

The Smoothness of Convolutions of Singular Orbital Measures on Complex Grassmannians

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Abstract. It is well known that if G/K is any irreducible symmetric space and μ_a is a continuous orbital measure supported on the double coset KaK , then the convolution product, μ_a^k , is absolutely continuous for some suitably large $k \leq \dim G/K$. The minimal value of k is known in some symmetric spaces and in the special case of compact groups or rank one compact symmetric spaces it has even been shown that μ_a^k belongs to the smaller space L^2 for some k . Here we prove that this L^2 property holds for all the compact, complex Grassmannian symmetric spaces, $SU(p+q)/S(U(p) \times U(q))$. Moreover, for the orbital measures at a dense set of points a , we prove that $\mu_a^2 \in L^2$ (or $\mu_a^3 \in L^2$ if $p = q$).

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1. Introduction

Suppose G is a semi-simple Lie group with Cartan involution θ and K is the compact subgroup of G fixed by θ . In a now classical paper, [18], Ragozin proved that if the symmetric space, G/K , is irreducible, then any convolution product of $\dim G/K$, continuous, K -bi-invariant measures on G is absolutely continuous with respect to any Haar measure on G . In the case that G is a non-compact group, Ragozin's result was significantly improved by Graczyk and Sawyer in [5] who showed that $\text{rank}(G/K) + 1$ convolutions would suffice. The building blocks for all K -bi-invariant measures on G/K are the so-called orbital measures, μ_a , supported on the double cosets, KaK . Building on the work of Graczyk and Sawyer in [5]–[8], the authors in [12] improved this result by characterizing the convolution products of orbital measures that are absolutely continuous, for any classical, non-compact symmetric space.

For the compact symmetric spaces G_c/K for G_c compact, the authors in [11] proved that any convolution product of $2\text{rank}(G_c/K) + 1$, continuous, K -bi-invariant measures is absolutely continuous. Any compact Lie group, G_c , can be regarded as a compact symmetric space, namely $G_c \times G_c/K$ with $K = \{(g, g) : g \in G_c\}$. The K -bi-invariant, orbital measures are the G_c -invariant measures supported on the conjugacy classes of the group. In this special case, it is known that μ_a^k belongs to

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the (smaller) space $L^2(G_c)$ if and only if μ_a^k is absolutely continuous (equivalently, belongs to $L^1(G_c)$). Moreover, the minimal exponent, $k(a)$, such that $\mu_a^{k(a)} \in L^2$ has even been determined for each a ; see [10, 14, 20]. This $L^1 - L^2$ dichotomy was shown to be false in the compact symmetric space $SU(2)/SO(2)$, although it is still true that $\mu_a^3 \in L^2$ for all continuous orbital measures in any rank one compact symmetric space; see [2], [13]. These L^2 results were all found by studying the decay properties of characters or spherical functions.

In [1], Al-Hashami and Anouchche studied the analogous problem for the compact, complex Grassmannian symmetric spaces G_c/K , where $G_c = SU(p+q)$ and $K = S(U(p) \times U(q))$. They proved that for the dense set of ‘regular’ points $a \in G_c$, $\mu_a^k \in L^2$ for a choice of k which is much smaller than $\dim G_c/K$; see (6). They did this by using the Berezin-Karpelevich formula for the spherical functions and obtaining rates of decay for these functions at the regular elements. Using these estimates and the Sobolev embedding theorem, they also proved that $\mu_a^k \in C^s(G_c)$ for regular elements a and suitably large choices of k , depending on s .

In this note, we find estimates on the rates of decay for the spherical functions at *all* points $a \notin N_{G_c}(K)$, these being the points which give rise to the continuous orbital measures μ_a . Obtaining such estimates at non-regular points is quite delicate for then the singularities of the spherical functions must be understood. With these estimates we are able to prove that $\mu_a^k \in L^2$ for all $a \notin N_{G_c}(K)$ whenever

$$k > \max(p, 2(p-q) + 3) \text{ if } p > q \text{ and } k > \max(2p, 6) \text{ if } p = q.$$

For the special case of the regular elements a , we show that $\mu_a^2 \in L^2$ if $p > q$ and $\mu_a^3 \in L^2$ otherwise. This significantly improves the work in [1] as their exponent k was unbounded in p . We also show that $\mu_a^k \in C^s$ for all $a \notin N_{G_c}(K)$ for suitable $k = k(s, a)$.

Although smaller than $\dim G_c/K$, our minimal exponent k is typically much larger than $2\text{rank}(G_c/K) + 1$. It would be interesting to know what the sharp exponent is for the L^2 problem and whether the $L^1 - L^2$ dichotomy holds.

2. Notation and preliminaries

2.1. Orbital measures and their smoothness

Suppose G_c/K is an irreducible symmetric space, where G_c is a compact semi-simple Lie group with Cartan involution θ and $K = \{g \in G_c : \theta(g) = g\}$. Denote its non-compact dual space by G/K and suppose the Lie algebra of G is \mathfrak{g} . We fix the Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ and the maximal abelian subspace \mathfrak{a} of \mathfrak{p} as in [15]. We identify the Lie algebra \mathfrak{g}_c of G_c with the subspace $\mathfrak{g}_c = \mathfrak{t} \oplus i\mathfrak{p}$ of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and let $A = \exp i\mathfrak{a} \subseteq G_c$. Always $G_c = KAK$. For any $a \in G_c$, the set KaK is called a double coset and has Haar measure zero.

Definition 2.1. Given any $a = \exp(iX) \in A$, we define the *orbital measure* μ_a on G_c , with support KaK , by

$$\int f d\mu_a = \int_K \int_K f(k_1 a k_2) dk_1 dk_2$$

for all continuous functions f . (Here dk_1, dk_2 denote normalized Haar measure on K .)

