

Cayley Graph Characterization of Geometric Reflections

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Abstract. We combine the covering theory of graphs introduced by Malnic, Nedela and Skoviera, the notion of a Cayley graph and the theory of reflection systems in order to obtain a new characterization of geometric reflections in the theory of extended affine Weyl groups. As an immediate byproduct, we recover that an extended affine Weyl group of nullity greater than one is not a Coxeter group, with respect to any minimal generating set.

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1. Introduction

In the past three decades there has been an intensive investigation on the theory of extended affine Lie algebras and related objects such as root systems and Weyl groups, see for example [1], [2], [4], [7], [8], [16], [21], [22]. Extended affine Lie algebras can be thought of as algebras obtained from finite dimensional simple Lie algebras with an extension of Kac's construction of untwisted and twisted affine Lie algebras.

Root systems and Weyl groups occupy a big portion of the theory of extended affine Lie algebras; in addition to their importance in the study of the structure of Lie algebras and their classification, they are of much interest because of their combinatorial nature and independent applications in other branches of mathematics and theoretical physics. In the case of finite and affine theory there is a huge related literature and a solid framework for the study of root systems and Weyl groups. However, for the cases other than finite and affine, despite several works in this direction (see for example [2], [3], [5], [6], [7], [8] and [22]), there is not yet a satisfactory framework of study. To achieve this, a natural procedure should be inspired by an extension of finite and affine theory in such a generality that it covers all nullities, and perhaps to cover more general Weyl group(oid)s such as [15].

Weyl groups are a subclass of groups generated by (geometric) reflections. In this work we present a new characterization of geometric reflections by merging the theory of extended affine Weyl groups, the covering theory of Cayley graphs in the sense of [20] and [13] and the theory of reflection systems and Coxeter systems introduced by Dyer [12].

Let R be an extended affine root system of type X and nullity ν . One knows that extended affine root systems of nullities 0 and 1 correspond to finite and affine root systems, respectively. It is known that the Weyl group of an extended affine root system is a Coxeter group if and only if $\nu \leq 1$, [16]. If $\nu > 1$ the Weyl group enjoys some generalized Coxeter presentations, see for example [22], [3], [6] and [7]. Despite the given presentations for higher nullities there is not yet a suitable framework for a unified study of extended affine Weyl groups.

In [13], the authors give a new characterization of Coxeter groups by using a refined notion of a Cayley graph, introduced in 2000 by Malnic, Nedela and Skoviera [20]. Such a graph is a 4-tuple (V, D, ι, λ) in which V is a nonempty set of vertices, D is a set of darts, $\iota : D \rightarrow V$ is a map, called the initial vertex map, and λ is a permutation of order 2 on D , called reverse dart map. A graph in this sense might consist of darts without any end point; for example when Γ is a monopole, that is Γ is a graph with one vertex and a numbers of darts, each dart with $\lambda(d) = d$ is called a semi-edge and it is a dart without any terminal vertex.

An application of this new notion of graph appears in the theory of Cayley graphs. For example Gross and Tucker [14] show that in the classical graph theory there exist Cayley graphs which are not regular 1-covers of bouquets of circles, while in 2007, Gramlich, Hofmann and Neeb used the new notion of graph to show that any Cayley graph is a regular 1-cover of a monopole and vice versa [13].

Furthermore, the classic definition of a graph does not apply to the covering theory of graphs with semi-edges. In [13], the authors studied Cayley graphs, in its new notion, as 1-covers of monopoles admitting semi-edges only. They obtained a new characterization of Coxeter groups in terms of their Cayley graphs and showed that if $\pi : \Gamma \rightarrow (\{v\}, D, \iota, \lambda)$ is a 1-covering of a monopole admitting semi-edges only, then Γ is a Cayley graph of a Coxeter group if and only if π is regular and any deck transformation in $\Delta(\pi)$ that interchanges two neighbouring vertices of Γ is a (Cayley) graph-reflection. When we work with a connected graph Γ as a regular 1-cover of the monopole $(\{v\}, D, \iota, \lambda)$ via the 1-covering map π , we see that Γ is the Cayley graph of a group $G \cong \Delta(\pi)$ with respect to generating multiset D , [13, Theorem 5.6, Corollary 5.9]. In particular if Γ is the Cayley graph of a Coxeter group, for example when Γ is the Cayley graph of a finite or affine Weyl group, this gives a nice characterization of these groups in terms of 1-covers. Moreover, in [13] it is shown that (G, X) is a Coxeter system if and only if the elements of X are (Cayley) graph-reflections of the Cayley graph of (G, X) .

To achieve a unified framework for the study of extended affine Weyl groups, in this work we combine the works mentioned in the preceding paragraph to obtain a new characterization of geometric reflections in terms of the reflections appearing in the theory of Cayley graphs (in the sense of [20]) associated to extended affine Weyl groups. Here we present an overview of each section.

In Section 2, the following concepts are recalled from appropriate references and some details which are needed in the sequel are presented; the notion of an extended affine root system and its structure in terms of (translated) semilattices, the notion of an extended affine Weyl group; in particular the structure of an extended affine Weyl group of type A_1 , the concept of a reflectable set (base), and the concept of a graph in the sense of [20]. Since each extended affine root system of nullity ν

can be viewed as a union of extended affine root systems of type A_1 of the same nullity ν (Lemma 2.1), and this fact plays an important role in our characterization of reflections, we put an emphasis on the type A_1 . In particular, we obtain a precise characterization of reflectable bases of type A_1 in terms of a set of coset representatives of the involved semilattices (Lemma 2.3).

In Section 2, we also introduce reflection systems and consider Coxeter systems as a subclass of reflection systems. We discuss which subgroups of reflection systems and Coxeter groups are reflection subgroups. An important part of this section is devoted to the concept of a graph. We use the notion of a graph and the theory of 1-coverings of graphs as in [13] and [20]. The notion of a (Cayley) graph-reflection is introduced by means of normalized darts. The section is concluded by recalling two theorems from [13] concerning the Cayley graph characterization of finite and affine Coxeter groups which will be crucial in the rest of work.

In Section 3, we first present some general properties of normalized darts of involutions of the group of automorphisms of the Cayley graph of an arbitrary group G . Then we analyze normalized darts of geometric reflections on the Cayley graph of an extended affine Weyl group of type A_1 . This enables us to compute the set of normalized darts of associated geometric reflections. The key result of this work states that geometric reflections associated to a reflectable base coincide with Cayley graph-reflections of the corresponding Cayley graph if and only if $\nu \leq 1$ (Theorem 4.1).

For an extended affine root system of type A_1 of nullity $\nu > 1$ and a fixed reflectable base Π , let $\Gamma = \text{Cay}(\mathcal{W}, \Pi)$ be the Cayley graph associated the Weyl group \mathcal{W} . In Section 4, we show that any two vertices of $\Gamma = \text{Cay}(\mathcal{W}, \Pi)$ can be connected by a path without any normalized darts of the corresponding geometric reflections. This in particular shows that when $\nu > 1$, the corresponding A_1 -type extended affine Weyl group is not a Coxeter group with respect to any reflectable base (Theorem 4.3). Finally, we extend this result to arbitrary extended affine Weyl groups of arbitrary nullity ν (Theorem 4.4). This is achieved by some results concerning reflection subgroups and by observing that, we may embed a suitable extended affine Weyl group of type A_1 of nullity ν inside the Weyl group under consideration.

To conclude, in Section Appendix, we have provided several examples of Cayley graphs, elaborating on the results given in the paper.

2. Preliminary concepts

2.1. Extended affine root systems and semilattices. We start this section by giving the definition of an extended affine root system. All vector spaces are finite dimensional and considered over the field \mathbb{R} . For a vector space \mathcal{U} , by \mathcal{U}^* , we mean the dual space of \mathcal{U} . If \mathcal{U} is equipped with a symmetric bilinear form, we denote the radical of the form by \mathcal{U}^0 , and for a subset R of \mathcal{U} , we set $R^0 := R \cap \mathcal{U}^0$ and $R^\times := R \setminus R^0$. Also for a subset S of \mathcal{U} , we denote by $\langle S \rangle$, the additive subgroup of \mathcal{U} generated by S . We use the symbol \uplus to indicate the disjoint union of sets.

Let \mathcal{V} be a vector space equipped with a positive semidefinite symmetric bilinear form (\cdot, \cdot) . Suppose R is a subset of \mathcal{V} . Following [1, Definition II.2.1], we say R is an *irreducible reduced extended affine root system* if

1. $0 \in R$.
2. $R = -R$.
3. R spans \mathcal{V} .
4. If $\alpha \in R^\times$, then $2\alpha \notin R$.
5. R is discrete in \mathcal{V} .
6. If $\alpha \in R^\times$ and $\beta \in R$, then there exist $d, u \in \mathbb{Z}_{\geq 0}$ satisfying

$$\{\beta + n\alpha \mid n \in \mathbb{Z}\} \cap R = \{\beta - d\alpha, \dots, \beta + u\alpha\} \quad \text{and} \quad d - u = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$

This is called the *root string property*.

7. R^\times can not be decomposed as $R_1 \uplus R_2$, where R_1 and R_2 are nonempty subsets of R^\times satisfying $(R_1, R_2) = \{0\}$. Here R is called *connected*.
8. If $\sigma \in R^0$, then there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.

Axioms (7) and (8) together are referred to as *irreducibility* of the root system.

Elements of R^\times are called *non-isotropic roots* and elements of R^0 are called *isotropic roots*. The *nullity* of R , denoted by ν , is defined to be $\dim \mathcal{V}^0$. One can check that $R^0 = \{\alpha \in R \mid (\alpha, \alpha) = 0\}$ and $R^\times = \{\alpha \in R \mid (\alpha, \alpha) \neq 0\}$. It is clear from axioms that irreducible reduced finite root systems are extended affine root systems of nullity zero. The interested reader can see [1, Chapter II] for more details about extended affine root systems and their structures.

In general R can be written in the form

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E), \quad (1)$$

where the involved terms are explained below. It follows that the image \bar{R} of R in $\bar{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$ under the canonical map is a finite irreducible root system in $\bar{\mathcal{V}}$. Then one can find an appropriate preimage \dot{R} of \bar{R} such that \dot{R} is an irreducible finite root system in $\dot{\mathcal{V}} := \text{span}_{\mathbb{R}} \dot{R}$ isomorphic to \bar{R} . Then we write $\dot{R} = \dot{R}_{sh} \cup \dot{R}_{lg} \cup \dot{R}_{ex} \cup \{0\}$ where \dot{R}_{sh} , \dot{R}_{lg} and \dot{R}_{ex} are the sets of short, long and extra long roots of \dot{R} respectively, (if \dot{R} is of simply laced types we assume by convention that $\dot{R} \setminus \{0\} = \dot{R}_{sh}$). The type X of $\dot{R} \equiv \bar{R}$ which is an isomorphism invariant of R is called the *type* of R . Also, the rank ℓ of \dot{R} is called the *rank* of R . The sets S and L which are called *semilattices* and E which is called a *translated semilattice* in \mathcal{V}^0 are defined as follows

$$\begin{aligned} S &= \{\sigma \in \mathcal{V}^0 \mid \sigma + \alpha \in R \text{ for some } \alpha \in \dot{R}_{sh}\}, \\ L &= \{\sigma \in \mathcal{V}^0 \mid \sigma + \alpha \in R \text{ for some } \alpha \in \dot{R}_{lg}\}, \\ E &= \{\sigma \in \mathcal{V}^0 \mid \sigma + \alpha \in R \text{ for some } \alpha \in \dot{R}_{ex}\}. \end{aligned}$$

We refer the interested reader to [1, II.§1] for details about (translated) semilattice. Semilattices also appear in the study of symmetric spaces in Geometry under the name reflection spaces, see [19]. The sets S , L and E satisfy some further algebraic properties which we record here. If R is simply laced of rank > 1 , or is of type C_ℓ , $\ell \geq 3$, F_4 or G_2 , then S is a lattice. If R is of type B_ℓ , $\ell \geq 3$, F_4 or G_2 , then L is a lattice. We set

$$k = \begin{cases} 3 & \text{if } X = G_2, \\ 2 & \text{if } X = \text{other non-simply laced types.} \end{cases} \quad (2)$$

By [1, Proposition II.2.23], we have

$$\begin{aligned} L + kS \subseteq L \subseteq S + L \subseteq S, & \quad (\text{if } \dot{R}_{lg} \neq \emptyset) \\ E + 2L \subseteq E \subseteq L + E \subseteq L, & \quad (\text{if } \dot{R}_{lg} \neq \emptyset, \dot{R}_{ex} \neq \emptyset) \\ 4S + E \subseteq E \subseteq S \quad \text{and} \quad 2S \cap E = \emptyset, & \quad (\text{if } \dot{R}_{ex} \neq \emptyset). \end{aligned} \tag{3}$$

Now (3) implies that $k\langle S \rangle \subseteq \langle L \rangle \subseteq \langle S \rangle$; (4)

and so $\langle S \rangle / \langle L \rangle$ is a finite dimensional vector space over the finite field \mathbb{Z}_k . The integer $0 \leq t \leq \nu$ satisfying $|\langle S \rangle / \langle L \rangle| = k^t$ is called the *twist number* of R . We make the convention that if R is of simply laced type, $t = 0$.

