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Algebraically Independent Generators for the Algebra of Invariant Differential Operators on $\mathbf{SL}_n(\mathbb{R})/\mathbf{SO}_n(\mathbb{R})$

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Abstract. We provide an explicit set of algebraically independent generators for the algebra of invariant differential operators on the Riemannian symmetric space associated with $\mathrm{SL}_n(\mathbb{R})$.

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1. Introduction

Let $\mathrm{Pos}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})$ be the space of positive definite real $n \times n$-matrices and $\mathrm{SPos}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ its subset of elements of determinant 1. Both spaces are Riemannian symmetric spaces whose algebras $\mathbb{D}(\mathrm{Pos}_n(\mathbb{R}))$, respectively $\mathbb{D}(\mathrm{SPos}_n(\mathbb{R}))$, of invariant differential operators are commutative. The Harish-Chandra isomorphism together with Chevalley's Theorem shows that these algebras are isomorphic to the polynomial algebras in n, respectively $n-1$, variables. For $\mathbb{D}(\mathrm{Pos}_n(\mathbb{R}))$ one finds various explicit sets of algebraically independent generators in the literature. One example of such a set is given by the Maass-Selberg operators $\delta_1, \ldots, \delta_n$ described in detail in [9]. It is much harder to track down an explicit set of algebraically independent generators for $\mathbb{D}(\mathrm{SPos}_n(\mathbb{R}))$ in the literature. In [6, § 2] one finds a set of algebraically independent generators for the center $\mathcal{Z}$ of the universal enveloping algebra of $\mathfrak{sl}_n(\mathbb{C})$, which is known to be isomorphic to the algebra of invariant differential operators on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ (see e.g. [7, Prop. II.5.32 & Exer. II.D.3]). In [11, § 7] the author follows a similar line giving the generators (without proof) as coefficients in a characteristic polynomial.

In this paper we show how to derive algebraically independent generators for $\mathbb{D}(\mathrm{SPos}_n(\mathbb{R}))$ from the Maass-Selberg operators as the $\mathrm{SPos}_n(\mathbb{R})$-parts (constructed in Lemma 3.2) of the Maass-Selberg operators $\delta_2, \ldots, \delta_n$ (Theorem 4.3). The arguments given can be applied in more generality as we will explain in Section 5. The key input of the paper comes from the master theses [2, 3] written by the first two authors under the direction of the third author.

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2. Preliminaries on invariant differential operators

For a real reductive Lie group G in the Harish-Chandra class [5] and a maximal compact subgroup $K \leq G$ we denote by

$$\mathbb{D}_K(G) = \{D \in \mathbb{D}(G) \mid \forall\, k \in K : \mathrm{Ad}(k)D = D\}$$

the K-invariant elements of $\mathbb{D}(G)$, i.e., the differential operators that are G-invariant from the left and K-invariant from the right. We also declare

$$\mathbb{D}(G)\mathfrak{k} = \langle D\tilde{X} \mid D \in \mathbb{D}(G), X \in \mathfrak{k}\rangle_{\mathbb{C}-\text{vector space}}$$

to be the left ideal generated by $\mathfrak{k}$ in $\mathbb{D}(G)$. Here $\tilde{X}$ is the left-invariant vector field on G associated with X.

Proposition 2.1 ([7, Thm. II.4.6]). *Let $\pi : G \to G/K$ be the canonical projection. Then the map*

$$\mu : \mathbb{D}_K(G) \to \mathbb{D}(G/K),\ (\mu(D)f)(gK) = D(f \circ \pi)(g), \quad f \in C^\infty(G/K), g \in G$$

is an algebra epimorphism with kernel $\mathbb{D}_K(G) \cap \mathbb{D}(G)\mathfrak{k}$. Moreover, we have an isomorphism of algebras

$$\mu : \mathbb{D}_K(G)/(\mathbb{D}_K(G) \cap \mathbb{D}(G)\mathfrak{k}) \to \mathbb{D}(G/K).$$