This probability measure is K -bi-invariant. It is continuous (meaning as a measure on G_c/K , hence $\mu_a(xK) = 0$ for all $x \in G$) precisely when $a \notin N_{G_c}(K)$ and is purely singular with respect to Haar measure.

The measure μ on G is said to be absolutely continuous if it is absolutely continuous with respect to Haar measure m , meaning that $\mu(E) = 0$ whenever $m(E) = 0$. In [18], Ragozin essentially proved the following geometric characterization of absolute continuity.

Proposition 2.2. *Let $a_i \in A$, $i = 1, \dots, k$. Then $\mu_{a_1} * \dots * \mu_{a_k}$ is absolutely continuous if and only if the product of double cosets $Ka_1K \dots Ka_kK$ has non-empty interior in G_c . Moreover, if $a_i \notin N_{G_c}(K)$ and $k \geq \dim(G_c/K)$, then $\mu_{a_1} * \dots * \mu_{a_k}$ is absolutely continuous.*

Ragozin used geometric methods to show that $Ka_1K \dots Ka_kK$ has non-empty interior for $k = \dim G_c/K$ if $a_i \notin N_{G_c}(K)$ for all i . Notice that if $a \in N_{G_c}(K)$, then $(KaK)^k = Ka^kK$ and hence has Haar measure zero. Thus μ_a^k is purely singular to Haar measure for all k . Ragozin’s geometric characterization was verified using algebraic methods by the authors in [11] when they improved upon his result, showing that $k = 2\text{rank}(G_c/K) + 1$ suffices. This exponent is close to sharp as there are continuous orbital measures (in some compact symmetric spaces) with $\mu_a^{2\text{rank}(G_c/K)-1}$ singular; [10].

A measure μ is absolutely continuous with respect to Haar measure if and only if its density function (or Radon Nikodym derivative) belongs to L^1 . If the density function belongs to the properly smaller space L^2 (or to C^s), we will write $\mu \in L^2$ (resp., C^s). Ragozin’s geometric approach is not helpful in studying the problem of determining if $\mu_a^k \in L^2$. Instead, a harmonic analysis approach has been taken: estimates are made on the rate of decay of the Fourier transform of the measure and then the Plancherel theorem is invoked. This approach has been applied very successfully when the symmetric space is a compact group or a rank one compact symmetric space, and is the approach we will take in this paper to study the L^2 problem for the complex Grassmannian symmetric spaces.

2.2. The symmetric space $SU(p + q)/S(U(p) \times U(q))$

For the remainder of this paper we will focus on the case of the complex Grassmannian symmetric space, G_c/K , where $G_c = SU(p + q)$, $K = S(U(p) \times U(q))$ and $p \geq q \geq 2$. This compact symmetric space has rank q and dimension $2pq$. The non-compact dual space is the symmetric space G/K , where G has Lie algebra $\mathfrak{g} = \mathfrak{su}(p, q)$. With Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, we can take as the maximal abelian subspace \mathfrak{a} the $(p + q) \times (p + q)$ matrices of the form

$$X = X(t_1, \dots, t_q) = \begin{bmatrix} 0 & 0 & M \\ 0 & 0 & 0 \\ M' & 0 & 0 \end{bmatrix}$$

where M, M' are $q \times q$ anti-diagonal matrices, with t_1, \dots, t_q and t_q, \dots, t_1 respectively, on the anti-diagonal. As above, we let $A = \exp i\mathfrak{a} \subseteq G_c$. We can identify $X(t_1, \dots, t_q) \in \mathfrak{a}$ by $(t_1, \dots, t_q) \in \mathbb{R}^q$.

The restricted roots and the highest spherical weights will be very important in our work. The positive restricted roots can be taken to be

$$\{2e_k, e_k, e_i \pm e_j : 1 \leq i < j \leq q, 1 \leq k \leq q\}$$

(where $\{e_j\}$ are the usual basis vectors for \mathbb{R}^q) with multiplicities $m_{2e_k} = 1$, $m_{e_k} = 2(p - q)$ (so these are not present if $p = q$) and $m_{e_i \pm e_j} = 2$. We also view the roots as acting on A by the (well defined) rule $\alpha(\exp iX) \equiv \alpha(X) \pmod{\pi}$. The reader is referred to [15] for further background information.

Suppose $a \in A$. Then $a \in N_{G_c}(K)$, the normalizer of K , if and only if we have $Ad(a)(X_\alpha + \theta X_\alpha) \in \mathfrak{k}$ for all X_α , root vectors of the restricted roots α . An easy calculation shows that if $a = \exp iH$ for $H \in \mathfrak{a}$, then

$$Ad(a)(X_\alpha + \theta X_\alpha) = (X_\alpha + \theta X_\alpha) \cos \alpha(H) + i(X_\alpha - \theta X_\alpha) \sin \alpha(H).$$

Therefore $Ad(a)(X_\alpha + \theta X_\alpha) \in \mathfrak{k}$ if and only if $\sin \alpha(H) = 0$. This shows that $a = \exp iH \in N_{G_c}(K) \cap A$ if and only if $\alpha(H) \equiv 0 \pmod{\pi}$ for all restricted roots α .

An element $X \in \mathfrak{a}$ or $\exp iX \in A$ is said to be *regular* if $\alpha(X) \not\equiv 0 \pmod{\pi}$ for each restricted root α , thus $X = X(t_1, \dots, t_q)$ is regular precisely when $t_j \not\equiv \pm t_k \pmod{\pi}$ for all $j \neq k$ and $t_k \not\equiv 0, \pi/2 \pmod{\pi}$ for all k . Such elements are dense in \mathfrak{a} or A respectively. When X is regular, $\mu_{\exp iX}^2$ is absolutely continuous (whatever the compact symmetric space); [9].

