenerate typical modules (see Section 5 for more details). The methods in this section are different from the ones in Section 3 and Section 4. We use the reduction procedure and count lattice points in convex simplex-like polytopes.

From the first papers on PBW degenerations for classical Lie algebras, there is a large variety of various applications, influences and appearances. We discuss briefly where to go from here. Feigin introduced the PBW degenerate flag variety, considering the orbit of the degenerate SL_n on the PBW degenerate module (see [8, 9]). In here the links to combinatorics and quiver Grassmannian, linear degenerations of flag varieties (see [3, 5, 4]) have been already provided. The combinatorics of the monomial bases is providing the link to the theory of crystal bases (see [20, 21]), toric degenerations of flag varieties (see [6]), Newton-Okounkov bodies (see [12]) and discrete geometry (see [1]). We would like to think of the present paper as being the starting point of a similar study for super flag varieties (see for example [23]). Does a PBW degenerate super flag variety would also have such an impact? How to interpret the combinatorics of the monomial bases? Would a notion of super marked chain polytopes be reasonable here? We restrict ourselves for this paper to the PBW filtration and monomial bases and postpone these questions to a forthcoming publication.

The paper is structured as follows: in Section 2 we recall the most important definitions for Lie superalgebras and their typical representations, as well as the PBW filtration. In Section 3, we consider the type I case, focusing on $A(m,n)$ and $C(n)$. In Section 4 we analyze the $\mathfrak{osp}(1,2n)$-case, providing the monomial bases and essential polytope. In the Appendix 5, we consider the exceptional cases $F(4), G(3)$ and $D(2,1;\alpha)$.

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2. Lie superalgebras, their representations and the PBW filtration

We denote the set of complex numbers by $\mathbb{C}$ and, respectively, the set of integers, non-negative integers, and positive integers by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$. Unless otherwise stated, all the vector spaces considered in this paper are $\mathbb{C}$-vector spaces and $\otimes$ stands for $\otimes_{\mathbb{C}}$.

Finite-dimensional simple complex Lie superalgebras were classified by Kac [17] and are divided into two groups: the classical Lie superalgebras (the even part is a reductive Lie algebra) and the Cartan type Lie superalgebras. In this paper we consider basic classical Lie superalgebras, i.e. classical Lie superalgebras on which there exists a non-degenerate, invariant, even bilinear form. A basic classical Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is said to be of type II if $\mathfrak{g}_{\bar{1}}$ is an irreducible $\mathfrak{g}_{\bar{0}}$-module via the adjoint action, and is said to be of type I if $\mathfrak{g}_{\bar{1}}$ is a direct sum of two irreducible $\mathfrak{g}_{\bar{0}}$-modules. The following table gives all isoclasses of basic classical Lie superalgebras, which are not simple Lie algebras:

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	
$A(m,n),\ m > n \geq 0$	$A_m \oplus A_n \oplus \mathbb{C}$	type I
$A(n,n),\ n \geq 1$	$A_n \oplus A_n$	type I
$B(m,n),\ m \geq 0, n \geq 1$	$B_m \oplus C_n$	type II
$C(n),\ n \geq 2$	$C_{n-1} \oplus \mathbb{C}$	type I
$D(m,n),\ m \geq 2, n \geq 1$	$D_m \oplus C_n$	type II
$D(2,1;\alpha),\ \alpha \neq 0,-1$	$A_1 \oplus A_1 \oplus A_1$	type II
$F(4)$	$A_1 \oplus B_3$	type II
$G(3)$	$A_1 \oplus G_2$	type II

In terms of matrices we have (all entries are in the field of complex numbers) $A(m,n) = \mathfrak{sl}(m+1,n+1)$, $A(n,n) = \mathfrak{psl}(n+1,n+1)$, $B(m,n) = \mathfrak{osp}(2m+1,2n)$, $C(n) = \mathfrak{osp}(2,2n-2)$, $D(m,n) = \mathfrak{osp}(2m,2n)$. *For technical reasons we want to leave out the Lie superalgebra $A(n,n)$ in the rest of this paper, but we can extend without any difficulties our results for the central extension $\mathfrak{sl}(n+1,n+1)$ of $A(n,n)$.*

We fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, which is by definition a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. For $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h,x] = \alpha(h)x,\ \forall h \in \mathfrak{h}\}$$

the root space associated to α and $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$ be the root system of $\mathfrak{g}$.

We obtain

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \tag{1}$$

where each root space $\mathfrak{g}_\alpha$ in (1) is one-dimensional [18, Proposition 1.3].

We define the even and odd roots to be

$$R_{\bar{0}} = \{\alpha \in R \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0\},\ R_{\bar{1}} = \{\alpha \in R \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0\}.$$

Let E be the real vector space spanned by R equipped with a total ordering $\succ$ compatible with the real vector space structure. We denote by $R^+ = \{\alpha \in R \mid \alpha \succ 0\}$ and $R^- = \{\alpha \in R \mid \alpha \prec 0\}$ respectively the set of positive roots and negative roots respectively. We fix a subset $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq R^+$ of simple roots, which by definition means that $\alpha \in \Delta$ cannot be written as a sum of two positive roots. We denote by $I = \{1, \ldots, r\}$ the corresponding index set. Let $\rho_{\bar{0}}$ (respectively $\rho_{\bar{1}}$) be the half-sum of all the even (respectively odd) positive roots and set $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$. We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,\ \text{where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha.$$

The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called the Borel subalgebra of $\mathfrak{g}$ corresponding to the positive system R^+. We emphasize that unlike in the setting of semi-simple Lie algebras, Borel subalgebras need not to be conjugate. Most of the theory heavily depends on the choice of a simple system, but we will see later that there is a canonical choice which is called distinguished simple system. We set

$$\mathfrak{n}^\pm_{\bar{i}} = \mathfrak{n}^\pm \cap \mathfrak{g}_{\bar{i}},\quad 0 \leq i \leq 1.$$

Example 2.1. We discuss the properties of the Lie superalgebra

$$\mathfrak{g} = \mathfrak{sl}(m,n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{n\times m}, D \in \mathbb{C}^{n\times n}, str(A) = 0 \right\},$$

where $str(A) := tr(A) - tr(D)$ denotes the supertrace of A. The subspace consisting of matrices where $B = C = 0$ determines a reductive Lie algebra isomorphic to $\mathfrak{g}_{\bar{0}}$ and since $\mathfrak{g}_{\bar{1}}$ decomposes into two irreducible $\mathfrak{g}_{\bar{0}}$-modules we have that $\mathfrak{g}$ is a basic classical Lie superalgebra of type I. The non-degenerate bilinear form is given by the formula

$$(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, (X_1, X_2) \mapsto str(X_1 X_2).$$

The diagonal matrices in $\mathfrak{g}$ form a Cartan algebra and the corresponding roots are given by

$$R_{\bar{0}} = \{\epsilon_i - \epsilon_j,\ \delta_r - \delta_s : 1 \leq i \neq j \leq m,\ 1 \leq r \neq s \leq n\},\quad \text{and}$$

$$R_{\bar{1}} = \{\pm(\epsilon_i - \delta_j) : 1 \leq i \leq m,\ 1 \leq j \leq n\},$$

where for $h = \mathrm{diag}(a_1, \ldots, a_m, b_1, \ldots, b_n)$, $\epsilon_i(h) = a_i$ and $\delta_i(h) = b_i$. There are several choices for simple systems, but the most canonical one is the following with exactly one odd root:

$$\Delta = \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m\}. \tag{2}$$

A postive root system is called distinguished if the corresponding system of simple roots contains exactly one odd root; for example the set of simple roots in (2) is distinguished. *From now on we fix a distinguished positive root system for* $\mathfrak{g}$ *with*

Cartan matrix $A = (a_{i,j})_{i,j\in I}$ whose Dynkin diagram S is given as in [18, Table 1]. We denote by s the unique node such that α_s is odd. The following table gives an explicit description of the distinguished simple system Δ.

$\mathfrak{g}$	Δ
$A(m-1,n-1)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n-\epsilon_1,\epsilon_1-\epsilon_2,\dots,\epsilon_{m-1}-\epsilon_m$
$B(m,n)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n-\epsilon_1,\epsilon_1-\epsilon_2,\dots,\epsilon_{m-1}-\epsilon_m,\epsilon_m$
$B(0,n)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n$
$C(n)$	$\epsilon-\delta_1,\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,2\delta_n$
$D(m,n)$	$\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_n-\epsilon_1,\epsilon_1-\epsilon_2,\dots,\epsilon_{m-1}-\epsilon_m,\epsilon_{m-1}+\epsilon_m$
$D(2,1;\alpha)$	$\epsilon_1-\epsilon_2-\epsilon_3,2\epsilon_2,2\epsilon_3$
$F(4)$	$\frac{1}{2}(\delta-\epsilon_1-\epsilon_2-\epsilon_3),\epsilon_3,\epsilon_2-\epsilon_3,\epsilon_1-\epsilon_2$
$G(3)$	$\delta+\epsilon_3,\epsilon_1,\epsilon_2-\epsilon_1$

If $\Delta = \{\alpha_1,\dots,\alpha_r\}$ is a distinguished simple system of a basic classical Lie superalgebra of type I (resp. type II), then $\Delta_{\bar 0} = \{\alpha_i : i \neq s\}$ (resp. $\Delta_{\bar 0} = \{\alpha_i, \gamma : i \neq s\}$) is a simple system for $\mathfrak{g}_{\bar 0}$, where $\gamma = \sum_{i=s}^r c_i\alpha_i$ with labels c_i as in [18, Table 1]. We have that γ is the longest simple root of C_n in the case of $B(m,n)$ and $D(m,n)$ and the simple root of A_1 in the case of $F(4), G(3)$ and $D(2,1;\alpha)$.