There exists a unique linear isomorphism, called the *symmetrization* of $\mathbb{D}(G)$,

$$\lambda_G = \lambda : \mathcal{S}(\mathfrak{g}_\mathbb{C}) \to \mathbb{D}(G)$$

satisfying $\lambda(X^m) = \tilde{X}^m$ for each $X \in \mathfrak{g}$, where $\mathcal{S}(\mathfrak{g}_\mathbb{C})$ is the symmetric algebra of $\mathfrak{g}_\mathbb{C}$. If $X_1, \dots, X_n$ is a basis of $\mathfrak{g}$, then under the identification

$$\mathbb{C}[X_1, \dots, X_n] \cong \mathcal{S}(\mathfrak{g}_\mathbb{C}),$$

we obtain for each $P \in \mathcal{S}(\mathfrak{g}_\mathbb{C})$

$$\lambda(P)f(g) = P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right)\Bigg|_{t=0} f(g \exp(t_1X_1 + \dots + t_nX_n)),\ f \in C^\infty(G),\ g \in G, \quad (1)$$

where the suffix $|_{t=0}$ means the evaluation in $t = (t_1, \dots, t_n) = 0$ after differentiation ([7, Thm. II.4.3]). Furthermore, for any $Y_1, \dots, Y_r \in \mathfrak{g}$ we have

$$\lambda(Y_1 \cdots Y_r) = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} \tilde{Y}_{\sigma(1)} \cdots \tilde{Y}_{\sigma(r)}, \quad (2)$$

where $\mathcal{S}_r$ is the symmetric group of permutations on r elements.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition and $I(\mathfrak{p}_\mathbb{C})$ the space of complex K-invariant polynomials on $\mathfrak{p}_\mathbb{C}$.

Proposition 2.2 ([7, Thm. II.4.9]). *The map*

$$\mu \circ \lambda_G : I(\mathfrak{p}_\mathbb{C}) \to \mathbb{D}(G/K), \quad P \mapsto D_P = \mu(\lambda_G(P))$$

is a linear isomorphism. Moreover, for each $P_1, P_2 \in I(\mathfrak{p}_\mathbb{C})$ there exists a $Q \in I(\mathfrak{p}_\mathbb{C})$ with degree smaller than the sum of the degrees of P_1, P_2 and

$$D_{P_1P_2} = D_{P_1}D_{P_2} + D_Q.$$

From this we can see that, if $P_1, \dots, P_n$ are generators of $I(\mathfrak{p}_\mathbb{C})$, then $D_{P_1}, \dots, D_{P_n}$ are generators of $\mathbb{D}(G/K)$.

Remark 2.3. If $D \in \mathbb{D}(G/K)$ and $X_1, \dots, X_r$ is a basis of $\mathfrak{p}$, then there exists a polynomial $P \in \mathcal{S}(\mathfrak{p}_\mathbb{C})$, such that

$$Df(gH) = P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}\right)\Bigg|_{t=0} f(g\exp(t_1X_1 + \ldots + t_rX_r)K), \ f \in C^\infty(G/K),$$

which is uniquely determined, since $(t_1, \dots, t_r) \mapsto g\exp(t_1X_1 + \ldots + t_rX_r)K$ is a local chart around gK. Thus, Proposition 2.2 implies that P is automatically K-invariant, i.e., $P \in I(\mathfrak{p}_\mathbb{C})$.

3. Maass-Selberg operators and their $\mathrm{SPos}_n(\mathbb{R})$-radial parts

Let $\mathrm{Sym}_n(\mathbb{R})$ be the set of symmetric $n \times n$-matrices. Then the k-th *Maass-Selberg operator* $\delta_k \in \mathbb{D}(\mathrm{Pos}_n(\mathbb{R}))$ is given by

$$\delta_k f(g\mathrm{O}_n(\mathbb{R})) = \mathrm{tr}\left(\left(\frac{\partial}{\partial X}\right)^k\right)\Bigg|_{X=0} f(g\exp(X)\mathrm{O}_n(\mathbb{R})), \ f \in C^\infty(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})),$$

where

$$\frac{\partial}{\partial X} = \begin{pmatrix} \frac{\partial}{\partial x_{11}} & \cdots & \frac{1}{2}\frac{\partial}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{1}{2}\frac{\partial}{\partial x_{1n}} & \cdots & \frac{\partial}{\partial x_{nn}} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} \in \mathrm{Sym}_n(\mathbb{R}).$$