The highest spherical weights are given by

$$\Lambda_{sph} = \left\{ \sum_{j=1}^q 2m_j e_j : m_j \in \mathbb{N}, m_1 \geq m_2 \cdots \geq m_q \geq 0 \right\}.$$

We denote the spherical function corresponding to $\lambda \in \Lambda_{sph}$ by ϕ_λ . Put

$$r = p - q + 1 \quad \text{and} \quad n_j = m_j + q - j, j = 1, \dots, q$$

so that $n_1 > n_2 > \dots > n_q \geq 0$. We denote the normalized Jacobi polynomials by

$$\widetilde{P}_n(x) = \frac{P_n^{(p-q,0)}(x)}{P_n^{(p-q,0)}(1)}, n \in \mathbb{N}.$$

According to the Berezin-Karpelevich formula (see [3] or [4])

$$\phi_\lambda(\exp iX(t_1, \dots, t_q)) = \frac{C_{p,q} \det \left(\widetilde{P}_{n_j}(\cos 2t_k) \right)_{j,k=1}^q}{\prod_{1 \leq j < k \leq q} (\cos 2t_j - \cos 2t_k)(n_j(n_j + r) - n_k(n_k + r))} \tag{1}$$

where
$$C_{p,q} = 2^{q(q-1)/2} \prod_{j=1}^{q-1} j!(j + p - q)^{q-j}$$

and the quotient should be understood in the limiting sense if some $\cos 2t_j = \cos 2t_k$. Of course, this situation occurs precisely if some $t_j \equiv \pm t_k \pmod{\pi}$.

An elementary, but useful, observation is that

$$n_j(n_j + r) - n_k(n_k + r) \geq n_j + r \text{ when } j < k. \tag{2}$$

3. Decay of spherical functions

The objective of this section is to obtain estimates on the decay of the spherical functions. We will find estimates that hold for all $a \notin N_{G_c}(K)$ and better estimates for the regular elements.

Theorem 3.1. *Suppose that $\lambda = \sum_{j=1}^q 2m_j e_j$ is a highest spherical weight and $n_j = m_j + q - j$.*

(i) *If $\exp iX \in A \setminus N_{G_c}(K)$, then*

$$|\phi_\lambda(\exp iX)| \leq C \begin{cases} \prod_{j=1}^{q-1} (n_j + 1)^{-1} & \text{if } p > q \\ \prod_{j=1}^{q-1} (n_j + 1)^{-1/2} & \text{if } p = q \end{cases}$$

where C is a constant that depends on p, q and X , but not λ .

(ii) *If $\exp iX$ is regular, then there is a constant $C = C(p, q, X)$ such that*

$$|\phi_\lambda(\exp iX)| \leq C \prod_{j=1}^q (n_j + 1)^{-p+j-1/2}.$$

Remark 3.2. Note that when $X = X(t_1, \dots, t_q)$ is regular, $\cos 2t_j \neq \cos 2t_k$ for any $j \neq k$. This is very significant as it means we do not have the complication of having to understand $\phi_\lambda(\exp iX)$ through the limiting process. In [1], the decay in $\phi_\lambda(\exp iX)$ was studied for this special case.¹ They obtained the bound,

$$|\phi_\lambda(\exp iX)| \leq C \prod_{j=1}^q (n_j + 1)^{-p+q/2}. \quad \blacksquare$$

In our proof, the constants C which appear may vary from one occurrence to another, but will always be independent of λ . We will frequently write $\phi_\lambda(X)$ as shorthand for $\phi_\lambda(\exp iX)$. When we say $f \sim g$ for functions f, g defined on \mathbb{N} , we mean there are constants $A, B > 0$ such that $Ag(n) \leq f(n) \leq Bg(n)$ for all n . We begin by collecting useful facts about Jacobi polynomials, c.f. [19].

Lemma 3.3. *The following are well known facts about Jacobi polynomials:*

(i) $P_n^{(a,b)}(1) = \binom{n+a}{n} \sim (n+1)^a; P_n^{(a,b)}(-1) = (-1)^n \binom{n+b}{n};$

(ii) $\left| P_n^{(a,b)}(x) \right| \leq \frac{C_{a,b,x}}{\sqrt{n+1}}$ when $x \neq \pm 1$;

(iii) $\left| P_n^{(a,0)}(x) \right| \leq C_{x,a} \left| P_n^{(a,0)}(1) \right|$ so $\left| \widetilde{P}_n(x) \right| \leq C_{x,p,q};$

(iv) $P_0^{(a,b)}(x) = 0$ for all x ;

(v) $\frac{d}{dx} P_n^{(a,b)}(x) = \frac{1}{2}(n+a+b+1)P_{n-1}^{(a+1,b+1)}(x)$ if $n \neq 0$.

We will first prove part (ii) of the Theorem, the special case when X is regular.

¹ The authors do not make this assumption explicitly in the statement of their theorem, but this is indicated in Section 4 and the properties of a regular element are used extensively in their proof.

Proof of Theorem 3.1(ii). Assume that $X = X(t_1, \dots, t_q)$ is regular. Then

$$(e_j \pm e_k)(X) = t_j \pm t_k \neq 0 \pmod{\pi} \text{ for any } j \neq k$$

and that means $\cos 2t_j \neq \cos 2t_k$ for any $j \neq k$. Furthermore, $2e_k(X) = 2t_k \neq 0 \pmod{\pi}$ and that implies $\cos(2t_k) \neq \pm 1$ for any k . This latter fact ensures that

$$\left| \widetilde{P}_n(\cos(2t_k)) \right| = \left| \frac{P_n^{(p-q,0)}(\cos(2t_k))}{P_n^{(p-q,0)}(1)} \right| \leq \frac{C}{(n_j + 1)^{p-q+1/2}}.$$

Consequently, formula (1) implies, as claimed,

$$\begin{aligned} |\phi_\lambda(X)| &\leq \frac{C}{\prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r)) \prod_{1 \leq j \leq q} (n_j + 1)^{p-q+1/2}} \\ &\leq \frac{C}{\prod_{j=1}^{q-1} (n_j + r)^{q-j} \prod_{j=1}^q (n_j + 1)^{p-q+1/2}}. \quad \blacksquare \end{aligned}$$

For general X we must consider the possibility that the formula (1) for $\phi_\lambda(X)$ has to be understood through the limiting process. The next several lemmas will help with this. The proof of the first lemma also introduces a reduction technique that will be frequently used throughout the remainder of the proof of the theorem.

