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Generalizations of Weak Commutativity for Lie Algebras

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Abstract. We study weak commutativity constructions in the category of Lie algebras.

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1. Introduction

In this paper we study Lie algebra versions of the weak commutativity group constructions $\mathcal{X}(G)$, $\mathcal{E}(G)$ and $\nu(G)$. These three constructions are related and sometimes share common properties. A Lie algebra version of the construction $\mathcal{X}(G)$ was recently defined by Mendonça in [27]. Lie algebra versions of the construction $\mathcal{E}(G)$ and $\nu(G)$ are defined in this paper. The Lie algebra version of the group $\nu(G)$ is strongly related to the notion of the non-abelian tensor square of a Lie algebra that was first defined by Ellis in [15].

In this paper all Lie algebras $\mathfrak{g}$ are over a fixed field K and an ideal of $\mathfrak{g}$ generated by I is denoted by $\langle\langle I \rangle\rangle$. All commutators are left normed i.e. $[A, B, C] = [[A, B], C]$. We explain our results and their backgrounds in the following subsections of the introduction, separating one subsection for each one of the constructions $\mathcal{X}(\mathfrak{g})$, $\nu(\mathfrak{g})$ and $\mathcal{E}(\mathfrak{g})$. In the final subsection of the introduction we compare the constructions $\mathcal{X}(\mathfrak{g})$, $\nu(\mathfrak{g})$ and $\mathcal{E}(\mathfrak{g})$.

1.1. $\mathcal{X}(G)$, $\mathcal{X}(\mathfrak{g})$ and Theorem A

In [31] for every group G and isomorphism of groups $\psi : G \to G^\psi$ Sidki defined a new group

$$\mathcal{X}(G) = \langle G, G^\psi \mid [g, g^\psi] = 1 \text{ for } g \in G\rangle.$$

If G is polycyclic-by-finite Lima and Oliveira showed in [25] that $\mathcal{X}(G)$ is polycyclic-by-finite and in [31] Sidki proved that for a finite group G the group $\mathcal{X}(G)$ is again finite. In [7] Bridson and Kochloukova showed that if G is finitely presented then $\mathcal{X}(G)$ is finitely presented by proving first the result for free groups of finite rank.

The construction $\mathcal{X}(G)$ has some surprising homological properties. There are two important abelian normal subgroups $W(G)$, $R(G)$ of $\mathcal{X}(G)$ for which Rocco proved

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in [28] that $W(G)/R(G) \simeq H_2(G,\mathbb{Z})$. By definition a group G is of homological type FP_m if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution with finitely generated projective modules in dimension $\leq m$. In [21] Kochloukova and Sidki proved that if G is soluble of homological type FP_∞ then $\mathcal{X}(G)$ is soluble of homological type FP_∞. Still by [7] for G a free group of finite rank at least 2, the group $\mathcal{X}(G)$ is not of homological type FP_3 though G is of homological type FP_3 (actually it is of type FP_∞ in this case).

Let $\mathfrak{g}$ be a Lie algebra. Fix an isomorphism of Lie algebras $\psi : \mathfrak{g} \to \mathfrak{g}^\psi$ and consider the following Lie algebra given by presentation in terms of generators and relations

$$\mathcal{X}(\mathfrak{g}) = \langle \mathfrak{g}, \mathfrak{g}^\psi \mid [g, g^\psi] = 0 \text{ for } g \in \mathfrak{g}\rangle.$$

Thus $\mathcal{X}(\mathfrak{g})$ is the quotient of the free product of Lie algebras $\mathfrak{g} * \mathfrak{g}^\psi$ by the ideal $\langle\langle\{[g, g^\psi] \mid g \in \mathfrak{g}\}\rangle\rangle$. Here if $\mathfrak{g} = \langle X \mid R\rangle$ and $\mathfrak{g}^\psi = \langle X^\psi \mid R^\psi\rangle$ are presentations of Lie algebras by generators and relations, the free product of Lie algebras $\mathfrak{g} * \mathfrak{g}^\psi$ is given by the presentation $\langle X \cup X^\psi \mid R \cup R^\psi\rangle$. In [27] Mendonça developed a structural theory of $\mathcal{X}(\mathfrak{g})$ in the case $char(K) \neq 2$ and he showed that if $\mathfrak{g}$ is a finitely presented Lie algebra then $\mathcal{X}(\mathfrak{g})$ is finitely presented too.

Let L_0 be the ideal of the free product $\mathfrak{g} * \mathfrak{g}^\psi$ generated by the elements of the form $a - a^\psi$ for $a \in \mathfrak{g}$ and D_0 be the ideal of $\mathfrak{g} * \mathfrak{g}^\psi$ generated by the elements of the form $[a, b^\psi]$ for $a, b \in \mathfrak{g}$. One of the important properties of $\mathcal{X}(\mathfrak{g})$ is that for the ideals $L = L(\mathfrak{g})$ and $D = D(\mathfrak{g})$ of $\mathcal{X}(\mathfrak{g})$ which are the images of L_0 and D_0 under the canonical epimorphism $\mathfrak{g} * \mathfrak{g}^\psi \to \mathcal{X}(\mathfrak{g})$ by [27, Lemma 3.2] we have that $[L, D] = 0$. This property is the base of the whole theory developed in [27] and uses the fact proved in [27, Lemma 3.1] that when $char(K) \neq 2$ the ideal L is generated as a Lie algebra by the set $\{a - a^\psi \mid a \in \mathfrak{g}\}$. This explains the condition $char(K) \neq 2$ used in all results in [27].
Our main result about the Lie algebra $\mathcal{X}(\mathfrak{g})$ is the following theorem that we prove by homological methods.

Theorem A. *Let $\mathfrak{g}$ be a finitely generated, nilpotent Lie algebra (resp. nilpotent-by-finite dimensional) over a field K of characteristic* $\mathrm{char}(K) \neq 2$. *Then $\mathcal{X}(\mathfrak{g})$ is nilpotent (resp. is nilpotent-by-finite dimensional).*

In the case of groups, the fact that for a finitely generated nilpotent group G the group $\mathcal{X}(G)$ is nilpotent was proved by Gupta, Rocco and Sidki in [16]. In [8] Bridson and Kochloukova showed a new proof of this result and extended it to virtually nilpotent groups. Our proof of Theorem A modifies the proof from [8] in the category of Lie algebras.