We record some further properties of semilattices here. The \mathbb{Z} -span of a semilattice is a lattice. A semilattice with \mathbb{Z} -span Λ is called a semilattice in Λ . Let Λ be a lattice in \mathcal{V}^0 and S be a semilattice in Λ ; then Λ admits a \mathbb{Z} -basis $B = \{\sigma_1, \dots, \sigma_\nu\}$ consisting of elements of S [1, Proposition II.1.11], and $S = \bigcup_{i=0}^m (\tau_i + 2\Lambda)$, where τ_i 's represent distinct cosets of 2Λ in Λ . The integer m is called the *index* of S , denoted by $\text{ind}(S)$ [2, Definition 1.9]. Throughout this work, we fix the basis B of Λ . For $\sigma = \sum_{i=1}^\nu n_i \sigma_i \in \Lambda$, we write $\text{Supp}(\sigma) := \{i \mid n_i \in 2\mathbb{Z} + 1\}$. Then we have $\sigma = \sum_{i \in \text{Supp}(\sigma)} \sigma_i \pmod{2\Lambda}$. For a semilattice $S = \bigcup_{i=0}^m (\tau_i + 2\Lambda)$ in the above form, the collection

$$\text{Supp}(S) = \{\text{Supp}(\tau_i) \mid 0 \leq i \leq m\},$$

is called the *supporting class* of S with respect to the basis B . The supporting class determines S uniquely. In fact we have

$$S = \bigsqcup_{J \in \text{Supp}(S)} (\tau_J + 2\Lambda), \quad \text{where } \tau_J := \sum_{r \in J} \sigma_r. \tag{5}$$

We conclude this section by a result which will be used in the sequel and shows that each extended affine root system can be thought of a union of A_1 -type extended affine root systems.

Lemma 2.1. *Let R be an extended affine root system of type X of nullity ν and $\alpha \in R^\times$. Set $S_\alpha := \{\sigma \in \mathcal{V}^0 \mid \alpha + \sigma \in R\}$ and $R_\alpha := (S_\alpha + S_\alpha) \cup (\pm\alpha + S_\alpha)$. Then R_α is an extended affine root system of type A_1 of nullity ν and $R = \bigcup_{\alpha \in R^\times} R_\alpha$.*

Proof. From root string property, one can easily see that if $\alpha \in R^\times$, $\sigma \in \mathcal{V}^0$ and $\alpha + \sigma \in R$, then $\sigma \in R^0$. Thus $S_\alpha \subseteq R^0$. In particular, S_α is discrete. Clearly $0 \in S_\alpha$, moreover if $\sigma, \tau \in S_\alpha$, then as $w_{\alpha+\sigma}(\alpha + \tau) \in R$, it follows that $S_\alpha \pm 2S_\alpha \subseteq S_\alpha$. Thus S_α is a semilattice in its \mathbb{Z} -span. Now by [1, Theorem II.2.37] any set of the form $(S_\alpha + S_\alpha) \cup (\pm\alpha + S_\alpha)$, where S_α is a semilattice, is an extended affine root system of type A_1 in $\mathbb{R}\alpha + \text{span}_{\mathbb{R}} S_\alpha$. One also can show that $2\Lambda \subseteq \langle S_\alpha \rangle \subseteq \Lambda$ and so R_α has the same nullity as R .

For the second assertion, it is clear from definition of R_α , $\alpha \in R^\times$, that $R^\times \subseteq \bigcup_{\alpha \in R^\times} R_\alpha \subseteq R$. Thus it remains to show that R^0 is contained in the union of R_α 's. For this let $\sigma \in R^0$. Since isotropic roots are non-isolated (axiom (8)), there exists $\alpha \in R^\times$, with $\alpha + \sigma \in R$. Then $\sigma \in R_\alpha$, and we are done. ■

2.2. Extended affine Weyl groups. Let R be an extended affine root system in \mathcal{V} . Let $\dot{\Pi}$, \dot{R} , $\dot{\mathcal{V}}$ and \mathcal{V}^0 be as in Section 2. Then $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$. Set $\tilde{\mathcal{V}} := \dot{\mathcal{V}} \oplus \mathcal{V}^0 \oplus (\mathcal{V}^0)^*$ as a real vector space of dimension $\ell + 2\nu$. We normalize the form on $\dot{\mathcal{V}}$ such that

$$(\dot{\alpha}, \dot{\alpha}) := 2 \text{ for } \dot{\alpha} \in \dot{R}_{sh}.$$

We extend the form (\cdot, \cdot) on $\dot{\mathcal{V}}$ to $\tilde{\mathcal{V}}$ as follow, and denote it again by (\cdot, \cdot) ;

$$\begin{aligned} (\dot{\mathcal{V}}, (\mathcal{V}^0)^*) &= ((\mathcal{V}^0)^*, (\mathcal{V}^0)^*) := \{0\}, \\ (\sigma, \lambda) &= \lambda(\sigma), \text{ for } \sigma \in \mathcal{V}^0, \lambda \in (\mathcal{V}^0)^*. \end{aligned}$$

This forces the form on $\tilde{\mathcal{V}}$ to be non-degenerate. For $\alpha \in \mathcal{V}$ with $(\alpha, \alpha) \neq 0$ set $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. The *extended affine Weyl group* (EAWG for short) \mathcal{W} of R is defined to be the subgroup of $GL(\tilde{\mathcal{V}})$ generated by reflections $w_\alpha : \beta \mapsto \beta - (\beta, \alpha^\vee)\alpha$, $\alpha \in R^\times$. Since $0 \in L \subseteq S$, we have $\dot{R}_{sh} \cup \dot{R}_{lg} \subseteq R$, and so we may identify $\dot{\mathcal{W}}$, the Weyl group of \dot{R} , as a subgroup of \mathcal{W} . We denote by $\mathcal{Z}(\mathcal{W})$, the center of \mathcal{W} .

Next, we define *Eichler-Siegel* transformations which turn out to be very important in the description of EAWGs. For $\alpha \in \mathcal{V}$ and $\sigma \in \mathcal{V}^0$, we define $T_\alpha^\sigma \in \text{End}(\tilde{\mathcal{V}})$ by

$$T_\alpha^\sigma(u) := u - (\sigma, u)\alpha + (\alpha, u)\sigma - \frac{(\alpha, \alpha)}{2}(\sigma, u)\sigma,$$

and set $\mathcal{H} := \langle T_\alpha^\sigma \mid \alpha \in \mathcal{V}, \sigma \in \mathcal{V}^0 \rangle \cap \mathcal{W}$. Then we have $\mathcal{W} = \dot{\mathcal{W}} \ltimes \mathcal{H}$, [2, Proposition 3.27]. It follows that each $w \in \mathcal{W}$ has a unique expression of the form

$$w = \dot{w}Tz, \tag{6}$$

where $\dot{w} \in \dot{\mathcal{W}}$, $T \in \langle T_{\dot{\alpha}^r}^{\sigma_r} \mid \dot{\alpha} \in \dot{R}, 1 \leq r \leq \nu \rangle$, and $z \in \mathcal{Z}(\mathcal{W})$. When $\dot{R} = \{\pm\epsilon, 0\}$ is a finite root system of type A_1 , the expression (6) can be formulated as

$$w = w_\epsilon^n \prod_{r=1}^\nu t_r^{m_r} z, \quad n \in \{0, 1\}, m_r \in \mathbb{Z}, z \in \mathcal{Z}(\mathcal{W}), \tag{7}$$

where $t_r := T_\epsilon^{\sigma_r}$. One can easily compute that $t_r = w_{\epsilon+\sigma_r} w_\epsilon$.

To describe the center of \mathcal{W} , we set $c_{r,s} := T_{\sigma_r}^{\sigma_s}$ and define

$$z_J := \begin{cases} \prod_{\{r,s \in J \mid r < s\}} c_{r,s} & J \in \text{Supp}(S), \\ c_{r,s}^2 & J = \{r, s\} \notin \text{Supp}(S), \\ 1 & \text{otherwise.} \end{cases} \tag{8}$$

If nullity ν is greater than or equal 2, then from [7, Proposition 2.2(vi)] we have $\mathcal{Z}(\mathcal{W}) = \langle z_J \mid J \subseteq \{1, \dots, \nu\} \rangle$.

In the sequel, to indicate that a group G is considered with respect to a fixed generating set S , we denote G by (G, S) . For a subset Π of R^\times , we set $S_\Pi = \{w_\alpha \mid \alpha \in \Pi\}$, and we let \mathcal{W}_Π be the subgroup of \mathcal{W} generated by S_Π . For simplicity, we write (\mathcal{W}, Π) instead of (\mathcal{W}, S_Π) and (\mathcal{W}, R) instead of $(\mathcal{W}, S_{R^\times})$.

A subset Π of R^\times is called a *reflectable set* if $\mathcal{W}_\Pi \Pi = R^\times$. A reflectable set Π is called a *reflectable base* if it is minimal with respect to inclusion, namely no proper subset Π' of Π satisfies $\mathcal{W}_{\Pi'} \Pi' = R^\times$. In the theory of EARSs of nullity $\nu \geq 2$, reflectable bases are natural substitutes for bases in the theory of finite and affine root systems.

2.3. Reflectable bases of type A_1 . In this subsection, we let R be an extended affine root system of type A_1 . Then R has the form

$$R = (S + S) \cup (\pm\epsilon + S),$$

where S is a semilattice in Λ . Reflectable bases are characterized in [9], in particular the following characterization is given there for type A_1 :

Theorem 2.2. [9, Theorem 3.1] *A subset Π of non-isotropic roots of an affine reflection system R of type A_1 is a reflectable base if and only if Π satisfies*

$$R^\times = \cup_{\alpha \in \Pi} (\alpha + 2\langle R \rangle) \cap R^\times \tag{9}$$

and no proper subset of Π satisfies (9).

We also recall from [2, Theorem 4.22], that if τ_J 's are as in (5), then the set

$$\{\tau_J - \epsilon \mid J \in \text{supp}(S)\} \tag{10}$$

is a reflectable base for R .

Lemma 2.3. *Let $\mathcal{T} = \{\tau_0, \dots, \tau_m\}$, $m = \text{ind}(S)$, be a set of coset representatives for S , namely $S = \uplus_{i=0}^m (\tau_i + 2\Lambda)$. Let $r_i, s_i \in \{\pm 1\}$ and $\Pi = \{r_i\tau_i + s_i\epsilon \mid 0 \leq i \leq m\}$. Then Π is a reflectable base for R if and only if $\langle \Pi \rangle = \langle R \rangle$. Moreover, any reflectable base for R is of this form, in particular any reflectable base has cardinality $m + 1$.*

Proof. Let Π be as in the statement. Suppose Π is a reflectable base of R , then $R = \mathcal{W}_\Pi \Pi \subseteq \langle \Pi \rangle$, and so $\langle R \rangle = \langle \Pi \rangle$.

Conversely, suppose $\Pi = \{r_i\tau_i + s_i\epsilon \mid 0 \leq i \leq m\}$ and $\langle \Pi \rangle = \langle R \rangle$. We need to show that Π satisfies conditions of Theorem 2.2.

As $R^\times = \pm\epsilon + S$, $\tau_i + 2\Lambda = -\tau_i + 2\Lambda$ and $\epsilon + 2\langle R \rangle = -\epsilon + 2\langle R \rangle$, we have

$$\begin{aligned} R^\times &= \pm\epsilon + S = \cup_{i=0}^m (\pm\epsilon + \tau_i + 2\Lambda) \\ &\subseteq \cup_{i=0}^m (\pm\epsilon + r_i\tau_i + 2\Lambda) \subseteq \cup_{i=0}^m (s_i\epsilon + r_i\tau_i + 2\langle R \rangle) \cap R^\times \\ &= \cup_{\alpha \in \Pi} (\alpha + 2\langle R \rangle) \cap R^\times. \end{aligned}$$

We now check the minimality of Π . It is enough to show that if $\alpha' = s_i\epsilon + r_i\tau_i \in \Pi$,

then
$$\alpha' \notin \bigcup_{\alpha \in \Pi \setminus \{\alpha'\}} (\alpha + 2\langle R \rangle) \cap R^\times.$$

Otherwise, $\alpha' = s_i\epsilon + r_i\tau_i = s_j\epsilon + r_j\tau_j \pmod{2\langle R \rangle}$, for some $j \neq i$. This then forces $r_i\tau_i = r_j\tau_j \pmod{2\Lambda}$ which is a contradiction.