Let $D = \operatorname{diag}(d_i)_{i\in I}$ and $B = (b_{i,j})_{i,j\in I}$ be diagonal and symmetric matrices such that $A = DB$. We recall the notion of a Chevalley basis; for more details we refer the reader to [13] and [16]. By [16, Theorem 3.9] we can choose for any basic classical Lie superalgebra a homogeneous vector space basis $\{x_\alpha, h_i : i \in I, \alpha \in R\}$ consisting of root vectors $x_\alpha \in \mathfrak{g}_\alpha, \alpha \in R$ such that the following holds:

$$\begin{gathered}\{h_1,\dots,h_r\} \text{ is a basis of } \mathfrak{h} \text{ with } \alpha_i(h_j) = a_{j,i},\\ [h_i,h_j]=0,\quad [h_i,x_\alpha]=\alpha(h_i)x_\alpha,\quad [x_\alpha,x_{-\alpha}]=\sigma_\alpha h_\alpha,\quad \forall i,j\in I,\ \alpha\in R\\ [x_\alpha,x_\beta]=C_{\alpha,\beta}\,x_{\alpha+\beta},\quad \forall\alpha,\beta\in R\ \text{ with } \alpha+\beta\in R \text{ and } C_{\alpha,\beta}\in\mathbb{Z}\backslash\{0\},\end{gathered} \tag{3}$$

where σ_α, h_α are defined as follows. We have $\sigma_\alpha = -1$ if $\alpha \in R_{\bar 1}^-$ and $\sigma_\alpha = 1$ otherwise. For a root $\alpha = \sum_{i=1}^r k_i\alpha_i \in R$ we define its coroot

$$h_\alpha = d_\alpha \sum_{i=1}^r k_i d_i^{-1} h_i, \quad \text{where} \quad d_\alpha = \begin{cases} \frac{2}{(\alpha,\alpha)}, & \text{if } (\alpha,\alpha)\neq 0\\ d_s, & \text{if } (\alpha,\alpha)=0.\end{cases}$$

There are some further restrictions on the structure constants $C_{\alpha,\beta}$ which are not important in the remainder of this paper; see for example [13, Definition 3.3] or [16, Theorem 3.9].

We recall the Poincaré-Birkoff-Witt theorem for Lie superalgebras. We denote by $\mathbf{T}(\mathfrak{g})$ the tensor superalgebra and let $\mathbf{J}$ the ideal of $\mathbf{T}(\mathfrak{g})$ generated by the elements of the form

$$[x,y] - x\otimes y + (-1)^{|x||y|} y\otimes x, \quad x,y\in\mathfrak{g},$$

where $|x|$ denotes the parity of x. The universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ is defined as the quotient $\mathbf{T}(\mathfrak{g})/\mathbf{J}$. Note that the ideal $\mathbf{J}$ is graded and hence $\mathbf{U}(\mathfrak{g})$ is an associative superalgebra. If $x_1, \ldots, x_p$ is a vector space basis of $\mathfrak{g}_{\bar{0}}$ and $y_1, \ldots, y_q$ a vector space basis of $\mathfrak{g}_{\bar{1}}$, then the PBW theorem says that the set of monomials

$$y_{i_1} \cdots y_{i_\ell} x_1^{k_1} \cdots x_p^{k_p}, \quad k_i \geq 0, \quad 1 \leq i_1 < \cdots < i_\ell \leq q$$

forms a basis of $\mathbf{U}(\mathfrak{g})$. We define a filtration on $\mathbf{U}(\mathfrak{g})$:

$$\mathbf{U}(\mathfrak{g})_0 \subseteq \mathbf{U}(\mathfrak{g})_1 \subseteq \cdots \subseteq \mathbf{U}(\mathfrak{g})_p \subseteq \cdots \tag{4}$$

where $\mathbf{U}(\mathfrak{g})_0 = \mathbb{C} \cdot 1$ and $\mathbf{U}(\mathfrak{g})_p$ is generated by the products of the form

$$a_1 \cdots a_m, \quad 0 \leq m \leq p, \ a_i \in \mathfrak{g}.$$

We call (4) the PBW filtration of the universal enveloping algebra and the associated graded space with respect to (4)

$$\bigoplus_{k \in \mathbb{Z}} \mathbf{U}(\mathfrak{g})_k / \mathbf{U}(\mathfrak{g})_{k-1}$$

admits an obvious $\mathbb{Z}$-graded algebra structure obtained from that in $\mathbf{U}(\mathfrak{g})$ by going to the quotients. The algebra can be realized as follows. Define the symmetric superalgebra

$$\Lambda = S(\mathfrak{g}_{\bar{0}}) \otimes \Lambda(\mathfrak{g}_{\bar{1}}),$$

to be the tensor product of the symmetric algebra of $\mathfrak{g}_{\bar{0}}$ with the exterior algebra of $\mathfrak{g}_{\bar{1}}$ with grading

$$\Lambda_n := \bigoplus_{m=0}^{n} \left(S^m(\mathfrak{g}_{\bar{0}}) \otimes \Lambda^{(n-m)}(\mathfrak{g}_{\bar{1}}) \right).$$

The following is a straightforward consequence of the PBW theorem and can be found for example in [25, Chapter 2].

Proposition 2.2. *The associated graded space of* $\mathbf{U}(\mathfrak{g})$ *with respect to the PBW filtration* (4) *is isomorphic to the algebra* $S(\mathfrak{g}_{\bar{0}}) \otimes \Lambda(\mathfrak{g}_{\bar{1}})$ *as* $\mathbb{Z}$*-graded superalgebras.*

Now we discuss finite-dimensional representations of basic classical Lie superalgebras and the PBW filtration. For $\lambda \in \mathfrak{h}^*$ we define a one-dimensional irreducible $\mathfrak{b}$-module $\mathbb{C}_\lambda := \mathbb{C}v_\lambda$ by

$$\mathfrak{n}^+ v_\lambda = 0, \ h v_\lambda = \lambda(h) v_\lambda, \ \forall h \in \mathfrak{h}.$$

The module $L(\lambda) := \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{b})} \mathbb{C}_\lambda$ contains a unique maximal submodule $J(\lambda)$. It is clear that the quotient $V(\lambda) = L(\lambda)/J(\lambda)$ is an irreducible representation with $V = \mathbf{U}(\mathfrak{n}^-) v_\lambda$ (for simplicity we denote the highest weight vector $1 \otimes v_\lambda$ also by v_λ). The following proposition is stated in [18, Proposition 2.2].

Proposition 2.3. *Let* V *be a finite-dimensional irreducible module for the basic classical Lie superalgebra* $\mathfrak{g}$*. Then there exists* $\lambda \in \mathfrak{h}^*$ *such that* $V \cong V(\lambda)$*. Moreover,* $V(\lambda) \cong V(\mu)$ *if and only if* $\lambda = \mu$*.*

The construction of finite-dimensional irreducible representations for basic classical Lie superalgebras is quite similar as for simple Lie algebras, but the parametrizing set again depends on the choice of the Borel subalgebra.

Since we have assumed in this paper that $\mathfrak{b}$ is distinguished we can describe the Zariski dense set $P^+ = \{\lambda \in \mathfrak{h}^* \mid \dim V(\lambda) < \infty\}$ as follows. Let $P_{\bar{0}}^+$ be the set of dominant integral weights for $\mathfrak{g}_{\bar{0}}$. One of the necessary conditions for $V(\lambda)$ to be finite-dimensional is that $\lambda(h_\alpha) \in \mathbb{Z}_+$ for all $\alpha \in R_{\bar{0}}^+$, i.e. $P^+ \subseteq P_{\bar{0}}^+$. For the special linear Lie superalgebra this condition is also sufficient, but there are a few extra conditions in the remaining types. All characterizing properties can be found in [18, Proposition 2.3].

We give a more explicit construction of the finite-dimensional irreducible $\mathfrak{g}$-modules in the typical case. Recall that $V(\lambda)$ is called typical if $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in R_{\bar{1}}^+$. For a complete and explicit characterization of typical representations we refer to [18, pg. 620-622]. The following proposition stated in [18, Theorem 1] gives generators and relations for typical finite-dimensional irreducible $\mathfrak{g}$-modules. Recall the definition of γ from Section 2.