1.2. $\nu(G)$, $\nu(\mathfrak{g})$, Theorem B and Theorem C

In [29] Rocco defined the group

$$\nu(G) = \langle G, G^\psi \mid [g_1, g_2^\psi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\psi] = [g_1, g_2^\psi]^{g_3^\psi}, g_1, g_2, g_3 \in G\rangle.$$

and showed that the subgroup $[G, G^\psi]$ of $\nu(G)$ is the non-abelian tensor product $G \otimes G$. The non-abelian tensor product of two groups acting compatibly on each other was defined in [9] and [10] in relation with problems in algebraic topology. In [13] Ellis showed that if G is a finite group then $G \otimes G$ is finite, too.

The groups $\mathcal{X}(G)$ and $\nu(G)$ are related in the following way:

$$\mathcal{X}(G)/R(G) \cong \nu(G)/\Delta(G),$$

where $\Delta(G)$ is a central subgroup of $\nu(G)$. In [21] Kochloukova and Sidki showed that if G is finitely presented then $\nu(G)$ is finitely presented too.

We define a Lie algebra version $\nu(\mathfrak{g})$ of $\nu(G)$ for a Lie algebra $\mathfrak{g}$ as

$$\nu(\mathfrak{g}) = \langle \mathfrak{g} * \mathfrak{g}^{\psi} \mid [a_1, a_2^{\psi}, a_3] = [a_1, a_2, a_3^{\psi}] = [a_1, a_2^{\psi}, a_3^{\psi}] \text{ for } a_1, a_2, a_3 \in \mathfrak{g} \rangle.$$

We will explain in section 4 why the above definition is a natural translation of the defining relations of the group $\nu(G)$ in the category of Lie algebras and we will develop a structure theory for the Lie algebra $\nu(\mathfrak{g})$.

Non-abelian tensor products of Lie algebras were defined in [15], here we state the definition only in the particular case of the non-abelian tensor square of a Lie algebra. Recall that all Lie algebras we consider are over a fixed field K.

Definition 1.1. ([15]) Let $\mathfrak{g}$ be a Lie algebra. The non-abelian tensor square $\mathfrak{g} \otimes \mathfrak{g}$ is the Lie algebra generated by the symbols $a \otimes b$ with $a, b \in \mathfrak{g}$ subject to the following relations where $\lambda \in K$, $a, b, a', b' \in \mathfrak{g}$:

(i) $\lambda(a \otimes b) = \lambda a \otimes b = a \otimes \lambda b$,

(ii) $(a + a') \otimes b = a \otimes b + a' \otimes b$,

(iii) $a \otimes (b + b') = a \otimes b + a \otimes b'$,

(iv) $[a, a'] \otimes b = a \otimes [a', b] - a' \otimes [a, b]$,

(v) $a \otimes [b, b'] = [b', a] \otimes b - [b, a] \otimes b'$.

(vi) $[a \otimes b, a' \otimes b'] = -[b, a] \otimes [a', b']$.

Theorem B. *Let $\mathfrak{g}$ be a Lie algebra. Then the non-abelian tensor square $\mathfrak{g} \otimes \mathfrak{g}$ is isomorphic to the ideal $[\mathfrak{g}, \mathfrak{g}^{\psi}]$ of $\nu(\mathfrak{g})$.*

In the case when the derived Lie subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent Salemkar, Tavallaee, Mohammadzadeh and Edalatzadeh proved in [30] that the non-abelian tensor square $\mathfrak{g} \otimes \mathfrak{g}$ is a nilpotent Lie algebra. Here we generalise this to the construction $\nu(\mathfrak{g})$, when $\mathfrak{g}$ is a nilpotent Lie algebra. Similar results hold for $\nu(G)$ and $\mathcal{X}(G)$ for a finitely generated nilpotent group G, the case of $\mathcal{X}(G)$ was proved in [16] and the case of $\nu(G)$ follows from the fact for $\mathcal{X}(G)$. We also use $\nu(\mathfrak{g})$ to exhibit examples of finitely presented Lie algebras which are not of type FP_3. By definition a Lie algebra $\mathfrak{g}$ is of homological type FP_m if the trivial $U(\mathfrak{g})$-module K has a projective resolution with finitely generated projective modules in dimension $\leq m$.

Theorem C. *Let $\mathfrak{g}$ be a Lie algebra. Then*

(a) *if $\mathfrak{g}$ is finitely presented then $\nu(\mathfrak{g})$ is finitely presented;*

(b) *if $\mathfrak{g}$ is nilpotent then $\nu(\mathfrak{g})$ is nilpotent;*

(c) *if $\mathfrak{g}$ is finitely generated, free, non-abelian then $\nu(\mathfrak{g})$ is not of type* FP_3.

Note that we give two proofs of Theorem C, one using properties of the construction $\mathcal{X}(\mathfrak{g})$ and one without. The first method has the restriction that it requires that $\mathrm{char}(K) \neq 2$ as the theory of $\mathcal{X}(\mathfrak{g})$ is developed in [27] under the restriction that $\mathrm{char}(K) \neq 2$ (and as explained before there this condition is significantly used).

Here it is possible to avoid this condition since in general $\nu(\mathfrak{g})$ has simpler structure than $\mathcal{X}(\mathfrak{g})$. The proof of Theorem C, b) uses the notion of the external square $\mathfrak{g}\wedge\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ and the Ellis formula $H_2(\mathfrak{g},K) \simeq \ker(\mathfrak{g}\wedge\mathfrak{g} \to \mathfrak{g})$ from [14], where $\mathfrak{g}\wedge\mathfrak{g}$ is the non-abelian exterior square of $\mathfrak{g}$.

1.3. $\mathcal{E}(G)$, $\mathcal{E}(\mathfrak{g})$, Theorem D and Theorem E

In [26] Lima and Sidki defined the group

$$\mathcal{E}(G) = \langle G, G^{\psi} \mid [[g_1, g_2^{\psi}], g_3^{-1} g_3^{\psi}] = 1 \text{ for } g_1, g_2, g_3 \in G\rangle.$$