To construct the $\mathrm{SPos}_n(\mathbb{R})$-parts of the δ_k we consider the maps

$$i : \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}), \quad g\mathrm{SO}_n(\mathbb{R}) \mapsto g\mathrm{O}_n(\mathbb{R})$$

and

$$p : \mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}), \quad g\mathrm{O}_n(\mathbb{R}) \mapsto \det(g)^{-\frac{1}{n}} g\mathrm{SO}_n(\mathbb{R}),$$

where $\mathrm{GL}_n^+(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det(g) > 0\}$. Note that

$$\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}), \quad g\mathrm{SO}_n(\mathbb{R}) \mapsto g\mathrm{O}_n(\mathbb{R})$$

is a diffeomorphism. We use it to identify the two spaces. For $g \in \mathrm{SL}_n(\mathbb{R})$ let τ_g denote the translation by g on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$.

Proposition 3.1. *The maps i and p are $\mathrm{SL}_n(\mathbb{R})$-equivariant, i.e.*

$$\forall g \in \mathrm{SL}_n(\mathbb{R}) : \quad p \circ \tau_g = \tau_g \circ p \text{ and } i \circ \tau_g = \tau_g \circ i.$$

Proof. Let $h\mathrm{O}_n(\mathbb{R}) \in \mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})$ and $g \in \mathrm{SL}_n(\mathbb{R})$. We can assume that $\det(h) > 0$, so

$$p(g.h\mathrm{O}_n(\mathbb{R})) = \det(gh)^{-\frac{1}{n}} gh\mathrm{SO}_n(\mathbb{R}) = g.\det(h)^{-\frac{1}{n}} h\mathrm{SO}_n(\mathbb{R}) = g.p(h\mathrm{O}_n(\mathbb{R})).$$

For $h\mathrm{SO}_n(\mathbb{R}) \in \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ we have

$$i(g.h\mathrm{SO}_n(\mathbb{R})) = gh\mathrm{O}_n(\mathbb{R}) = g.h\mathrm{O}_n(\mathbb{R}) = g.i(h\mathrm{SO}_n(\mathbb{R})).$$ ■

Lemma 3.2. *The map* $P : \mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})) \to \mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ *defined by*

$$P(D)f = D(f \circ p) \circ i$$

for $f \in C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ *is a morphism of algebras. Furthermore, it satisfies*

$$(P(D)f) \circ p = D(f \circ p) \tag{3}$$

for $D \in \mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$ *and* $f \in C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$.

We call $P(D)$ the $\mathrm{SPos}_n(\mathbb{R})$-*radial part* of D.

Proof. First we check that the image of P consists of invariant differential operators on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. If $f \in C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$, then $f \circ p$ is smooth since p is. As D is a differential operator and i is smooth, we have

$$P(D)f = D(f \circ p) \circ i \in C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})).$$

The linearity of $P(D)$ is an easy consequence of the linearity of D. Moreover, D satisfies $\mathrm{supp}(D(f \circ p)) \subseteq \mathrm{supp}(f \circ p)$. Precomposing by i we find

$$\mathrm{supp}(P(D)f) = \mathrm{supp}(D(f \circ p) \circ i) \subseteq \mathrm{supp}(f \circ p \circ i) = \mathrm{supp}(f),$$

so that Peetre's Theorem shows that $P(D)$ is a differential operator on the space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. From the invariance of $D \in \mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$ and Proposition 3.1 we obtain

$$P(D)(f \circ \tau_g) = D(f \circ \tau_g \circ p) \circ i = D(f \circ p) \circ i \circ \tau_g = P(D)f \circ \tau_g$$

for $g \in \mathrm{SL}_n(\mathbb{R})$, so $P(D) \in \mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$.