Lemma 3.4. Fix $X = X(t_1, \dots, t_q)$ and suppose there is an index k_0 such that $\cos 2t_{k_0} \neq \cos 2t_k$ for any $k \neq k_0$. Then

$$|\phi_\lambda(X(t_1, \dots, t_q))| \leq C \prod_{j=1}^{q-1} (n_j + 1)^{-1}.$$

Proof. First, suppose $q = 2$. In this case,

$$|\phi_\lambda(X(t_1, t_2))| = \frac{C |\det M|}{|\cos 2t_1 - \cos 2t_2| (n_1(n_1 + r) - n_2(n_2 + r))}$$

where from Lemma 3.3(iii) we see that M is a 2×2 matrix with entries bounded independent of the choice of λ . As $\cos 2t_1 \neq \cos 2t_2$, it follows that

$$|\phi_\lambda(X(t_1, t_2))| \leq \frac{C}{n_1 + r} \leq C(n_1 + 1)^{-1}$$

as we desired to show.

Now assume $q > 2$. Expand the determinant in (1) along column k_0 to obtain

$$\begin{aligned} \left| \det \left(\widetilde{P}_{n_j}(\cos 2t_k) \right)_{j,k=1}^q \right| &= \left| \sum_{i=1}^q (-1)^i \widetilde{P}_{n_i}(\cos 2t_{k_0}) \det \left(\widetilde{P}_{n_j}(\cos 2t_k) \right)_{j \neq i, k \neq k_0} \right| \\ &\leq q \max_i \left| \widetilde{P}_{n_i}(\cos 2t_{k_0}) \det \left(\widetilde{P}_{n_j}(\cos 2t_k) \right)_{j \neq i, k \neq k_0} \right|. \end{aligned}$$

Assume the maximum occurs at index $i = j_0$. Then $|\phi_\lambda(X(t_1, \dots, t_q))| \leq CI_1 \cdot I_2$,

$$\text{where } I_1 = \frac{\left| \widetilde{P}_{n_{j_0}}(\cos 2t_{k_0}) \right|}{\prod_{k \neq k_0} |\cos 2t_k - \cos 2t_{k_0}| \prod_{j \neq j_0} |n_j(n_j + r) - n_{j_0}(n_{j_0} + r)|}$$

and
$$I_2 = \frac{\left| \det \left(\widetilde{P}_{n_j}(\cos 2t_k) \right)_{j \neq j_0, k \neq k_0} \right|}{\prod_{\substack{1 \leq j < k \leq q \\ j, k \neq k_0}} |\cos 2t_j - \cos 2t_k| \prod_{\substack{1 \leq j < k \leq q \\ j, k \neq j_0}} (n_j(n_j + r) - n_k(n_k + r))}$$

(with I_2 understood in the limiting sense if some $\cos 2t_j = \cos 2t_k$).

As $\cos 2t_k \neq \cos 2t_{k_0}$ when $k \neq k_0$, applying property (2) gives

$$I_1 \leq \frac{C}{\prod_{j \neq j_0} |n_j(n_j + r) - n_{j_0}(n_{j_0} + r)|} \leq \frac{C}{(n_1 + r) \cdots (n_{j_0-1} + r)(n_{j_0} + r)^{q-j_0}} \leq C \prod_{j=1}^{q-1} (n_j + r)^{-1}$$

since $n_1 > n_2 > \cdots > n_q$. In order to bound I_2 , we will use a reduction argument. Consider the symmetric space $G'/K' = SU((p-1) + (q-1))/S(U(p-1) \times U(q-1))$ where $p-1 \geq q-1 \geq 2$, and the spherical representation $\lambda' = \sum_{i=1}^{q-1} 2m'_i e_i$ where e_i are the standard basis vectors for \mathbb{R}^{q-1} and

$$m'_i = \begin{cases} n_i - (q-1) + i & \text{for } i = 1, \dots, j_0 - 1 \\ n_{i+1} - (q-1) + i & \text{for } i = j_0, \dots, q-1. \end{cases}$$

This choice is made so that $n'_i = m'_i + (q-1) - i$ satisfies the conditions $n'_i = n_i$ for $i = 1, \dots, j_0 - 1$ and $n'_i = n_{i+1}$ for $i = j_0, \dots, q-1$. Let $t' = (t_1, \dots, \widehat{t_{k_0}}, \dots, t_q)$ belong to \mathfrak{a}' for G'/K' (here the notation $\widehat{}$ means the element is not present) and $\exp iX' \in A' \subseteq G'$ for $X' = X'(t')$. Notice that for any x ,

$$\widetilde{P}_n(x) = \frac{P_n^{((p-1)-(q-1), 0)}(x)}{P_n^{((p-1)-(q-1), 0)}(1)},$$

so $I_2 = C |\phi_{\lambda'}(X')| \leq C$ since all spherical functions are bounded by 1. Thus

$$|\phi_\lambda(X(t_1, \dots, t_q))| \leq C I_1 \cdot I_2 \leq C \prod_{j=1}^{q-1} (n_j + r)^{-1} \quad \blacksquare$$

