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The Liouville Theorem of a Torsion System and its Application to the Symmetry Group of a Porous Medium Type Equation on Symmetric Spaces

Xiao-Peng Chen, Shi-Zhong Du, Tian-Pei Guo

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Abstract. We first prove a Liouville theorem to the torsion system

$$\begin{cases} \xi^i_i = \lambda(x) \pm \dfrac{2x^k\xi^k}{|x|^2+1}, & \forall i = 1, 2, \cdots, n \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases}$$

for $(\xi, \lambda) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R})$. As an application, complete resolutions of symmetry groups to the porous medium equation

$$u_t - \triangle_g(u^m) = u^p, \quad \forall (x, t) \in M \times \mathbb{R}$$

of Fujita type are obtained, where M is the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ or hyperbolic space $\mathbb{H}^n$ with canonical metric g.

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1. Introduction

A classical method to find symmetry reductions of PDEs is the Lie group method [21]. The classical symmetries of the general nonlinear heat equation was considered by Clarkson and Mansfield [3]. They gave a catalogue of symmetry reductions. The classical Lie group method has been generalized to the porous medium equation [25]. Gandarias [8] applied the Lie-group formalism to deduce symmetries of the porous medium equation. Franco [6] globalize the symmetry group of the n-dimensional nonlinear porous medium equation. Some other results concerning the symmetry groups of (1) may refer to [7, 10, 9, 22] for one spatial dimensional case, and refer to [1, 2, 4, 11] for higher dimensional case.

In this paper, we will firstly study a system of differential equation

$$\begin{cases} \xi^i_i = \lambda(x) \pm \frac{2x^k\xi^k}{|x|^2+1}, & \forall i = 1, 2, \cdots, n \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases} \tag{1}$$

for $$(\xi, \lambda) = ((\xi^1, \cdots, \xi^n, \lambda) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}),$$

and then prove the following Liouville property:

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Theorem 1.1. *The unique solution of* (1) *is given by* $\lambda(x) \equiv 0$ *and*

$$\xi^i = \sum_{j \neq i} a^i_j x^j + b^i, \quad a^i_j = -a^j_i, \quad \forall i \neq j, \tag{2}$$

where $a^i_j, i, j = 1, 2, \cdots, n$ *and* $b^i, i = 1, 2, \cdots, n$ *are both constants.*

The system (1) comes frequently from the studying of Lie's theory to PDEs on symmetric spaces. As an application, we derive classification results of symmetry groups to a porous medium equation

$$u_t - \triangle_g u^m = u^p, \quad \forall (x, t) \in M \times \mathbb{R} \tag{3}$$

of Fujita type, where n is assumed to be greater than one and (M, g) is a complete Riemannian manifold equipped with metric g.

The classical frame of porous medium type equation was set up after landmark works of [5, 18, 20, 23, 24, 25]. Although the applications of Lie's theory and the analyses of qualitative behaviors of solutions on Riemannian manifolds [13, 14, 15, 16, 17, 25] are known for porous medium type equation, the interaction of both roles is not clear yet. It is the main purpose of this paper to clarify the symmetry group for equation on Riemmanian manifold.

If one sets $v(x, t) = u^m(x, t/m)$ and $r = (1 - m)/m, q = p/m$, (3) changes to a semilinear equations in form of

$$u_t = u^{-r}(\triangle_g u + u^q). \tag{4}$$

For the sake of simplicity, we turn to classify the symmetry groups of (4) with constants $q \neq 0, r \neq -1$ instead of (3), and assume that $n \geq 3$.
In case of $M = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, we will prove the following result.

Theorem 1.2. *When* $M = \mathbb{S}^n \subset \mathbb{R}^{n+1}$*, after rotation on circle and the stereo polar projection from north polar, the symmetry groups of* (2) *are generated*

(1) *by* (43)*, in case* $q, r + 1, 1$ *don't equal mutually,*

(2) *by* (50)–(51)*, in case of* $q = r + 1 \neq 1$*,*

(3) *by* (57)*, in case of* $q = 1 \neq r + 1$*,*

(4) *by* (62)*, in case of* $r + 1 = 1 \neq q$*,*

(5) *by* (72)*, in case of* $q = r + 1 = 1$*.*

And in case of $M = \mathbb{H}^n$, the following characterization result was shown.

Theorem 1.3. *When* $M = \mathbb{H}^n$*, the symmetry groups of* (2) *are generated*

(1) *by* (80)*, in case* $q, r + 1, 1$ *don't equal mutually,*

(2) *by* (87)–(88)*, in case of* $q = r + 1 \neq 1$*,*

(3) *by* (94)*, in case of* $q = 1 \neq r + 1$*,*

(4) *by* (99)*, in case of* $r + 1 = 1 \neq q$*,*

(5) *by* (109)*, in case of* $q = r + 1 = 1$*.*

Since there is no asymptotic assumption was imposed, our classification results Theorem 1.2–1.3 are complete. The contents of this paper are organized as follows. In Section 2, we recall some basic facts about the Lie's theory for PDEs. Next, the Liouville property will be proven for torsion system (1) in Section 3. Finally, as an application of Theorem 1.1, we give the proofs of Theorem 1.2 in Section 4 and Theorem 1.3 in Section 5.

2. Preliminary facts to Lie's theorem on manifolds

Let's first recall some facts of Lie's theorem (see for example the elegant book by Olver [21]) to partial differential equation. Here, we consider a parabolic partial differential equation

$$F(x,t,u,Du,D^2u,u_t)=0 \tag{5}$$

of second order, and suppose that

$$\overrightarrow{v}=\xi^i(x,t,u)\frac{\partial}{\partial x^i}+\eta(x,t,u)\frac{\partial}{\partial t}+\phi(x,t,u)\frac{\partial}{\partial u}$$

is an infinitesimal generator of one-parameter group action $g(\varepsilon),\varepsilon\in\mathbb{R}$. By the prolongation formula in [21] (Theorem 2.36, Page 110)

$$\begin{aligned}pr^{(2)}\overrightarrow{v}=&\ \xi^i\frac{\partial}{\partial x^i}+\eta\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}+\phi^i\frac{\partial}{\partial u_i}+\phi^t\frac{\partial}{\partial u_t}\\&+\ \phi^{ij}\frac{\partial}{\partial u_{ij}}+\phi^{it}\frac{\partial}{\partial u_{it}}+\phi^{tt}\frac{\partial}{\partial u_{tt}},\end{aligned} \tag{6}$$

where

$$\begin{aligned}\phi^i&=D_i(\phi-\xi^j u_j-\eta u_t)+\xi^j u_{ij}+\eta u_{it}\\&=\phi_i+\phi_u u_i-(\xi^j_i+\xi^j_u u_i)u_j-(\eta_i+\eta_u u_i)u_t,\\\phi^t&=D_t(\phi-\xi^j u_j-\eta u_t)+\xi^j u_{jt}+\eta u_{tt}\\&=\phi_t+\phi_u u_t-(\xi^j_t+\xi^j_u u_t)u_j-(\eta_t+\eta_u u_t)u_t,\quad\text{and}\end{aligned}$$

$$\begin{aligned}\phi^{ij}&=D_{ij}(\phi-\xi^k u_k-\eta u_t)+\xi^k u_{ijk}+\eta u_{ijt}\\&=D_j\Big\{\phi_i+\phi_u u_i-(\xi^k_i+\xi^k_u u_i)u_k-\xi^k u_{ik}-(\eta_i+\eta_u u_i)u_t-\eta u_{it}\Big\}+\xi^k u_{ijk}+\eta u_{ijt}\\&=\phi_{ij}+\phi_{iu}u_j+(\phi_{uj}+\phi_{uu}u_j)u_i+\phi_u u_{ij}-\Big\{\xi^k_{ij}+\xi^k_{iu}u_j+(\xi^k_{uj}+\xi^k_{uu}u_j)u_i\Big\}u_k\\&\quad-(\xi^k_i+\xi^k_u u_i)u_{jk}-(\xi^k_j+\xi^k_u u_j)u_{ik}-\Big\{\eta_{ij}+\eta_{iu}u_j+(\eta_{uj}+\eta_{uu}u_j)u_i\Big\}u_t\\&\quad-(\eta_i+\eta_u u_i)u_{jt}-(\eta_j+\eta_u u_j)u_{it}.\end{aligned}$$