The linearity of P is clear. Next we prove equation (3). Let $g\mathrm{O}_n(\mathbb{R}) \in \mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})$ be arbitrary and assume that $\det(g) > 0$. Define $h = \det(g)^{-\frac{1}{n}}\mathbb{1}_n$. Then by the invariance of $D \in \mathbb{D}(\mathrm{GL}_n(\mathbb{R}))/\mathrm{O}_n(\mathbb{R}))$ and Proposition 3.1 we obtain

$$\begin{aligned} P(D)f \circ p)(g\mathrm{O}_n(\mathbb{R})) &= (D(f \circ p) \circ i)(\det(g)^{-\frac{1}{n}} g\mathrm{SO}_n(\mathbb{R})) \\ &= D(f \circ p)(\det(g)^{-\frac{1}{n}} g\mathrm{O}_n(\mathbb{R})) = (D(f \circ p) \circ \tau_h)(g\mathrm{O}_\mathrm{n}(\mathbb{R})) \\ &= D(f \circ p \circ \tau_h)(g\mathrm{O}_n(\mathbb{R})). \end{aligned}$$

Since τ_h only scales the determinant we have $p \circ \tau_h = p$, this implies (3).

Let $D_1, D_2 \in \mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}))$. Together with equation (3), we conclude

$$P(D_1D_2)f = (D_1D_2(f \circ p)) \circ i = D_1(P(D_2)f \circ p) \circ i = P(D_1)P(D_2)f.$$

As $P(\mathrm{id}_{\mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))}) = \mathrm{id}_{\mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))}$ is obvious, this concludes the proof. ∎

By Proposition 2.2 for each differential operator $D \in \mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$ there exists a polynomial Q_D on $\mathrm{Sym}_n(\mathbb{R})$, such that

$$Df(g\mathrm{O}_n(\mathbb{R})) = Q_D\left(\frac{\partial}{\partial X}\right)\bigg|_{X=0} f(g\exp(X)\mathrm{O}_n(\mathbb{R})),$$

where $X \in \mathrm{Sym}_n(\mathbb{R})$ and $f \in C^\infty(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$. This leads to another representation of the morphism P.

Proposition 3.3. *For each* $f \in C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$

$$P(D)f(g\mathrm{SO}_n(\mathbb{R})) = Q_D\left(\frac{\partial}{\partial X}\right)\bigg|_{X=0} f\left(g\exp\left(X - \frac{1}{n}\mathrm{tr}(X)\mathbb{1}_n\right)\mathrm{SO}_n(\mathbb{R})\right).$$

Proof.
$$\begin{aligned} P(D)f(g\mathrm{SO}_n(\mathbb{R})) &= D(f\circ p)(g\mathrm{O}_n(\mathbb{R}))) \\ &= Q_D\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} f(p(g\exp(X)\mathrm{O}_n(\mathbb{R}))) \\ &= Q_D\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} f(\det(\exp(X))^{-1/n} g\exp(X)\mathrm{SO}_n(\mathbb{R})) \\ &= Q_D\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} f\left(g\exp\left(-\frac{1}{n}\mathrm{tr}(X)\mathbb{1}_n\right)\exp(X)\mathrm{O}_n(\mathbb{R})\right) \\ &= Q_D\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} f\left(g\exp\left(X-\frac{1}{n}\mathrm{tr}(X)\mathbb{1}_n\right)\cdot\mathrm{SO}_n(\mathbb{R})\right). \quad\blacksquare \end{aligned}$$

In fact, the next lemma shows that P is surjective. Thus, the images $P(\delta_1),...,P(\delta_n)$ of the Maas-Selberg operators are generators of $\mathbb{D}(\mathrm{SPos}_n(\mathbb{R}))$ and we will show that $P(\delta_1)=0$, so that it remains to show that $P(\delta_2),\dots,P(\delta_n)$ are algebraically independent.