Little is known for the group construction $\mathcal{E}(G)$. The defining relations imply that the identity on G and G^{ψ} induces an epimorphism $\mathcal{E}(G) \to \mathcal{X}(G)$. In [26] Lima and Sidki proved that if G is polycyclic then $\mathcal{E}(G)$ is polycyclic if and only if G/G' is finite. This was generalised later by Kochloukova in [17] where it was shown that if $\mathcal{E}(G)$ is finitely presented then G/G' is finite. The proof of this result uses geometric ideas that can be traced back to a method introduced by Bieri and Strebel in [5], thus the original proof in [17] cannot be modified in the case of Lie algebras, furthermore if the field K is infinite there are no finite Lie algebras. Still there is a natural Lie algebra analogue $\mathcal{E}(\mathfrak{g})$ of the construction $\mathcal{E}(G)$.
For a Lie algebra $\mathfrak{g}$ we define $\mathcal{E}(\mathfrak{g}) = \langle \mathfrak{g}, \mathfrak{g}^{\psi} \mid [L_0, D_0] = 0 \rangle$, where L_0 is the ideal of the free product $\mathfrak{g} * \mathfrak{g}^{\psi}$ generated by the elements of the form $a - a^{\psi}$ for $a \in \mathfrak{g}$ and D_0 is the ideal of $\mathfrak{g} * \mathfrak{g}^{\psi}$ generated by the elements of the form $[a, b^{\psi}]$ for $a, b \in \mathfrak{g}$. The ideals $\mathcal{L}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$ of $\mathcal{E}(\mathfrak{g})$ which are the images of L_0 and D_0 under the canonical epimorphism $\mathfrak{g} * \mathfrak{g}^{\psi} \to \mathcal{E}(\mathfrak{g})$ play a prominent role in the theory we develop. In the theory of $\mathcal{E}(\mathfrak{g})$ that we develop no restriction on the characteristic of the field K will appear, as the important property $[\mathcal{L}(\mathfrak{g}), \mathcal{D}(\mathfrak{g})] = 0$ that is the base of the whole theory follows from the definition of $\mathcal{E}(\mathfrak{g})$. In comparison the fact that $[D, L] = 0$ in $\mathcal{X}(\mathfrak{g})$ requires that $char(K) \neq 2$.

Free products with amalgamation and HNN extensions are important notions in combinatorial group theory. Lie algebra versions of these notions exist, with free product with amalgamation of Lie algebras better known than HNN extension of Lie algebras. The latter was introduced independently by Lichtman and Shirvani in [24] and by Wasserman in [32]. In the proof of the following result we will use as an important tool HNN extensions of Lie algebras.

Theorem D. *Let $\mathfrak{g}$ be a Lie algebra such that $\mathcal{E}(\mathfrak{g})$ is finitely presented. Then $\mathfrak{g}$ is perfect i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

As a corollary of the proof of Theorem D we obtain a characterization of the Lie algebras $\mathcal{E}(\mathfrak{g})$ that are finitely presented and of homological type FP_m. This characterisation depends on the ideals $\mathcal{W}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})$ and $\mathcal{L}(\mathfrak{g})$ of $\mathcal{E}(\mathfrak{g})$. For simplicity we write $\mathcal{W}$ and $\mathcal{L}$ for $\mathcal{W}(\mathfrak{g})$ and $\mathcal{L}(\mathfrak{g})$.

Theorem E. *Let $\mathfrak{g}$ be a Lie algebra which is finitely presented and of homological type FP_m. Then the following conditions are equivalent:*

(1) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$;
(2) $\mathcal{W}$ *is finite dimensional and* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$;
(3) $\mathcal{L}/[\mathcal{L}, \mathcal{L}]$ *is finite dimensional;*
(4) $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$;
(5) $\mathcal{E}(\mathfrak{g})$ *is finitely presented and of homological type* FP_m.

By the definition of finite presentability and of the homological type FP_2 we have that if $\mathfrak{g}$ is a finitely presented Lie algebra it is always of homological type FP_2. Thus in the case $m = 2$ Theorem E is a criterion when $\mathcal{E}(\mathfrak{g})$ is finitely presented. We note that the assumption in Theorem E that $\mathfrak{g}$ is finitely presented and of type FP_m is not restrictive since the fact that $\mathfrak{g}$ is a retract of $\mathcal{E}(\mathfrak{g})$ implies that when $\mathcal{E}(\mathfrak{g})$ is finitely presented and of homological type FP_m then $\mathfrak{g}$ is automatically finitely presented and of homological type FP_m too.

1.4. Similarities and differences between $\mathcal{X}(\mathfrak{g})$, $\nu(\mathfrak{g})$ and $\mathcal{E}(\mathfrak{g})$

Finally we compare the Lie algebras $\mathcal{X}(\mathfrak{g})$, $\nu(\mathfrak{g})$ and $\mathcal{E}(\mathfrak{g})$ in the case $char(K) \neq 2$. We show in Lemma 5.1 that the identity on $\mathfrak{g}$ and $\mathfrak{g}^{\psi}$ induces an epimorphism

$$\mathcal{E}(\mathfrak{g}) \to \mathcal{X}(\mathfrak{g}) \tag{1}$$

and that $\mathcal{E}(\mathfrak{g})/\mathcal{W} \simeq \mathcal{X}(\mathfrak{g})/W$ is a subdirect sum in $\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$, i.e. is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ that maps surjectively on each summand $\mathfrak{g}$. Since $[\mathcal{L}, \mathcal{D}] = 0$ in $\mathcal{E}(\mathfrak{g})$ the ideal $\mathcal{W} = \mathcal{L} \cap \mathcal{D}$ is abelian, thus the kernel of the epimorphism $\mathcal{E}(\mathfrak{g}) \to \mathcal{X}(\mathfrak{g})$ is an abelian Lie algebra.

After the proof of Proposition 4.7 we construct an epimorphism of Lie algebras

$$\delta : \nu(\mathfrak{g}) \to \mathcal{X}(\mathfrak{g})/R,$$

where $R = R(\mathfrak{g}) = [\mathfrak{g}, L, \mathfrak{g}^{\psi}]$ and we show in Lemma 4.8 that

$$\Delta := Ker(\delta) \subseteq \nu(\mathfrak{g})' \cap Z(\nu(\mathfrak{g})).$$

In particular we have the isomorphism $\nu(\mathfrak{g})/\Delta \simeq \mathcal{X}(\mathfrak{g})/R$. Furthermore by Lemma 2.3 Δ is a quotient of $\mathrm{H}_2(\nu(\mathfrak{g})/\Delta, K) \simeq \mathrm{H}_2(\mathcal{X}(\mathfrak{g})/R, K)$. Note that we have a short exact sequence of Lie algebras

$$0 \to W/R \to \mathcal{X}(\mathfrak{g})/R \to \mathcal{X}(\mathfrak{g})/W \to 0, \quad \text{where} \quad W/R \simeq H_2(\mathfrak{g}, K)$$

by [27]. Thus the structure of $\nu(\mathfrak{g})$ is quite well understood in the case when $char(K) \neq 2$. It is plausible that the case when $char(K) = 2$ is not much different but in this case we cannot use the Lie algebra $\mathcal{X}(\mathfrak{g})$ and the structure theory developed in [27].