Next we note from (10) that R has a reflectable base Π with $|\Pi| = m + 1$. Thus if Π' is another reflectable bases for R , we have from Theorem 2.2 that

$$R^\times = \cup_{\alpha \in \Pi} (\alpha + 2\langle R \rangle) \cap R^\times = \cup_{\alpha' \in \Pi'} (\alpha' + 2\langle R \rangle) \cap R^\times \tag{11}$$

and both Π and Π' are minimal with this property. Therefore for each $\alpha' \in \Pi'$ there exists a unique $\alpha \in \Pi$ with $\alpha' + 2\langle R \rangle = \alpha + 2\langle R \rangle$, and vice versa. Thus $|\Pi'| = |\Pi| = m + 1 = \text{ind}(S) + 1$, namely any two reflectable bases of R have the same cardinality $m + 1$.

Finally suppose $\Pi' = \{\alpha'_0, \dots, \alpha'_m\}$ is a reflectable base for R and Π be a reflectable base of the form given in the statement. Then from (11), we get for each i , after possibly a change of indices, $\alpha'_i = s_i\epsilon + r_i\tau_i + 2\sigma_i$ for some $\sigma_i \in \Lambda$. Setting $\tau'_i := r_i\tau_i + 2\sigma_i$, we have $\tau_i + 2\Lambda = \tau'_i + 2\Lambda$ for each i , and so $\Pi' = \{s_i\epsilon + \tau'_i \mid 0 \leq i \leq m\}$ is of the form given in the statement as required. ■

2.4. Reflection systems and reflection subgroups. We will continue this section by recording the notion of a “*reflection system*”, in the sense of [12], which generalizes the concept of a Coxeter system. Coxeter systems form a subclass of reflection systems.

Let G be a group, $X \subseteq G \setminus \{1\}$ and $G = \langle X \rangle$. Set $T = \bigcup_{g \in G} gXg^{-1}$, the elements of T are called *reflections* of (G, X) . We denote by $\text{ord}(g)$, the order of an element g in G , and by \mathbb{Z}_m the abelian group of congruence classes of integers modulo integer $m > 0$. By convention, we set $\mathbb{Z}_\infty := \mathbb{Z}$. Also for every $g, g' \in G$, we set $g \cdot g' := gg'g^{-1}$.

Next, let $M(G, X)$ be the direct sum of family $\{\mathbb{Z}_{\text{ord}(t)}\}_{t \in T}$, namely

$$M(G, X) = \{\sum_{t \in T} a_t t \mid a_t \in \mathbb{Z}_{\text{ord}(t)}, a_t = 0 \text{ for almost all } t\}.$$

The abelian group $M(G, X)$ is in fact a G -module by the action

$$g \cdot \left(\sum_{t \in T} a_t t\right) = \sum_{t \in T} a_t (g \cdot t).$$

Definition 2.4. (i) Let G be a group, $X \subseteq G \setminus \{1\}$ and $G = \langle X \rangle$. The pair (G, X) is called a *reflection system*, if there exists a function $N : G \rightarrow M(G, X)$ such that

- $N(gh) = N(g) + g \cdot N(h)$, for all $g, h \in G$,
- $N(x) = 1x$, for all $x \in X$.

The function N is called the *reflection cocycle* of the reflection system (G, X) .

(ii) A subgroup G_1 of a reflection system (G, X) is called a *reflection subgroup*, if G_1 is generated by a subset of reflections of (G, X) , that is $G_1 = \langle G_1 \cap T \rangle$.

Remark 2.5. (i) If G_1 is a reflection subgroup of a reflection system (G, X) , then it is not in general true that $(G_1, G_1 \cap T)$ is a reflection system. The following theorem is a consequence of [12, Remark 2.11] and [12, Theorem 3.3, Proposition 3.5] and explains how one can find a suitable subset X_1 of $G_1 \cap T$ such that (G_1, X_1) is a reflection system. In this case (G_1, X_1) is called a *reflection subsystem* of (G, X) . We first need to define some notations. For $g \in G$, let us denote $N(g)$ by $\sum_{t \in T} g_t t \in M(G, X)$. Then for a subgroup G_1 of G we set

$$\chi(G_1) := \{a \in T \mid \sum_{t \in G_1 \cap T} a_t t = \{1a\}\}. \tag{12}$$

Note that $\chi(G_1) \subseteq G_1 \cap T$ and $G_1 \cap X = \chi(G_1) \cap X$.

(ii) Let G be a reflection system with respect to two different generating sets X and X' . Suppose both (G, X) and (G, X') have the same set of reflections, say T . Since by definition a subgroup G_1 of G is a reflection subgroup if and only if $G_1 = \langle G_1 \cap T \rangle$, then (G, X) and (G, X') have the same reflection subgroups.

Theorem 2.6. *Let (G, X) be a reflection system, G_1 be a reflection subgroup of G and $T' \subseteq T$;*

- (i) $(G_1, \chi(G_1))$ is a reflection system with the corresponding cocycle $N_1 : G_1 \rightarrow M(G_1, X_1)$ given by $N_1(g) = \sum_{t \in G_1 \cap T} g_t t$.
- (ii) $T' = \chi(\langle T' \rangle)$ if and only if $\chi(\langle t_1, t_2 \rangle) = \{t_1, t_2\}$ for all $t_1, t_2 \in T'$. In particular if $T' = \chi(\langle T' \rangle)$, then $(\langle T' \rangle, T')$ is a reflection subsystem of (G, X) .
- (iii) For any non-empty subset Y of X , $\chi(\langle Y \rangle) = Y$ and $(\langle Y \rangle, Y)$ is a reflection subsystem of (G, X) .

2.5. Coxeter groups. In what follows G is a group with identity element 1, and $X \subseteq G \setminus \{1\}$ is a generating set of G in which every element of X is of order 2, thus $X = X^{-1}$. For $x, x' \in X$, suppose $m(x, x')$ is the order of element xx' in the group G . Note that $m(x, x) = 1$ for every $x \in X$. We recall that the pair (G, X) is called a *Coxeter system* if G is defined by generators X , subject to relations $(xx')^{m(x, x')} = 1$ for all $x, x' \in X$ with $m(x, x') < \infty$. In this case, G is called a *Coxeter group* with generating set X . By [10, Chapter IV, §8, Corollary 3], if (G, X) is a Coxeter system, then X is a minimal generating set for G . Note that when we simply say a group G is a Coxeter group, we mean that there exists a minimal generating subset X of $G \setminus \{1\}$ such that (G, X) is a Coxeter system. For more details on Coxeter groups, we refer the interested reader to [10] and [18].

Suppose (G, X) is a Coxeter system and $\ell(g)$ is the length of $g \in G$, with respect to X . We recall that the pair (G, X) is said to satisfy *exchange condition* if the following holds; let $g \in G$ and $x \in X$ such that $\ell(xg) \leq \ell(g)$ and suppose $g = x_1 \cdots x_n$, $x_i \in X$, is a reduced expression for g , that is $\ell(g) = n$; then there exists a unique integer $1 \leq j \leq n$ such that $xx_1 \cdots x_{j-1} = x_1 \cdots x_{j-1}x_j$ and then

$$xg = x_1 \cdots \hat{x}_j \cdots x_n := x_1 \cdots x_{j-1}x_{j+1} \cdots x_n.$$

It is known from [10, Chapter IV, Theorem 1] that the pair (G, X) is a Coxeter system if and only if it satisfies the exchange condition. The following is a consequence of [12, Lemma 2.3, Corollary 2.6] and [12, Proposition 2.10, Theorem 3.3].

Theorem 2.7. *Let G be a group with a generating set X .*

- (i) *A pair (G, X) is a Coxeter system if and only if it is a reflection system and X consists of involutions.*
- (ii) *If (G, X) is a Coxeter system, its cocycle is defined by $N(g) := \sum_{t \in T} g_t t$, where*

$$g_t = \begin{cases} 1 & \text{if } \ell(tg) < \ell(g) \\ 0 & \text{otherwise.} \end{cases}$$

- (iii) *Suppose (G, X) is a Coxeter system and G' is a reflection subgroup of G ; then $(G', \chi(G'))$ is a Coxeter system. Namely any reflection subgroup of a Coxeter group is a Coxeter group.*

Remark 2.8. (i) In this remark, we explain why the map N given in Part (ii) of Theorem 2.7 is well-defined. Suppose (G, X) is a Coxeter system. By part (i) of the theorem, X consists of involutions and (G, X) is equipped with a cocycle

$N : G \rightarrow M(G, X)$ satisfying, for $g \in G$, $N(g) = \sum_{t \in T} g_t t$, $g_t \in \mathbb{Z}_{\text{ord}(t)}$ and $g_t = 0$ for almost all t . In this case, g_t is called the *multiplicity* of t for g and is denoted by $\text{mult}_t(g)$. As $\text{ord}(t) = 2$ for every $t \in T$, $\text{mult}_t(g) \in \mathbb{Z}_2$ and so

$$N(g) = \sum_{\substack{t \in T \\ \text{mult}_t(g)=1}} t;$$

thus the set $\{t \in T \mid \text{mult}_t(g) = 1\}$ is a finite set. Let $x_1 x_2 \cdots x_n$ be a reduced expression of g ; one can easily check that $N(g) = t_1 + \cdots + t_n$ where $t_1 = x_1$ and $t_i = x_1 \cdots x_{i-1} \cdot x_i$ for $2 \leq i \leq n$. On the other hand according to [12, Lemma 2.3] and its proof, $\{t \in T \mid \ell(tg) < \ell(g)\} = \{t \in T \mid \text{mult}_t(g) \neq 0\} = \{t_1, \dots, t_n\}$. Thus $\{t \in T \mid \ell(tg) < \ell(g)\}$ is a finite set and so the map N is well-defined.

(ii) In [11], the author has independently worked on reflection subgroups of Coxeter groups in terms of properties of their root systems and proved that any reflection subgroup of a Coxeter group is a Coxeter group. Deodhar has presented a geometric proof and calculated the set of Coxeter generators for every reflection subgroup. One can check that the Coxeter generators introduced in Theorem 2.7(iii) coincide to those described in [11].

Example 2.9. Let R be an extended affine root system of nullity $\nu \in \{0, 1\}$; namely R is either an irreducible finite or an affine root system. Let \mathcal{W} be its Weyl group, and Π be a fundamental system for R . It is known that \mathcal{W} is a Coxeter group with respect to Π . Set $S := \{w_\alpha \mid \alpha \in \Pi\}$; then $T = \bigcup_{w \in \mathcal{W}} w S w^{-1} = \{w_\alpha \mid \alpha \in R^\times\}$. Suppose $S' \subseteq S$ and $\mathcal{W}' := \langle S' \rangle$, then (\mathcal{W}', S') is a Coxeter system, by Theorem 2.6(iii).

2.6. The concept of a graph. We recall the concept of a graph in the sense of [20]. A graph Γ is a 4-tuple (V, D, ι, λ) where V is a non-empty set of *vertices*, D is a set, which might be empty, called the set of *darts*. Also $\iota : D \rightarrow V$ is a map and $\lambda : D \rightarrow D$ is a permutation of order 2. For every dart d , $\iota(d)$ is called the *initial vertex* of d and $\lambda(d)$, denoted by d^{-1} , is called the *reverse* of d . The vertex $\iota(d^{-1})$ is called *terminal vertex* of d . The orbits of λ are called *edges*. Note that edges consist of one or two darts. An edge is called a *semi-edge* if its cardinality is one, a *loop* if its cardinality is two and both darts contained in this edge have the same initial vertex, and a *link* otherwise. A *monopole* is a graph consisting of one vertex and a number of darts. A *path* γ of length t is a sequence of t darts d_1, \dots, d_t such that the terminal vertex of d_k coincides with the initial vertex of d_{k+1} for all $1 \leq k \leq t-1$; in this case we write $\gamma := d_1 \cdots d_t$. By convention, a vertex is a path of length 0. The *initial vertex* of γ , denoted by $\iota(\gamma)$, is the initial vertex of d_1 and the *terminal vertex* of γ is the terminal vertex of d_t , denoted by $\iota(\gamma^{-1})$. If a path γ has initial vertex x , then we say γ is a path based on x . If its initial and terminal vertices coincide, then γ is called a *cycle* or a *closed path*.