Lemma 3.4. *The morphism P is surjective and $P(\delta_1)=0$.*

Proof. Let $D\in\mathbb{D}(\mathrm{Pos}_n(\mathbb{R}))$ be given by a unique polynomial Q on $\mathrm{Sym}_n(\mathbb{R})$, i.e.

$$Df(g\mathrm{O}_n(\mathbb{R})) = Q\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} f(g\exp(X)\mathrm{O}_n(\mathbb{R}))$$

holds for every $f\in C^\infty(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$. Denote by $\langle X,Y\rangle=\mathrm{tr}(XY)$ the inner product on $\mathrm{Sym}_n(\mathbb{R})$. Then it is easy to verify that for $A\in\mathrm{Sym}_n(\mathbb{R})$ we have

$$\frac{\partial}{\partial x_{ii}}e^{\langle X,A\rangle}=a_{ii}e^{\langle X,A\rangle},\quad \frac{1}{2}\frac{\partial}{\partial x_{ij}}e^{\langle X,A\rangle}=a_{ij}e^{\langle X,A\rangle}. \tag{4}$$

Let Q_1 be the unique polynomial on $\mathrm{SSym}_n(\mathbb{R})$ so that for each function f of $C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ we have

$$P(D)f(g\mathrm{SO}_n(\mathbb{R})) = Q_1\left(\frac{\partial}{\partial Y}\right)\Big|_{Y=0} f(g\exp(Y)\mathrm{SO}_n(\mathbb{R})).$$

We associate to a matrix $A\in\mathrm{SSym}_n(\mathbb{R})$ the smooth function

$$f_A:\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})\to\mathbb{R},\quad \exp(Y)\mathrm{SO}_n(\mathbb{R})\mapsto e^{\langle Y,A\rangle}\text{ with }Y\in\mathrm{SSym}_n(\mathbb{R}).$$

We can observe with Proposition 3.3 and equation (4) that

$$\begin{aligned} Q(A) &= Q\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} e^{\langle X,A\rangle} = Q\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} e^{\mathrm{tr}\left(\left(X-\frac{\mathrm{tr}(X)}{n}\mathbb{1}_n\right)A\right)} \\ &= Q\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} f_A\left(\exp\left(X-\frac{\mathrm{tr}(X)}{n}\mathbb{1}_n\right)\mathrm{SO}_n(\mathbb{R})\right) \\ &= P(D)f_A(\mathbb{1}_n\mathrm{SO}_n(\mathbb{R})) \\ &= Q_1\left(\frac{\partial}{\partial Y}\right)\Big|_{Y=0} f_A\left(\exp(Y)\mathrm{SO}_n(\mathbb{R})\right) = Q_1\left(\frac{\partial}{\partial Y}\right)\Big|_{Y=0} e^{\langle Y,A\rangle} = Q_1(A). \end{aligned}$$

Thus Q_1 is the restriction of Q to $\mathrm{SSym}_n(\mathbb{R})$, so P is surjective. Moreover, for $D=\delta_1$ we have $Q=\mathrm{tr}$ and therefore $Q_1=0$, hence $P(\delta_1)=0$. ∎

To summarize, we have shown the following theorem.

Theorem 3.5. *Let $\delta_1,\ldots,\delta_n$ be the Maass-Selberg operators. Then $P(\delta_1)=0$ and*

$$P(\delta_k)f(g\mathrm{SO}_n(\mathbb{R}))=\mathrm{tr}\left(\left(\frac{\partial}{\partial X}\right)^k\right)\Bigg|_{X=0} f\left(g\exp\left(X-\frac{1}{n}\mathrm{tr}(X)\mathbb{1}_n\right)\cdot\mathrm{SO}_n(\mathbb{R})\right),$$

where $\dfrac{\partial}{\partial X}=\begin{pmatrix}\frac{\partial}{\partial x_{11}} & \cdots & \frac{1}{2}\frac{\partial}{\partial x_{1n}}\\ \vdots & \ddots & \vdots\\ \frac{1}{2}\frac{\partial}{\partial x_{1n}} & \cdots & \frac{\partial}{\partial x_{nn}}\end{pmatrix}$ *and* $X=\begin{pmatrix}x_{11} & \cdots & x_{1n}\\ \vdots & \ddots & \vdots\\ x_{1n} & \cdots & x_{nn}\end{pmatrix}\in\mathrm{Sym}_n(\mathbb{R})$ *are generators for the algebra $\mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$.*

Using the proof of Lemma 3.4 we can express the $P(\delta_k)$ in terms of a local chart for $\mathrm{SPos}_n(\mathbb{R})=\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$.