Proposition 2.4. *Let $\lambda \in P^+$ a typical weight. We have an isomorphism of $\mathfrak{g}$-modules*

$$V(\lambda) \cong \mathbf{U}(\mathfrak{g})/M(\lambda),$$

where $M(\lambda)$ is the left ideal generated by $\mathfrak{n}^+$, $(h - \lambda(h) \cdot 1)$ for all $h \in \mathfrak{h}$ and

$$\begin{cases} (x_{-\alpha_i})^{\lambda(h_i)+1} \text{ for } i \neq s, & \text{if } \mathfrak{g} \text{ is of type I} \\ (x_{-\alpha_i})^{\lambda(h_i)+1} \text{ for } i \neq s,\ (x_{-\gamma})^{2\frac{(\lambda,\gamma)}{(\gamma,\gamma)}+1}, & \text{if } \mathfrak{g} \text{ is of type II.} \end{cases}$$

Moreover,

$$\dim V(\lambda) = 2^{|R_{\bar{1}}^+|} \prod_{\alpha \in R_{\bar{0}}^+} \frac{(\lambda + \rho, \alpha)}{(\rho_0, \alpha)}. \tag{5}$$

Remark 2.5. (1) All finite-dimensional irreducible representations of $B(0,n)$ are typical.

(2) There is a distinguished $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ on $\mathfrak{g}$ by putting

$$\deg(h_i) = 0, \deg(e_i) = 0 = \deg(f_i), \quad i \neq s, \quad \deg(e_s) = -\deg(f_s) = 1.$$

Note that this grading is compatible with the $\mathbb{Z}_2$-grading, i.e. $\mathfrak{g}_{\bar{0}}$ is the direct sum over all even homogeneous spaces. Furthermore, $\mathfrak{g}_i = 0$ for all $|i| > 1$ (resp. $|i| > 2$) if $\mathfrak{g}$ is of type I (resp. type II). Set $\mathfrak{g}_+ = \bigoplus_{i \in \mathbb{Z}_+} \mathfrak{g}_i$ and let $V_{\mathfrak{g}_0}(\lambda)$ be the finite-dimensional irreducible $\mathfrak{g}_0$-module of highest weight λ. We extend $V_{\mathfrak{g}_0}(\lambda)$ to a $\mathbf{U}(\mathfrak{g}_+)$-module by requiring that $\mathfrak{g}_i$ acts as zero for all $i > 0$. The Kac module is then defined as

$$K(\lambda) \cong \operatorname{Ind}_{\mathfrak{g}_+}^{\mathfrak{g}} V_{\mathfrak{g}_0}(\lambda)/M,$$

where $M = 0$ if $\mathfrak{g}$ is of type I and $M = \mathbf{U}(\mathfrak{g})(x_{-\gamma})^{2\frac{(\lambda,\gamma)}{(\gamma,\gamma)}+1} v_\lambda$ otherwise. One of the fundamental results in the representation theory of basic classical Lie superalgebras is that $K(\lambda)$ is irreducible if and only if λ is typical.

(3) For atypical representations we have in general that $\dim V(\lambda)$ is strictly less than the right hand side of (5). The equality in (5) is essential for the rest of the paper.

Recall the PBW filtration from (4). We define an induced filtration on $V(\lambda), \lambda \in P^+$ as follows:

$$\mathbf{U}(\mathfrak{n}^-)_0 v_\lambda \subseteq \mathbf{U}(\mathfrak{n}^-)_1 v_\lambda \subseteq \cdots \subseteq \mathbf{U}(\mathfrak{n}^-)_p v_\lambda \subseteq \cdots \subseteq V(\lambda). \tag{6}$$

The associated graded space with respect to (6) is defined as

$$\operatorname{gr} V(\lambda) := \bigoplus_{k \in \mathbb{Z}} V^k(\lambda)/V^{k-1}(\lambda), \quad V^k(\lambda) := \mathbf{U}(\mathfrak{n}^-)_k v_\lambda$$

The following lemma is straightforward.

Lemma 2.6. *The action of* $\mathbf{U}(\mathfrak{n}^-)$ *on* $V(\lambda)$ *induces an action of* $S(\mathfrak{n}^-_{\bar{0}}) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})$ *on* $grV(\lambda)$*, which turns* $\operatorname{gr} V(\lambda)$ *into a cyclic representation. Moreover there is an induced action of* $\mathbf{U}(\mathfrak{n}^+)$ *on the associated graded space* $\operatorname{gr} V(\lambda)$*.*

Let $\mathbf{I}(\lambda)$ the left ideal of $S(\mathfrak{n}^-_{\bar{0}}) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})$ such that

$$\operatorname{gr} V(\lambda) \cong S(\mathfrak{n}^-_{\bar{0}}) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})/\mathbf{I}(\lambda). \tag{7}$$

We will make the action of $\mathbf{U}(\mathfrak{n}^+)$ on $\operatorname{gr} V(\lambda)$ via the identification (7) more explicit. Define differential operators ∂_α, $\alpha \in R^+$ on $S(\mathfrak{n}^-_{\bar{0}}) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})$ by:

$$\partial_\alpha x_{-\beta} := \begin{cases} x_{-\beta+\alpha}, & \text{if } \beta - \alpha \in R^+ \\ 0, & \text{else.} \end{cases}$$

Then x_α, $\alpha \in R^+$ acts on the associated graded space via the differential operator ∂_α. The goal of this paper is to make a first step towards understanding the structure of $\operatorname{gr} V(\lambda)$ provided that the structure of $\operatorname{gr} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$ is known, where $V_{\mathfrak{g}_{\bar{0}}}(\lambda)$ is the finite-dimensional irreducible $\mathfrak{g}_{\bar{0}}$-module and $\operatorname{gr} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$ is defined as in [10]. The structure of $\operatorname{gr} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$ including a computation of a monomial basis parametrized by the lattice points of a convex polytope has been worked out in [10] for type A_n, in [11] for type C_n, in type B_3 in [2] and for type G_2 in [15].

As a first step we will reduce the problem of finding a PBW basis into several pieces. We denote by $\mathfrak{g}(1), \ldots \mathfrak{g}(p)$ the simple finite-dimensional Lie algebras which appear in the semi-simple part of the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$ and let $R^+(i) \subset R^+_{\bar{0}}$ be the corresponding set of positive roots. Let $\mathfrak{g}$ be a Lie superalgebra of type II. We will assume without loss of generality that $\gamma \in R^+(1)$ (e.g. if $\mathfrak{g}$ is of type $B(m,n)$, then $\mathfrak{g}(1)$ is of type C_n and $\mathfrak{g}(2)$ of type B_m). Since $P^+ \subseteq P^+_{\bar{0}}$ we can write any $\lambda \in P^+$ as a linear combination

$$\lambda = \lambda_1 + \cdots + \lambda_p, \tag{8}$$

where λ_i is a dominant integral weight for the simple Lie algebra $\mathfrak{g}(i)$. The following lemma is straightforward to check by using the fact that

$$(\rho_{\bar{1}}, \alpha) = 0, \quad \forall \alpha \in R^+(i), \ i \neq 1.$$

It gives a factorization of the dimension formula (5).

Lemma 2.7. *Let $\lambda \in P^+$ be a typical weight, let $\lambda = \lambda_1 + \cdots + \lambda_p$ be a decomposition as in* (8)*, and let $V(\lambda)$ be the corresponding Lie superalgebra representation. Then*

$$\dim V(\lambda) = \dim V(\lambda_1) \left(\prod_{i=2}^{p} \dim V_{\mathfrak{g}(i)}(\lambda_i) \right).$$

We will need some more notation to state our reduction result for type II Lie superalgebras (see Proposition 2.8). Let $\mathfrak{n}^-(i)$ be the Lie superalgebra generated by the root vectors $x_{-\alpha}$, $\alpha \in R^+(i)$. Since v_λ satisfies the relations of $V_{\mathfrak{g}(i)}(\lambda_i)$ by Proposition 2.4, we immediatly obtain

$$V_{\mathfrak{g}(i)}(\lambda_i) \cong \mathbf{U}(\mathfrak{n}^-(i))v_\lambda \subseteq V(\lambda). \tag{9}$$

Moreover, it is not hard to check that $(\mathfrak{n}^-(1) \oplus \mathfrak{n}^-_{\bar{1}})$ is a Lie superalgebra. We also get

$$V(\lambda_1) \cong \mathbf{U}(\mathfrak{n}^-(1) \oplus \mathfrak{n}^-_{\bar{1}})v_\lambda \subseteq V(\lambda). \tag{10}$$

Proposition 2.8. *Let $\mathcal{B}_i$ be a PBW basis for the space* $\operatorname{gr} V_{\mathfrak{g}(i)}(\lambda_i)$ *for $2 \leq i \leq n$ and $\mathcal{B}_1$ a basis for the space* $\operatorname{gr} V(\lambda_1)$*. Then*

$$\{b_1 \cdots b_p v_\lambda : b_i \in \mathcal{B}_i\} \tag{11}$$

forms a basis for $\operatorname{gr} V(\lambda)$*.*

Proof. The cardinality of (11) coincides with $\dim V(\lambda)$ by Lemma 2.7. It remains to show that the above set is a generating set. Clearly, the set of elements

$$u_1 \cdots u_p v_\lambda, \quad \text{where } u_i \in \mathbf{U}(\mathfrak{n}^-(i)) \text{ is a monomial}$$

spans $\operatorname{gr} V(\lambda)$. From (9) and (10) we obtain that

$$\operatorname{gr} V_{\mathfrak{g}(i)}(\lambda_i) \cong S(\mathfrak{n}^-(i))v_\lambda, \ (2 \leq i \leq n), \quad \operatorname{gr} V(\lambda_1) \cong \left(S(\mathfrak{n}^-(1)) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})\right) v_\lambda.$$

Now we can rewrite $u_p v_\lambda$ as a linear combination of elements $b_p v_\lambda$, $b_p \in \mathcal{B}_p$. By the commutativity we can move u_{p-1} through all elements in $\mathcal{B}_p$ and write $u_{p-1}v_\lambda$ in the basis $\mathcal{B}_{p-1}$. The proposition by using this procedure inductively. ■

We emphasize the importance of Proposition 2.8. In order to determine a PBW basis for $\operatorname{gr} V(\lambda)$ we only have to compute a PBW basis for the representations of the underlying simple Lie algebras (most of the cases are known in the literature; see for example [2, 10, 11, 15]) and a PBW basis for $\operatorname{gr} V(\lambda_1)$, where λ_1 is a dominant integral weight for the simple Lie algebra $\mathfrak{g}(1)$. In other words, we can lift a PBW basis of $\operatorname{gr} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$ to a PBW basis of $\operatorname{gr} V(\lambda)$ provided that the structure of $\operatorname{gr} V(\lambda_1)$ is known. An application of this important result can be found for example in the Appendix, when we contruct a basis for the exceptional cases.