A graph $\Gamma = (V, D, \iota, \lambda)$ is called *connected*, if for each pair x and y of vertices of Γ , there exists a path from x to y . We say that a vertex y is a *neighbour* of a vertex x if there exists a dart d in Γ with $\iota(d) = x$ and $\iota(d^{-1}) = y$. A *graph morphism* from $\Gamma_1 = (V_1, D_1, \iota_1, \lambda_1)$ to $\Gamma_2 = (V_2, D_2, \iota_2, \lambda_2)$ is a pair $\phi = (\phi_1, \phi_2)$ of maps $\phi_1 : V_1 \rightarrow V_2$ and $\phi_2 : D_1 \rightarrow D_2$ such that $\phi_2 \iota_1 = \iota_2 \phi_2$ and $\phi_1 \lambda_1 = \lambda_2 \phi_2$.

In fact a graph morphism is a map between graphs which preserves initial vertex and reverse of each dart. An automorphism of order 2 of a graph Γ is called an *involution*. A graph epimorphism $\pi : \hat{\Gamma} = (\hat{V}, \hat{D}, \hat{\iota}, \hat{\lambda}) \rightarrow \Gamma = (V, D, \iota, \lambda)$ is called a *1-covering* if for every vertex $x \in \hat{\Gamma}$, the map π sends the set of darts of $\hat{\Gamma}$ with initial vertex x bijectively onto the set of darts of Γ with initial vertex $\pi(x)$. In this case, $\hat{\Gamma}$ is called a *1-cover* of Γ and we say $(\hat{\Gamma}, \pi)$ is a 1-covering of Γ . The set $\pi^{-1}(x)$, $x \in V \cup D$ is called a *vertex fiber* if $x \in V$ and a *dart fiber* if $x \in D$.

Let $\pi : \hat{\Gamma} \rightarrow \Gamma$ be a 1-covering of graphs and let ϕ be an automorphism of Γ . We say ϕ *lifts to* $\hat{\Gamma}$, if there exists an automorphism $\hat{\phi}$ of $\hat{\Gamma}$ such that $\pi\hat{\phi} = \phi\pi$. If ϕ lifts, then so does ϕ^{-1} . If all automorphisms in a subgroup A of $\text{Aut}\Gamma$ lift, then all those lifts form a subgroup of $\text{Aut}(\hat{\Gamma})$, denoted $\hat{A} := \text{lift}A$ (see [20, Theorem 5.3] for conditions under which lifts exist). The identity automorphism of Γ always lifts. The group of all lifts of identity is called the group of *deck transformations* and denoted by $\Delta(\pi)$, that is

$$\Delta(\pi) = \{\hat{\phi} \in \text{Aut}(\hat{\Gamma}) \mid \pi\hat{\phi} = \pi\}.$$

A *group action* of an arbitrary group G on graph $\Gamma = (V, D, \iota, \lambda)$ is a group homomorphism $\alpha : G \rightarrow \text{Aut}(\Gamma)$. The action α is called *free* if no vertex of Γ is fixed by a group element other than the identity. This implies that α is injective, so we may think of G as a subgroup of $\text{Aut}(\Gamma)$. If $\pi : \hat{\Gamma} \rightarrow \Gamma$ is a 1-covering of connected graphs, then $\Delta(\pi)$ acts *semi-regularly* on $\hat{\Gamma}$, that is, $\Delta(\pi)$ acts fixed point-freely on the set of vertices and the set of darts of $\hat{\Gamma}$ [13, Proposition 2.4]. If moreover, $\Delta(\pi)$ acts transitively on some, whence each vertex fiber, then π is called *regular*. In what follows we assume $(\hat{\Gamma}, \pi)$ is a regular 1-covering of a connected graph Γ .

2.7. Reflections. For an automorphism σ of a connected graph $\Gamma = (V, D, \iota, \lambda)$ set $\text{Fix}_\sigma(V) := \{v \in V \mid \sigma(v) = v\}$ and $\text{Norm}_\sigma(D) := \{d \in D \mid d \neq \sigma(d) = d^{-1}\}$. The sets $\text{Fix}_\sigma(V)$ and $\text{Norm}_\sigma(D)$ are called the *set of fixed vertices* and the set of *normalized darts* of Γ with respect to the automorphism σ , respectively.

Definition 2.10. An involution σ of a connected graph $\Gamma = (V, D, \iota, \lambda)$ is called a *reflection on* Γ , if $\text{Fix}_\sigma(V) = \emptyset$ and the graph $\Gamma_\sigma = (V, D_\sigma, \iota_\sigma, \lambda_\sigma)$ with $D_\sigma = D \setminus \text{Norm}_\sigma(D)$ and $\iota_\sigma = \iota|_{D_\sigma}$, $\lambda_\sigma = \lambda|_{D_\sigma}$, is disconnected.

One can easily check that an involution σ is a reflection on a graph $\Gamma = (V, D, \iota, \lambda)$ if and only if $\phi\sigma\phi^{-1}$ is a reflection on Γ for every $\phi \in \text{Aut}(\Gamma)$. In this case we have $\text{Fix}_{\phi\sigma\phi^{-1}}(V) = \phi(\text{Fix}_\sigma(V))$ and $\text{Norm}_{\phi\sigma\phi^{-1}}(D) = \phi(\text{Norm}_\sigma(D))$.

2.8. Cayley graphs. Let G be a group and X be a *generating symmetric multiset* of G , that is, $X = X^{-1}$ and there exists a map $\alpha : X \rightarrow G$ with $\langle \alpha(X) \rangle = G$ and $\alpha(x^{-1}) = (\alpha(x))^{-1}$ for all $x \in X$. The *Cayley graph* $\text{Cay}(G, X)$ is the 4-tuple $(G, G \times X, \iota, -1)$ where $\iota(g, x) := g$ and $(g, x)^{-1} = (g\alpha(x), x^{-1})$. If $X \subseteq G$ is a generating symmetric multiset, we consider α as the inclusion map, then $(g, x)^{-1} = (gx, x^{-1})$. If (g, x) is a dart in $\text{Cay}(G, X)$ with initial vertex $g_1 := g$ and terminal vertex $g_2 := gx$, then $g_1^{-1}g_2 = x \in X$, thus the definition of a Cayley graph used here coincides with its classic definition.

The following theorems give new characterizations of a Coxeter group in terms of its Cayley graph.

Theorem 2.11. [13, Theorem 7.6] *The following statements are equivalent:*

- (i) (G, X) is a Coxeter system.
- (ii) (G, X) satisfies the exchange condition.
- (iii) The elements of X act as reflections on $\text{Cay}(G, X)$.

Theorem 2.12. [13, Theorem 8.1] *Let $\pi : \Gamma \rightarrow (V, D, \iota, \lambda)$ be a 1-covering of a monopole admitting semi-edges only. The graph Γ is the Cayley graph of a Coxeter group if and only if π is regular and any deck transformation in $\Delta(\pi)$ that interchanges two neighbouring vertices of Γ acts as a reflection on Γ .*

Remark 2.13. In this work the term “reflection” is used in three different concepts, one as an element of the Weyl group and one in the sense of Definition 2.10 and the other in the sense of [12]. In fact one of the key results of this work is to show that the two first terms do not coincide in general, see Theorem 4.1. In the sequel in order to avoid ambiguity, we use the term “a geometric reflection” when we refer to a Weyl group element, and use the term “a (Cayley) graph-reflection” when we refer to the reflection defined in Definition 2.10, and “a reflection” when we refer to [12].

3. Normalized darts

3.1. General properties of normalized darts. In this section, we investigate the normalized darts of the Cayley graph associated to an extended affine Weyl group of type A_1 . Before that, we establish some general facts about normalized darts of Cayley graphs.

Suppose G is a group with a symmetric generating set $X \subseteq G$, namely $1 \notin X$ and every element of X is of order 2. Let $\Gamma := \text{Cay}(G, X)$ be the Cayley graph of (G, X) . The group G acts on Γ by left multiplication and this action is regular, so we can consider G as a subgroup of $\text{Aut}(\Gamma)$. Since a morphism must preserve the initial vertex and the reverse of every dart, one can check that the right action of G on Γ does not guarantee that the elements of G will be morphism in the sense of [13].

Suppose $1 \neq \sigma \in \text{Aut}(\Gamma)$ is an involution. From Definition 2.10, we have

$$\begin{aligned} \text{Norm}_\sigma(G \times X) &= \{(g, x) \in G \times X \mid (g, x) \neq \sigma(g, x) = (g, x)^{-1}\} \\ &= \{(g, x) \in G \times X \mid (g, x) \neq \sigma(g, x) = (gx, x)\}. \end{aligned} \quad (13)$$

We note that $d \in \text{Norm}_\sigma(G \times X)$ if and only if $d^{-1} \in \text{Norm}_\sigma(G \times X)$.

Lemma 3.1. *Let $x' \in X$. Then $(g, x) \in \text{Norm}_{x'}(G \times X)$ if and only if $x'g = gx$.*

Proof. If $(g, x) \in \text{Norm}_{x'}(G \times X)$, then by (13), we have $x'(g, x) = (x'g, x) = (gx, x)$. On the other hand $(x'g, x)$ is a dart in $\Gamma = \text{Cay}(G, X)$ with initial vertex $x'g$ and terminal vertex $x'gx$, and (gx, x) is a dart with initial vertex gx and terminal vertex x . Since Γ has no multiple darts, we must have $x'g = gx$. That is the initial vertices must coincide.

Conversely suppose (g, x) is a dart in Γ and $x'g = gx$. We have $x'(g, x) = (x'g, x) = (gx, x) = (g, x)^{-1}$. Since $x' \neq 1$, we have $x'(g, x) \neq (g, x)$ and so (g, x) is a normalized dart of x' . ■

Lemma 3.2. *Suppose g is an arbitrary vertex of the Cayley graph $\Gamma = \text{Cay}(G, X)$. Then with respect to an involution $x \in X$, there is at most one normalized dart in Γ with initial vertex g .*

Proof. Assume $d_1 = (g, x_1)$ and $d_2 = (g, x_2)$ are two normalized darts of involution x with initial vertex g , then by Lemma 3.1 we must have $gx_1 = gx_2$, forcing $x_1 = x_2$. ■

3.2. Normalized darts of a Cayley graph of an EAWG of type A_1 . In what follows, we assume that R is an extended affine root system of type A_1 and nullity $\nu > 0$, let \mathcal{W} be the Weyl group of R ; we use the same notation as in Section 2 without further reference. In particular, we have

- $R = (S + S) \cup (\dot{R} + S)$, where $\dot{R} = \{0, \pm\epsilon\}$ with $(\epsilon, \epsilon) = 2$,
- S is a semilattice in \mathcal{V}^0 ,
- $S = \cup_{i=0}^m (\tau_i + 2\Lambda)$ where $m := \text{ind}(S)$ and τ_i 's represent distinct cosets of 2Λ in Λ ,
- $\Lambda := \langle S \rangle$ has a \mathbb{Z} -basis $= \{\sigma_1, \dots, \sigma_\nu\}$, satisfying $\sigma_r \in S$ for all r , accordingly we have $\tau_i = \sum_{r=1}^\nu n_{i,r} \sigma_r$, $i = 1, \dots, m$, $n_{i,r} \in \{0, 1\}$,
- $\Pi = \{\alpha_i = r_i \tau_i + s_i \epsilon \mid 0 \leq i \leq m\}$ with $r_i, s_i \in \{\pm 1\}$, is a reflectable base of R ,
- $S_\Pi := \{w_\alpha \mid \alpha \in \Pi\}$.

Note that $2\Lambda \subseteq S$ and $\langle \Pi \rangle = \langle R \rangle = \mathbb{Z}\epsilon + \Lambda$, thus without loss of generality we assume that $\tau_0 = 0$ and τ_1, \dots, τ_m generate Λ , and $\alpha_0 = \epsilon$. Also as a reflectable base will remain a reflectable base under arbitrary sign change of its elements, and that $\tau_i + 2\Lambda = -\tau_i + 2\Lambda$, we may assume that any Π is of the form

$$\Pi = \{\alpha_0 := \epsilon, \alpha_i = \tau_i - \epsilon \mid 1 \leq i \leq m\}. \tag{14}$$

Now let $\alpha, \beta \in \Pi$. By Lemma 3.1, $(w, w_\alpha) \in \text{Norm}_{w_\beta}(\mathcal{W} \times S_\Pi)$ if and only if $w_\beta w = w w_\alpha$. Thus (w, w_α) is a normalized dart of w_β if and only if $w(\alpha) = \pm\beta$. So to characterize the normalized darts of geometric reflections, we need to compute the orbits of elements of Π under the Weyl group action.