Corollary 3.6. *For $f\in C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ and $Y\in\mathrm{SSym}_n(\mathbb{R})$ we have*

$$P(\delta_k)f(g\mathrm{SO}_n(\mathbb{R}))=\mathrm{tr}\left(\left(\frac{\partial}{\partial Y}\right)^k\right)\Bigg|_{Y=0} f(g\exp(Y)\cdot\mathrm{SO}_n(\mathbb{R})). \tag{5}$$

It is possible to prove the surjectivity of P by giving an explicit algebra splitting. In fact, for $g\mathrm{SO}_n(\mathbb{R})\in\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ one can define the map

$$\varphi_g:\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})\to\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}),\quad h\mathrm{SO}_n(\mathbb{R})\to\det(g)^{\frac{1}{n}}h\mathrm{SO}_n(\mathbb{R}),$$

which only depends on the coset $g\mathrm{O}_n(\mathbb{R})$. Using φ_g and the identification

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})=\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$$

one obtains an algebra morphism $I:\mathbb{D}(\mathrm{SPos}_n(\mathbb{R}))\to\mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$ via

$$I(D)f(g\mathrm{O}_n(\mathbb{R}))=(D(f\circ\varphi_g\circ i)\circ p)(g\mathrm{O}_n(\mathbb{R}))$$

for $f\in C^\infty(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$ and $D\in\mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$. This morphism satisfies

$$I(D)f\circ i=D(f\circ i),\quad P\circ I=\mathrm{id}_{\mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))}.$$

For more details we refer to [2, Lemma 4.5, Theorem 4.6].

4. Algebraic independence

In this section we prove the algebraic independence of $P(\delta_2),\ldots,P(\delta_n)$. We start with a general lemma on polynomial algebras.

Lemma 4.1. *Let k be a field and A an (associative) commutative unital algebra over k. Assume that A contains algebraically independent generators $x_1,\ldots,x_n\in A$. Then we have*

(i) *If* $y_1, \ldots, y_m \in A$ *generate* A*, then* $m \geq n$*.*

(ii) *If* $y_1, \ldots, y_n \in A$ *generate* A*, then* $y_1, \ldots, y_n$ *are algebraically independent.*

(iii) *The number* n *is independent of the choice of algebraically independent generators.*

Proof. Since $x_1, \ldots, x_n \in A$ are algebraically independent generators of A, we have an algebra isomorphism

$$k[X_1, \ldots, X_n] \to A, \quad X_i \mapsto x_i.$$

So we may assume that $A = k[X_1, \ldots, X_n]$ is the ring of polynomials in n-indeterminants and $x_i = X_i$.

(i): We define a morphism of algebras by

$$k[Y_1, \ldots, Y_m] \to k[X_1, \ldots, X_n], \quad Y_i \mapsto y_i.$$

The morphism φ is surjective as $y_1, \ldots, y_m$ are generators of $A = k[X_1, \ldots, X_n]$. If dim denotes the Krull-dimension (cf. [1, Def. 2.5.3]), then the surjectivity of φ implies $\dim(k[Y_1, \ldots, Y_m]) \geq \dim(k[X_1, \ldots, X_n])$ as preimages of prime ideals are prime (cf. [1, Prop. 2.5.5]). Now [1, Corollary 2.25.2] implies $\dim(k[Y_1, \ldots, Y_m]) = m$ and $\dim(k[X_1, \ldots, X_n]) = n$, and hence the claim.