3. Lifting PBW bases: type I case

The answer to the natural question whether we can lift a PBW basis of $\operatorname{gr} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$ to a PBW basis of $\operatorname{gr} V(\lambda)$ for type I basic classical Lie superalgebras turns out to be quite easy.

We enumerate the positive odd roots $R^+_{\bar{1}} = \{\beta_1, \ldots, \beta_d\}$. The following theorem gives a PBW basis for the associated graded space.

Theorem 3.1. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra of type I and $\lambda \in P^+$ a typical weight. Further, let $\mathcal{B}=\{b_1, ..., b_\ell\}$ a PBW basis of* $\operatorname{gr} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$. *Then the set*

$$\mathcal{B} = \{x^{\mathbf{r}} b_i v_\lambda : \quad 1 \leq i \leq \ell, \quad \mathbf{r} \in \{0,1\}^d\}$$

forms a PBW basis of $\operatorname{gr} V(\lambda)$, *where* $x^{\mathbf{r}} := x_{-\beta_1}^{r_1} \cdots x_{-\beta_d}^{r_d}$.

Proof. Since $V(\lambda) \cong \mathbf{U}(\mathfrak{g}_{-1}) \otimes V_{\mathfrak{g}_{\bar{0}}}(\lambda)$ as a vector space by Remark 2.5, we have the correct cardinality of $\mathcal{B}$. Hence it will be enough to show that the elements of $\mathcal{B}$ are linearly independent. Assume by contradiction that we have a linearly dependence in $\operatorname{gr} V(\lambda)$:

$$\sum_{i,\mathbf{r}} \lambda_{i,\mathbf{r}}\, x^{\mathbf{r}} b_i v_\lambda = 0, \quad \lambda_{i,\mathbf{r}} \in \mathbb{C}, \tag{12}$$

where we can assume without loss of generality that all summands have the same PBW degree, i.e. the linear dependence is in the space $V^k(\lambda)/V^{k-1}(\lambda)$ for some $k \in \mathbb{N}$. This means that the left hand side of (12) is contained in $V^{k-1}(\lambda)$, say it equals an element of the form $Z = \sum_j \mu_j z_j v_\lambda$ with $z_j \in \mathbf{U}(\mathfrak{n}^-)_{k-1}, \mu_j \in \mathbb{C}$. Writing each z_j in PBW order (i.e. as a product of elements in $\mathfrak{n}^-_{\bar{1}}$ followed by a product of elements in $\mathfrak{n}^-_{\bar{0}}$) we can assume that Z is of the following form

$$Z = \sum_{\mathbf{p}} \mu_{\mathbf{p}}\, x^{\mathbf{p}} v_{\mathbf{p}} v_\lambda, \quad \mu_{\mathbf{p}} \in \mathbb{C}$$

for some elements $v_{\mathbf{p}} \in \mathbf{U}(\mathfrak{n}^-_{\bar{0}})$. Now using the fact that

$$V(\lambda) \cong \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{g}_+)} V_{\mathfrak{g}_{\bar{0}}}(\lambda) \cong \mathbf{U}(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$$

we get

$$0 = \sum_{i,\mathbf{r}} \lambda_{i,\mathbf{r}}\, x^{\mathbf{r}} \otimes b_i v_\lambda - \sum_{\mathbf{p}} \mu_{\mathbf{p}}\, x^{\mathbf{p}} \otimes v_{\mathbf{p}} v_\lambda.$$

Since $\{x^{\mathbf{r}}\}_{\mathbf{r} \in \{0,1\}^d}$ is a linearly independent subset of $\mathbf{U}(\mathfrak{g}_{-1})$ we obtain by collecting all the coefficients of $x^{\mathbf{r}}$ that $\lambda_{1,\mathbf{r}} b_1 + \cdots + \lambda_{\ell,\mathbf{r}} b_\ell = \mu_{\mathbf{r}} v_{\mathbf{r}}$, which means that we have $\lambda_{1,\mathbf{r}} b_1 + \cdots + \lambda_{\ell,\mathbf{r}} b_\ell = 0$ in the space $\operatorname{gr} V_{\mathfrak{g}_{\bar{0}}}(\lambda)$. This is a contradiction and the claim follows. ∎

Recall from (7) the definition of the left ideal $\mathbf{I}(\lambda)$.

Corollary 3.2. *We have that* $\mathbf{I}(\lambda)$ *is the left ideal in* $S(\mathfrak{n}^-_{\bar{0}}) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})$ *generated by the elements*

$$\mathbf{U}(\mathfrak{n}^+_{\bar{0}}) \circ x_{-\alpha}^{2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}+1}, \ \alpha \in R^+_{\bar{0}}. \tag{13}$$

Moreover, for all typical weights $\lambda, \mu \in P^+$ we have

$$\operatorname{gr} V(\lambda + \mu) \cong S(\mathfrak{n}^-_{\bar{0}}) \otimes \Lambda(\mathfrak{n}^-_{\bar{1}})(v_\lambda \otimes v_\mu) \subseteq \operatorname{gr} V(\lambda) \otimes \operatorname{gr} V(\mu).$$

Proof. The first part of the corollary follows by combining [10, Theorem A] and [11, Theorem A] with Theorem 3.1. We denote by $S_{\mathfrak{g}_{\bar{0}}}(\lambda)$ the lattice points of the convex polytope constructed in [10, Definition 2] and [11, Section 2] respectively. Recall the important Minkowski sum property of the polytope:

$$S_{\mathfrak{g}_{\bar{0}}}(\lambda + \mu) = S_{\mathfrak{g}_{\bar{0}}}(\lambda) + S_{\mathfrak{g}_{\bar{0}}}(\mu).$$

By (13) we immediately obtain a surjective map

$$\operatorname{gr} V(\lambda+\mu) \twoheadrightarrow S(\mathfrak{n}_{\bar{0}}^{-}) \otimes \Lambda(\mathfrak{n}_{\bar{1}}^{-})(v_\lambda \otimes v_\mu) \tag{14}$$

and the injectivity would follow from the fact that the set

$$\{x^{\mathbf{r}} x^{\mathbf{s}} v_\lambda : \quad \mathbf{s} \in S_{\mathfrak{g}_{\bar{0}}}(\lambda+\mu), \quad \mathbf{r} \in \{0,1\}^d\}$$

is linearly independent in the right hand side of (14). The linearly independence can be obtained using the same idea and strategy as [10, Proposition 6]: we refine the homogeneous filtration to a homogeneous lexicographic order $\leq$. For any $V(\lambda)$, a basis of the associated graded module can be uniquely parametrized by normalized monomials in $U(\mathfrak{n}^-)$. Following [12], we call these monomials essential for $V(\lambda)$ and it has been shown in [12], that this is compatible with tensor products. Namely, if $x^{\mathbf{r}}$ is essential for $V(\lambda)$ and $x^{\mathbf{s}}$ is essential for $V(\mu)$, then $x^{\mathbf{r}+\mathbf{s}}$ is essential for $V(\lambda+\mu)$. For the relevant types, A and C, an order is provided in [10] and [11], and the set of essential monomials is precisely $S_{\mathfrak{g}_{\bar{0}}}(\lambda)$. Using the Minkowski property, we see that any essential monomial for $V(\lambda+\mu)$ is a product of essentials for $V(\lambda)$ and $V(\mu)$. This implies that the set is linearly independent for the associated graded module with respect to the homogeneous order and hence also in $\operatorname{gr} V(\lambda+\mu)$. ∎

Remark 3.3. Another way to construct a basis for $V(\lambda)$ would be to compute the $\mathfrak{g}_{\bar{0}}$ decomposition and to take a basis for each direct summand. We emphasize that this basis will generically not be compatible with the PBW filtration, i.e. the image of this basis in $\operatorname{gr} V(\lambda)$ will not be a basis. For example, let $\mathfrak{g} = \mathfrak{sl}(3,1)$ and consider the typical weight $\lambda = \epsilon_1$. Then $x_{-(\epsilon_1-\delta_1)} x_{-(\epsilon_3-\delta_1)} v_\lambda$ will be a $\mathfrak{g}_{\bar{0}}$ highest weight vector (up to a filtration) of weight $\epsilon_1+\epsilon_2$. Hence $x_{-(\epsilon_2-\epsilon_3)} x_{-(\epsilon_1-\delta_1)} x_{-(\epsilon_3-\delta_1)} v_\lambda$ will be a basis vector of $V(\lambda)$, but it vanishes in the associated graded space since

$$x_{-(\epsilon_2-\epsilon_3)} x_{-(\epsilon_1-\delta_1)} x_{-(\epsilon_3-\delta_1)} v_\lambda = x_{-(\epsilon_1-\delta_1)} x_{-(\epsilon_2-\delta_1)} v_\lambda.$$

For another example consider the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1,4)$ and $\lambda = \delta_1 + \delta_2$. Then $x_{-\delta_2} v_\lambda$ is a highest weight vector of highest weight δ_1 and hence $x_{-(\delta_1-\delta_2)} x_{-\delta_2} v_\lambda$ would be a basis vector of $V(\lambda)$. But since $x_{-(\delta_1-\delta_2)} x_{-\delta_2} v_\lambda = x_{-\delta_1} v_\lambda$ we see that this vector vanishes in the associated graded space.