Moreover, $g(\cdot)$ is a one-parameter symmetry group of (5) if and only if

$$pr^{(2)}\overrightarrow{v}F(x,t,u,Du,D^2u,u_t)=0 \tag{7}$$

holds for any $u^{(2)}\equiv(u,Du,D^2u,u_t)$ satisfying

$$F(x,t,u^{(2)})=0, \tag{8}$$

where u,Du,D^2u,u_t are regarded as independent variables as usually.
When we consider the solution u of (4) on Riemannian manifold (M,g) which can be parametrized by global coordinates $x\in\mathbb{R}^n$, we have

$$F(x,t,u,Du,D^2u,u_t)\equiv u_t-u^{-r}\Big\{g^{ij}\Big(D_{ij}u-\Gamma^k_{ij}D_k u\Big)+u^q\Big\}, \tag{9}$$

where $\Gamma^k_{ij},i,j,k=1,2,\cdots,n$ are the Christoffel symbols of M.

3. Proof of the Liouville theorem for the torsion system (1)

In this section, we will prove that the Theorem 1.1 holds true for torsion system (1). Without some specification, we may only give the proof to

$$\xi_i^i = \lambda(x) + \frac{2x^k\xi^k}{|x|^2+1},$$

the minus case is similar.
At first, multiplying the second identity of (1) by x^ix^j, one concludes that

$$x^jD_j(x^i\xi^i) + x^iD_i(x^j\xi^j) = 0, \quad \forall i \neq j. \tag{10}$$

One also has for all $i = 1, \ldots, n$

$$x^iD_i(x^i\xi^i) = |x^i|^2D_i\xi^i - x^i\xi^i = \lambda|x^i|^2 + \frac{2x^k\xi^k}{|x|^2+1}|x^i|^2 - x^i\xi^i, \tag{11}$$

by the first formula in (1). Adding (10) by two times of (11) and then summing over all indices $i, j = 1, 2, \cdots$, one obtains that

$$x \cdot D(x\cdot\xi) = \lambda|x|^2 + \frac{2(x\cdot\xi)}{|x|^2+1}|x|^2 - (x\cdot\xi). \tag{12}$$

Using polar coordinates (r, θ), $r \equiv |x| \geq 0$, $\theta \equiv \frac{x}{|x|} \in \mathbb{S}^{n-1}$, and expressing $x \cdot \xi = \varphi(r, \theta)$, we have

$$\begin{cases} r\varphi_r = \lambda(r,\theta)r^2 + \frac{r^2-1}{r^2+1}\varphi, & \forall r \geq 0 \\ \varphi(0,\theta) = \varphi_0(\theta), \quad \varphi_r(0,\theta) = 0, & \forall \theta \in \mathbb{S}^{n-1}. \end{cases} \tag{13}$$

Solving the above first order ODE, one concludes that

$$\varphi(r,\theta) = \frac{r^2+1}{r}\Big[\int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds + C(\theta)\Big]. \tag{14}$$

Using the initial condition of φ, there must be $C(\theta) = 0$, $\forall \theta \in \mathbb{S}^{n-1}$. Thus,

$$x\cdot\xi(x) = \varphi(r,\theta) \equiv \frac{|x|^2+1}{|x|}\int_0^{|x|} \frac{s^2}{s^2+1}\lambda(s,\theta)ds, \forall x \in \mathbb{R}^n. \tag{15}$$

Next, summing (10) for all $j \neq i$ and then adding it by (11), we obtain that

$$x\cdot D(x^i\xi^i) + x^iD_i(x\cdot\xi) = 2\lambda|x^i|^2 + \frac{4x^k\xi^k}{|x|^2+1}|x^i|^2 - 2x^i\xi^i,$$

or equivalently

$$\begin{aligned} x\cdot D(x^i\xi^i) + 2x^i\xi^i \; &= \\ &= -x^iD_i\Big\{\frac{r^2+1}{r}\int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds\Big\} + 2\lambda|x^i|^2 + \frac{4|x^i|^2}{r}\int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds \\ &= \Big(\frac{3}{r} + \frac{1}{r^3}\Big)|x^i|^2\int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds + |x^i|^2\lambda(r,\theta). \end{aligned} \tag{16}$$

Setting $\psi(r,\theta) \equiv x^i\xi^i$, $r \equiv |x|$, $\theta \equiv \frac{x}{|x|}$, one gets

$$r\psi_r + 2\psi = \Big(3r + \frac{1}{r}\Big)\cos^2\theta_i \int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds + r^2\cos^2\theta_i\lambda(r,\theta). \tag{17}$$

Solving this ODE, we obtain that

$$x^i\xi^i = \psi(r,\theta) = \Big(r + \frac{1}{r}\Big)\cos^2\theta_i \int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds$$
$$\Longleftrightarrow \ \xi^i = \frac{r^2+1}{r^3}x_i \int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds. \tag{18}$$

As a result,

$$x^jD_j(x^i\xi^i) = \frac{x_i^2x_j^2}{r^3}\Bigg\{ -\Big(1+\frac{3}{r^2}\Big)\int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds + r\lambda(r,\theta)\Bigg\}. \tag{19}$$

Substituting into (10), one concludes that

$$\int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds = \frac{r^3}{r^2+3}\lambda(r,\theta). \tag{20}$$

Therefore, λ is the solution to

$$\begin{cases} \partial_r\lambda = -\dfrac{4r}{(r^2+1)(r^2+3)}\lambda, & \forall r>0 \\ \lambda(0,\theta) = \lambda_0, \quad \partial_r\lambda(0,\theta) = 0, \end{cases} \tag{21}$$

which is given by
$$\lambda(r,\theta) = \lambda_0\frac{r^2+3}{r^2+1}. \tag{22}$$

Taking derivative on second identity of (18), and comparing both sides of (1), we derive that

$$\frac{1}{r^3}\Bigg\{1 - r^2 - x_ir - \frac{3x_i}{r}\Bigg\}\int_0^r \frac{s^2}{s^2+1}\lambda(s,\theta)ds = \frac{r-x_i}{r}\lambda(r,\theta) \tag{23}$$

for the function λ given by (22). So, there must be $\lambda_0 = 0$ and thus $\lambda(x) \equiv 0$, $\forall x \in \mathbb{R}^n$. Now, the conclusion of Theorem 1.1 follows from the following Lemma.

Lemma 3.1. *The unique solution* $(\lambda,\xi) \in \mathbb{R}\times C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ *of*

$$\begin{cases} \xi_i^i = \lambda + \frac{2x^k\xi^k}{|x|^2+1}, & \forall i \\ \xi_j^i + \xi_i^j = 0, & \forall i\neq j \end{cases} \tag{24}$$

are given by $\lambda = 0$ *and*
$$\xi^i = \sum_{j\neq i} a_j^ix^j + b^i, \quad a_j^i = -a_i^j, \quad \forall i\neq j. \tag{25}$$

Remark 3.2. Taking derivatives on (24), one gets

$$\begin{cases} \xi_{ii}^j = -2D_j\Big(\frac{x^k\xi^k}{|x|^2+1}\Big), & \forall i\neq j \\ \xi_{jj}^j = 2D_j\Big(\frac{x^k\xi^k}{|x|^2+1}\Big), & \forall j \end{cases} \tag{26}$$

and hence
$$\triangle\xi^j + (n-2)D_j^2\xi^j = 0, \quad \forall j. \tag{27}$$

However, since there is no asymptotic assumption of ξ at infinity, it would be difficulty to classify all solutions ξ using harmonicity formula (27). Fortunately, using the special structure of system of differential equations (24), one can prove the Lemma 3.1.