The structure of the Lie algebras $\mathcal{E}(\mathfrak{g})$ and $\mathcal{X}(\mathfrak{g})$ is more complicated because we do not have a good understanding of the abelian Lie algebras W and $\mathcal{W}$. In general they do not need be finitely generated and they should be viewed as modules over $\mathcal{U}(\mathcal{X}(\mathfrak{g}))$ and $\mathcal{U}(\mathcal{E}(\mathfrak{g}))$ respectively via the adjoint action, where for a Lie algebra $\mathfrak{h}$ we denote by $\mathcal{U}(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}$. But as $D+L$ and $\mathcal{D}+\mathcal{L}$ act trivially on W and $\mathcal{W}$ respectively, W and $\mathcal{W}$ can be viewed as modules over $\mathcal{U}(\mathcal{X}(\mathfrak{g})/(D+L))$ and $\mathcal{U}(\mathcal{E}(\mathfrak{g})/(\mathcal{D}+\mathcal{L}))$ respectively. It is easy to see that

$$\mathcal{X}(\mathfrak{g})/(D+L) \simeq \mathfrak{g}/\mathfrak{g}' \simeq \mathcal{E}(\mathfrak{g})/(\mathcal{D}+\mathcal{L})$$

is an abelian Lie algebra and the epimorphism (1) induces an epimorphism $\mathcal{W} \to W$ of $\mathcal{U}(\mathfrak{g}/\mathfrak{g}')$-modules.

2. Preliminaries on Lie algebras

Homology and homological finiteness conditions for Lie algebras can be defined in a similar way as in the case of groups but using universal enveloping algebras instead of group algebras. In this Section we recall a few basic definitions and properties and fix some notation. Recall that K is a field and Lie algebra means Lie algebra over the field K.

2.1. On homological properties of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra. This is an augmented algebra, i.e., there is a map $\epsilon : U(\mathfrak{g}) \to K$ and its kernel is the augmentation ideal $\operatorname{Aug} U(\mathfrak{g}) = \mathfrak{g}U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{g}$. We define the n-th homology of $\mathfrak{g}$ with coefficients in a $U(\mathfrak{g})$-module V as

$$\mathrm{H}_n(\mathfrak{g}, V) := \mathrm{T}or_n^{U(\mathfrak{g})}(K, V)$$

where K denotes the trivial $U(\mathfrak{g})$-module i.e. the module on which $\mathfrak{g}$ acts as 0.

We have
$$\mathrm{H}_1(\mathfrak{g}, K) \simeq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

As for groups, having an explicit presentation, or an explicit expression of a Lie algebra as a quotient of a free Lie algebra yields the beginning of a projective resolution as we shall see now. As usual, we will be denoting by $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the commutator and by $\mathfrak{g}_{ab} = \mathfrak{g}/\mathfrak{g}'$ the abelianization of a given Lie algebra $\mathfrak{g}$.

The proof of the following lemma can be found in the proof of Lemma 3.1 from [18].

Lemma 2.1. *Let $\mathfrak{g}$ be a Lie algebra given as $\mathfrak{g} = F(X)/I$, where $F(X)$ is the free Lie algebra with a free basis X and I is an ideal of $F(X)$. Then there is an exact sequence of right $U(\mathfrak{g})$-modules*

$$0 \to I_{ab} \to \oplus_{x \in X} xU(\mathfrak{g}) \xrightarrow{\delta_1} U(\mathfrak{g}) \to K \to 0,$$

where I_{ab} is the abelianization $I/[I, I] = I/I'$ and the map δ_1 maps $xf \in xU(\mathfrak{g})$ to $xf \in \operatorname{Aug} U(\mathfrak{g})$.

Lemma 2.2. (Hopf formula) [3], [22] *Let $\mathfrak{g}$ be a Lie algebra given as $\mathfrak{g} = F/I$ where F is a free Lie algebra and I is an ideal of F. Then*

$$\mathrm{H}_2(\mathfrak{g}, K) \simeq (I \cap F')/[I, F].$$

As for groups, $\mathrm{H}_2(\mathfrak{g}, K)$ is called the Schur multiplier. A stem extension of Lie algebras is a short exact sequence of Lie algebras $0 \to I \to \mathfrak{h} \to \mathfrak{g} \to 0$ such that $I \leq Z(\mathfrak{h}) \cap \mathfrak{h}'$, where $Z(\mathfrak{h})$ is the center of $\mathfrak{h}$. The following Lemma is standard.

Lemma 2.3. *For any stem extension of Lie algebras $0 \to I \to \mathfrak{h} \to \mathfrak{g} \to 0$ there is an epimorphism $\mathrm{H}_2(\mathfrak{g}, K) \twoheadrightarrow I$.*

Proof. Let F be a free Lie algebra and let J be an ideal of F such that $F/J \cong \mathfrak{h}$. Let I_1 be the ideal of F such that I_1/J gets identified with I. Then the fact that $I \leq \mathfrak{h}'$ implies that $I_1 \leq F' + J$ and the fact that $I \leq Z(\mathfrak{h})$ implies that $[I_1, F] \leq J$.

Then
$$(I_1 \cap F') + J = (F' + J) \cap I_1 = I_1$$

and $[I_1, F] \leq (I_1 \cap F') \cap J = F' \cap J$.

Therefore there is an epimorphism

$$H_2(\mathfrak{g}, K) \cong (I_1 \cap F')/[I_1, F] \twoheadrightarrow (I_1 \cap F')/(J \cap F'),$$

where

$$(I_1 \cap F')/(J \cap F') = (I_1 \cap F')/((I_1 \cap F') \cap J) \cong ((I_1 \cap F') + J)/J = I_1/J \cong I. \qquad \blacksquare$$

Recall that a Lie algebra $\mathfrak{g}$ is said to be of type FP_m if there is a projective resolution of the trivial $U(\mathfrak{g})$-module K

$$\mathcal{P} : \ldots \to P_i \to P_{i-1} \to \ldots \to P_1 \to P_0 \to K \to 0$$

such that P_i is a finitely generated projective $U(\mathfrak{g})$-module for $i \leq m$. As in the group case if there is a projective resolution with this property then there is a free resolution with the same property so one can assume that each P_i is a finitely generated free $U(\mathfrak{g})$-module for $i \leq m$. We do not know much about the homological property FP_m for Lie algebras, although there are many analogies with the group case. In general, the Lie algebra versions of group theoretical results are easy to prove as long as there are algebraic proofs available for group case that do not involve any geometry. For example we have (see [4, Prop. 2.1,Prop. 2.2] for the group version) that a Lie algebra $\mathfrak{g}$ is of type FP_1 if and only if $\mathfrak{g}$ is finitely generated as a Lie algebra.