4. Lifting PBW bases: the $\mathfrak{osp}(1,2n)$ case

Here we consider the Lie superalgebra $\mathfrak{osp}(1,2n)$ where $\mathfrak{g}_{\bar{0}}$ is the symplectic Lie algebra of rank n and $\mathfrak{g}_{\bar{1}}$ is the irreducible $\mathfrak{g}_{\bar{0}}$-representation of dimension $2n$. Recall that the set of simple roots for $\mathfrak{g}_{\bar{0}}$ is given by $\{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = \delta_i - \delta_{i+1}$, $1 \leq i < n$ and $\alpha_n = 2\delta_n$. All positive roots of $\mathfrak{g}_{\bar{0}}$ are given by

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \ 1 \leq i \leq j \leq n,$$

$$\alpha_{i,\bar{j}} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n + \alpha_{n-1} + \ldots + \alpha_j, \ 1 \leq i \leq j \leq n.$$

Recall also the set of positive odd roots $R_{\bar{1}}^{+} = \{\delta_1, \ldots, \delta_n\}$ and let $\{\varpi_1, \ldots, \varpi_n\}$ the set of fundamental weights for $\mathfrak{g}_{\bar{0}}$.

We introduce the notion of an orthosymplectic Dyck path by slightly modifying the definition of a symplectic Dyck path from [11, Definition 1.2].
Let $J = \{1, \dots, n, \overline{n-1}, \dots, \overline{1}\}$ with order

$$1 < \cdots < n-1 < n < \overline{n-1} < \cdots < \overline{1}.$$

Recall that a sequence $\mathbf{q} = (q(0), q(1), \dots, q(k)), k \geq 0$ of positive roots in $R_{\bar{0}}^+$ is called a symplectic Dyck path if the first root is simple, the last root is simple or of the form $2\delta_i$ for some $1 \leq i \leq n$ and the following property holds: If $q(s) = \alpha_{r,q}$ with $r, q \in J$, then $q(s+1) = \alpha_{r,q+1}$ or $q(s+1) = \alpha_{r+1,q}$, where $x+1$ denotes the smallest element in J which is bigger than x.

Definition 4.1. An *orthosymplectic Dyck path* is a sequence

$$\mathbf{p} = (p(0), p(1), \dots, p(k)),\ k \geq 0$$

of positive roots in R^+ which satisfies one of the following conditions:

(a) The sequence $\mathbf{p}$ is a symplectic Dyck path satisfying $p(0) = \alpha_i$ and $p(k) = \alpha_j$ for some $1 \leq i \leq j < n$;

(b) The sequence $(p(0), p(1), \dots, p(k-1))$ is a symplectic Dyck path satisfying $p(0) = \alpha_i$ and $p(k-1) = 2\delta_j$ for some $i \leq j \leq n$ and $p(k) = \delta_i$.

Denote by $\mathcal{D}$ the set of all orthosymplectic Dyck paths. For a dominant integral $\mathfrak{g}_{\bar{0}}$ weight $\lambda = \sum_{i=1}^n m_i \varpi_i$ and $\mathbf{p} \in \mathcal{D}$ we set

$$M_{\mathbf{p}}(\lambda) = \begin{cases} m_i + \cdots + m_j, & \text{if } \mathbf{p} \text{ satisfies (a)} \\ m_i + \cdots + m_n, & \text{otherwise.} \end{cases}$$

Denote by $P(\lambda) \subseteq \mathbb{R}_{\geq 0}^{n(n+1)}$ the polytope

$$P(\lambda) := \Big\{ (s_\alpha)_{\alpha \in R^+} \mid \forall \mathbf{p} \in \mathcal{D} : s_{p(0)} + \ldots + s_{p(k)} \leq M_{\mathbf{p}}(\lambda),\ \forall \beta \in R_{\bar{1}}^+ : s_\beta \in \{0,1\} \Big\}, \quad (15)$$

and let $S(\lambda)$ be the set of integral points in $P(\lambda)$. The main theorem of this section is the following.

Theorem 4.2. *Let* $\lambda = \sum_{i=1}^n m_i \varpi_i \in P^+$.

(1) *The vectors* $x^{\mathbf{s}}.v_\lambda$, $\mathbf{s} \in S(\lambda)$ *form a basis of* $\operatorname{gr} V(\lambda)$ *where* $x^{\mathbf{s}} := \prod_{\alpha \in R^+} x_{-\alpha}^{s_\alpha}$.

(2) $\mathbf{I}(\lambda)$ *is the left ideal in* $S(\mathfrak{n}_{\bar{0}}^-) \otimes \Lambda(\mathfrak{n}_{\bar{1}}^-)$ *generated by the elements*

$$\mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-(\delta_i - \delta_j)}^{m_i + \cdots + m_j + 1},\ 1 \leq i < j \leq n, \quad \mathbf{U}(\mathfrak{n}^+) \circ x_{-2\delta_i}^{m_i + \cdots + m_n + 1},\ 1 \leq i \leq n. \quad (16)$$

(3) *For* $\lambda, \mu \in P^+$ *we have*

$$\operatorname{gr} V(\lambda + \mu) \cong S(\mathfrak{n}_{\bar{0}}^-) \otimes \Lambda(\mathfrak{n}_{\bar{1}}^-)(v_\lambda \otimes v_\mu) \subseteq \operatorname{gr} V(\lambda) \otimes \operatorname{gr} V(\mu).$$

The proof of the above theorem will be given in the rest of this section.

Remark 4.3. Note that the elements $x^{\mathbf{s}}$ in Theorem 4.2 are only well-defined up to a sign. We could avoid this by choosing a total order on the set of odd roots and order the odd root vectors in $x^{\mathbf{s}}$ with respect to this order. For simplicity we will ignore this and write $x^{\mathbf{s}}$ without further comment.

In order to show that the lattice points of (15) parametrize a basis of $\operatorname{gr} V(\lambda)$, we will need a staightening law similar to the one given in [11, Theorem 2.4]. Recall the total order $\succ$ on the set of monomials in $S(\mathfrak{n}_{\bar{0}}^-)$ from [11, Definition 2.6]. Similarly we can define a total order $>$ on the non-zero monomials in $\Lambda(\mathfrak{n}_{\bar{1}}^-)$. Let

$$x_{-\delta_1} > x_{-\delta_2} > \cdots > x_{-\delta_n}$$

and let $>$ the induced homogeneous lexicographic ordering (which we also denote by $>$ for simplicity). We extend both to a total order on the set of monomials in $S(\mathfrak{n}_{\bar{0}}^-) \otimes \Lambda(\mathfrak{n}_{\bar{1}}^-)$ as follows. We define for a multi-exponent $\mathbf{s} \in \mathbb{Z}_+^{n^2} \times \{0,1\}^n$ the following elements

$$x^{\mathbf{s}} := \prod_{\alpha \in R^+} x_{-\alpha}^{s_\alpha}, \quad x^{\mathbf{s}_{\bar{0}}} := \prod_{\alpha \in R_{\bar{0}}^+} x_{-\alpha}^{s_\alpha}, \quad x^{\mathbf{s}_{\bar{1}}} := \prod_{\alpha \in R_{\bar{1}}^+} x_{-\alpha}^{s_\alpha}.$$

We say $x^{\mathbf{s}} \succ x^{\mathbf{t}}$ if one of the following conditions holds

- the total degree of $x^{\mathbf{s}}$ is greater than the total degree of $x^{\mathbf{t}}$
- both have the same total degree, but $x^{\mathbf{s}_{\bar{1}}} > x^{\mathbf{t}_{\bar{1}}}$.
- both have the same total degree, $s_\alpha = t_\alpha$ for all $\alpha \in R_{\bar{1}}^+$, but $x^{\mathbf{s}_{\bar{0}}} \succ x^{\mathbf{t}_{\bar{0}}}$.

We will need the following lemma later.