Lemma 3.3. *Considering the unique expression of an element $w \in \mathcal{W}$ of the form $w = w_\epsilon^n \prod_{r=1}^\nu t_r^{m_r} z$ given in (7), we have*

$$w(\alpha_i) = \begin{cases} (-1)^n \epsilon + \sum_{r=1}^\nu 2m_r \sigma_r & i = 0, \\ -(-1)^n \epsilon + \sum_{r=1}^\nu (n_{i,r} - 2m_r \sigma_r) & 1 \leq i \leq m. \end{cases}$$

Proof. Suppose $w = w_\epsilon^n \prod_{r=1}^\nu t_r^{m_r} z$ is of the form given in the statement. Since, $\alpha_i \in \dot{\mathcal{V}} \oplus \mathcal{V}^0$, $0 \leq i \leq m$, we have using (8) that $z(\alpha_i) = \alpha_i$. Also for every $1 \leq r \leq \nu$, we compute easily that $t_r^{m_r}(\epsilon) = \epsilon + 2m_r \sigma_r$ and then $\prod_{r=1}^\nu t_r^{m_r}(\epsilon) = \epsilon + \sum_{r=1}^\nu 2m_r \sigma_r$. As $\alpha_0 = \epsilon$, we have

$$w(\alpha_0) = w_\epsilon^n \prod_{r=1}^\nu t_r^{m_r} z(\epsilon) = w_\epsilon^n \prod_{r=1}^\nu t_r^{m_r}(\epsilon) = w_\epsilon^n (\epsilon + \sum_{r=1}^\nu 2m_r \sigma_r) = (-1)^n \epsilon + \sum_{r=1}^\nu 2m_r \sigma_r.$$

Now assume $1 \leq i \leq m$, since $\alpha_i = \tau_i - \epsilon$ and $\tau_i = \sum_{r=1}^{\nu} n_{i,r} \sigma_r$ by the same argument as above we have

$$\begin{aligned} w(\alpha_i) &= w_{\epsilon}^n \prod_{r=1}^{\nu} t_r^{m_r} z(\tau_i - \epsilon) = -(-1)^n \epsilon - \sum_{r=1}^{\nu} 2m_r \sigma_r + \tau_i \\ &= -(-1)^n \epsilon - \sum_{r=1}^{\nu} 2m_r \sigma_r + \sum_{r=1}^{\nu} n_{i,r} \sigma_r = -(-1)^n \epsilon + \sum_{r=1}^{\nu} (n_{i,r} - 2m_r) \sigma_r. \quad \blacksquare \end{aligned}$$

Theorem 3.4. *Suppose \mathcal{W} is an extended affine Weyl group of type A_1 with nullity ν , and Γ is the Cayley graph of \mathcal{W} with respect to the generating set S_{Π} , then for $0 \leq i \leq m$, we have,*

$$\begin{aligned} \text{Norm}_{w_{\alpha_i}}(\mathcal{W} \times S_{\Pi}) &= \{(w, w_{\alpha_i}) \mid w \in \mathcal{Z}(\mathcal{W}) \text{ or } w = w_{\epsilon} \prod_{r=1}^{\nu} t_r^{n_{i,r}} z\} \\ &= \{(w, w_{\alpha_i}) \mid w \in w_{\alpha_i}^n \mathcal{Z}(\mathcal{W}), n = 0, 1\}. \end{aligned}$$

Proof. Let $\beta \in \Pi$. By Lemma 3.1, $(w, w_{\alpha}) \in \text{Norm}_{w_{\beta}}(\mathcal{W} \times S_{\Pi})$ if and only if $w_{\beta} w = w w_{\alpha}$. Thus (w, w_{α}) is a normalized dart of w_{β} if and only if $w(\alpha) = \pm \beta$. Therefore, we first check that if $\alpha \in \Pi$, then for which $w \in \mathcal{W}$ the equation

$$w(\alpha) \in \pm \Pi, \tag{15}$$

holds. By (7), each $w \in \mathcal{W}$ has a unique expression of the form $w = w_{\epsilon}^n \prod_{r=1}^{\nu} t_r^{m_r} z$. For $0 \leq i \leq m$, consider the formulas given in Lemma 3.3.

Suppose $i = 0$; we want to have $w(\alpha_0) = (-1)^n \epsilon + \sum_{r=1}^{\nu} 2m_r \sigma_r \in \pm \Pi$. By considering the elements of Π , two cases may happen,

Case 1: $m_r = 0$ for all $1 \leq r \leq \nu$, then $w(\alpha_0) = \pm \epsilon = \pm \alpha_0 \in \pm \Pi$.

Case 2: There exists at least one $1 \leq r \leq \nu$ such that $m_r \neq 0$. Then $w(\alpha_0) \in \pm \Pi$ forces that $w(\alpha_0) = \pm \alpha_i$ for some $1 \leq i \leq m$, implying that

$$(-1)^n \epsilon + \sum_{r=1}^{\nu} 2m_r \sigma_r = \pm(\tau_i - \epsilon).$$

Linearly independence of σ_r 's implies that $n_{i,r} = \pm 2m_r$ for each $1 \leq r \leq \nu$, but as $n_{i,r} \in \{0, 1\}$ and by our assumption $m_r \neq 0$ for at least one r , this gives a contradiction. Therefore if $i = 0$, (15) holds only if $m_r = 0$ for all r , namely $w = z$ or $w = w_{\epsilon} z$ for some $z \in \mathcal{Z}(\mathcal{W})$ and then $w(\alpha_0) = \pm \alpha_0$.

Next suppose $1 \leq i \leq m$. We want to have $w(\alpha_i) = -(-1)^n \epsilon + \sum_{r=1}^{\nu} (n_{i,r} - 2m_r) \sigma_r \in \pm \Pi$. Let $n = 0$; three cases may happen:

- (1) $w(\alpha_i) = -\epsilon = -\alpha_0$,
- (2) $w(\alpha_i) = -\epsilon + \sum_{r=1}^{\nu} (n_{i,r} - 2m_r) \sigma_r = \alpha_j = \tau_j - \epsilon$ for some $j \neq i$,
- (3) $w(\alpha_i) = \alpha_i$.

In the first case, we get $\sum_{r=1}^{\nu} (n_{i,r} - 2m_r) \sigma_r = 0$; but this implies that $n_{i,r} = 2m_r$ for every $1 \leq r \leq \nu$ and then the only possible case is $n_{i,r} = m_r = 0$ for every r and then $\alpha_i = \alpha_0$ which is absurd.

In the second case, as $\tau_j = \sum_{r=1}^{\nu} n_{j,r} \sigma_r$ and σ_r 's are linearly independent, we conclude that $\tau_i = \tau_j$; this is a contradiction again. Thus for $1 \leq i \leq m$, the only possible case when $n = 0$ is $w(\alpha_i) = \alpha_i$. But this is possible only if $m_r = 0$ for every $1 \leq r \leq \nu$, namely $w = z$ for some $z \in \mathcal{Z}(\mathcal{W})$.

If $n = 1$, by a similar argument as above we conclude that the only possible case is $w(\alpha_i) = -\alpha_i$ and that $m_r = n_{i,r}$ for every $1 \leq r \leq \nu$ and so $w = w_\epsilon \prod_{r=1}^\nu t_r^{n_{i,r}} z$.

Next, we fix $0 \leq i \leq m$. If (w, w_α) is a normalized dart of w_{α_i} then the above argument implies that $w = z$ or $w = w_\epsilon \prod_{r=1}^\nu t_r^{n_{i,r}} z$. Then $w(\alpha_i) = \pm\alpha_i$ and for each $\alpha \in \Pi \setminus \{\alpha_i\}$, $w(\alpha) \neq \pm\alpha_i$. Then by using the characterization of normalized darts of w_{α_i} given in Lemma 3.1, the only possible choice for α is $\alpha = \alpha_i$.

Furthermore for every $0 \leq i \leq m$, $w_\epsilon \prod_{r=1}^\nu t_r^{n_{i,r}}(\alpha_i) = -\alpha_i$ and w_{α_i} is the unique geometric reflection with this property, then $w = w_\epsilon \prod_{r=1}^\nu t_r^{n_{i,r}} z = w_{\alpha_i} z$. In fact $(w, w_\alpha) \in \text{Norm}_{w_{\alpha_i}}(\mathcal{W} \times S_\Pi)$ only if $\alpha = \alpha_i$ and $w = w_{\alpha_i}^n z$ for $n = 0, 1$ and $z \in \mathcal{Z}(\mathcal{W})$. This gives the equation in the statement and completes the proof. \blacksquare

4. Geometric reflections via Cayley graph-reflections

In this section we use the theory of normalized darts in Cayley graphs to show that in extended affine Weyl groups, two concepts of geometric reflections and Cayley graph-reflections on minimal generating sets of Weyl groups coincide if and only if the nullity of the corresponding root system is less than or equal one. More precisely, we show that in type A_1 , if $\nu > 1$, then no geometric reflection based on an element of a reflectable base of R is a Cayley graph-reflection on the corresponding Cayley graph, and in other types, with respect to any minimal generating set, one can always find some geometric reflections which are not Cayley graph-reflections. We start with an A_1 -type extended affine Weyl group and then extend it to the general case.

Theorem 4.1. *Let $R = (S + S) \cup (\dot{R}^\times + S)$ be an extended affine root system of type A_1 of nullity ν , Π be a reflectable base for R and $\alpha \in \Pi$. Then the geometric reflection w_α is a graph-reflection of the Cayley graph of (\mathcal{W}, Π) if and only if $\nu \leq 1$.*

Proof. Let $\nu > 1$ and $\alpha \in \Pi$. Assume by contradiction that w_α acts as a (Cayley) graph-reflection on $\Gamma = \text{Cay}(\mathcal{W}, S_\Pi)$. In accordance with the notation of Section 3.2, we may assume that $\Pi = \{\alpha_0 = \epsilon, \alpha_1 = \tau_1 - \epsilon, \dots, \alpha_m = \tau_m - \epsilon\}$, where $S = \cup_{i=0}^m (\tau_i + 2\Lambda)$, $m = \text{ind}(S)$, $\tau_0 = 0$ and τ_i 's represent distinct cosets of 2Λ in Λ . Since $\text{ind}(S) = m \geq \nu > 1$, we can choose $\tau_r, \tau_s \in \{\tau_1, \dots, \tau_m\}$ such that $1 \leq r \neq s \leq m$. By [8, Lemma 2.1], we know that $z := (w_\epsilon w_{\tau_r - \epsilon} w_{\tau_s - \epsilon})^2$ is a central element of \mathcal{W} . For simplicity, we set $w_0 := w_\epsilon$ and $w_i := w_{\alpha_i} = w_{\tau_i - \epsilon}$. Then,

$$z = w_\epsilon w_{\tau_r - \epsilon} w_{\tau_s - \epsilon} w_\epsilon w_{\tau_r - \epsilon} w_{\tau_s - \epsilon} = w_0 w_r w_s w_0 w_r w_s,$$

and since $w_s = z w_s z^{-1}$, then

$$w_s = w_0 w_r w_s w_0 w_r w_s w_r w_0 w_s w_r w_0. \tag{16}$$

This then gives $w_r = w_s w_0 w_r w_s w_r w_0 w_s w_r w_0 w_s w_0. \tag{17}$

Now based on relation (17) we select the path

$$\gamma_{r,z} := (z, w_s)(z w_s, w_0)(z w_s w_0, w_r) \cdots (z w_s w_0 w_r \cdots w_s, w_0), \tag{18}$$

in the Cayley graph of (\mathcal{W}, Π) from z to $z w_s w_0 w_r w_s w_r w_0 w_s w_r w_0 w_s w_0 = z w_r$.