But sometimes things are more intricate. Bryant and Groves classified the finitely presented metabelian Lie algebras in [11], [12] and from their classification it follows that for metabelian Lie algebras finite presentability and type FP_2 are the same. The classificaton of finitely presented metabelian groups was established by Bieri and Strebel in [5]. Subdirect products of groups and their homotopical and homological properties (including finite presentability and FP_2) were studied by Baumslag, Bridson, Howie, Kochloukova, Kuckuck, Lima, Miller, Short in [2], [6], [19], [23]. A Lie algebra version was studied by Kochloukova and Martínez-Pérez in [20]. For the applications it is important to study subdirect sums of Lie algebras that map surjectively on pairs.

Theorem 2.4. [20, Corollary D] *Let $\mathfrak{g}$ be a subdirect Lie sum of $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ with $\mathfrak{g} \cap \mathfrak{g}_i \neq 0$ and $\mathfrak{g}_i$ a finitely presented Lie algebra for all $1 \leq i \leq k$. Assume that $p_{i,j}(\mathfrak{g}) = \mathfrak{g}_i \oplus \mathfrak{g}_j$ for all $1 \leq i < j \leq k$, where $p_{i,j} : \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k \to \mathfrak{g}_i \oplus \mathfrak{g}_j$ is the canonical projection. Then $\mathfrak{g}$ is a finitely presented Lie algebra.*

2.2. On HNN-extensions of Lie algebras

Let $\mathfrak{h}$ be a Lie algebra and take $\mathfrak{a} \leq \mathfrak{h}$ a Lie subalgebra. We follow Lichtman and Shirvani [24] and Wassermann [32] to define a new Lie algebra $\mathfrak{g}$ called an HNN extension of $\mathfrak{h}$ with associated Lie subalgebra $\mathfrak{a}$ and stable letter t. A *derivation* $d : \mathfrak{a} \to \mathfrak{h}$ is a K-linear map such that

$$d([a, b]) = [a, d(b)] + [d(a), b]$$

for any $a, b \in \mathfrak{a}$. Given a derivation d we set

$$\mathfrak{g} = \langle \mathfrak{h}, t \mid [t, a] = d(a) \text{ for } a \in \mathfrak{a} \rangle. \tag{2}$$

Lemma 2.5. [32, Cor. 5.3] *Let x be an element in $\mathfrak{h}$ but not in $\mathfrak{a}$. Then t and x are a free basis for the subalgebra generated by them in the HNN extension $\mathfrak{g}$.*

Lemma 2.6. [24] *Let L_0 be a Lie subalgebra of $\mathfrak{g}$ that intersects $\mathfrak{a}$ trivially and such that $L_0 \cap \mathfrak{h}$ is a free Lie algebra. Then L_0 is a free Lie algebra.*

Theorem 2.7. [32, Thm. 9.1] *Let $\mathfrak{h}$ be a finitely presented Lie algebra and let I be an ideal of $\mathfrak{h}$ of co-dimension 1. Let a be an element in $\mathfrak{h} \setminus I$. Then there exist finitely generated Lie subalgebras S and B of I with $S \subseteq B$ such that the map $d : S \to B$ given by $d(s) = [a, s]$ is a derivation and such that the inclusion of B in $\mathfrak{h}$ and the assignment $t \to a$ induce an isomorphism*

$$\psi : \langle B, t \mid [t, s] = d(s) \text{ for all } s \in S \rangle \to \mathfrak{h}$$

between $\mathfrak{h}$ and the HNN extension defined by B and the derivation d.

3. On $\mathcal{X}(\mathfrak{g})$ for $\mathfrak{g}$ nilpotent-by-finite dimensional

In this paper we denote by $\langle\langle Y \rangle\rangle$ the ideal generated by the set Y and it will be clear from the context in which Lie algebra this ideal lives.

In this section we assume that $\mathrm{char}(K) \neq 2$. There is already a Lie algebra version of the construction $\mathcal{X}(G)$ defined by Mendonça in [27]. For every Lie algebra $\mathfrak{g}$ there is a Lie algebra

$$\mathcal{X}(\mathfrak{g}) = \langle \mathfrak{g}, \mathfrak{g}^{\psi} \mid [a, a^{\psi}] \text{ for every } a \in \mathfrak{g} \rangle.$$

Consider the following ideals of $\mathcal{X}(\mathfrak{g})$:

$$L(\mathfrak{g}) = \langle\langle \{x - x^{\psi} \mid x \in \mathfrak{g}\} \rangle\rangle = \text{kernel of the map } \alpha \text{ defined below,}$$
$$D(\mathfrak{g}) = \langle\langle \{[x, y^{\psi}] \mid x, y \in \mathfrak{g}\} \rangle\rangle = \text{kernel of the map } \beta \text{ defined below,}$$
$$W(\mathfrak{g}) = L(\mathfrak{g}) \cap D(\mathfrak{g}) = \text{kernel of the map } \rho \text{ defined below.}$$

For simplicity, we will denote them by L, D and W respectively. The maps α, β, ρ are the following homomorphisms of Lie algebras defined for $x \in \mathfrak{g}$ by

$$\alpha : \mathcal{X}(\mathfrak{g}) \to \mathfrak{g}, \quad x \mapsto x, \ x^{\psi} \mapsto x;$$
$$\beta : \mathcal{X}(\mathfrak{g}) \to \mathfrak{g} \oplus \mathfrak{g}, \quad x \mapsto (x, 0), \ x^{\psi} \mapsto (0, x)$$

and
$$\rho : \mathcal{X}(\mathfrak{g}) \to \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}, \quad x \mapsto (x, x, 0), \ x^{\psi} \mapsto (0, x, x).$$

Thus we have a short exact sequence of Lie algebras

$$0 \to W \to \mathcal{X}(\mathfrak{g}) \to Im(\rho) \to 0 \tag{3}$$

and $\quad Im(\rho) = \{(a_1, a_2, a_3) \in \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \mid a_1 - a_2 + a_3 \in [\mathfrak{g}, \mathfrak{g}]\}$.

As shown by Mendonça in [27] for $\mathrm{char}(K) \neq 2$, L is in fact the Lie subalgebra generated by the elements $x - x^{\psi}$ for $x \in \mathfrak{g}$. He also showed that $[L, D] = 0$, thus W is abelian and central in the Lie algebra $\langle L, D \rangle$ generated by L and D (see [27] for details). Mendoça also showed ([27, Theorem 1.5]) that when $\mathfrak{g}$ is finitely presented then $\mathcal{X}(\mathfrak{g})$ is finitely presented and that if $\mathfrak{g}$ is finite dimensional then $\mathcal{X}(\mathfrak{g})$ is finite dimensional. A key step for the first result is the use of Theorem 2.4 to show:

Lemma 3.1. [27, Proposition 3.3] *Via ρ, $\mathcal{X}(\mathfrak{g})/W$ is a subdirect Lie sum in $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. As a consequence if $\mathfrak{g}$ is finitely presented, then so is $\mathcal{X}(\mathfrak{g})/W$.*

Note that $V = L/[L,L]$ is a right $\mathcal{X}(\mathfrak{g})$-module via the adjoint action on the right that factors through $\mathcal{X}(\mathfrak{g})/L \simeq \mathfrak{g}$ (in fact it factors through $\mathcal{X}(\mathfrak{g})/\langle L, D\rangle \simeq \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$). Thus we view V as a right $U(\mathfrak{g})$-module.