Proof of Lemma 3.1. Multiplying the second identity of (24) by $x^i x^j$, one gets

$$x^j D_j(x^i\xi^i) + x^i D_i(x^j\xi^j) = 0, \quad \forall i \neq j. \tag{28}$$

On the other hand, we have for all $i = 1, \dots, n$

$$x^i D_i(x^i\xi^i) = |x^i|^2 D_i\xi^i - x^i\xi^i = \lambda|x^i|^2 + \frac{2x^k\xi^k}{|x|^2+1}|x^i|^2 - x^i\xi^i, \tag{29}$$

by the first formula in (24). Adding (26) by two times of (27) and then summing over all indices i, j, one obtains that

$$x \cdot D(x\cdot\xi) = \lambda|x|^2 + \frac{2(x\cdot\xi)}{|x|^2+1}|x|^2 - (x\cdot\xi). \tag{30}$$

Using the polar coordinates (r,θ), $r \equiv |x| \geq 0$, $\theta \equiv \frac{x}{|x|} \in \mathbb{S}^{n-1}$ and expressing $x\cdot\xi = \varphi(r,\theta)$, we have

$$\begin{cases} r\varphi_r = \lambda r^2 + \frac{r^2-1}{r^2+1}\varphi, & \forall r \geq 0 \\ \varphi(0,\theta) = \varphi_0(\theta), \quad \varphi_r(0,\theta) = 0, & \forall\theta \in \mathbb{S}^{n-1}. \end{cases} \tag{31}$$

Solving the above first order ODE, one concludes that

$$\varphi(r,\theta) = \frac{r^2+1}{r}\Big[\lambda\Big(r - \arctan r\Big) + C(\theta)\Big]. \tag{32}$$

Using the initial condition of φ, there must be $C(\theta) = 0$, $\forall\theta \in \mathbb{S}^{n-1}$. Thus,

$$x\cdot\xi(x) = \varphi(r,\theta) \equiv \lambda\frac{|x|^2+1}{|x|}(|x| - \arctan|x|), \forall x \in \mathbb{R}^n. \tag{33}$$

It is clear that for the solution ξ of (24), (26) and (27) also hold true. Therefore, one concludes that

$$\begin{aligned} &\left(\frac{\partial^2}{\partial(x^i)^2} + \frac{\partial^2}{\partial(x^j)^2}\right)D_j\left(\frac{x\cdot\xi}{|x|^2+1}\right) = 0 \\ \Longleftrightarrow &\left(\frac{\partial^2}{\partial(x^i)^2} + \frac{\partial^2}{\partial(x^j)^2}\right)\left\{-\lambda\left(\frac{1}{|x|(|x|^2+1)} - \frac{\arctan|x|}{|x|^2}\right)\frac{x^j}{|x|}\right\} = 0. \end{aligned} \tag{34}$$

Using Taylor's expansion for the function

$$f(r) \equiv -\frac{1}{r}\left\{\frac{1}{r(r^2+1)} - \frac{\arctan r}{r^2}\right\} = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{2k+2}{2k+3}r^{2k},$$

we know that $F(x^i,x^j) \equiv -\lambda\left(\frac{1}{|x|(|x|^2+1)} - \frac{\arctan|x|}{|x|^2}\right)\frac{x^j}{|x|} \in C^\infty(\mathbb{R}^2)$ is a smooth harmonic function on $(x^i,x^j) \in \mathbb{R}^2$. Noting that it is also bounded, a famous Liouville theorem [12] for harmonic function gives the unique possibility by $\lambda = 0$ and thus

$$x \cdot \xi(x) = 0, \quad \forall x \in \mathbb{R}^n. \tag{35}$$

Combining 35 with (26), we obtain

$$\xi^i = \sum_j a^i_j x^j + b^i \quad \text{and} \quad a^i_j = -a^j_i, \quad a^i_i = 0, \quad \forall i \neq j.$$

completing the proof of the Lemma 3.1. Thus, we have also completed the proof of Theorem 1.1. ■

4. Symmetry group on $\mathbb{S}^n$

Let's start with a well known Hairy ball theorem (refer to a analytic proof by Milnor [19]).

Theorem 4.1. *Suppose that n is even, then there is no non-vanishing tangential vector field on $\mathbb{S}^n$.*

As a corollary, we have the following Brouwer type fix point theorem on sphere.

Corollary 4.2. *Suppose that n is even. Then for any continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ satisfying*

$$f(x) \neq -x, \quad \forall x \in \mathbb{S}^n,$$

there is at least one fix point $x_0 \in \mathbb{S}^n$.

Proof. Suppose on the contrary, there is a continuous mapping $f : \mathbb{S}^n \to \mathbb{S}^n$ satisfying

$$f(x) \neq x, -x, \quad \forall x \in \mathbb{S}^n.$$

Setting

$$v(x) \equiv f(x) - (f(x), x)x, \quad \forall x \in \mathbb{S}^n,$$

we have v is a tangential vector field on $\mathbb{S}^n$. Moreover, v does not vanish everywhere on $\mathbb{S}^n$ due to $|(f(x), x)| < 1$ for all $x \in \mathbb{S}^n$, which contradicts with the hairy ball Theorem 4.1. The proof is complete. ■

Remark 4.3. The assumption $f(x) \neq -x$, $\forall x \in \mathbb{S}^n$ can not be removed, since the inversion transformation is a continuous mapping of $\mathbb{S}^n$ which has no fix point.

By Corollary 4.2, we may assume that the symmetry group action has at least one fix point for n even, as well as for n odd after a rotation $SO(1)$. Without loss of generality, we may also assume that the fix point is $(0, 1) \in \mathbb{R}^n \times \mathbb{R}$. Letting $X = (y, z) \in \mathbb{R}^n \times \mathbb{R}$ be the coordinate representation of $\mathbb{S}^n$, we have the spherical polar projection $P : \mathbb{S}^n \to \mathbb{R}^n$ given by

$$P(y, z) \equiv x = \frac{y}{1 - z}, \quad \forall (y, z) \in \mathbb{S}^n \setminus (0, 1).$$

Regarding $x \in \mathbb{R}^n$ as local coordinates of $\mathbb{S}^n \setminus (0,1)$, one can express X in terms of x by

$$X = \left(\frac{2x}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1} \right).$$

Therefore, upon this local coordinates, the induced metric g of $\mathbb{S}^n$ is given by

$$g_{ij} \equiv \left(\frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right) = \frac{4\delta_{ij}}{(|x|^2+1)^2}$$

and the Christoffel symbol

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) = \frac{2}{|x|^2+1} \Big(- x_i \delta_{jk} - x_j \delta_{ik} + x_k \delta_{ij} \Big).$$