Lemma 3.5. *Suppose (upon a suitable reordering of coefficients, if necessary) $X = X(b_1, \dots, b_s)$ where each $b_j = (t_1^{(j)}, \dots, t_{L_j}^{(j)})$ with $\cos(2t_i^{(j)}) = \cos(2t_1^{(j)})$ for all $i = 1, \dots, L_j$, $\cos(2t_1^{(i)}) \neq \cos(2t_1^{(j)})$ if $i \neq j$, $L_j \geq 2$ for all j and $s \geq 2$. Then*

$$|\phi_\lambda(X)| \leq C \prod_{j=1}^q (n_j + r)^{-1}.$$

Proof. We will give the details for $s = 2$, but it will be clear how the method generalizes. Thus assume $X = X(t_1, \dots, t_{L_1}, t_{L_1+1}, \dots, t_q)$ where $\cos(2t_1) = \cos(2t_j)$ for $j = 2, \dots, L_1$, $\cos(2t_q) = \cos(2t_k)$ for $k = L_1 + 1, \dots, q-1$, $\cos(2t_1) \neq \cos(2t_q)$ and $L_1, L_2 = q - L_1 \geq 2$. A basic property of the determinant is

$$\det \left(\widetilde{P}_{n_j}(\cos 2t_k) \right)_{j,k=1}^q = \sum_{\sigma} c_{\sigma} A_{\sigma} B_{\sigma}$$

where the sum is over all choices, σ , of L_1 indices, say $\sigma : j_1 < \dots < j_{L_1}$, A_σ is the determinant of the $L_1 \times L_1$ matrix $(\widetilde{P_{n_{j_i}}(\cos 2t_k)})_{i,k=1}^{L_1}$, B_σ is the determinant of the $L_2 \times L_2$ submatrix of $(\widetilde{P_{n_j}(\cos 2t_k)})$ formed by the remaining rows and columns (the remaining columns being columns $L_1 + 1, \dots, q$), and c_σ is a suitable choice of ± 1 . This gives the bound

$$|\phi_\lambda(X)| \leq C \sum_{\sigma} I_\sigma J_\sigma K_\sigma, \text{ where}$$

$$I_\sigma = \frac{|A_\sigma|}{\prod_{1 \leq j < k \leq L_1} |\cos 2t_j - \cos 2t_k| \prod_{1 \leq i < l \leq L_1} (n_{j_i}(n_{j_i} + r) - n_{j_l}(n_{j_l} + r))}$$

$$J_\sigma = \frac{|B_\sigma|}{\prod_{L_1+1 \leq j < k \leq q} |\cos 2t_j - \cos 2t_k| \prod_{\substack{1 \leq j < k \leq q \\ j, k \neq j_1, \dots, j_{L_1}}} (n_j(n_j + r) - n_k(n_k + r))},$$

$$K_\sigma = \frac{1}{\prod_{\substack{1 \leq j \leq L_1 \\ L_1+1 \leq k \leq 1}} |\cos 2t_j - \cos 2t_k| \prod_{\substack{1 \leq i \leq L_1 \\ k \neq j_1, \dots, j_{L_1}}} (n_{j_i}(n_{j_i} + r) - n_k(n_k + r))}.$$

(As usual, I_σ, J_σ should be understood in the limiting sense.) Since

$$|n_{j_i}(n_{j_i} + r) - n_k(n_k + r)| \geq \max(n_j + r, n_k + r) \geq \sqrt{(n_j + r)(n_k + r)},$$

it follows that

$$\prod_{\substack{(i,k): 1 \leq i \leq L_1 \\ k \neq j_1, \dots, j_{L_1}}} (n_{j_i}(n_{j_i} + r) - n_k(n_k + r)) \geq \prod_{\substack{(i,k): 1 \leq i \leq L_1 \\ k \neq j_1, \dots, j_{L_1}}} \sqrt{(n_{j_i} + r)(n_k + r)}$$

$$= \prod_{j \in \{j_1, \dots, j_{L_1}\}} (n_j + r)^{L_2/2} \prod_{k \notin \{j_1, \dots, j_{L_1}\}} (n_k + r)^{L_1/2}$$

$$\geq \prod_{i=1}^q (n_i + r)^{\min(L_1, L_2)/2} \geq \prod_{i=1}^q (n_i + r).$$

As $\cos 2t_j \neq \cos 2t_k$ whenever $j \leq L_1$ and $K \geq L_1 + 1$, we see that

$$K_\sigma \leq C \prod_{i=1}^q (n_i + r)^{-1}.$$

Similar to the previous lemma, I_σ and J_σ are, up to a constant, the modulus of spherical functions on Grassmannian symmetric spaces of rank L_1 and L_2 respectively. Indeed, $I_\sigma = |\phi_{\lambda'}(X')|$ on $SU(p+q)/S(U(p+q-L_1) \times U(L_1))$ where λ' is identified with $(n_{j_1}, \dots, n_{j_{L_1}})$ and $X' = X'(t_1, \dots, t_{L_1})$, and similarly for J_σ . Hence both are bounded and therefore $\phi_\lambda(X)$ has the claimed bound. ■

Lemma 3.6. *Suppose $X = X(t_1, \dots, t_q)$ where $\cos(2t_j) = \cos(2t_k)$ for all j, k , but $\cos(2t_j) \neq \pm 1$. Then*

$$|\phi_\lambda(X)| \leq C \prod_{j=1}^{q-1} (n_j + r)^{-p+q-1/2} (n_q + 1)^{-p+q}.$$