Lemma 4.4. (1) *Let* $\mathbf{s}$ *a multi-exponent such that* $s_{-\delta_i} = 0$ *for all* $1 \le i \le n$*. Moreover, assume that* $\mathbf{s}$ *is only supported on a orthosymplectic Dyck path* $\mathbf{p}$*. Then there exists a homogeneous expression of the form*

$$x^{\mathbf{s}} + \sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} x^{\mathbf{t}} \in \mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-\tau}^{|\mathbf{s}|}, \quad c_{\mathbf{t}} \in \mathbb{C},$$

where $\tau = 2\delta_i$ *if* $\mathbf{p}$ *is a Dyck path satisfying* (b) *and* $\tau = \delta_i - \delta_j$ *otherwise.*

(2) *For every* $\ell \in \mathbb{N}$ *and* $1 \le i \le n$ *we have the inclusion*

$$\left(\mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-2\delta_i}^{\ell}\right) x_{-\delta_i} \subseteq \mathbf{U}(\mathfrak{n}^+) \circ x_{-2\delta_i}^{\ell+1} + \sum_{p=i+1}^{n} \left(\mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-2\delta_i}^{\ell}\right) x_{-\delta_p}.$$

Proof. The first part of the lemma has been proved in [11, Theorem 2.4]. In order to prove the second part let $\beta_1, \ldots, \beta_r$ a sequence of positive even roots. Since $\delta_i - \alpha \notin R^+$ for all $\alpha \in R_{\bar{0}}^+$ unless $\alpha = \delta_i - \delta_p$ for some $p > i$ we obtain

$$\partial_{\beta_1} \cdots \partial_{\beta_r}(x_{-2\delta_i}^k x_{-\delta_i}) - \left(\partial_{\beta_1} \cdots \partial_{\beta_r}(x_{-2\delta_i}^k)\right) x_{-\delta_i} \in \sum_{p=i+1}^{n} \left(\mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-2\delta_i}^k\right) x_{-\delta_p}.$$

Since $x_{-2\delta_i}^k x_{-\delta_i} = \partial_{\delta_i} x_{-2\delta_i}^{k+1}$ the proof of the lemma is finished. ∎

We claim that the first part of the above lemma holds even without the additional assumption $s_{-\delta_i} = 0$. So let $\mathbf{s}$ a multi-exponent such that $s_{-\delta_i} \neq 0$ for some $i \in \{1, \ldots, n\}$ and assume that $\mathbf{s}$ is supported on a Dyck path $\mathbf{p}$ with $p(0) = \alpha_i$, $p(k-1) = 2\delta_j$ and $p(k) = \delta_i$. If we set $\mathbf{s}'$ as the multi-exponent obtained from $\mathbf{s}$ by setting $s_{-\delta_i} = 0$ we obtain with Lemma 4.4 the existence of a homogeneous expression

$$x^{\mathbf{s}'} + \sum_{\mathbf{t} \prec \mathbf{s}'} c_{\mathbf{t}} x^{\mathbf{t}} \in \mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-2\delta_i}^{|\mathbf{s}'|}. \tag{17}$$

If we multiply (17) by $x_{-\delta_i}$ and use Lemma 4.4(2) we obtain

$$x^{\mathbf{s}} + \sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} x^{\mathbf{t}} \in \mathbf{U}(\mathfrak{n}^+) \circ x_{-2\delta_i}^{|\mathbf{s}|} + \sum_{p=i+1}^{n} \left(\mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-2\delta_i}^{|\mathbf{s}'|} \right) x_{-\delta_p}.$$

But the order on the monomials implies that each summand of $\left(\mathbf{U}(\mathfrak{n}_{\bar{0}}^+) \circ x_{-2\delta_i}^{k}\right) x_{-\delta_p}$ is less than $x^{\mathbf{s}}$ and we obtain the desired straightening law which we summarize in the following corollary.

Corollary 4.5. *There exists a total order $\succ$ on the monomials in $S(\mathfrak{n}_{\bar{0}}^-) \otimes \Lambda(\mathfrak{n}_{\bar{1}}^-)$ such that for any $\mathbf{s} \notin S(\lambda)$ there exists a homogeneous expression of the form*

$$x^{\mathbf{s}} - \sum_{\mathbf{t} \prec \mathbf{s}} c_{\mathbf{t}} x^{\mathbf{t}} \tag{18}$$

which is contained in the left ideal generated by the elements (16).

Proof. If $\mathbf{s}$ is supported on a Dyck path, the statement is clear by the above discussion. Otherwise we use the fact that $\prec$ is a monomial order. ∎

Here we can adapt the proofs from [2, 10, 11] for type A_n, C_n and several further cases in other types, to show that the set $\{x^{\mathbf{s}} : \mathbf{s} \in S(\lambda)\}$ spans $S(\mathfrak{n}_{\bar{0}}^-) \otimes \Lambda(\mathfrak{n}_{\bar{1}}^-)/\mathbf{I}(\lambda)$ and hence the quotient space $\operatorname{gr} V(\lambda)$; we omit the details. In order to finish the proof of Theorem 4.2 we are left to show that $\{x^{\mathbf{s}} : \mathbf{s} \in S(\lambda)\}$ is a linearly independent subset of $\operatorname{gr} V(\lambda)$ (this shows part (1) and part (2) of the theorem) and that $\{x^{\mathbf{s}}(v_\lambda \otimes v_\mu) : \mathbf{s} \in S(\lambda+\mu)\}$ is a linearly independent subset of $S(\mathfrak{n}_{\bar{0}}^-) \otimes \Lambda(\mathfrak{n}_{\bar{1}}^-)(v_\lambda \otimes v_\mu)$ (this shows part (3)). This would follow from the following proposition; see for example the results of [12].

Proposition 4.6. *Let $\lambda = \sum_{i=1}^{n} m_i \varpi_i \in P^+$.*

(1) *Let i maximal with the property that $m_i \neq 0$. Then we have*

$$S(\lambda) \subseteq S(\lambda - \varpi_i) + S(\varpi_i).$$

In particular, for $\lambda, \mu \in P^+$ we have an equality

$$S(\lambda + \mu) = \big(S(\lambda) + S(\mu)\big) \cap \big(\mathbb{Z}_+^{n^2} \times \{0,1\}^n\big).$$

(2) *For each $i \in \{1, \ldots, n\}$ the cardinality of $S(\varpi_i)$ is equal to $dim V(\varpi_i)$.*

The proof of the above proposition can be found in the following two paragraphs. Given $\lambda = \sum_{i=1}^{n} m_i \varpi_i$ and i maximal with the property that $m_i \neq 0$, we shall provide a procedure how to decompose a lattice point in the polytopes $S(\lambda)$ as a sum of two elements in $S(\lambda - \varpi_i)$ and $S(\varpi_i)$ respectively. The procedure will depend on a total order. The equality in part (1) of the proposition is then an immediate consequence.

Proof of Proposition 4.6(1). Define the total order on the set of positive roots

$$\alpha_{n,n} > \delta_n > \alpha_{n-1,\overline{n-1}} > \alpha_{n-1,n} > \alpha_{n-1,n-1} > \delta_{n-1} > \alpha_{n-2,\overline{n-2}} > \cdots \\ \cdots > \alpha_{n-2,n-2} > \delta_{n-2} > \ldots > \alpha_{1,1} > \delta_1 \tag{19}$$

and consider the induced lexicographic ordering on the multi-exponents. Given a multi-exponent $\mathbf{s} \in S(\lambda)$ we define

$$\mathbf{s}^1 := \max\{\mathbf{t} \in S(\varpi_i) \mid t_\alpha \leq s_\alpha, \text{ for all } \alpha \in R^+\},$$

where the maximum is taken with respect to the total order (19). We claim that $\mathbf{s} - \mathbf{s}^1 \in S(\lambda - \varpi_i)$ which would finish the proof of the first part of the proposition. Let $\mathbf{p}$ be an orthosymplectic Dyck path and assume first that $\mathbf{p}$ starts and ends at the simple even roots, α_s and α_e with $s \leq e$, different from $2\delta_n$. We have to show that

$$\sum_{\beta \in \mathbf{p}} (s_\beta - s^1_\beta) \leq m_s + \ldots + m_e - \delta_{i,\{s,\ldots,e\}}, \tag{20}$$

where $\delta_{i,\{s,\ldots,e\}} = 1$ if $i \in \{s, \ldots, e\}$ and 0 otherwise. If $e < i$, there is nothing to be checked as the right hand side won't be changed by turning from λ to $\lambda - \varpi_i$. Since i is maximal with $m_i \neq 0$ we can assume that $e = i$. Let σ be the maximal root (with respect to (19)) such that σ appears in the sequence $\mathbf{p}$ and $s_\sigma \neq 0$. If $s^1_\sigma \neq 0$, then we immediately get (20); so assume without loss of generality that $s^1_\sigma = 0$.