By Theorem 3.4, the only darts in $\gamma_{r,z}$ which can be normalized by w_r , are

$$d = (z w_s w_0, w_r), \quad d' = (z w_s w_0 w_r w_s, w_r), \quad d'' = (z w_s w_0 w_r w_s w_r w_0 w_s, w_r).$$

But

$$\begin{aligned} zw_s w_0(\alpha_r) &= -\epsilon + \tau_r + 2\tau_s \neq \pm\alpha_r, \\ zw_s w_0 w_r w_s(\alpha_r) &= -\epsilon + 2\tau_r \neq \pm\alpha_r, \\ zw_s w_0 w_r w_s w_r w_0 w_s(\alpha_r) &= \epsilon - 3\tau_r - 2\tau_s \neq \pm\alpha_r. \end{aligned}$$

So none of d , d' and d'' are normalized darts of w_r .

Next, suppose γ is a path in $\Gamma = \text{Cay}(\mathcal{W} \times S_\Pi)$ from $1_{\mathcal{W}}$ to an arbitrary vertex w . If there exists a normalized dart d_z of w_r with initial vertex z in the path γ , we replace this dart by the path $\gamma_{r,z}$ which is a path without any normalized dart. We do this for all normalized darts in the path γ to obtain a path η from $1_{\mathcal{W}}$ to w . Clearly η contains no normalized dart of w_r . Then after deleting normalized darts of w_r from the Cayley graph Γ , the graph Γ_{w_r} introduced in Definition 2.10, is a connected graph. This shows that any two vertex of Γ can be connected by a path without any normalized darts of w_r . So w_r is not a Cayley graph-reflection. This contradiction shows that $\nu \leq 1$.

Conversely, let $\nu \leq 1$. If $\nu = 0$, then without loss of generality we may assume that $\Pi = \{\epsilon\}$. The Cayley graph of \mathcal{W} with respect to this reflectable base is the 4-tuple $\Gamma = (\{1, w_\epsilon\}, \{(1, w_\epsilon), (w_\epsilon, w_\epsilon)\}, \iota, \lambda)$ where ι, λ are defined as in Section 3.1. Since $\nu = 0$, $\mathcal{Z}(\mathcal{W}) = \{1\}$ and so by Theorem 3.4,

$$\text{Norm}_{w_\epsilon}(\mathcal{W} \times S_\Pi) = \{(1, w_\epsilon), (w_\epsilon, w_\epsilon)\}.$$

Clearly when we delete the normalized darts of w_ϵ from Γ , the graph Γ_{w_ϵ} will be disconnected, that is w_ϵ is a Cayley graph-reflection.

Next suppose $\nu = 1$; then $\text{ind}(S) = 1$, $\Pi = \{\alpha_0 = \epsilon, \alpha_1 = \tau_1 - \epsilon\}$ and $\mathcal{Z}(\mathcal{W}) = \{1\}$. Let $\Gamma = \text{Cay}(\mathcal{W}, S_\Pi)$ and $w_i := w_{\alpha_i}$, $i = 0, 1$. From Theorem 3.4,

$$\text{Norm}_{w_i}(\mathcal{W} \times S_\Pi) = \{(1, w_i), (w_i, w_i)\}$$

for $i = 0, 1$. Suppose $i = 0$ and consider the graph Γ_{w_0} introduced in Definition 2.10. We claim that this is a disconnected graph. We prove this by showing that there is no path in Γ_{w_0} from 1 to w_0 . Suppose to the contrary that γ is such a path. The path γ must be of the following form,

$$\gamma = (1, w_1)(w_1, w_0)(w_1 w_0, w_1) \dots (w_1 w_0 \dots, w_1),$$

then we have $\iota(\gamma^{-1}) = w_1 w_0 \dots w_0 w_1 = w_0 = \iota((1, w_0)^{-1})$.

This implies that $\text{ord}(w_1 w_0) < \infty$ and this is a contradiction; thus Γ_{w_0} is disconnected and so w_0 is a Cayley graph-reflection. Using a similar argument we see that w_1 is a Cayley graph-reflection. ■

In the proof of Theorem 4.1, we provided a path $\gamma_{r,z}$ from $1_{\mathcal{W}}$ to w_r , without any normalized darts of w_{α_r} . In the following example we show that this path is not unique.

Example 4.2. Let $R = \Lambda \cup (\pm\epsilon + \Lambda)$, where $\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$. We know R is an extended affine root system of type A_1 with nullity $\nu = 2$, and that

$$\Pi = \{\alpha_0 := \epsilon, \alpha_1 := \sigma_1 - \epsilon, \alpha_2 := \sigma_2 - \epsilon, \alpha_3 := \sigma_1 + \sigma_2 - \epsilon\}$$

is a reflectable base for R . Let \mathcal{W} be the Weyl group of R and consider $\Gamma = \text{Cay}(\mathcal{W}, S_\Pi)$. According to Theorem 4.1, w_{α_i} 's are not Cayley graph-reflections. In what follows, we provide a path, different from the one given in the proof of Theorem 4.1, from $1_{\mathcal{W}}$ to an arbitrary vertex of Γ , without any normalized darts of w_{α_1} .

Set $w_i := w_{\alpha_i}$ for $0 \leq i \leq 3$. By [3, Section 2], the element $z := w_3w_2w_0w_1$ is a central element of \mathcal{W} and so

$$w_1 = w_3w_1w_0w_2w_3w_2w_0 \tag{19}$$

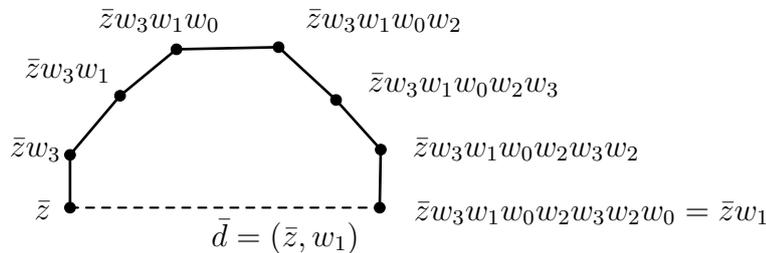
is an expression for w_1 in terms of roots in Π . Let γ be a path from $1_{\mathcal{W}}$ to an arbitrary vertex $\bar{w} \in \mathcal{W}$. Assume that \bar{d} is a normalized dart of w_1 in γ . By Theorem 3.4, $\bar{d} \in \{(\bar{z}, w_{\alpha_1}), (\bar{z}w_{\alpha_1}, w_{\alpha_1})\}$ where $\bar{z} \in \mathcal{Z}(\mathcal{W})$. We now want to replace this dart by a path γ_1 such that no darts in γ_1 is a normalized dart of w_1 . We set

$$\gamma_1 := d_1d_2d_3d_4d_5d_6d_7,$$

$$d_1 := (\bar{z}, w_3), \quad d_2 := (\bar{z}w_3, w_1), \quad d_3 := (\bar{z}w_3w_1, w_0), \quad d_4 := (\bar{z}w_3w_1w_0, w_2),$$

$$d_5 := (\bar{z}w_3w_1w_0w_2, w_3), \quad d_6 := (\bar{z}w_3w_1w_0w_2w_3, w_2), \quad d_7 := (\bar{z}w_3w_1w_0w_2w_3w_2, w_0).$$

By Theorem 3.4, the only dart in γ_1 which can be a normalized dart of w_1 is $d_2 = (\bar{z}w_3, w_1)$. But in this case, by Lemma 3.1 we must have $\bar{z}w_3(\alpha_1) = \pm\alpha_1$ which does not hold. Therefore d_2 is not a normalized dart of w_1 . Next, the diagram



shows that the path γ_1 connects the initial vertex of $\bar{d} = (\bar{z}, w_1)$ to its terminal vertex without passing from any normalized dart of w_1 . So by replacing the normalized dart \bar{d} by the path γ_1 , we can reduce the number of normalized darts in the path γ . By repeating this process, we get a substitute path with no normalized dart of w_1 from $1_{\mathcal{W}}$ to \bar{w} .

Theorem 4.3. *Let R be an extended affine root system of type A_1 of nullity ν , and \mathcal{W} be its corresponding Weyl group. Assume Π is a reflectable base of R . Then (\mathcal{W}, Π) is a Coxeter system if and only if $\nu \leq 1$.*

Proof. Suppose Π is a reflectable base of R . If $\nu \leq 1$, then as a consequence of Theorems 2.11 and 4.1, \mathcal{W} is a Coxeter group with respect to Π .

Now suppose $\nu > 1$, Π is a reflectable base and to the contrary assume that (\mathcal{W}, Π) is a Coxeter system. Now by Theorem 2.11, (\mathcal{W}, Π) is a Coxeter system if and only if the geometric reflections w_α , $\alpha \in \Pi$ and Cayley graph-reflections of $\Gamma = \text{Cay}(\mathcal{W}, S_\Pi)$ coincide, but by Theorem 4.1 this happens if and only if $\nu \leq 1$. ■

Theorem 4.4. *Let R be an extended affine root system of type $X \neq BC_1$ and nullity ν , with extended affine Weyl group \mathcal{W} . Then \mathcal{W} is a Coxeter group if and only if $\nu \leq 1$.*

Proof. It is known that Weyl groups of finite and affine root systems (extended affine roots systems of nullities 0 and 1) are Coxeter groups with respect to any fundamental system. Next, to the contrary assume that $\nu > 1$, $X \neq BC_1$, and Π is a subset of R^\times such that (\mathcal{W}, Π) is a Coxeter system. The set of all reflections

of (\mathcal{W}, Π) is the set T consisting of reflections w_α , $\alpha \in \tilde{R}^\times := \mathcal{W}\Pi$. By [16, Proposition 5.9], $\tilde{R} := \tilde{R}^\times \cup ((\tilde{R}^\times - \tilde{R}^\times) \cap \mathcal{V}^0)$ is an extended affine root system. Since $\text{span}_{\mathbb{R}} \tilde{R}^\times = \text{span}_{\mathbb{R}} R^\times = \text{span}_{\mathbb{R}} \Pi$, \tilde{R} has the same rank and nullity as R . We note that \tilde{R} and R have the same Weyl group \mathcal{W} , thus as $\mathcal{W} = \langle w_\alpha \mid \alpha \in \tilde{R}^\times \rangle$, \mathcal{W} is a reflection subgroup of (\mathcal{W}, Π) ; now by Theorem 2.7(iii), \mathcal{W} is a Coxeter group with respect to the Coxeter generators $\chi(\mathcal{W})$ introduced in (12). By Remark 2.5(i), $S_\Pi = \mathcal{W} \cap S_\Pi = \chi(\mathcal{W}) \cap S_\Pi$. In particular, $S_\Pi \subseteq \chi(\mathcal{W})$. Now the minimality of $\chi(\mathcal{W})$ implies that $\chi(\mathcal{W}) = S_\Pi$. Thus if \tilde{T} is the set of all reflections of $(\mathcal{W}, \chi(\mathcal{W}))$, we have $\tilde{T} = T = S_{\tilde{R}^\times}$.

We now fix $\alpha \in \tilde{R}^\times$. We take $\alpha \in \tilde{R}_{sh}$ if \tilde{R} is of reduced type and $\alpha \in \tilde{R}_{lg}$ if \tilde{R} is of type $BC_\ell (\ell > 1)$. Let \tilde{R}_α be the extended affine root system of nullity ν and type A_1 constructed in Lemma 2.1, with Weyl group \mathcal{W}_α . As $\tilde{R}_\alpha^\times \subseteq \tilde{R}^\times$, \mathcal{W}_α is a reflection subgroup of \mathcal{W} and so by Theorem 2.7(iii), \mathcal{W}_α is a Coxeter group with respect to the set of Coxeter generators $\chi(\mathcal{W}_\alpha) \subseteq \mathcal{W}_\alpha \cap T$. Let $\tilde{T}_\alpha = \{w_\beta \mid \beta \in \tilde{R}_\alpha^\times\}$.

Claim: $\mathcal{W}_\alpha \cap T = \tilde{T}_\alpha$. We only need to show that $\mathcal{W}_\alpha \cap T \subseteq \tilde{T}_\alpha$. Let $w \in \mathcal{W}_\alpha \cap T$, then there exist $\beta \in \tilde{R}^\times$ and $\beta_1, \dots, \beta_n \in \tilde{R}_\alpha^\times$ such that $w = w_\beta = w_{\beta_1} \cdots w_{\beta_n}$. It follows that n is an odd integer. As for each i , $\beta_i \in \tilde{R}_\alpha^\times = \pm\alpha + S_\alpha$, by changing β_i to $-\beta_i$ if necessary, we may assume that $\beta_i = \alpha + \sigma_i$ for some $\sigma_i \in S_\alpha$. Now $w_\beta = w_{\beta_1} \cdots w_{\beta_n}$ implies that,

$$-\beta = w_{\beta_1} \cdots w_{\beta_n}(\beta) = \beta + r(\alpha + \sigma)$$

where $\sigma \in \langle S_\alpha \rangle$ and $r = (\beta, \check{\alpha})$. Since $\beta = -\frac{1}{2}r(\alpha + \sigma)$, it follows from our choice of α ($\alpha \in \tilde{R}_{lg}$ for type BC and $\alpha \in \tilde{R}_{sh}$ for other types) that $-\frac{1}{2}r = \pm 1$ and then $\beta = \pm(\alpha + \sigma) \in \tilde{R}_\alpha^\times$, so $w_\beta \in \tilde{T}_\alpha$, and the claim is established.