Thus, the Laplace-Beltrami operator becomes

$$\begin{aligned} \triangle_g &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right) = g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial}{\partial x^k} \right) \\ &= \frac{(|x|^2+1)^2}{4} \triangle - (n-2) \frac{|x|^2+1}{2} x \cdot \nabla, \end{aligned}$$

where
$$\triangle \equiv \sum_i \frac{\partial^2}{\partial (x^i)^2}, \quad \nabla \equiv \left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \right)$$

are the canonical Laplace operator and gradient operator of $\mathbb{R}^n$ under flat metric, respectively. Suppose that

$$\overrightarrow{v} = \xi^i(x,t,u) \frac{\partial}{\partial x^i} + \eta(x,t,u) \frac{\partial}{\partial t} + \phi(x,t,u) \frac{\partial}{\partial u}$$

to be an infinitesimal generator of one-parameter symmetry group $g(\varepsilon), \varepsilon \in \mathbb{R}$, we have the prolongation formula is given by (6). Using spherical polar projection, (9) changes to

$$u^r u_t = \frac{(|x|^2+1)^2}{4} \triangle u - (n-2) \frac{|x|^2+1}{2} x \cdot \nabla u + u^q. \tag{36}$$

Therefore, the group action $g(\cdot)$ is a one-parameter symmetry group of (36) if and only if

$$\begin{aligned} r u^{r-1} u_t \phi + u^r \phi^t &= (|x|^2+1) x^i \xi^i \triangle u + \frac{(|x|^2+1)^2}{4} \phi^{ii} \\ -(n-2) &\left\{ x^i \xi^i x \cdot \nabla u + \frac{|x|^2+1}{2} u_i \xi^i + \frac{|x|^2+1}{2} x^i \phi^i \right\} + q u^{q-1} \phi. \end{aligned} \tag{37}$$

Comparing the like terms on both sides of (37), one gets

$$\begin{gathered} \eta = \eta(t), \quad \xi_u^k = 0 \ \forall k, \quad \phi_{uu} = 0 \\ r u^{-1} \phi - \eta_t - \frac{4x^k}{|x|^2+1} \xi^k + 2\xi_i^i = 0 \ \ \forall i, \quad \xi_j^i + \xi_i^j = 0 \ \ \forall i \neq j \end{gathered}$$

$$\frac{2}{|x|^2+1}u^r\xi_t^k+\frac{|x|^2+1}{2}(2\phi_{ku}-\triangle\xi^k)-ru^{-1}\phi x^k$$
$$+\Big(\eta_t+(1-n)\phi_u\Big)x^k+(2-n)\Big\{\frac{2x^i}{|x|^2+1}\xi^i x^k+\xi^k-\xi_i^k x^i\Big\}=0\quad\forall k$$

$$(r-q)u^{q-1}\phi+u^r\phi_t+(\phi_u-\eta_t)u^q-\frac{(|x|^2+1)^2}{4}\triangle\phi-(2-n)\frac{|x|^2+1}{2}x^i\phi_i=0,$$

or equivalently $\qquad \xi^k=\xi^k(x,t),\quad \eta=\eta(t),\quad \phi=\alpha(x,t)u$

$$2\xi_i^i=-r\alpha(x,t)+\eta_t+\frac{4x^k\xi^k}{|x|^2+1}\quad\forall i,\quad \xi_j^i+\xi_i^j=0\quad\forall i\neq j$$

$$\Big[(r+1-q)\alpha(x,t)-\eta_t\Big]u^q+\alpha_t(x,t)u^{r+1}-u\triangle_g\alpha=0 \tag{38}$$

$$u^r\partial_t\xi^k-\triangle_g\xi^k+\frac{(|x|^2+1)^2}{2}D_k\alpha+\frac{|x|^2+1}{2}\Big\{\eta_t+(1-n-r)\alpha\Big\}x^k$$
$$+(2-n)\Big\{(x^i\xi^i)x^k+\frac{|x|^2+1}{2}\xi^k\Big\}=0\quad\forall k$$

So, our situation can be divided into five cases:

Case 1: ($q, r+1, 1$ pairwise not equal) By the 4th and 5th identity in (38), we get

$$\alpha=\frac{\beta}{r+1-q},\quad \eta(t)=\beta t+\gamma,\quad \xi^k=\xi^k(x) \tag{39}$$

and

$$\begin{cases}\xi_i^i=-\frac{\beta(1-q)}{2(r+1-q)}+\frac{2x^k}{|x|^2+1}\xi^k, & \forall i\\[2ex] \xi_j^i+\xi_i^j=0, & \forall i\neq j\end{cases} \tag{40}$$

and, where $\xi\equiv(\xi^1,\cdots,\xi^n)$,

$$\triangle_g\xi-(2-n)\frac{\xi}{1-z}=\Big\{\frac{\beta(2-n-q)}{r+1-q}+(2-n)y\cdot\xi\Big\}\frac{y}{(1-z)^2}. \tag{41}$$

Applying Theorem 1.1 to (39) and (40), one gets

$$\xi^i=\sum_{j\neq i}a_j^i x^j+b^i,\quad \beta=0.$$

Substituting into (41) and noting that $(n-2)\sum_{j=1}^{n}\Big\{b^i+(b^jy^j)x^i\Big\}=0$ for all i, we conclude that

$$\alpha=0,\quad \eta(t)=\gamma,\quad \xi^i=\sum_{j\neq i}a_j^i x^j,\quad a_j^i=-a_i^j\quad\forall i\neq j. \tag{42}$$

Therefore the infinitesimal generators are spanned by

$$\vec{v}_1=\partial_t\,,\qquad \vec{v}_2=\sum_{j\neq i}a_j^i x^j\partial_{x^i}\,,$$

and their corresponding group actions are generated by

$$\begin{aligned} g_1(\varepsilon) &: (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (x, t+\varepsilon, u) \\ g_2(\varepsilon) &: (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (A_\varepsilon x, t, u), \quad A_\varepsilon \in SO(n) \end{aligned} \tag{43}$$

corresponding to translation in time and rotation on space.

Case 2: $(q = r+1 \neq 1)$ By the 4th identity in (38), one obtains

$$\alpha(x,t) = \eta(t) + \beta(x), \quad \eta(t) = \kappa e^{rt} - \gamma, \quad \xi^k = \xi^k(x), \quad \triangle_g \beta = 0 \tag{44}$$

and

$$\begin{cases} \xi_i^i = \frac{r}{2}\Big[\gamma - \beta(x)\Big] + \frac{2x^k}{|x|^2+1}\xi^k, & \forall i \\ \xi_j^i + \xi_i^j = 0, & \forall i \neq j \end{cases} \tag{45}$$

where $D\beta \equiv (\beta_1, \cdots, \beta_n)$,

$$\begin{aligned} &\triangle_g \xi - (2-n)\frac{\xi}{1-z} \\ = &\Big\{(1-n)\kappa e^{rt} + (1-n-r)\Big[\beta(x) - \gamma\Big] + (2-n) y \cdot \xi\Big\}\frac{y}{(1-z)^2} + \frac{2D\beta}{(1-z)^2}. \end{aligned} \tag{46}$$

Applying Theorem 1.1 to (45), one concludes that $\beta = \gamma$ and

$$\xi^k = a_j^k x^j + b^k, \quad \forall k. \tag{47}$$

So, we get $\alpha = \kappa e^{rt}, \quad \eta = \kappa e^{rt} - \gamma, \quad \xi^k = \sum_{j\neq k} a_j^k x^j. \quad a_j^i = -a_i^j, \quad \forall i \neq j.$ (48)

Thus, the infinitesimal generators are spanned by

$$\overrightarrow{v}_1 = e^{rt}\partial_t + e^{rt}u\partial_u, \quad \overrightarrow{v}_2 = \partial_t, \quad \overrightarrow{v}_3 = \sum_{j \neq i} a_j^i x^j \partial_{x^i}, \tag{49}$$

and their corresponding group actions are generated by

$$g_1(\varepsilon): \begin{pmatrix} x \\ t \\ u \end{pmatrix} \to \begin{pmatrix} \widetilde{x} \\ \widetilde{t} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} x \\ -\frac{1}{r}\ln(e^{-rt} - r\varepsilon) \\ u(e^{-rt} - r\varepsilon)^{-\frac{1}{r}} e^{-t} \end{pmatrix} \tag{50}$$

and

$$\begin{aligned} &g_2(\varepsilon): (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (x, t+\varepsilon, u), \\ &g_3(\varepsilon): (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (A_\varepsilon x, t, u), \quad A_\varepsilon \in SO(n). \end{aligned} \tag{51}$$