By the maximality of $\mathbf{s}^1$, this is only possible if there exists a root $\tau > \sigma$ such that $s^1_\tau \neq 0$ and the multi-exponent $\mathbf{s}''$ with $s''_\tau = s''_\sigma = 1$ and $s''_\alpha = 0$ for $\alpha \notin \{\sigma, \tau\}$ not contained in $S(\varpi_i)$. In other words, there is a Dyck path which contains both roots τ and σ. So we can replace the path $\mathbf{p}$ by a Dyck path $\mathbf{p}'$ which ends at $p'(k) = \delta_s$ (recall Definition 4.1(b)) with the property $\tau \in \mathbf{p}'$ and $\beta \in \mathbf{p}'$ for all $\beta \in \mathbf{p}$ with $s_\beta \neq 0$. Then

$$\sum_{\beta \in \mathbf{p}} s_\beta < \sum_{\beta \in \mathbf{p}} s_\beta + s_\tau \leq \sum_{\beta \in \mathbf{p}'} s_\beta \leq m_s + \ldots + m_{i-1} + m_i$$

and the claim follows. Similarly, we can treat paths ending in $2\delta_j$. As before, we consider the maximal element on the path which is non-zero. Then either the corresponding element in $\mathbf{s}^1$ is also non-zero or there is a non-zero element in $\mathbf{s}^1$ which corresponds to a larger element than this maximal element. ■

Remark 4.7. Let $\mathbf{s} \in S(\lambda)$ and let $\tilde{\mathbf{s}} \in S(\lambda)$ the multi-exponent defined by

$$\tilde{s}_\alpha = \begin{cases} s_\alpha, & \text{if } \alpha \in R^+_{\bar{0}} \\ 0, & \text{else.} \end{cases}$$

Hence by [11, Lemma 4.5] (the same result holds for i maximal) there exists $\tilde{\mathbf{s}}^1 \in S(\varpi_i)$ with $\tilde{s}^1_\alpha = 0$ for all $\alpha \in R^+_{\bar{1}}$ such that $\tilde{\mathbf{s}} - \tilde{\mathbf{s}}^1 \in S(\lambda - \varpi_i)$. Our desired element $\mathbf{s}^1 \in S(\varpi_i)$ from Proposition 4.6(1) is then given by

$$s^1_\alpha = \begin{cases} \tilde{s}^1_\alpha, & \text{if } \alpha \in R^+_{\bar{0}} \\ 1, & \text{if } s_\alpha \neq 0 \text{ and } \alpha \in \{\delta_p : p \geq k+1\} \\ 0, & \text{else} \end{cases}$$

where $k \in \{1, \dots, n\}$ is the maximal column (in the Hasse diagram) such that there is a non-zero entry in $\tilde{\mathbf{s}}^1$.

Proof of Proposition 4.6(2). Here we prove that the cardinality of $S(\varpi_i)$ coincides with $\dim V(\varpi_i)$. For this we will use the notion of KT-tableaux; see for example [19, Section 4]. Let $J = \{\#_1, 1, \bar{1}, \#_2, 2, \bar{2}, \dots, \#_n, n, \bar{n}\}$ with ordering

$$\#_1 < 1 < \bar{1} < \#_2 < 2 < \bar{2} < \cdots < \#_n < n < \bar{n}.$$

A KT-tableaux of shape $\epsilon\varpi_i$ ($\epsilon = 2$ if $i = n$ and $\epsilon = 1$ otherwise) is a column with i boxes filled with entries from J such that the entries from top to bottom are strictly increasing and the entry in row r is greater or equal to $\#_r$. We denote the set of KT-tableaux of shape $\epsilon\varpi_i$ by $KT(\epsilon\varpi_i)$. For $\mathbf{s} \in S(\varpi_i)$ we define a KT-tableaux $T(\mathbf{s}) \in KT(\epsilon\varpi_i)$ as follows. Since $\mathbf{s} \in S(\varpi_i)$ we must have the following: if $s_\alpha \neq 0$, then $\alpha = \alpha_{r,\ell}$ or $\alpha = \alpha_{r,\bar{\ell}}$ for some $r \leq i$. We take the tableaux

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline i \\ \hline\end{array}$$

and replace r by $\ell+1$ if $s_{\alpha_{r,\ell}} \neq 0$ (we understand $n+1 = \bar{n}$) and r by $\bar{\ell}$ if $s_{\alpha_{r,\bar{\ell}}} \neq 0$. We denote the resulting tableaux (after permuting the entries in increasing order) by $Q(\mathbf{s})$. If $s_{\delta_j} \neq 0$, it is clear that $j \leq i$ and $s_{\alpha_{r,\bullet}} = 0$ for all $r \geq j$. Hence there exists an entry j in $Q(\mathbf{s})$. We replace j by $\#_j$ in $Q(\mathbf{s})$ and define the resulting tableaux as $T(\mathbf{s})$. We claim that $T(\mathbf{s})$ is a KT-tableaux. We need to check that the entry in row r is greater than or equal to $\#_r$. For that it will be enough to prove that the entry in row j of $Q(\mathbf{s})$ is contained in the set $\{j, \bar{j}, \dots, n, \bar{n}\}$ for all $1 \leq j \leq i$. But this is a straightforward calculation. ∎

Lemma 4.8. *The map* $S(\omega_i) \to KT(\epsilon\omega_i)$, $\mathbf{s} \mapsto T(\mathbf{s})$, *is injective.*

Proof. Assume that $\mathbf{s} \neq \mathbf{s}'$ and $T(\mathbf{s}) = T(\mathbf{s}')$. It follows immediately that

$$|\{1 \leq r \leq n : s_{\delta_r} = 1\}| = |\{1 \leq r \leq n : s'_{\delta_r} = 1\}|. \tag{21}$$

We suppose that there exists $j \in \{1, \dots, n\}$ such that $s_{\delta_j} = 1$ but $s'_{\delta_j} = 0$ and assume that j is minimal with this property. Since $Q(\mathbf{s})$ contains j (which will be removed by s_{δ_j}) there must exist a root of the form $\alpha_{j,\bullet}$ such that $s'_{\alpha_{j,\bullet}} \neq 0$ (in order to remove entry j from $Q(\mathbf{s}')$).

Since $\mathbf{s}' \in S(\varpi_i)$ this is only possible if $s'_{\delta_1} = \cdots = s'_{\delta_j} = 0$.

By the minimality of j we must have $s_{\delta_1} = \cdots = s_{\delta_{j-1}} = 0$ and by (21) we get the existence of $r \in \{j+1, \ldots, n\}$ such that $s'_{\delta_r} = 1$ but $s_{\delta_r} = 0$. Again we can repeat the above argument. Since $Q(\mathbf{s}')$ contains r (which will be removed by s'_{δ_r}) there must exist a root of the form $\alpha_{r,\bullet}$ such that $s_{\alpha_{r,\bullet}} \neq 0$ (in order to remove entry r from $Q(\mathbf{s})$). This is a contradiction to $s_{\delta_j} \neq 0$ since $\mathbf{s} \in S(\varpi_i)$. Hence $s_{\delta_j} = s'_{\delta_j}$ for all $1 \leq j \leq n$. This implies $Q(\mathbf{s}) = Q(\mathbf{s}')$ and a straightforward calculation shows now $\mathbf{s} = \mathbf{s}'$, which is a contradiction. ■

The proof of $|S(\varpi_i)| = \dim V(\varpi_i)$ is now finished as follows. By Lemma 4.8, we have $|S(\varpi_i)| \leq |KT(\epsilon\varpi_i)|$ and by [19, Proposition 4.2] we have that $|KT(\epsilon\varpi_i)|$ is equal to the dimension of the irreducible $\mathfrak{so}_{2n+1}$ representation of highest weight $\epsilon\varpi_i$. Now, using [24, Theorem 2.1] we get $\dim V(\varpi_i) = \dim V_{\mathfrak{so}_{2n+1}}(\epsilon\varpi_i)$. Putting all together, we get

$$|S(\varpi_i)| \leq |KT(\epsilon\varpi_i)| = \dim V_{\mathfrak{so}_{2n+1}}(\epsilon\varpi_i) = \dim V(\varpi_i).$$

The converse direction follows from the spanning property.

5. Appendix: The exceptional cases

Here we will consider the exceptional cases $D(2,1;\alpha)$, $F(4)$ and $G(3)$ and construct a PBW basis parametrized by the lattice points of a polytope for a class of dominant integral typical weights. We will need a combinatorial result first whose proof is straightforward and will be omitted.

Lemma 5.1. *The cardinality of the set*

$$\{(a_0, a_1, \ldots, a_\ell) \in \mathbb{Z}_+ \times \{0,1\}^\ell : \sum_{i=0}^{\ell} a_i \leq m\}$$

is given by $2^{\ell-1}(2m - \ell + 2)$ *provided that* $m \geq \ell - 1$.

We define $$P_d^+ = \{\lambda \in P^+ : \lambda \text{ typical}, k \geq |R_{\bar{1}}^+| - 1\},$$

where $k := 2\frac{(\lambda,\gamma)}{(\gamma,\gamma)}$ is explicitly computed in [18, Table 2]. The fact that $\lambda \in P^+$ is typical gives already some restrictions on k, e.g. in $D(2,1;\alpha)$ we must have $k \geq 2$ and in $F(4)$ we must have $k \geq 4$ and in $G(3)$ we get $k \geq 3$.

Note that $k \geq |R_{\bar{1}}| - 1$ is slightly stronger, e.g. in $D(2,1;\alpha)$ we have four positive odd roots.

Consider the Lie superalgebra $D(2,1;\alpha)$ with distinguished simple system

$$\{\alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_2 = 2\epsilon_2, \alpha_3 = \epsilon_3\},$$

where $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ is an orthogonal basis such that $(\epsilon_1, \epsilon_1) = \frac{-(1+\alpha)}{2}$, $(\epsilon_2, \epsilon_2) = \frac{1}{2}$, $(\epsilon_3, \epsilon_3) = \frac{\alpha}{2}$. The set of positive roots is given by

$$R_{\bar{0}}^+ = \{2\epsilon_1, 2\epsilon_2, 2\epsilon_3\}, \quad R_{\bar{1}}^+ = \{\epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}.$$

We fix a dominant integral typical weight $\lambda \in P_d^+$.