Proof. Here we must understand $\phi_\lambda(X)$ as

$$\lim_{\substack{x_i \rightarrow \cos(2t_i) \\ i=1, \dots, q}} \frac{\det \left(\widetilde{P}_{n_j}(x_k) \right)_{j,k=1, \dots, q}}{\prod_{1 \leq j < k \leq q} (x_j - x_k) \prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r))}$$

where x_i are distinct and not equal to $\cos(2t_i)$. According to [16, Thm. 1.24], this limit is equal to

$$\phi_\lambda(X) = C \frac{\det \left(\widetilde{P}_{n_j}^{(k-1)}(\cos(2t_1)) \right)_{j,k=1, \dots, q}}{\prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r))} \tag{3}$$

where $C = C(q) = (-1)^{q(q-1)/2} / (q-1)!$ and $\widetilde{P}_{n_j}^{(k-1)}$ means the $(k-1)$ 'st derivative of \widetilde{P}_{n_j} . Applying Lemma 3.3(v) repeatedly, we see that

$$\widetilde{P}_n^{(k-1)}(x) = \frac{2^{-(k-1)}(n+p-q+1) \cdots (n+p-q+k-1) P_{n-(k-1)}^{(p-q+k-1, k-1)}(x)}{P_n^{(p-q, 0)}(1)},$$

where we recall that $P_m^{(a,b)}(x) = 0$ if $m \leq 0$. So the main task is to bound

$$\left| \det \left((n_j + p - q + 1) \cdots (n_j + p - q + k - 1) P_{n_j - (k-1)}^{(p-q+k-1, k-1)}(\cos(2t_1)) \right)_{j,k} \right|.$$

Using the permutation method for taking determinants, one can easily see this determinant is bounded in modulus by

$$q! \max_{\pi} \left| \prod_{j=1}^q (n_j + p - q + 1) \cdots (n_j + p - q + \pi(j) - 1) P_{n_j - (\pi(j) - 1)}^{(p-q+\pi(j)-1, \pi(j)-1)}(\cos(2t_1)) \right| \tag{4}$$

where the maximum is taken over all permutations π of $\{1, \dots, q\}$. If all $n_j \leq q-1$, this expression is clearly bounded (independently of the choice (n_1, \dots, n_q)), so assume otherwise, say

$$n_q < \cdots < n_s \leq q-1 < n_{s-1} < \cdots < n_1,$$

for some $s = 1, \dots, q$. Put $s = q+1$ if all $n_j > q-1$.

Now, $j \leq s-1$ implies $n_j > q-1 \geq \pi(j)-1$ and thus $n_j - (\pi(j)-1) \geq (n_j+1)/(q+1)$. Since $\cos(2t_1) \neq \pm 1$, Lemma 3.3(ii) together with this observation yields the bound

$$\left| P_{n_j - (\pi(j) - 1)}^{(p-q+\pi(j)-1, \pi(j)-1)}(\cos 2t_1) \right| \leq \frac{C(t_1, p, q)}{\sqrt{n_j + 1}}$$

for these j . Moreover, $n_j + p - q + \pi(j) - 1 \leq C_{p,q}(n_j + 1)$ for these j .

Since always $P_n^{(a,b)}(\cos 2t_1)$ is uniformly bounded over n (with constant depending on t_1, a, b) it follows that the terms

$$(n_j + p - q + 1) \cdots (n_j + p - q + \pi(j) - 1) P_{n_j - (\pi(j) - 1)}^{(p-q+\pi(j)-1, \pi(j)-1)}(\cos(2t_1))$$

are uniformly bounded when $n_j \leq q-1$.

Consequently, the expression in (4) for any fixed π is bounded by

$$C \prod_{j=1}^{s-1} \frac{(n_j + p - q + 1) \cdots (n_j + p - q + \pi(j) - 1)}{\sqrt{n_j + 1}} \leq C \prod_{j=1}^{s-1} (n_j + 1)^{\pi(j)-1-1/2}$$

where C does not depend on the choice of n_j . Since the terms n_j are strictly increasing, this is maximized with the permutation π that maps j to $q - j + 1$. Hence

$$(4) \leq C \prod_{j=1}^{s-1} (n_j + 1)^{q-j-1/2}.$$

Recalling the formula (3), we have

$$\begin{aligned} |\phi_\lambda(X)| &\leq \frac{C \prod_{j=1}^{s-1} (n_j + 1)^{q-j-1/2}}{\prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r)) \prod_{j=1}^q P_{n_j}^{(p-q,0)}(1)} \\ &\leq \frac{C \prod_{j=1}^{s-1} (n_j + 1)^{q-j-1/2}}{\prod_{j=1}^{q-1} (n_j + 1)^{q-j} \prod_{j=1}^q (n_j + 1)^{p-q}} = \frac{C \prod_{j=1}^{s-1} (n_j + 1)^{q-j-1/2}}{\prod_{j=1}^q (n_j + 1)^{p-j}}. \end{aligned}$$

When $s = q + 1$ (all $n_j > q - 1$), this simplifies to $C \prod_{j=1}^q (n_j + 1)^{-p+q-1/2}$, which is even smaller than the valued stated in the lemma. Otherwise, $s - 1 \leq q - 1$ and for $j \leq q - 1$, $q - j - 1/2 > 0$, thus

$$\prod_{j=1}^{s-1} (n_j + 1)^{q-j-1/2} \leq \prod_{j=1}^{q-1} (n_j + 1)^{q-j-1/2}.$$

We immediately deduce that $\phi_\lambda(X)$ is bounded as stated in the lemma. ■

Lemma 3.7. *Suppose $X = (t_1, \dots, t_q)$ where $\cos(2t_j) = -1$ for all j . Then*

$$|\phi_\lambda(X)| \leq C \prod_{j=1}^q (n_j + r)^{-p+q}.$$

Remark 3.8. We note that when $p = q$, $\exp iX \in N_{G_c}(K)$ for such X .

Proof. Similar arguments to the previous proof show that

$$\begin{aligned} \phi_\lambda(X) &= \lim_{\substack{x_i \rightarrow -1 \\ x_i \text{ distinct}}} \frac{\det \left(\widetilde{P}_{n_j}(x_k) \right)_{j,k=1,\dots,q}}{\prod_{1 \leq j < k \leq q} (x_j - x_k) \prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r))} \\ &= C \frac{\det \left(\widetilde{P}_{n_j}^{(k-1)}(-1) \right)_{j,k=1,\dots,q}}{\prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r))} \end{aligned}$$

In this situation, bounding the determinant by a multiple of the largest term in the sum that arises from the permutation method for calculating the determinant will not give us a good enough estimate.