In this section we add to the list of the properties of $\mathcal{X}(\mathfrak{g})$ that if $\mathfrak{g}$ is a finitely generated nilpotent Lie algebra then $\mathcal{X}(\mathfrak{g})$ is nilpotent. In order to prove Theorem A we need the following simple lemma that holds for groups too (see [8]).

Lemma 3.2. *Assume that* $\mathrm{char}(K) \neq 2$. *Let* $\mathfrak{g}$ *be a Lie algebra and let* L *be the ideal of* $\mathcal{X}(\mathfrak{g})$ *defined before. Then* $\mathfrak{g}$ *acts nilpotently on* $V = L/[L,L]$ *i.e. there is* $m \geq 1$ *such that for the adjoint action extended to the universal enveloping algebra* $U(\mathfrak{g})$ *on* V *(denoted by* $\circ$*) we have* $V \circ \mathrm{Aug}(U(\mathfrak{g}))^m = 0$.

Proof. We will show that m can be chosen to be 2. By the proof of Proposition 4.2 in [27], for every $a \in \mathfrak{g}$ we have that $(L/[L,L]) \circ a^2 = 0$. Then since $(a+b)^2 = a^2 + b^2 + ab + ba$ we have for every $v \in V, a, b \in \mathfrak{g}$ that $v \circ ab = -v \circ ba$. As well since D and L commute in $\mathcal{X}(\mathfrak{g})$ and $\mathcal{X}(\mathfrak{g})/\langle L, D\rangle \simeq \mathfrak{g}/\mathfrak{g}'$ we have that for the adjoint action of $U(\mathfrak{g})$ on V the action of the Lie subagebra $\mathfrak{g}'$ is trivial i.e. for every $v \in V, a, b \in \mathfrak{g}$ we have that $v \circ ab = v \circ ba$. Since $\mathrm{char}(K) \neq 2$ we deduce that $v \circ ab = 0$ for every $v \in V, a, b \in \mathfrak{g}$. ∎

Proof of Theorem A. Consider the following exact sequence which is part of the 5-term exact sequence in homology associated to the central extension of Lie algebras $0 \to W \to L \to L/W \to 0$:

$$\ldots \to \mathrm{H}_2(L/W, K) \to \mathrm{H}_1(W,K)^L = W \to \mathrm{H}_1(L,K) = L/[L,L] \to \ldots$$

This yields an epimorphism $\mathrm{H}_2(L/W, K) \to W \cap [L,L]$.

Suppose now that $\mathfrak{g}_1$ is a nilpotent ideal in $\mathfrak{g}$ of finite codimension. Consider the short exact sequence of Lie algebras induced by (3)

$$0 \to W \to \rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1) \to \mathrm{Im}(\rho) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1) \to 0. \quad (4)$$

We aim to show that

$$\rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1) \text{ acts nilpotently on } W \text{ via the adjoint action.} \quad (5)$$

Note that since W is abelian the adjoint action of $\rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)$ factors through $\rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)/W \simeq \mathrm{Im}(\rho) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)$.

Also, $\rho(L) \simeq L/W \subseteq \mathcal{X}(\mathfrak{g})/W \simeq \mathrm{Im}(\rho) \subseteq \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. Then

$$[\mathrm{Im}(\rho), \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1] \subseteq [\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1] \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1$$

and since $\mathfrak{g}_1$ is nilpotent we deduce that $\mathrm{Im}(\rho) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)$ acts nilpotently on $\mathrm{Im}(\rho)$, hence it acts nilpotenly on $\rho(L) \simeq L/W$ via the adjoint action (on the right). Thus $\mathrm{Im}(\rho) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)$ acts nilpotently on $H_2(L/W, K)$ and its quotient $W \cap [L,L]$ i.e.

$$\rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1) \text{ acts nilpotently on } W \cap [L,L]. \quad (6)$$

On the other hand by Lemma 3.2 $\mathfrak{g}$ acts nilpotently on $V = L/[L,L]$, hence it acts nilpotently on $W/(W \cap [L,L])$. Note that the $\mathfrak{g}$-action in Lemma 3.2 is induced by the adjoint $\mathcal{X}(\mathfrak{g})$-action on L and the induced $\mathcal{X}(\mathfrak{g})$-action on V factors through $\mathcal{X}(\mathfrak{g})/\langle D, L\rangle \simeq \mathfrak{g}/\mathfrak{g}'$. Thus $\mathcal{X}(\mathfrak{g})$ acts nilpotently on $W/(W \cap [L, L])$. In particular

$$\rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1) \text{ acts nilpotently on } W/(W \cap [L, L]). \tag{7}$$

Thus (6) and (7) imply (5).

Observe that in the short exact sequence (4) W is abelian and $\mathrm{Im}(\rho) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)$ is nilpotent. This combined with (5) implies that $\rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)$ is a nilpotent Lie algebra. Finally

$$\mathcal{X}(\mathfrak{g})/\rho^{-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1) \simeq \mathrm{Im}(\rho)/\,\mathrm{Im}(\rho) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1)$$

is finite dimensional and in particular is zero if $\mathfrak{g}_1 = \mathfrak{g}$. ■

4. The Lie algebra $\nu(\mathfrak{g})$ and the non-abelian tensor square $\mathfrak{g} \otimes \mathfrak{g}$

In this section we define a Lie algebra version of the group

$$\nu(G) = \langle G, G^\psi \mid [g_1, g_2^\psi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\psi] = [g_1, g_2^\psi]^{g_3^\psi} \text{ for } g_1, g_2, g_3 \in G\rangle$$

defined by Rocco in [29]. Recall that $\psi : \mathfrak{g} \to \mathfrak{g}^\psi$ is an isomorphism of Lie algebras that sends $a \in \mathfrak{g}$ to $a^\psi \in \mathfrak{g}^\psi$.