The claim now shows that $\chi(\mathcal{W}_\alpha) \subseteq \tilde{T}_\alpha$ and so $\chi(\mathcal{W}_\alpha) = S_{\tilde{\Pi}_\alpha}$ for some subset $\tilde{\Pi}_\alpha$ of \tilde{R}_α^\times . Then $\mathcal{W}_\alpha = \langle \chi(\mathcal{W}_\alpha) \rangle$ (see [12, Theorem 3.3](i)) and

$$\tilde{T}_\alpha = \bigcup_{w \in \mathcal{W}_\alpha, \beta \in \tilde{\Pi}_\alpha} w w_\beta w^{-1}.$$

This in particular gives $\tilde{R}_\alpha^\times \subseteq \mathcal{W}_\alpha \tilde{\Pi}_\alpha$, that is $\tilde{\Pi}_\alpha$ is a reflectable set of \tilde{R}_α and since any set of Coxeter generators is a minimal set of generators, $\tilde{\Pi}_\alpha$ is a reflectable base of \tilde{R}_α . Thus by Theorem 2.7(iii), $(\mathcal{W}_\alpha, \tilde{\Pi}_\alpha)$ is a Coxeter system, contradicting Theorem 4.3. ■

Corollary 4.5. *Let R be an extended affine root system of type $X \neq BC_1$ and nullity $\nu > 1$, with extended affine Weyl group \mathcal{W} and assume $\Pi \subseteq R^\times$ such that S_Π is a generating set of \mathcal{W} . Then there exist geometric reflections in S_Π , which are not Cayley graph-reflections on $\text{Cay}(\mathcal{W}, \Pi)$.*

Proof. The proof is a consequence of Theorems 2.11 and 4.4. ■

Remark 4.6. Theorem 4.4 is a byproduct of our general results on reflections, in particular it recovers [16, Theorem 3.6] (or [17, Theorem 2.9]). We emphasize that we have excluded type BC_1 in this theorem, since this case needs a special treatment and a more careful consideration. In this context, we should also mention that the statement and the proof of Lemma [16, Lemma 5.7] needs to be more precise for the case $\ell = 1$.

5. Appendix: Examples

This section is devoted to some examples elaborating on the results in the previous sections. We provide several examples of extended affine Weyl groups of low nullities and discuss the behaviour of geometric reflections and Cayley graph-reflections. We begin with some details about Cayley graphs.

Let G be a group and $S \subseteq G \setminus \{1\}$ be a generating set of G such that $s^2 = 1$ for every $s \in S$. Set $\Gamma := \text{Cay}(G, S)$ and $n := |S|$. By [13, Theorem 5.6], Γ is a regular 1-cover of the monopole $(\{G\}, S, \iota, \lambda)$ and so is a n -regular connected graph which is undirected. Every relation $s_1 s_2 \cdots s_n = 1$ in G , creates a n -gon in Γ . Let $\Gamma = \text{Cay}(G, S)$, we say, the dart d is of class s and write $d \in [s]$, if $d = (g, s)$ for some $g \in G$. As $s^2 = 1$ for every $s \in S$, $d \in [s]$ if and only if $d^{-1} \in [s]$.

Example 5.1. (Finite type A_1). We know that the Weyl group of a finite root system of type A_1 is isomorphic to $\mathcal{W} = (x \mid x^2)$. The Cayley graph of \mathcal{W} with respect to $X = \{x\}$ is of the form:

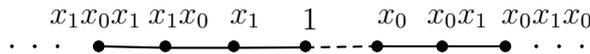


By Theorem 3.4, $\text{Norm}_x(\mathcal{W} \times X) = \{(1, x), (x, x)\}$ and then the graph Γ_x is a disconnected graph, that is the geometric reflection x is a Cayley graph-reflection.

Example 5.2. (Affine type A_1). According to [6, Remark 4.6], with respect to any reflectable base, \mathcal{W} has the presentation:

$$\mathcal{W} = D_\infty = (x_0, x_1 \mid x_0^2 = x_1^2).$$

The Cayley graph of \mathcal{W} with respect to $S = \{x_0, x_1\}$ is

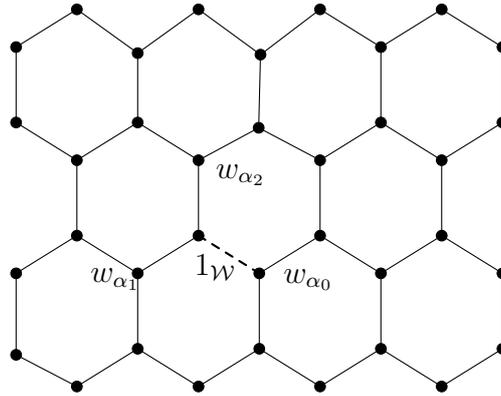


As \mathcal{W} has trivial center, by Theorem 3.4, $\text{Norm}_{x_i}(\mathcal{W} \times S) = \{(1, x_i), (x_i, x_i)\}$ for $i = 0, 1$. Clearly the graphs Γ_{x_0} and Γ_{x_1} are disconnected and then x_0 and x_1 are Cayley graph-reflections in the sense of Definition 2.10. In the above graph, normalized darts of x_0 are shown in dashed line.

Example 5.3. (Baby extended affine of type A_1 , nullity 2). Let R be an A_1 -type extended affine root system of nullity 2, such that the involved semilattice (see (1)) is not a lattice. Let $S = \{x_0, x_1, x_2\}$ be a set with three elements. According to [6, Theorem 5.3], with respect to any reflectable base, \mathcal{W} has the presentation

$$\mathcal{W} = (S \mid x^2, (xyz)^2, x, y, z \in S). \tag{20}$$

In what follows, we show that the Cayley graph $\Gamma = \text{Cay}(\mathcal{W}, S)$ has the following graph, in particular it is 3-regular graph:



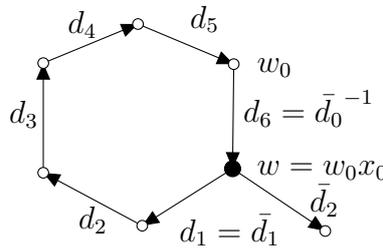
We start by considering an arbitrary element $w \in \mathcal{W}$ as a vertex of Γ and fix it. Let $w = x_{i_1} \cdots x_{i_n}$ where $i_j \in \{0, 1, 2\}$. Without loss of generality, we may assume that $x_{i_n} = x_0$ (the argument for $x_{i_n} = x_1$ or x_2 is the same). Set $w_0 = x_{i_1} \cdots x_{i_{n-1}}$; then we have $w = w_0x_0$. We know there exist only three darts in Γ with initial vertex w , these darts are $\bar{d}_0 = (w, x_0)$, $\bar{d}_1 = (w, x_1)$ and $\bar{d}_2 = (w, x_2)$. We want to introduce three closed paths γ_1 , γ_2 and γ_3 of length 6 based on the vertex w .

The first closed path is $\gamma_1 = d_1d_2d_3d_4d_5d_6$ in which

$$d_1 := (w_0x_0, x_1), \quad d_2 := (w_0x_0x_1, x_2), \quad d_3 := (w_0x_0x_1x_2, x_0), \quad d_4 := (w_0x_0x_1x_2x_0, x_1),$$

$$d_5 := (w_0x_0x_1x_2x_0x_1, x_2), \quad d_6 := (w_0x_0x_1x_2x_0x_1x_2, x_0) = (w_0, x_0).$$

Note that $d_1 = \bar{d}_1$ and $d_6 = \bar{d}_0^{-1}$. The following graph shows the path γ_1 .



The second closed path is $\gamma_2 = d'_1d'_2d'_3d'_4d'_5d'_6$ in which

$$d'_1 := (w_0x_0, x_2), \quad d'_2 := (w_0x_0x_2, x_1), \quad d'_3 := (w_0x_0x_2x_1, x_0), \quad d'_4 := (w_0x_0x_2x_1x_0, x_2),$$

$$d'_5 := (w_0x_0x_2x_1x_0x_2, x_1), \quad d'_6 := (w_0x_0x_2x_1x_0x_2x_1, x_0) = (w_0, x_0).$$

Note that $d'_1 = \bar{d}_2$ and $d'_6 = d_6 = \bar{d}_0^{-1}$.

Finally the third closed path of length 6 is $\gamma_3 = d''_1d''_2d''_3d''_4d''_5d''_6$ where

$$d''_1 := (w_0x_0, x_1), \quad d''_2 := (w_0x_0x_1, x_0), \quad d''_3 := (w_0x_0x_1x_0, x_2), \quad d''_4 := (w_0x_0x_1x_0x_2, x_1),$$

$$d''_5 := (w_0x_0x_1x_0x_2x_1, x_0), \quad d''_6 := (w_0x_0x_1x_0x_2x_1x_0, x_2).$$

The terminal vertex of d''_6 is $w_0x_0x_1x_0x_2x_1x_0x_2 = w_0x_0 = w$; also $d''_1 = d_1 = \bar{d}_1$ and $d''_6 = d''_1^{-1} = \bar{d}_2^{-1}$.

One notes that as any other relation in \mathcal{W} is a consequence of relations x_i^2 and $(x_ix_jx_k)^2$, the corresponding graphs are embedded in the given hexagons.

(ii) Let $\Gamma = \text{Cay}(\mathcal{W}, S_\Pi)$ where \mathcal{W} is the Weyl group of an extended affine root system R of nullity ν , and Π is a reflectable base of R . Assume that $\alpha \in \Pi$. By Lemma 3.1, we have

$$\{(z, w_\alpha), (zw_\alpha, w_\alpha) \mid z \in \mathcal{Z}(\mathcal{W})\} \subseteq \text{Norm}_{w_\alpha}(\mathcal{W} \times S_\Pi). \tag{21}$$

Now Theorem 3.4 implies that the inclusion (21) is in fact an equality if R is of type A_1 . However this is not the case in general, as from Example 5.4 and part (i) of this remark, we see that $\text{Norm}_{w_\alpha}(\mathcal{W} \times S_\Pi) \cap [w_\beta] \neq \emptyset$ for $\beta \in \Pi$ with $\beta \neq \alpha$.

(iii) Assume that a geometric reflection w_α is a Cayley graph-reflection; then as a consequence of [13, Proposition 4.1, Corollary 4.2], the graph Γ_{w_α} consists of two isomorphic connected components. Furthermore one can check that, in all of the above examples, the connected components of Γ_{w_α} and Γ_{w_β} for $\alpha, \beta \in \Pi$, have the same diagram.

Example 5.6. This example extends Example 5.3 to simply laced extended affine Weyl groups of rank and nullity > 1 , namely it shows that any geometric reflection corresponding to the considered underlying reflectable base, is not a Cayley graph-reflection. To show this, let R be an extended affine root system of simply laced type X , rank $\ell > 1$ and nullity $\nu > 1$. We know that $R = \dot{R} + \Lambda$ where \dot{R} is an irreducible finite root system of type X and Λ is a lattice of rank ν . We fix a basis $\dot{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}$ of \dot{R} and a \mathbb{Z} -basis $\{\sigma_1, \dots, \sigma_\nu\}$ of Λ . Set $\alpha := \alpha_i$ for some $1 \leq i \leq \ell$ and fix it. From [2, Lemma 4.24] (also see [9, Lemma 1.21(i)]), we know that

$$\Pi(X) := \{\alpha_1, \dots, \alpha_\ell, \sigma_1 - \alpha, \dots, \sigma_\nu - \alpha\}, \tag{22}$$

is a reflectable base for R . We set $\sigma_0 = 0$, and

$$S := \cup_{i=0}^{\nu} (\sigma_i + 2\Lambda) \quad \text{and} \quad R_b = (S + S) \cup (\pm\alpha + S).$$

Then S is a semilattice in Λ with $\text{ind}(S) = \nu$, and R_b is an extended affine root system of type A_1 and nullity ν . By Lemma 2.3, $\Pi_b := \{\alpha, \sigma_1 - \alpha, \dots, \sigma_\nu - \alpha\}$ is a reflectable base for R_b . We denote the Weyl group of R_b by \mathcal{W}_b . Since $\mathcal{W}_b \subseteq \mathcal{W}$ and $\Pi_b \subseteq \Pi(X)$, the Cayley graph $\Gamma_b := \text{Cay}(\mathcal{W}_b, S_{\Pi_b})$ is a subgraph of the Cayley graph $\Gamma := \text{Cay}(\mathcal{W}, S_{\Pi(X)})$. Since $\nu > 1$, we see from Theorem 4.1 that for $\beta \in \Pi_b$ the geometric reflection w_β is not a Cayley graph-reflection of Γ_b ; thus if d is a normalized dart of w_β , then there is a path $\gamma = d_1 d_2 \cdots d_m$ in Γ_b from $\iota(d)$ to $\iota(d^{-1})$ without any normalized dart (of w_β in Γ_b).