(ii): If $y_1, \ldots, y_n$ are generators of $A = k[X_1, \ldots, X_n]$, then it suffices to show that the map φ from part (i) is injective, because the $Y_1, \ldots, Y_n$ are algebraically independent.

Assume that $0 \neq \ker(\varphi)$. Then we can find a polynomial $0 \neq f \in \ker(\varphi)$. Decompose $f = f_1 \cdots f_r$ into irreducible polynomials f_i. Then

$$0 = \varphi(f) = \varphi(f_1) \cdots \varphi(f_r)$$

leads to an irreducible polynomial $f_i \in \ker(\varphi)$, since $k[X_1, \ldots, X_n]$ has no zero divisors. Let $I \subseteq \ker(\varphi)$ be the ideal generated by f_i. Moreover, I is a prime ideal which does not contain any proper prime ideal except $\{0\}$. This means that the height $\mathrm{ht}(I)$ of I is 1 (cf. [1, Def. 2.5.4 & Prop. 2.25.3]), so that [1, Lemma 2.25.7] implies $\dim(k[Y_1, \ldots, Y_n]/I) = n - 1$. The morphism φ factors through I to a surjective morphism $\tilde{\varphi} : k[Y_1, \ldots, Y_n]/I \to k[X_1, \ldots, X_n]$. Again, we conclude $\dim(k[Y_1, \ldots, Y_n]/I) \geq \dim(k[X_1, \ldots, X_n])$ and obtain a contradiction proving the claim.

(iii): This follows applying parts (i) and (ii) to two different algebraically independent generators $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_m$ of A. ∎

Proposition 4.2. *Let* $\delta_1, \ldots, \delta_n$ *be the Maass-Selberg operators. Then the operators* $P(\delta_2), \ldots, P(\delta_2)$ *are algebraically independent.*

Proof. By the Harish-Chandra Isomorphism and Chevalley's Theorem about invariants of finite reflection groups, we know that $\mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ is generated by $n-1$ algebraically independent generators. Since $\delta_1, \ldots, \delta_n$ are generators of $\mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$ and $P : \mathbb{D}(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})) \to \mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ is an epimorphism of algebras and $P(\delta_1) = 0$ by Theorem 3.5, the operators $P(\delta_2), \ldots, P(\delta_n)$ generate $\mathbb{D}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$. Finally, Lemma 4.1 implies the algebraic independence. ∎

Combining Proposition 4.2 with the results of Section 3 we obtain

Theorem 4.3. *The operators* $P(\delta_2),\ldots,P(\delta_2)$ *given by* (5) *form an algebraically independent set of generators for* $\mathbb{D}(\mathrm{SPos}_n(\mathbb{R}))$.

5. Related results

The arguments given in this paper also work for different sets of algebraically independent generators for $\mathbb{D}(\mathrm{Pos}_n(\mathbb{R}))$.

Remark 5.1 ([2, Thm. 3.7&Prop. 4.13]). Let $X=(x_{ij})_{i,j}\in\mathfrak{p}$ be a symmetric matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$. Assume that $Y_1,\ldots,Y_n$ are the elementary symmetric polynomials in n indeterminants. Then one can see that the characteristic polynomial of X can be written as

$$\det(t\mathbb{1}_n - X)=\prod_{i=1}^{n}(t-\lambda_i)=t^n+\sum_{k=1}^{n}(-1)^k Y_k(\lambda_1,\ldots,\lambda_n)t^{n-k}. \tag{6}$$