We will actually directly compute the determinant, instead. As before, we have

$$\begin{aligned} & \det \left(\widetilde{P}_{n_j}^{(k-1)}(-1) \right) \\ &= \frac{\det(c_k(n_j + p - q + 1) \cdots (n_j + p - q + k - 1) P_{n_j - (k-1)}^{(p-q+k-1, k-1)}(-1))_{j,k}}{\prod_{j=1}^q P_{n_j}^{(p-q, 0)}(1)}. \end{aligned}$$

From Lemma 3.3(i),

$$\begin{aligned} P_{n_j - (k-1)}^{(p-q+k-1, k-1)}(-1) &= (-1)^{n_j - (k-1)} \binom{n_j}{n_j - (k-1)} \\ &= \frac{(-1)^{n_j - (k-1)} n_j \cdots (n_j - (k-1) + 1)}{(k-1)!}, \end{aligned}$$

thus we need to evaluate the determinant of the matrix whose j, k entry is given by

$$c'_k(n_j + p - q + 1) \cdots (n_j + p - q + k - 1) (-1)^{n_j - (k-1)} n_j \cdots (n_j - (k-1) + 1).$$

In other words, we want to find

$$\det \left((-1)^{n_1 + \cdots + n_q} \prod_{k=1}^q c'_k (-1)^{k-1} M_{j,k} \right)_{j,k}$$

where $M_{j,1} = 1$ and for $k > 1$,

$$\begin{aligned} M_{j,k} &= \prod_{i=0}^{k-2} (n_j + p - q + 1 + i)(n_j - i) \\ &= \prod_{i=0}^{k-2} (n_j + r + i)(n_j - i) = \prod_{i=0}^{k-2} (n_j(n_j + r) - i(r + i)). \end{aligned}$$

Put $M_{j,1}(x) = 1$ and $M_{j,k}(x) = \prod_{\ell=0}^{k-2} (x_j - \ell(r + \ell))$ and consider the multi-variable polynomial

$$F(x_1, \dots, x_q) = \det(M_{j,k}(x))_{j,k}.$$

Our desire is to compute $F(x)$ at $x = (x_j)_{j=1}^q$ with $x_j = n_j(n_j + r)$. Notice that if some coordinates $x_i = x_j$ for $i \neq j$, then the matrix $(M_{j,k}(x))_{j,k}$ has two identical rows and therefore $F(x) = 0$. Thus the polynomial $P(x) = \prod_{1 \leq i < j \leq q} (x_i - x_j)$ divides $F(x)$. Since both F and P are degree $q(q-1)$ polynomials, it must be that there is a constant c such that for all x ,

$$F(x) = c \prod_{1 \leq i < j \leq q} (x_i - x_j).$$

In particular,

$$\begin{aligned} F(n_1(n_1 + r), \dots, n_q(n_q + r)) &= \det(M_{j,k})_{j,k} \\ &= c \prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r)). \end{aligned}$$

Hence,
$$\phi_\lambda(X) = \frac{C \det(M_{j,k})_{j,k}}{\prod_{j=1}^q P_{n_j}^{(p-q,0)}(1) \prod_{1 \leq j < k \leq q} (n_j(n_j + r) - n_k(n_k + r))}$$

$$\leq C \prod_{j=1}^q (n_j + 1)^{-p+q}. \quad \blacksquare$$

We are now ready to complete the proof of the theorem for arbitrary X .

Proof of Theorem 3.1(i). Let $\exp iX \in A \backslash N_{G_c}(K)$ with $X = X(t_1, \dots, t_q)$. First, suppose there is a pair j, k such that $t_j \not\equiv \pm t_k \pmod{\pi}$. In this case, $\cos(2t_j) \neq \cos(2t_k)$. Lemmas 3.4 or 3.5 (depending on the situation) show that for such X , $\phi_\lambda(X)$ is bounded in modulus by $C \prod_{j=1}^{q-1} (n_j + 1)^{-1}$.

So we can assume that for every j, k , either $t_j \equiv t_k \pmod{\pi}$ or $t_j \equiv -t_k \pmod{\pi}$ (or both). If there is some j such that $t_j \equiv 0 \pmod{\pi}$, then this will be true for all t_k . Then $\alpha(X) \equiv 0 \pmod{\pi}$ for all restricted roots α and that contradicts the assumption that $\exp iX \notin N_{G_c}(K)$.

If some $t_j \equiv \pi/2 \pmod{\pi}$, then the same is true for all t_k and then $\cos(2t_j) = -1$ for all j . In the case that $p = q$, as the positive restricted roots are only of the form $\alpha = e_j \pm e_k$ and $2e_j$, we again have $\alpha(X) \equiv 0 \pmod{\pi}$ for all α , giving a contradiction. If $p > q$, Lemma 3.7 gives the bound

$$|\phi_\lambda(X)| \leq C \prod_{j=1}^q (n_j + r)^{-p+q} \leq C \prod_{j=1}^q (n_j + r)^{-1}.$$

Otherwise we must have $t_j \not\equiv 0, \pi/2 \pmod{\pi}$ for any j , but $\cos(2t_j) = \cos(2t_k)$ for all j, k , and then Lemma 3.6 says that $|\phi_\lambda(X)| \leq C \prod_{j=1}^{q-1} (n_j + 1)^{-1/2}$ when $p = q$ and

$$|\phi_\lambda(X)| \leq C \prod_{j=1}^{q-1} (n_j + 1)^{-3/2} (n_q + 1)^{-1} \leq C \prod_{j=1}^q (n_j + 1)^{-1}$$

for $p > q$. That completes the proof. \blacksquare

4. L^2 and other smoothness results

With these decay estimates we can determine when convolution powers of orbital measures have square integrable density functions.