Since $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ we must have

$$\lambda = m_1\varpi_1^1 + m_2\varpi_1^2 + m_3\varpi_1^3, \quad m_1, m_2, m_2 \in \mathbb{Z}_+$$

with some further restrictions on m_1, m_2, m_3; see for example [18, pg. 622] (in this case we have $m_1 = k$). We have with Lemma 2.7

$$\dim V(\lambda) = 16(m_1 - 1)(m_2 + 1)(m_3 + 1).$$

By the PBW theorem it is clear that the following elements span $\operatorname{gr} V(\lambda)$:

$$\left(\prod_{\beta \in R_{\bar 1}^+} x_{-\beta}^{s_\beta}\right) x_{-2\epsilon_1}^{s_{2\epsilon_1}} x_{-2\epsilon_2}^{s_{2\epsilon_2}} x_{-2\epsilon_3}^{s_{2\epsilon_3}}, \quad s_{2\epsilon_1} \le m_1,\ s_{2\epsilon_2} \le m_2,\ s_{2\epsilon_3} \le m_3,\ s_\beta \in \{0,1\}. \quad (22)$$

We will need one further relation in (22) in order to obtain a basis. Recall that $\gamma = 2\alpha_1 + \alpha_2 + \alpha_3$ and hence $x_{-2\epsilon_1}^{m_1+1} v_\lambda = 0$ in $\operatorname{gr} V(\lambda)$. By applying the operators $\partial_\beta, \beta \in R_{\bar 1}^+$ we obtain:

$$\begin{aligned} 0 &= \partial_{\epsilon_1-\epsilon_2-\epsilon_3}\partial_{\epsilon_1-\epsilon_2+\epsilon_3}\partial_{\epsilon_1+\epsilon_2-\epsilon_3}\partial_{\epsilon_1+\epsilon_2+\epsilon_3} x_{-2\epsilon_1}^{m_1+1} \\ &= x_{-2\epsilon_1}^{m_1-3} x_{-\epsilon_1-\epsilon_2-\epsilon_3} x_{-\epsilon_1-\epsilon_2+\epsilon_3} x_{-\epsilon_1+\epsilon_2-\epsilon_3} x_{-\epsilon_1+\epsilon_2+\epsilon_3} + x_{-2\epsilon_1}^{m_1-2} x_{-2\epsilon_2} x_{-\epsilon_1+\epsilon_2-\epsilon_3} x_{-\epsilon_1+\epsilon_2+\epsilon_3} \\ &\quad + x_{-2\epsilon_1}^{m_1-2} x_{-\epsilon_1-\epsilon_2+\epsilon_3} x_{-2\epsilon_3} x_{-\epsilon_1+\epsilon_2+\epsilon_3}. \end{aligned}$$

Now choosing an appropriate order on the set of positive roots, we can impose the following relation in (22)

$$\sum_{\beta \in R_{\bar 1}^+} s_\beta + s_{2\epsilon_1} \le m_1.$$

We claim that the set $\{x^{\mathbf{s}} v_\lambda : \mathbf{s} \in S(\lambda)\}$ is a basis of $\operatorname{gr} V(\lambda)$, where

$$S(\lambda) = \Big\{\mathbf{s} : \sum_{\beta \in R_{\bar 1}^+} s_\beta + s_{2\epsilon_1} \le m_1,\ s_{2\epsilon_2} \le m_2,\ s_{2\epsilon_3} \le m_3,\ \forall \beta \in R_{\bar 1}^+ :\ s_\beta \le 1\Big\}.$$

This follows immediately if we can show that the cardinality of $S(\lambda)$ equals the dimension of $V(\lambda)$. But this is clear with Lemma 5.1.

Here we consider the Lie superalgebra $F(4)$ with distinguished simple system

$$\{\alpha_1 = \frac{1}{2}(\delta - \epsilon_1 - \epsilon_2 - \epsilon_3), \alpha_2 = \epsilon_3, \alpha_3 = \epsilon_2 - \epsilon_3, \alpha_4 = \epsilon_1 - \epsilon_2\}$$

and positive roots

$$R_{\bar 0}^+ = \{\delta, \epsilon_1 \pm \epsilon_2, \epsilon_2 \pm \epsilon_3, \epsilon_1 \pm \epsilon_3, \epsilon_1, \epsilon_2, \epsilon_3\}, \quad R_{\bar 1}^+ = \{\frac{1}{2}(\delta \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3)\},$$

where $(\epsilon_i, \epsilon_j) = \delta_{i,j},\ (\epsilon_i, \delta) = 0,\ (\delta, \delta) = -3$. Again we fix a dominant integral weight $\lambda \in P_d^+$. Since $\mathfrak{g}_{\bar 0} \cong \mathfrak{sl}_2 \oplus \mathfrak{so}_7$ we must have

$$\lambda = m_1 \varpi_1^1 + k_1 \varpi_1^2 + k_2 \varpi_2^2 + k_3 \varpi_3^2, \quad m_1, k_1, k_2, k_3 \in \mathbb{Z}_+$$

with some further restrictions on m_1, k_1, k_2, k_3; see for example [18, pg. 622] (in this case we have $m_1 = k$). Setting $\lambda' = \lambda - m_1 \varpi_1^1$ we get with Lemma 2.7 that

$$\dim V(\lambda) = 2^8 (m_1 - 3) \dim V_{\mathfrak{so}_7}(\lambda').$$

Recall that a convex polytope parametrizing a basis of $\operatorname{gr} V_{\mathfrak{so}_7}(\lambda')$ has been determined in [2, Theorem 5.2]; we denote this polytope by $S_{\mathfrak{so}_7}(\lambda')$. In fact, we can choose any basis but we favor a basis parametrized by the lattice points of a polytope. A similar calculation as in the previous paragraph and Proposition 2.8 shows that the following set is a basis of $\operatorname{gr} V(\lambda)$:

$$\{x^{\mathbf{s}} v_\lambda : \mathbf{s} \in S(\lambda)\}, \tag{23}$$

where $$S(\lambda) = \Big\{\mathbf{s} : \sum_{\beta \in R_{\bar{1}}^+} s_\beta + s_\delta \leq m_1, \ \ (s_\alpha)_{\alpha \in R_{\bar{0}}^+ \setminus \{\delta\}} \in S_{\mathfrak{so}_7}(\lambda')\Big\}.$$

The fact that (23) is a spanning set is done in a similar fashion by applying suitable differential operators and choosing an appropriate order on the set of positive roots. The fact that the cardinality of $S(\lambda)$ is equal to dim $V(\lambda)$ follows again by Lemma 5.1.

Here we consider the Lie superalgebra $G(3)$ with distinguished simple system

$$\{\alpha_1 = \delta + \epsilon_3, \alpha_2 = \epsilon_1, \alpha_3 = \epsilon_2 - \epsilon_1\}.$$

The set of positive roots is given by

$$R_{\bar{0}}^+ = \{2\delta, \epsilon_1, \epsilon_2 - \epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2, 2\epsilon_1 + \epsilon_2, \epsilon_1 + 2\epsilon_2\}, \quad R_{\bar{1}}^+ = \{\delta, \ \delta \pm \epsilon_i : 1 \leq i \leq 3\},$$

where $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and $(\epsilon_i, \epsilon_j) = 1 - 3\delta_{i,j}$, $(\delta, \delta) = 2$, $(\epsilon_i, \delta) = 0$. Again we fix a dominant integral weight $\lambda \in P_d^+$. Since $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}_2 \oplus G_2$ we must have

$$\lambda = m_1 \varpi_1^1 + k_1 \varpi_1^2 + k_2 \varpi_2^2, \quad m_1, k_1, k_2 \in \mathbb{Z}_+$$

with some further restrictions on m_1, k_1, k_2; see for example [18, pg. 633] (in this case we have $m_1 = k$). Setting $\lambda' = \lambda - m_1 \varpi_1^1$ we get with Lemma 2.7 that

$$\dim V(\lambda) = 64(2m_1 - 5) \dim V_{G_2}(\lambda').$$

Recall that a convex polytope parametrizing a basis of $\operatorname{gr} V_{G_2}(\lambda')$ has been determined in [15, Theorem 1]; we denote this polytope by $S_{G_2}(\lambda')$. Again for our purposes we could choose any basis. A similar calculation as in the previous pargraphs and Proposition 2.8 shows that the following set is a basis of $\operatorname{gr} V(\lambda)$:

$$\{x^{\mathbf{s}} v_\lambda : \mathbf{s} \in S(\lambda)\}, \tag{24}$$

where

$$S(\lambda) = \Big\{\mathbf{s} : \sum_{\beta \in R_{\bar{1}}^+} s_\beta + s_{2\delta} \leq m_1, \ s_\beta \leq 1 \ \forall \beta \in R_{\bar{1}}^+, \ (s_\tau)_{\tau \in R_{\bar{0}}^+ \setminus \{2\delta\}} \in S_{G_2}(\lambda')\Big\}.$$

The fact that (24) is a spanning set is done by applying differential operators corresponding to positive odd roots to the element $x_{-2\delta}^{m_1+1}$ (recall that this element vanishes in $\operatorname{gr} V(\lambda)$). The fact that the cardinality of $S(\lambda)$ is equal to dim $V(\lambda)$ is done similarly by using Lemma 5.1.

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