Case 3: $(q = 1 \neq r+1)$ We have $\xi^k = \xi^k(x)$,

$$\alpha = \alpha(x), \quad \eta = \beta t + \gamma, \quad \xi^k = \xi^k(x), \quad \triangle_g \alpha - r\alpha + \beta = 0, \tag{52}$$

and

$$\begin{cases} 2\xi_i^i = -r\alpha(x) + \beta + \frac{4x^k\xi^k}{|x|^2+1}, & \forall i \\ \xi_j^i + \xi_i^j = 0, & \forall i \neq j \end{cases} \tag{53}$$

and

$$\triangle_g\xi-(2-n)\frac{\xi}{1-z}=\Big\{\beta+(2-n-r)\alpha+y\cdot\xi\Big\}\frac{y}{(1-z)^2}+\frac{2D\alpha}{(1-z)^2}. \tag{54}$$

By Theorem 1.1 and (53), one obtains that

$$\alpha=\frac{\beta}{r},\quad \eta=\beta t+\gamma,\quad \xi^k=\sum_{j\neq k}a^k_jx^j,\quad a^i_j=-a^j_i,\quad \forall i\neq j. \tag{55}$$

Thus, the infinitesimal generators are spanned by

$$\overrightarrow{v}_1=t\partial_t+u/r\partial_u,\quad \overrightarrow{v}_2=\partial_t,\quad \overrightarrow{v}_3=\sum_{j\neq i}a^i_jx^j\partial_{x^i}, \tag{56}$$

and their corresponding group actions are generated by

$$\begin{aligned} g_1(\varepsilon)\ &:\ (x,t,u)\to(\widetilde{x},\widetilde{t},\widetilde{u})=(x,e^{\varepsilon}t,e^{\varepsilon/r}u)\\ g_2(\varepsilon)\ &:\ (x,t,u)\to(\widetilde{x},\widetilde{t},\widetilde{u})=(x,t+\varepsilon,u)\\ g_3(\varepsilon)\ &:\ (x,t,u)\to(\widetilde{x},\widetilde{t},\widetilde{u})=(A_\varepsilon x,t,u),\quad A_\varepsilon\in SO(n). \end{aligned} \tag{57}$$

Case 4: $(r+1=1\neq q)$ We have $\xi^k=\xi^k(x,t)$ and

$$\begin{cases}(1-q)\alpha(x,t)=\eta_t\\ \alpha_t-\frac{(|x|^2+1)^2}{4}\triangle\alpha-\frac{|x|^2+1}{2}x\cdot\nabla\alpha=0\end{cases}\Longrightarrow\ \eta=\beta t+\gamma,\quad \alpha=\frac{\beta}{1-q} \tag{58}$$

and

$$\begin{cases}2\xi^i_i=\beta+\frac{4x^k}{|x|^2+1}\xi^k,\quad \forall i\\ \xi^i_j+\xi^j_i=0,\qquad \forall i\neq j\end{cases} \tag{59}$$

and

$$\partial_t\xi-\triangle_g\xi+(2-n)\frac{\xi}{1-z}=-\left\{\frac{\beta(2-n-q)}{1-q}+(2-n)y\cdot\xi\right\}\frac{y}{(1-z)^2}. \tag{60}$$

Applying Theorem 1.1 to (59), one concludes that

$$\alpha=0,\quad \eta=\gamma,\quad \xi^k=\sum_{j\neq k}a^k_j(t)x^j,\quad a^k_j(t)=-a^j_k(t),\quad \forall j\neq k. \tag{61}$$

Substituting into (60), there holds

$$\sum_{j\neq k}\partial_ta^k_j(t)x^j=0,\quad \forall k$$

or equivalently $\xi^k=\xi^k(x),\forall k$. So, the infinitesimal generators are spanned by

$$\overrightarrow{v}_1=\partial_t,\qquad \overrightarrow{v}_2=\sum_{j\neq i}a^i_jx^j\partial_{x^i},$$

and their corresponding group actions are generated by

$$\begin{aligned} g_1(\varepsilon) &: (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (x,t+\varepsilon,u) \\ g_2(\varepsilon) &: (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (Ax,t,u), \quad A \in SO(n) \end{aligned} \tag{62}$$

corresponding to translation in time and rotation on space respectively.

Case 5: $(q = r+1 = 1)$ We have

$$\alpha_t - \triangle_g \alpha = \eta_t \tag{63}$$

and

$$\begin{cases} 2\xi^i_i = \eta_t + \frac{4x^k}{|x|^2+1}\xi^k, & \forall i \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases} \tag{64}$$

and

$$\partial_t \xi - \triangle_g \xi + (2-n)\frac{\xi}{1-z} = -\Big\{\eta_t + (1-n)\alpha + (2-n) y\cdot\xi\Big\}\frac{y}{(1-z)^2} - \frac{2D\alpha}{(1-z)^2}. \tag{65}$$

Using again Theorem 1.1, we infer from (63)–(64) that

$$\alpha_t - \triangle_g \alpha = 0, \quad \eta = \gamma, \quad \xi^k = \sum_{j\neq k} a^k_j(t) x^j, \quad a^k_j(t) = -a^j_k(t), \quad \forall j \neq k. \tag{66}$$

Combining (65) with (66), one concludes that

$$D_k \alpha + \frac{(1-n)x^k}{|x|^2+1}\alpha = -\sum_{j\neq k} \frac{(|x|^2+1)^2}{2}\partial_t a^k_j(t) x^j. \tag{67}$$

Taking the second derivative with respect to x^l we obtain

$$\begin{aligned} D_{kl}\alpha &= -(1-n)\left\{\frac{\delta_{kl}}{|x|^2+1} - \frac{2x^k x^l}{(|x|^2+1)^2}\right\}\alpha - (1-n)\frac{x^k}{|x|^2+1}D_l\alpha \\ &-2(|x|^2+1)\sum_{j\neq k, j\neq l} x^l x^j \partial_t a^k_j(t) - \left\{\frac{(|x|^2+1)^2}{2} + 2(x^l)^2(|x|^2+1)\right\}\partial_t a^k_l(t). \end{aligned} \tag{68}$$

Substituting (67) into (68), one derives that

$$\begin{aligned} &D_{kl}\alpha + (1-n)\left\{\frac{\delta_{kl}}{|x|^2+1} + (n-3)\frac{x^k x^l}{(|x|^2+1)^2}\right\}\alpha \\ &= \frac{1-n}{2}(|x|^2+1)\sum_{j\neq k, j\neq l} x^k x^j \partial_t a^l_j(t) - 2(|x|^2+1)\sum_{j\neq k, j\neq l} x^l x^j \partial_t a^k_j(t) \\ &-\frac{1-n}{2}(|x|^2+1)(x^k)^2\partial_t a^k_k(t) - (|x|^2+1)\left\{\frac{|x|^2+1}{2} + 2(x^l)^2\right\}\partial_t a^k_l(t). \end{aligned}$$

Exchanging the indices k, l and using the symmetry of second derivatives, we get

$$\begin{aligned} &\frac{5-n}{2}\sum_{j\neq k, j\neq l} x^j\left\{x^k \partial_t a^l_j(t) - x^l \partial_t a^k_j(t)\right\} \\ &-\left\{(|x|^2+1) + \frac{5-n}{2}\Big[(x^k)^2 + (x^l)^2\Big]\right\}\partial_t a^k_l(t) = 0, \quad \forall k \neq l. \end{aligned}$$