The group $\nu(G)$ is related to both the $\mathcal{X}(G)$ construction and the non abelian tensor square defined in [9]. Note that in groups we have that $[a, b]^c = [a^c, b^c]$, where $a^c = c^{-1}ac$ and $[a, c] = a^{-1}c^{-1}ac$. But for Lie algebras we have the adjoint action and the Jacobi identity gives

$$[a, b, c] = [[a, c], b] + [a, [b, c]] = [a, c, b] - [b, c, a],$$

where all triple commutators are left normed. This can be interpreted by saying that the diagonal map for groups $\mathbb{Z}G \to \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G$ is given $g \to (g, g)$ but for Lie algebras the diagonal map $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes_K \mathcal{U}(\mathfrak{g})$ is given by $a \to 1 \otimes a + a \otimes 1$ for $a \in \mathfrak{g}$. This is the comultiplication structure for the respective Hopf algebras. Thus the relation in groups $[g_1, g_2^\psi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\psi]$ translates for Lie algebras as

$$[a_1, a_2^\psi, a_3] = [[a_1, a_3], a_2^\psi] + [a_1, [a_2^\psi, a_3^\psi]] = [a_1, a_3, a_2^\psi] + [a_3^\psi, a_2^\psi, a_1].$$

Therefore
$$[a_1, a_3, a_2^\psi] = [a_1, a_2^\psi, a_3] - [a_3^\psi, a_2^\psi, a_1].$$

The relation in groups $[g_1, g_2^\psi]^{g_3} = [g_1, g_2^\psi]^{g_3^\psi}$ translates for Lie algebras as

$$[a_1, a_2^\psi, a_3] = [a_1, a_2^\psi, a_3^\psi].$$

Note that these two relations together imply

$$[a_1, a_3, a_2^\psi] = [a_1, a_2^\psi, a_3] - [a_3^\psi, a_2^\psi, a_1] = [a_1, a_2^\psi, a_3^\psi] - [a_3^\psi, a_2^\psi, a_1] = [a_1, a_3^\psi, a_2^\psi],$$

where the last equality is the Jacobi identity. This justifies the following definition of $\nu(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$

$$\nu(\mathfrak{g}) = \langle \mathfrak{g} * \mathfrak{g}^\psi \mid [a_1, a_2^\psi, a_3] = [a_1, a_2, a_3^\psi] = [a_1, a_2^\psi, a_3^\psi] \text{ for } a_1, a_2, a_3 \in \mathfrak{g}\rangle. \tag{8}$$

Lemma 4.1. *The following relations hold in* $\nu(\mathfrak{g})$*:*

$$[a_1^\psi, a_2, a_3] = [a_1, a_2^\psi, a_3] = [a_1, a_2, a_3^\psi] = [a_1, a_2^\psi, a_3^\psi]$$
$$= [a_1^\psi, a_2, a_3^\psi] = [a_1^\psi, a_2^\psi, a_3].$$

Proof. To see it note that by the defining relations of $\nu(\mathfrak{g})$

$$[a_1^\psi, a_2, a_3] = -[a_2, a_1^\psi, a_3] = -[a_2, a_1, a_3^\psi] = [a_1, a_2, a_3^\psi],$$
$$[a_1^\psi, a_2, a_3^\psi] = -[a_2, a_1^\psi, a_3^\psi] = -[a_2, a_1, a_3^\psi] = [a_1, a_2, a_3^\psi],$$

and

$$[a_1^\psi, a_2^\psi, a_3] = -[a_3, a_1^\psi, a_2^\psi] - [a_2^\psi, a_3, a_1^\psi]$$
$$= -[a_3, a_1^\psi, a_2] - [a_2, a_3, a_1^\psi] = [a_1^\psi, a_2, a_3].$$

■

We recall the definition of the non-abelian tensor square of a Lie algebra. All Lie algebras we consider are over a fixed field K.

Definition 4.2. ([15]) Let $\mathfrak{g}$ be a Lie algebra. The non-abelian tensor square $\mathfrak{g}\otimes\mathfrak{g}$ is the Lie algebra generated by the symbols $a\otimes b$ with $a, b\in\mathfrak{g}$ subject to the following relations where $\lambda\in K$, $a, b, a', b'\in\mathfrak{g}$:

(i) $\lambda(a\otimes b) = \lambda a\otimes b = a\otimes\lambda b$,

(ii) $(a + a')\otimes b = a\otimes b + a'\otimes b$,

(iii) $a\otimes(b + b') = a\otimes b + a\otimes b'$,

(iv) $[a, a']\otimes b = a\otimes[a', b] - a'\otimes[a, b]$,

(v) $a\otimes[b, b'] = [b', a]\otimes b - [b, a]\otimes b'$.

(vi) $[a\otimes b, a'\otimes b'] = -[b, a]\otimes[a', b']$.

Lemma 4.3. *In the non-abelian tensor square* $\mathfrak{g}\otimes\mathfrak{g}$ *of a Lie algebra* $\mathfrak{g}$ *we have*

$$a\otimes[b, c] + [b, c]\otimes a = 0, \quad \textit{and}$$

$$a\otimes[b, c] + b\otimes[c, a] + c\otimes[a, b] = 0 = [a, b]\otimes c + [b, c]\otimes a + [c, a]\otimes b.$$

Proof. Using first (v) and then (iv) we obtain

$$a\otimes[b, c] + [b, c]\otimes a = [c, a]\otimes b - [b, a]\otimes c + [b, c]\otimes a$$
$$= (c\otimes[a, b] - a\otimes[c, b]) - (b\otimes[a, c] - a\otimes[b, c]) + (b\otimes[c, a] - c\otimes[b, a])$$
$$= 2(a\otimes[b, c] + b\otimes[c, a] + c\otimes[a, b]).$$

On the other hand using first (v) and then (iv) we have

$$a\otimes[b, c] = [c, a]\otimes b - [b, a]\otimes c$$
$$= (c\otimes[a, b] - a\otimes[c, b]) - (b\otimes[a, c] - a\otimes[b, c])$$
$$= 2a\otimes[b, c] + b\otimes[c, a] + c\otimes[a, b],$$

hence

$$a\otimes[b, c] + b\otimes[c, a] + c\otimes[a, b] = 0.$$

Similarly $0 = [a, b]\otimes c + [b, c]\otimes a + [c, a]\otimes b$. ■

Denote by $A \rightthreetimes B$ a semi-direct product of Lie algebras A and B i.e. we have a split short exact sequence of Lie algebras $0 \to A \to A \rightthreetimes B \to B \to 0$.