Theorem 4.1. *Let $a \in A \backslash N_{G_c}(K)$.*

- (i) *Then $\mu_a^k \in L^2$ provided $k > \max(p, 2(p - q) + 3)$ if $p > q$ or $k > \max(2p, 6)$ if $p = q$.*
- (ii) *If a is regular, then $\mu_a^2 \in L^2$ when $p > q$ and $\mu_a^3 \in L^2$ when $p = q$.*

Proof. We will use Plancherel’s theorem, which in this setting tells that

$$\|\mu_a^k\|_2^2 = \sum_\lambda d_\lambda \text{Tr} |\widehat{\mu}_a(\lambda)|^{2k} = \sum_{\lambda \in \Lambda_{sph}} d_\lambda |\phi_\lambda(a)|^{2k},$$

c.f. [2, Prop. 2]. As we already have estimates on the rate of decay of the spherical functions, the key additional idea needed to prove this result is a bound on the growth in the degree of λ as a function of (n_1, \dots, n_q) .

This uses the Weyl degree formula which states

$$\text{deg } \lambda = C \prod_{\alpha} \langle \alpha, \lambda + \rho \rangle$$

where ρ is half the sum of the positive roots. Here

$$\rho = \sum_{j=1}^q \left(\frac{r}{2} + q - j \right) 2e_j,$$

so
$$\lambda + \rho = \sum_{j=1}^q \left(m_j + q - j + \frac{r}{2} \right) 2e_j = \sum_{j=1}^q \left(n_j + \frac{r}{2} \right) 2e_j.$$

Thus

- (a) $\langle e_j + e_i, \lambda + \rho \rangle \geq n_j + n_i + r \geq n_j + 1$ if $j < i$ (and there are $q - j$ such roots, each with multiplicity 2);
- (b) $\langle e_j - e_i, \lambda + \rho \rangle \geq n_j - n_i \geq 1$ if $j < i$;
- (c) $\langle (2)e_j, \lambda + \rho \rangle \geq n_j + 1$ (with multiplicity 1 for the roots $2e_j$ and multiplicity $2(p - q)$ for the roots e_j).

Consequently,

$$\text{deg } \lambda \sim \prod_{j=1}^q (n_j + 1)^{2(q-j)+1+2(p-q)} = \prod_{j=1}^q (n_j + 1)^{2p-2j+1}. \tag{5}$$

Thus if $p > q$, Theorem 3.1(i) and the degree formula (5) tells us that

$$\begin{aligned} \|\mu_a^k\|_2^2 &\leq C \sum_{\lambda} \prod_{j=1}^q (n_j + 1)^{2p-2j+1} \prod_{j=1}^{q-1} (n_j + 1)^{-2k} \\ &\leq C \sum_{n_1 > n_2 > \dots > n_{q-1} \geq 1} \prod_{j=1}^{q-1} (n_j + 1)^{2p-2j+1-2k} \sum_{n_q < n_{q-1}} (n_q + 1)^{1+2(p-q)}. \end{aligned}$$

Since $\sum_{n_q < n_{q-1}} (n_q + 1)^t \leq C(n_{q-1} + 1)^{t+1}$ when $t \geq 0$, it follows that

$$\|\mu_a^k\|_2^2 \leq \sum_{n_1 > n_2 > \dots > n_{q-1} \geq 1} \prod_{j=1}^{q-2} (n_j + 1)^{2p-2j+1-2k} (n_{q-1} + 1)^{4p-4q-2k+5}.$$

This sum is clearly finite if $2p - 2j + 1 - 2k < -1$ for all $j = 1, \dots, q - 2$ (equivalently, $p < k$) and $4p - 4q - 2k + 5 < -1$ (i.e., $k > 2(p - q) + 3$), as we desired to show. Similar reasoning using the other formula from Theorem 3.1(i) for the case $p = q$ or the formula from part (ii) in the regular case, gives the other statements. ■

Remark 4.2. We remark that the conclusion $\mu_a^2 \in L^2$ for a regular is sharp since μ_a is singular. Previously, it was shown in [1] that when a is regular $\mu_a^k \in L^2$ if

$$k > \frac{1 + \binom{p+q}{2}}{2p - q}. \tag{6}$$

In [1], conditions are given under which the density function of μ_a^k belongs to $C^d(G_c)$. This is done by noting that if $H^s(G_c)$ is the Sobolev space of functions in L^2 with derivatives up to order s in L^2 , then the Sobolev embedding theorem [17, Theorem 1.2.1], implies that if $s > d + \dim G_c/2$, then $H^s(G_c) \subseteq C^d(G_c)$. Moreover, the H^s norm can be computed as

$$\|\mu_a^k\|_{H^s}^2 = \sum_{\lambda \in \Lambda_{sph}} d_\lambda (1 + \kappa_\lambda)^s |\phi_\lambda(a)|^{2k}$$

where κ_λ is the Casimir constant, $\kappa_\lambda = \langle \lambda + 2\rho, \lambda \rangle \sim \|\lambda\|^2 \sim n_1^2$.

They deduce that $\mu_a^k \in H^s$ for $k > \frac{1 + \binom{p+q}{2} + 2s}{2p-q}$ when a is regular.

With our better estimates on the decay of the spherical functions, we can obtain the following stronger results.

Proposition 4.3.

- (i) For any $a \notin N_{G_c}(K)$, $\mu_a^k \in H^s$ if $k > \max(s + p, 2(p - q) + 3)$ if $p > q$ and $k > \max(2s + 2p, 6)$ if $p = q$.
- (ii) If a is regular, then $\mu_a^k \in H^s$ if $k > (p + s)/(p - 1/2)$.

The proof involves the same types of calculations as in the proof of the theorem above.

Remark 4.4. It would be interesting to know if these results are sharp and also to determine for each a , the minimal exponent k such that $\mu_a^k \in L^2$ (or C^d).

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