Let $F_k(X)$ be the sum over the $k\times k$ principal minors of X, i.e.,

$$F_k(X)=\sum_{1\le i_1<\ldots<i_k\le n}\det\left((x_{i_j,i_\ell})_{j,\ell=1,\ldots,k}\right)=\sum_{1\le i_1<\ldots<i_k\le n}\sum_{\sigma\in\mathcal{S}_k}\mathrm{sgn}(\sigma)\prod_{j=1}^{k}x_{i_j i_{\sigma(j)}},$$

where $\mathrm{sgn}(\sigma)$ is the sign of σ. Now we also obtain (cf. [8, Formula 1.2.13]), that

$$\det(t\mathbb{1}_n - X)=t^n+\sum_{k=1}^{n}(-1)^k F_k(X)t^{n-k}. \tag{7}$$

Thus, we can see that for each orthogonal matrix $k\in K$ we have

$$F_j(kXk^{-1})=F_j(X),\quad j=1,\ldots,n, \tag{8}$$

since the characteristic polynomial is invariant under conjugations. Finally, combining (6) and (7),

$$F_j(X)=Y_j(\lambda_1,\ldots,\lambda_n),\quad j=1,\ldots,n. \tag{9}$$

The operators $\eta_1,\ldots,\eta_n$ given by

$$\eta_k f(g\mathrm{O}_n(\mathbb{R}))=F_k\left(\frac{\partial}{\partial X}\right)\Big|_{X=0} f(g\exp(X)\mathrm{O}_n(\mathbb{R})),\quad f\in C^\infty(\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}))$$

form an algebraically independent set of generators for $\mathbb{D}(\mathrm{Pos}_n(\mathbb{R})$. Then

$$P(\eta_k)f(g\mathrm{SO}_n(\mathbb{R}))=F_k\left(\frac{\partial}{\partial Y}\right)\Big|_{Y=0} f(g\exp(Y)\cdot\mathrm{SO}_n(\mathbb{R}))$$

holds for $f\in C^\infty(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ and $Y\in\mathrm{SSym}_n(\mathbb{R})$. Moreover, $P(\eta_1)=0$ and the $P(\eta_2),\ldots,P(\eta_n)$ form an algebraically independent set of generators for $\mathbb{D}(\mathrm{SPos}_n(\mathbb{R}))$.

Similar results hold for the symmetric spaces $\mathrm{Pos}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})/\mathrm{U}_n(\mathbb{C})$ of positive definite hermitian $n \times n$-matrices and $\mathrm{SPos}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n(\mathbb{C})$ of determinant 1 elements in $\mathrm{Pos}_n(\mathbb{C})$ (see [2, Theorems 3.8, 3.9, 4.17]).

Note that $\mathrm{Pos}_n(\mathbb{R})$ and $\mathrm{Pos}_n(\mathbb{C})$ are examples for symmetric cones (see [4]). Nomura gave algebraically independent generators $D_1, \ldots, D_r$ for the algebra of invariant differential operators on any symmetric cone Ω in [10]. Using the Jordan theoretic determinant one obtains the symmetric space $S\Omega$ associated with the derived groups $\mathrm{Str}(V)'$ of the structure group $\mathrm{Str}(V)$ of the euclidean Jordan algebra V defined by Ω (see [4] for the relevant definitions and constructions). The arguments presented in this paper can also be adapted to this situation to yield an algebra epimorphism $P : \mathbb{D}(\Omega) \to \mathbb{D}(S\Omega)$ via $\mathrm{Str}(V)'$-radial parts such that $\mathbb{C}D_1 = \ker P$. One obtains the following theorem.

Theorem 5.2. *Let Ω be a symmetric cone and $D_1, \ldots, D_r$ the set of Nomura's generators for $\mathbb{D}(\Omega)$. Then $P(D_2), \ldots, P(D_r)$ are algebraically independent generators for $\mathbb{D}(S\Omega)$.*

The classification of symmetric cones ([4, p. 97]) shows that this yields algebraically independent generators for the algebras of invariant differential operators for the following Riemannian symmetric spaces:

$$\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}), \quad \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}_n(\mathbb{C}), \quad \mathrm{SL}_n(\mathbb{H})/\mathrm{SU}_n(\mathbb{H}),$$
$$\mathrm{O}_{1,n-1}(\mathbb{R})/\mathrm{O}_{n-1}(\mathbb{R}), \quad E_{6(-26)}/F_4,$$

where $\mathbb{H}$ denotes the quaternions and $E_{6(-26)}/F_4$ is the space of positive definite 3×3-matrices of determinant 1 with entries in the octonians.

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