Consequently, we get
$$a_j^k(t) = a_j^k, \quad \forall j \neq k \tag{69}$$
and so
$$\alpha = \beta, \quad \eta = \gamma, \quad \xi^k = \sum_{j\neq k} a_j^k x^j, \quad a_j^k = -a_k^j, \quad \forall j \neq k. \tag{70}$$
Thus, the infinitesimal generators are spanned by
$$\overrightarrow{v}_1 = u\partial_u, \quad \overrightarrow{v}_2 = \partial_t, \quad \overrightarrow{v}_3 = \sum_{j\neq i} a_j^i x^j \partial_{x^i}, \tag{71}$$
and their corresponding group actions are generated by
$$\begin{aligned} g_1(\varepsilon) &: (x,t,u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (x, t, e^{\varepsilon} u) \\ g_2(\varepsilon) &: (x,t,u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (x, t+\varepsilon, u) \\ g_3(\varepsilon) &: (x,t,u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (A_\varepsilon x, t, u), \quad A_\varepsilon \in SO(n). \end{aligned} \tag{72}$$

5. Symmetry group on $\mathbb{H}^n$

As one knows, Hyperbolic space $\mathbb{H}^n, n \geq 2$ is a complete, simply connected Riemannian manifolds having constant sectional curvature -1. The most important models of hyperbolic spaces include the Poincaré balls, half-spaces and hyperboloids (or Lorentz model). In this section, we only discuss the Poincaré balls $B_1 \subset \mathbb{R}^n, n \geq 2$, which equipped with a complete metric
$$g_{ij} = \left(\frac{2}{1-|x|^2}\right)^2 \delta_{ij}, \quad \forall i,j = 1,2,\cdots,n, \quad \forall x \in B_1,$$
whose Christoffel symbol is given by
$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) = \frac{2}{1-|x|^2}\Big(x^i\delta_{jk} + x^j\delta_{ik} - x^k\delta_{ij}\Big).$$
Thus, the Laplace-Beltrami operator
$$\begin{aligned} \triangle_g &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij}\frac{\partial}{\partial x^j}\right) = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial}{\partial x^k}\right) \\ &= \frac{(1-|x|^2)^2}{4}\triangle + (n-2)\frac{1-|x|^2}{2} x\cdot\nabla, \quad \forall x \in B_1 \end{aligned}$$
and (9) changes to
$$u^r u_t = \frac{(1-|x|^2)^2}{4}\triangle u + (n-2)\frac{1-|x|^2}{2} x\cdot\nabla u + u^q. \tag{73}$$
As above, suppose that
$$\overrightarrow{v} = \xi^i(x,t,u)\frac{\partial}{\partial x^i} + \eta(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$$

is an infinitesimal generator of the one-parameter symmetry group $g(\varepsilon)$, $\varepsilon \in \mathbb{R}$, where the prolongation $pr^{(2)}\overrightarrow{v}$ is given by (6). Therefore, the group action $g(\cdot)$ is a one-parameter symmetry group of (73) if and only if

$$ru^{r-1}u_t\phi + u^r\phi^t = -(1-|x|^2)x^i\xi^i\triangle u + \frac{(1-|x|^2)^2}{4}\phi^{ii}$$
$$+(n-2)\Big\{-x^i\xi^i x\cdot\nabla u + \frac{1-|x|^2}{2}u_i\xi^i + \frac{1-|x|^2}{2}x^i\phi^i\Big\} + qu^{q-1}\phi. \quad (74)$$

Comparing the like terms on both sides of (74), we get

$$\eta = \eta(t), \quad \xi^k_u = 0 \ \ \forall k, \quad \phi_{uu} = 0,$$
$$ru^{-1}\phi - \eta_t + \frac{4x^k}{1-|x|^2}\xi^k + 2\xi^i_i = 0 \ \ \forall i, \quad \xi^i_j + \xi^j_i = 0 \ \ \forall i \neq j,$$
$$-\frac{2}{1-|x|^2}u^r\xi^k_t - \frac{1-|x|^2}{2}(2\phi_{ku} - \triangle\xi^k) - ru^{-1}\phi x^k$$
$$+\Big(\eta_t + (1-n)\phi_u\Big)x^k + (2-n)\Big\{-\frac{2x^i}{1-|x|^2}\xi^i x^k + \xi^k - \xi^k_i x^i\Big\} = 0 \ \ \forall k,$$
$$(r-q)u^{q-1}\phi + u^r\phi_t + (\phi_u - \eta_t)u^q - \frac{(1-|x|^2)^2}{4}\triangle\phi + (2-n)\frac{1-|x|^2}{2}x^i\phi_i = 0,$$

or equivalently $\quad xi^k = \xi^k(x,t), \quad \eta = \eta(t), \quad \phi = \alpha(x,t)u,$

$$2\xi^i_i = -r\alpha(x,t) + \eta_t - \frac{4x^k\xi^k}{1-|x|^2} \ \ \forall i, \quad \xi^i_j + \xi^j_i = 0 \ \ \forall i \neq j,$$
$$\Big[(r+1-q)\alpha(x,t) - \eta_t\Big]u^q + \alpha_t(x,t)u^{r+1} - u\triangle_g\alpha = 0, \quad (75)$$
$$u^r\partial_t\xi^k - \triangle_g\xi^k + \frac{(1-|x|^2)^2}{2}D_k\alpha - \frac{1-|x|^2}{2}\Big\{\eta_t + (1-n-r)\alpha\Big\}x^k$$
$$+(2-n)\Big\{(x^i\xi^i)x^k - \frac{1-|x|^2}{2}\xi^k\Big\} = 0 \ \ \forall k.$$

As in Section 4, our results are divided into five cases:

Case 1: ($q, r+1, 1$ pairwise not equal) By the 4th and the 5th identity in (75) one has

$$\alpha = \frac{\beta}{r+1-q}, \quad \eta(t) = \beta t + \gamma, \quad \xi^k = \xi^k(x) \quad (76)$$

and

$$\begin{cases} \xi^i_i = -\dfrac{\beta(1-q)}{2(r+1-q)} - \dfrac{2x^k}{1-|x|^2}\xi^k, & \forall i \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases} \quad (77)$$

and for $\xi \equiv (\xi^1, \cdots, \xi^n)$,

$$\triangle_g\xi - (2-n)\frac{\xi}{1-z} = \Big\{\frac{\beta(2-n-q)}{r+1-q} + (2-n)y\cdot\xi\Big\}\frac{y}{(1-z)^2}. \quad (78)$$

Applying Theorem 1.1 to (77), one concludes that

$$\alpha = 0, \quad \eta(t) = \gamma, \quad \xi^i = \sum_{j\neq i} a^i_j x^j, \quad a^i_j = -a^j_i, \quad \forall i \neq j. \tag{79}$$

Therefore the infinitesimal generators are spanned by

$$\overrightarrow{v}_1 = \partial_t, \qquad \overrightarrow{v}_2 = \sum_{j\neq i} a^i_j x^j \partial_{x^i},$$

and their corresponding group actions are generated by

$$\begin{aligned} g_1(\varepsilon) &: (x,t,u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (x, t+\varepsilon, u) \\ g_2(\varepsilon) &: (x,t,u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (Ax, t, u), \quad A \in SO(n). \end{aligned} \tag{80}$$

corresponding to translation in time and rotation on space.