We claim that γ is a path without any normalized dart of w_β in Γ . In fact, if $d_j \in \text{Norm}_{w_\beta}(\mathcal{W} \times S_{\Pi(X)})$ for some $1 \leq j \leq m$, then as $d_j \in \mathcal{W}_b \times S_{\Pi_b}$ and $d_j \notin \text{Norm}_{w_\beta}(\mathcal{W}_b \times S_{\Pi_b})$, we get from Theorem 3.4 that $d_j = (w', w_{\beta'})$ with $\beta' \neq \beta$ and $w'(\beta') = \pm\beta$; that is $\pm\beta$ is in the \mathcal{W}_b -orbit of β' , contradicting Lemma 3.3. Therefore γ is a path in the Cayley graph Γ without any normalized dart of w_β . This implies that w_β is not a Cayley graph-reflection of Γ .

Example 5.7. In this example we examine geometric reflections for a non-simply laced reduced extended affine root systems R of nullity $\nu \geq 3$. Let t be the twist number of R and \mathcal{W} be its extended affine Weyl group. From Section 2, we know that R is of the form

$$R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L),$$

where S and L are semilattices satisfying certain interrelations.

By [1, II. §4(b)], we have

$$S = S_1 \oplus \langle S_2 \rangle \quad \text{and} \quad L = k\langle S_1 \rangle \oplus S_2, \tag{23}$$

where k is as (2), moreover

$$S_1 = \bigoplus_{i=0}^{\text{ind}(S_1)} (\gamma_i + 2\langle S_1 \rangle) \quad \text{and} \quad S_2 = \bigoplus_{i=0}^{\text{ind}(S_2)} (\eta_i + 2\langle S_2 \rangle),$$

with $\gamma_0 = 0$ and $\eta_0 = 0$. Furthermore, γ_i 's and η_i 's can be chosen such that $\{\gamma_1, \dots, \gamma_t\}$ is a \mathbb{Z} -basis of $\langle S_1 \rangle$ and $\{\eta_1, \dots, \eta_{\nu-t}\}$ is a \mathbb{Z} -basis of $\langle S_2 \rangle$.

Let $\dot{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}$ be a fundamental system for \dot{R} . We may assume, without loss of generality, that $\alpha_1 \in \dot{R}_{sh}$ and $\alpha_2 \in \dot{R}_{lg}$. For each type X , we introduce a subset $\Pi(X)$ of R^\times as follows:

X	$\Pi(X)$
F_4, G_2	$\{\alpha_1, \dots, \alpha_\ell, \gamma_1 - \alpha_1, \dots, \gamma_t - \alpha_1, \eta_1 - \alpha_2, \dots, \eta_{\nu-t} - \alpha_2\}$
B_2	$\{\alpha_1, \alpha_2, \gamma_1 - \alpha_1, \dots, \gamma_{\text{ind}(S_1)} - \alpha_1, \eta_1 - \alpha_2, \dots, \eta_{\text{ind}(S_2)} - \alpha_2\}$
$B_\ell (\ell > 2)$	$\{\alpha_1, \dots, \alpha_\ell, \gamma_1 - \alpha_1, \dots, \gamma_{\text{ind}(S_1)} - \alpha_1, \eta_1 - \alpha_2, \dots, \eta_{\nu-t} - \alpha_2\}$
$C_\ell (\ell \geq 3)$	$\{\alpha_1, \dots, \alpha_\ell, \gamma_1 - \alpha_1, \dots, \gamma_t - \alpha_1, \eta_1 - \alpha_2, \dots, \eta_{\text{ind}(S_2)} - \alpha_2\}$

From [9, Section 3] we know that $\Pi(X)$ is a reflectable base for R . Now depending on the twist number t , we introduce \dot{R}' , Λ' and Π' as follows:

t	\dot{R}'	Π'	Λ'	S'
$t \leq 1$	$\{0, \pm\alpha_2\}$	$\{\alpha_2, \eta_1 - \alpha_2, \eta_2 - \alpha_2\}$	$\mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2$	$\cup_{i=0}^2 (\eta_i + 2\Lambda')$
$t > 1$	$\{0, \pm\alpha_1\}$	$\{\alpha_1, \gamma_1 - \alpha_1, \gamma_2 - \alpha_1\}$	$\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$	$\cup_{i=0}^2 (\gamma_i + 2\Lambda')$

Then $R_b := (S' + S') \cup (\dot{R}' + S') = \Lambda' \cup (\dot{R}' + S')$ is an A_1 -extended affine root system of nullity 2. Moreover, Π' is a reflectable base for R_b contained in $\Pi(X)$. Now the same argument given in the last paragraph of Example 5.6 shows that any geometric reflection based on an element of Π' is not a Cayley graph-reflection of $\Gamma = \text{Cay}(\mathcal{W}, S_{\Pi(X)})$.

Example 5.8. (Affine type $A_\ell, \ell \geq 2$). Let R be an affine root system of type $A_\ell, \ell \geq 2$, and \mathcal{W} be its weyl group. We compute normalized darts of a geometric reflection corresponding to a base of R . Let $\dot{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}$ be a base of \dot{R} , θ the highest root of \dot{R} , and $\dot{\mathcal{W}}$ be its Weyl group. We have $\Lambda = \mathbb{Z}\sigma$, for some $\sigma \in \mathcal{V}^0$.

The set $\Pi := \{\alpha_0 = \sigma - \theta, \alpha_1, \dots, \alpha_\ell\}$.

is a base of R . Each $w \in \mathcal{W}$ has a unique expression of the form

$$w = \dot{w} \prod_{i=1}^{\ell} t_i^{m_i}, \tag{24}$$

where $t_i = w_{\alpha_i + \sigma} w_{\alpha_i}$, $m_i \in \mathbb{Z}$ and $\dot{w} \in \dot{\mathcal{W}}$. For simplicity we set $s_i = w_{\alpha_i}$ for $0 \leq i \leq \ell$. It is known that \mathcal{W} is a Coxeter group with the presentation defined by

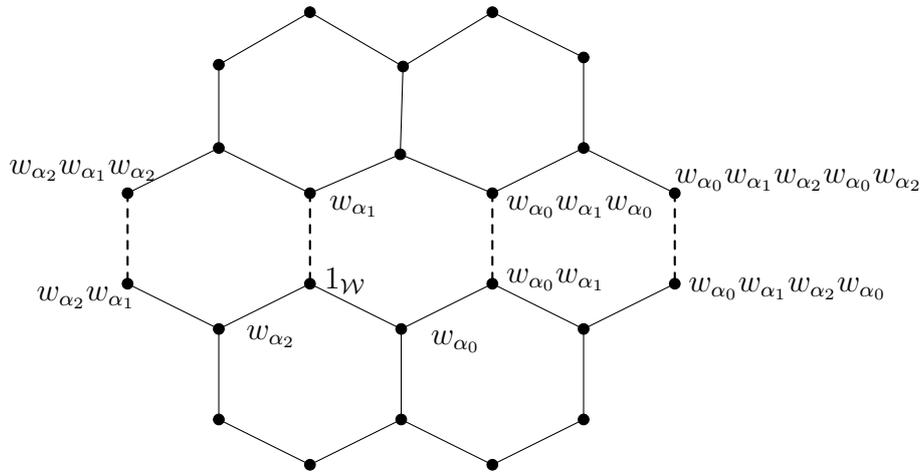
generators s_i , $0 \leq i \leq \ell$, and relations,

$$\begin{cases} s_i^2, & 0 \leq i \leq \ell, \\ (s_0 s_1)^3, (s_i s_{i+1})^3, (s_0 s_\ell)^3, & 1 \leq i \leq \ell - 1, \\ (s_0 s_i)^2, & 2 \leq i \leq \ell - 1, \\ (s_i s_j)^2, & 1 \leq i < j \leq \ell, j \neq i + 1. \end{cases}$$

Set $S = S_\Pi$ and let $\Gamma = \text{Cay}(\mathcal{W}, S)$. Using the Coxeter relations, we see that the graph Γ is characterized by

- Γ is an $(\ell + 1)$ -regular graph.
- Each vertex of Γ is a vertex of exactly $\ell+1$ cycles of length 6, and $(\ell-2)(\ell+1)/2$ cycles of length 4.

The situation is illustrated below for $\ell = 2$.



We now compute the normalized darts. Fix $\alpha \in \Pi$. By Lemma 3.1, (w, w_β) , $\beta \in \Pi$, is a normalized dart of w_α if and only if $w(\beta) = \pm\alpha$; thus we need to compute the orbits $\mathcal{W}\alpha$, $\alpha \in \Pi$. Set $m_0 = m_{\ell+1} = 0$. Then for $w \in \mathcal{W}$, using (24), we get

$$w(\alpha_j) = \begin{cases} -\dot{w}(\theta) + k_0\sigma & j = 0, \\ \dot{w}(\alpha_j) + k_j\sigma & 1 \leq j \leq \ell, \end{cases}$$

where $k_0 := 1 - m_1 - m_\ell$ and $k_j := 2m_j - m_{j-1} - m_{j+1}$, for $1 \leq j \leq \ell$.

Now, by using Lemma 3.1, the set of normalized darts of any elements of S is as follows:

$$\begin{aligned} \text{Norm}_{w_{\alpha_0}}(\mathcal{W} \times S) &= \{(w, w_{\alpha_0}) \in \mathcal{W} \times S \mid \dot{w}(\theta) = \pm\theta, k_0 = \pm 1\} \\ &\cup \{(w, w_{\alpha_j}) \in \mathcal{W} \times S \mid \dot{w}(\alpha_j) = \pm\theta, k_j = \mp 1, 1 \leq j \leq \ell\}, \end{aligned}$$

and for any $1 \leq i \leq \ell$ we have,

$$\begin{aligned} \text{Norm}_{w_{\alpha_i}}(\mathcal{W} \times S) &= \{(w, w_{\alpha_0}) \in \mathcal{W} \times S \mid \dot{w}(\theta) = \pm\alpha_i, k_0 = 0\} \\ &\cup \{(w, w_{\alpha_j}) \in \mathcal{W} \times S \mid \dot{w}(\alpha_j) = \pm\alpha_i, k_j = 0, 1 \leq j \leq \ell\}. \end{aligned}$$

In the case $\ell = 2$, normalized darts of the geometric reflection w_{α_1} are shown with dashed lines in the above graph. From left to right, they are as follows:

$$\{(w_{\alpha_2} w_{\alpha_1}, w_{\alpha_2}), (1_{\mathcal{W}}, w_{\alpha_1}), (w_{\alpha_0} w_{\alpha_1}, w_{\alpha_0}), (w_{\alpha_0} w_{\alpha_1} w_{\alpha_2} w_{\alpha_0}, w_{\alpha_2})\}.$$

Remark 5.9. Let R and $\Pi(X)$ be as in Examples 5.6. Let $\beta \in R^\times$. Since $R^\times = \cup_{\alpha \in \Pi(X)} \mathcal{W}\alpha$, we have $w_\beta = ww_\alpha w^{-1}$ for some $\alpha \in \Pi(X)$ and some $w \in \mathcal{W}$. It then follows from Example 5.6 and the paragraph after Definition 2.10 that w_α is not a Cayley graph-reflection on $\Gamma = \text{Cay}(\mathcal{W}, S_{\Pi(X)})$. The same reasoning also shows that if R and $\Pi(X)$ are as in Example 5.7 then for $\alpha \in \Pi(X)$ and $\beta \in \mathcal{W}\alpha$, the geometric reflection w_β is a Cayley graph-reflection on $\Gamma = \text{Cay}(\mathcal{W}, S_{\Pi(X)})$ if and only if w_α is a Cayley graph-reflection.

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