Case 2: $(q = r+1 \neq 1)$ By 4th identity in (75), we get

$$\alpha(x,t) = \eta(t) + \beta(x), \quad \eta(t) = \kappa e^{rt} - \gamma, \quad \xi^k = \xi^k(x), \quad \triangle_g \beta = 0 \tag{81}$$

and

$$\begin{cases} \xi^i_i = \dfrac{r}{2}\Big[\gamma - \beta(x)\Big] - \dfrac{2x^k}{1-|x|^2}\xi^k, & \forall i \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases} \tag{82}$$

and, where $D\beta \equiv (\beta_1, \cdots, \beta_n)$,

$$\begin{aligned} &\triangle_g \xi - (2-n)\frac{\xi}{1-z} \\ = &\Big\{(1-n)\kappa e^{rt} + (1-n-r)\Big[\beta(x) - \gamma\Big] + (2-n) y\cdot\xi\Big\}\frac{y}{(1-z)^2} + \frac{2D\beta}{(1-z)^2}. \end{aligned} \tag{83}$$

Applying Theorem 1.1 to (82), one concludes that $\beta = \gamma$ and

$$\xi^k = a^k_j x^j + b^k, \quad \forall k. \tag{84}$$

So, we get

$$\alpha = \kappa e^{rt}, \quad \eta = \kappa e^{rt} - \gamma, \quad \xi^k = \sum_{j\neq k} a^k_j x^j. \quad a^i_j = -a^j_i, \quad \forall i \neq j. \tag{85}$$

Thus, the infinitesimal generators are spanned by

$$\overrightarrow{v}_1 = e^{rt}\partial_t + e^{rt} u \partial_u, \quad \overrightarrow{v}_2 = \partial_t, \quad \overrightarrow{v}_3 = \sum_{j\neq i} a^i_j x^j \partial_{x^i}, \tag{86}$$

and their corresponding group actions are generated by

$$g_1(\varepsilon): \begin{pmatrix} x \\ t \\ u \end{pmatrix} \to \begin{pmatrix} \widetilde{x} \\ \widetilde{t} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} x \\ -\frac{1}{r}\ln(e^{-rt} - r\varepsilon) \\ u(e^{-rt} - r\varepsilon)^{-\frac{1}{r}} e^{-t} \end{pmatrix} \tag{87}$$

and

$$\begin{aligned} &g_2(\varepsilon): (x,t,u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (x, t+\varepsilon, u) \\ &g_3(\varepsilon): (x,t,u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (A_\varepsilon x, t, u), \quad A_\varepsilon \in SO(n). \end{aligned} \tag{88}$$

Case 3: $(q = 1 \neq r+1)$ We have $\xi^k = \xi^k(x)$ and

$$\alpha = \alpha(x), \quad \eta = \beta t + \gamma, \quad \xi^k = \xi^k(x), \quad \triangle_g \alpha - r\alpha + \beta = 0, \tag{89}$$

and
$$\begin{cases} 2\xi^i_i = -r\alpha(x) + \beta - \dfrac{4x^k\xi^k}{1-|x|^2}, & \forall i \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases} \tag{90}$$

and

$$\triangle_g \xi - (2-n)\frac{\xi}{1-z} = \Big\{\beta + (2-n-r)\alpha + y\cdot\xi\Big\}\frac{y}{(1-z)^2} + \frac{2D\alpha}{(1-z)^2}. \tag{91}$$

By Theorem 1.1 and (90), one obtains that

$$\alpha = \frac{\beta}{r}, \quad \eta = \beta t + \gamma, \quad \xi^k = \sum_{j\neq k} a^k_j x^j, \quad a^i_j = -a^j_i, \quad \forall i \neq j. \tag{92}$$

Thus, the infinitesimal generators are spanned by

$$\overrightarrow{v}_1 = t\partial_t + u/r\partial_u, \quad \overrightarrow{v}_2 = \partial_t, \quad \overrightarrow{v}_3 = \sum_{j\neq i} a^i_j x^j \partial_{x^i}, \tag{93}$$

and their corresponding group actions are generated by

$$\begin{array}{lcl} g_1(\varepsilon) & : & (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (x, e^{\varepsilon}t, e^{\varepsilon/r}u) \\ g_2(\varepsilon) & : & (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (x, t+\varepsilon, u) \\ g_3(\varepsilon) & : & (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (A_\varepsilon x, t, u), \quad A_\varepsilon \in SO(n). \end{array} \tag{94}$$

Case 4: $(r+1 = 1 \neq q)$ One has $\xi^k = \xi^k(x,t)$ and

$$\begin{cases} (1-q)\alpha(x,t) = \eta_t \\ \alpha_t - \dfrac{(1-|x|^2)^2}{4}\triangle\alpha - (n-2)\dfrac{1-|x|^2}{2}x\cdot\nabla\alpha = 0 \end{cases} \Longrightarrow \eta = \beta t + \gamma, \ \alpha = \frac{\beta}{1-q} \tag{95}$$

and
$$\begin{cases} 2\xi^i_i = \beta - \frac{4x^k}{1-|x|^2}\xi^k, & \forall i \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases} \tag{96}$$

and

$$\partial_t\xi - \triangle_g\xi + (2-n)\frac{\xi}{1-z} = -\left\{\frac{\beta(2-n-q)}{1-q} + (2-n)y\cdot\xi\right\}\frac{y}{(1-z)^2}. \tag{97}$$

Applying Theorem 1.1 to (96), one gets that

$$\alpha = 0, \quad \eta = \gamma, \quad \xi^k = \sum_{j\neq k} a^k_j(t)x^j, \quad a^k_j(t) = -a^j_k(t), \quad \forall j \neq k. \tag{98}$$

Substituting into (97), one gets

$$\sum_{j\neq k} \partial_t a^k_j(t) x^j = 0, \quad \forall k$$

or equivalently, $\xi^k = \xi^k(x)\ \forall k$. Therefore the infinitesimal generators are spanned by

$$\vec{v}_1 = \partial_t, \qquad \vec{v}_2 = \sum_{j\neq i} a^i_j x^j \partial_{x^i},$$

and their corresponding group actions are generated by

$$\begin{aligned} g_1(\varepsilon) &: (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (x, t+\varepsilon, u) \\ g_2(\varepsilon) &: (x,t,u) \to (\widetilde{x},\widetilde{t},\widetilde{u}) = (Ax, t, u), \quad A \in SO(n) \end{aligned} \tag{99}$$

corresponding to translation in time and rotation on space respectively.

Case 5: $(q = r+1 = 1)$ We have

$$\alpha_t - \triangle_g \alpha = \eta_t \tag{100}$$

and

$$\begin{cases} 2\xi^i_i = \eta_t - \dfrac{4x^k}{1-|x|^2}\xi^k, & \forall i \\ \xi^i_j + \xi^j_i = 0, & \forall i \neq j \end{cases} \tag{101}$$

and

$$\partial_t \xi - \triangle_g \xi + (2-n)\frac{\xi}{1-z} = -\Big\{\eta_t + (1-n)\alpha + (2-n)y\cdot\xi\Big\}\frac{y}{(1-z)^2} - \frac{2D\alpha}{(1-z)^2}. \tag{102}$$

By Theorem 1.1, we infer from (100)–(101) that

$$\alpha_t - \triangle_g \alpha = 0, \quad \eta = \gamma, \quad \xi^k = \sum_{j\neq k} a^k_j(t) x^j, \quad a^k_j(t) = -a^j_k(t), \quad \forall j \neq k. \tag{103}$$

Combining (102) with , one concludes that

$$D_k\alpha + \frac{(n-1)x^k}{1-|x|^2}\alpha = -\sum_{j\neq k}\frac{(1-|x|^2)^2}{2}\partial_t a^k_j(t) x^j. \tag{104}$$

Taking the second derivative with respect to x^l we obtain

$$\begin{aligned} D_{kl}\alpha = &-(n-1)\Big\{\frac{\delta_{kl}}{1-|x|^2} + \frac{2x^k x^l}{(1-|x|^2)^2}\Big\}\alpha - (n-1)\frac{x^k}{1-|x|^2}D_l\alpha \\ &+2(1-|x|^2)\sum_{j\neq k, j\neq l} x^l x^j \partial_t a^k_j(t) - \Big\{\frac{(1-|x|^2)^2}{2} - 2(x^l)^2(1-|x|^2)\Big\}\partial_t a^k_l(t). \end{aligned} \tag{105}$$

Substituting (104) into (105), we derive that

$$\begin{aligned} &D_{kl}\alpha + (n-1)\Big\{\frac{\delta_{kl}}{1-|x|^2} - (n-3)\frac{x^k x^l}{(1-|x|^2)^2}\Big\}\alpha \\ &= \frac{n-1}{2}(1-|x|^2)\sum_{j\neq k, j\neq l} x^k x^j \partial_t a^l_j(t) + 2(1-|x|^2)\sum_{j\neq k, j\neq l} x^l x^j \partial_t a^k_j(t) \\ &-\frac{n-1}{2}(1-|x|^2)(x^k)^2\partial_t a^k_k(t) - (1-|x|^2)\Big\{\frac{1-|x|^2}{2} - 2(x^l)^2\Big\}\partial_t a^k_l(t). \end{aligned}$$

Exchanging the indices k,l and using the symmetry of second derivatives, we receive

$$\frac{5-n}{2}\sum_{j\neq k,j\neq l} x^j\Big\{x^k\partial_t a^l_j(t)-x^l\partial_t a^k_j(t)\Big\}$$
$$+\Big\{(1-|x|^2)+\frac{5-n}{2}\Big[(x^k)^2+(x^l)^2\Big]\Big\}\partial_t a^k_l(t)=0,\quad \forall k\neq l.$$

Consequently, we have
$$a^k_j(t)=a^k_j,\quad \forall j\neq k \tag{106}$$

and so
$$\alpha=\beta,\quad \eta=\gamma,\xi^k=\sum_{j\neq k}a^k_j x^j,\quad a^k_j=-a^j_k,\quad \forall j\neq k. \tag{107}$$

Thus, the infinitesimal generators are spanned by

$$\overrightarrow{v}_1=u\partial_u,\quad \overrightarrow{v}_2=\partial_t,\quad \overrightarrow{v}_3=\sum_{j\neq i}a^i_j x^j\partial_{x^i}, \tag{108}$$

and their corresponding group actions are generated by

$$\begin{aligned} g_1(\varepsilon)\ &:\ (x,t,u)\to(\widetilde{x},\widetilde{t},\widetilde{u})=(x,t,e^{\varepsilon}u)\\ g_2(\varepsilon)\ &:\ (x,t,u)\to(\widetilde{x},\widetilde{t},\widetilde{u})=(x,t+\varepsilon,u)\\ g_3(\varepsilon)\ &:\ (x,t,u)\to(\widetilde{x},\widetilde{t},\widetilde{u})=(A_\varepsilon x,t,u),\quad A_\varepsilon\in SO(n). \end{aligned} \tag{109}$$

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References

[1] S. I. Betelú, D. G. Aronson, S. B. Angenent: *Renormalization study of two-dimensional convergent solutions of the porous medium equation*, Physica D 138 (2000) 344–359.

[2] K.-S. Chou, C. Z. Qu: *Optimal systems and group classification of* (1+2)*-dimensional heat equation*, Acta Appl. Math. 83 (2004) 257–287.

[3] P. A. Clarkson, E. L. Mansfield: *Symmetry reductions and exact solutions of a class of nonlinear heat equations*, Physica D 70 (1994) 250–288.

[4] J. Dávila, J. D. Rossi: *Self-similar solutions of the porous medium equation in a half-space with a nonlinear boundary condition: existence and symmetry*, J. Math. Anal. Appl. 296 (2004) 634–649.

[5] R. Ferreira, A. de Pablo, J. L. Vazquez: *classification of blow-up with nonlinear diffusion and localized reaction*, J. Diff. Equations 230 (2006) 195–211.

[6] J. A. Franco: *On the n-dimensional porous medium diffusion equation and global actions of the symmetry group*, SIGMA 9 (2013), art. no. 11.

[7] M. L. Gandarias: *Potential symmetries of a porous medium equation*, J. Physics A: Math. General 29 (1996) 5919–5934.

[8] M. L. Gandarias: *Classical point symmetries of a porous medium equation*, J. Physics A: Math. General 29 (1996) 607–633.

[9] M. L. Gandarias: *Nonclassical symmetries of a porous medium equation with absorption*, J. Physics A: Math. General 30 (1997) 6081–6091.

[10] M. L. Gandarias: *Conservations laws for a porous medium equation through noncloassical generators*, Comm. Nonlinear Sci. Numer. Simulation 19 (2014) 371–376.

[11] T. M. Garrido, R. de la Rosa, E. Recio, M. S. Bruzón: *Symmetries, solutions and conservation laws for the* $(2+1)$ *filtration-absorption model*, J. Math. Chemistry 57 (2019) 1301–1313.

[12] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer, New York (1983).

[13] G. Grillo, M. Muratori: *Smoothing effects for the porous medium equation on Cartan-Hadamard manifolds*, Nonlinear Analysis 131 (2016) 326–362.

[14] G. Grillo, M. Muratori, F. Punzo: *Blow-up and global existence for the porous medium equation with reaction on a class of Cartan-Hadamard manifolds*, J. Diff. Equations 266 (2019) 4305–4336.

[15] G. Grillo, M. Muratori, F. Punzo: *Porous medium equations on manifolds with critical negative curvature: unbounded initial data*, Appl. Analysis 98 (2019) 1756–1772.

[16] G. Grillo, M. Muratori, J. L. Vázquez: *The porous medium equation on Riemannian manifolds with negative curvature. The large-time behaviour*, Adv. Math. 314 (2017) 328–377.

[17] G. Grillo, M. Muratori, J. L. Vázquez: *The porous medium equation on Riemannian manifolds with negative curvature: the superquadratic case*, Math. Annalen 373 (2019) 119–153.

[18] H. A. Levine, P. E. Sacks: *Some existence and nonexistence theorems for solutions of degenerate parabolic equations*, J. Diff. Equations 52 (1984) 135–161.

[19] J. Milnor: *Analytic proof of the "Hairy Ball Theorem" and the Brouwer fixed point theorem*, Amer. Math. Monthly 85 (1978) 521–524.

[20] W. M. Ni, P. E. Sacks, J. Tavantzis: *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Diff. Equations 54 (1984) 97–120.

[21] P. J. Olver: *Applications of Lie Groups to Differential Equations*, 2nd edition, Graduate Texts in Mathematics 107, Springer, New York (1993).

[22] C. Z. Qu: *Group classification and generalized conditional symmetry reduction of the nonlinear diffusion convection equation with a nonlinear source*, Stud. Appl. Math. 99 (1997) 107–136.

[23] P. E. Sacks: *Continuity of solutions of a singular parabolic equation*, Nonlinear Analysis Theory Methods Appl. 7 (1983) 387–409.

[24] P. E. Sacks: *Global behavior for a class of nonlinear evolution equations*, SIAM J. Math. Analysis 16 (1985) 233–250.

[25] J. L. Vázquez: *The Porous Medium Equation: Mathematical Theory*, Oxford Mathematical Monographs, Oxford University Press, Oxford (2007).

Xiao-Peng Chen, Shi-Zhong Du (corresponding author), Tian-Pei Guo,
Dept. of Mathematics, Shantou University, Shantou 515063, P. R. China;
xpchen@stu.edu.cn, szdu@stu.edu.cn, 18tpguo@stu.edu.cn.

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