Definition 4.4. We define a Lie algebra $V(\mathfrak{g})$ as $((\mathfrak{g} \otimes \mathfrak{g}) \rightthreetimes \mathfrak{g}) \rightthreetimes \mathfrak{g}^{\psi}$, subject to the following rules:

(1) $\mathfrak{g}$, $\mathfrak{g}^{\psi}$ are Lie subalgebras of $V(\mathfrak{g})$ and the non-abelian square $\mathfrak{g} \otimes \mathfrak{g}$ is an ideal in $V(\mathfrak{g})$;

(2) For $a, b, c \in \mathfrak{g}$ we have: $[a, b^{\psi}] = a \otimes b$ and $[a \otimes b, c] = [a, b] \otimes c = [a \otimes b, c^{\psi}]$.

Lemma 4.5. *$V(\mathfrak{g})$ is a well-defined Lie algebra.*

Proof. To be sure that $V(\mathfrak{g})$ is well-defined it is necessary to check the Jacobi identity $[a_1, a_2, a_3] + [a_2, a_3, a_1] + [a_3, a_1, a_2] = 0$ for elements $a_1, a_2, a_3 \in \mathfrak{g} \cup \mathfrak{g}^{\psi} \cup (\mathfrak{g} \otimes \mathfrak{g})$ that is long and tedious and follows from the defining relations of the bracket in $V(\mathfrak{g})$ and the defining relations of the non-abelian tensor square $\mathfrak{g} \otimes \mathfrak{g}$. We consider several cases :

Case 1: Neither of the elements a_1, a_2, a_3 belongs to the non-abelian tensor square $\mathfrak{g} \otimes \mathfrak{g}$. Then without loss of generality we can assume that $a_1, a_2 \in \mathfrak{g}$ and $a_3 = b^{\psi} \in \mathfrak{g}^{\psi}$ (the case $a_1, a_2 \in \mathfrak{g}^{\psi}$ and $a_3 \in \mathfrak{g}$ is similar). Then

$$[a_1, a_2, a_3] = [a_1, a_2] \otimes b, \quad [a_2, a_3, a_1] = [a_2 \otimes b, a_1] = [a_2, b] \otimes a_1$$

and

$$[a_3, a_1, a_2] = -[a_1, a_3, a_2] = -[a_1 \otimes b, a_2] = -[a_1, b] \otimes a_2 = [b, a_1] \otimes a_2,$$

hence

$$[a_1, a_2, a_3] + [a_2, a_3, a_1] + [a_3, a_1, a_2] = [a_1, a_2] \otimes b + [a_2, b] \otimes a_1 + [b, a_1] \otimes a_2 = 0.$$

Case 2: Exactly one element is in $\mathfrak{g}$, in $\mathfrak{g}^{\psi}$ and in $\mathfrak{g} \otimes \mathfrak{g}$. Without loss of generality we can assume $a_1 = a \otimes b \in \mathfrak{g} \otimes \mathfrak{g}$, $a_2 \in \mathfrak{g}$ and $a_3 = c^{\psi} \in \mathfrak{g}^{\psi}$ (the case $a_1 \in \mathfrak{g} \otimes \mathfrak{g}$, $a_2 \in \mathfrak{g}^{\psi}$ and $a_3 \in \mathfrak{g}$ is similar). Then

$$[a_1, a_2, a_3] = [[a \otimes b, a_2], c^{\psi}] = [[a, b] \otimes a_2, c^{\psi}] = [a, b, a_2] \otimes c,$$

$$[a_2, a_3, a_1] = [[a_2, c^{\psi}], a \otimes b] = [a_2 \otimes c, a \otimes b] = [a_2, c] \otimes [a, b]$$

and

$$[a_3, a_1, a_2] = [c^{\psi}, a \otimes b, a_2] = -[a \otimes b, c^{\psi}, a_2] = -[[a, b] \otimes c, a_2] = -[a, b, c] \otimes a_2.$$

Hence for $v = [a, b]$ and by Lemma 4.3

$$[a_1, a_2, a_3] + [a_2, a_3, a_1] + [a_3, a_1, a_2] = [a, b, a_2] \otimes c + [a_2, c] \otimes [a, b] - [a, b, c] \otimes a_2$$

$$= [v, a_2] \otimes c + [a_2, c] \otimes v + [c, v] \otimes a_2 = 0.$$

Case 3: Exactly two elements belong to $\mathfrak{g} \otimes \mathfrak{g}$ and the third belongs to $\mathfrak{g}$ (the case when the third belongs to $\mathfrak{g}^{\psi}$ is similar). Thus we can assume that $a_1, a_2 \in \mathfrak{g} \otimes \mathfrak{g}$ and $a_3 \in \mathfrak{g}$, and it suffices to consider the case when $a_1 = a \otimes b, a_2 = c \otimes d$. Then

$$[a_1, a_2, a_3] = [a \otimes b, c \otimes d, a_3] = [[a, b] \otimes [c, d], a_3] = [[a, b], [c, d]] \otimes a_3,$$

$$[a_2, a_3, a_1] = [c \otimes d, a_3, a \otimes b] = [[c, d] \otimes a_3, a \otimes b] = [c, d, a_3] \otimes [a, b]$$

and $$[a_3, a_1, a_2] = [a_3, a \otimes b, c \otimes d] = -[a \otimes b, a_3, c \otimes d]$$
$$= -[[a, b] \otimes a_3, c \otimes d] = -[a, b, a_3] \otimes [c, d].$$

Hence for $v = [a, b]$, $w = [c, d]$ we have

$$[a_1, a_2, a_3] + [a_2, a_3, a_1] + [a_3, a_1, a_2]$$
$$= [[a, b], [c, d]] \otimes a_3 + [c, d, a_3] \otimes [a, b] - [a, b, a_3] \otimes [c, d]$$
$$= [v, w] \otimes a_3 + [w, a_3] \otimes v + [a_3, v] \otimes w = 0.$$

Case 4: Finally if all three elements belong to $\mathfrak{g} \otimes \mathfrak{g}$ we can use that we already know that $\mathfrak{g} \otimes \mathfrak{g}$ is a Lie algebra, hence the Jacobi identity holds. This completes the proof of the fact that $V(\mathfrak{g})$ is well-defined. ■

In the case of groups, Rocco has shown (see [29, Proposition 2.6]) that the non abelian tensor square $G \otimes G$ is isomorphic to the normal subgroup $[G, G^\psi]$ of $\nu(G)$. For Lie algebras we have

Lemma 4.6. *The Lie subalgebra $[\mathfrak{g}, \mathfrak{g}^\psi] = \langle [a, b^\psi] \mid a, b \in \mathfrak{g} \rangle$ is an ideal in $\nu(G)$.*

Proof. Let $a_1, a_2, a_3 \in \mathfrak{g}$. Since $[a_1, a_2^\psi, a_3] = [a_1, a_2, a_3^\psi] \in [\mathfrak{g}, \mathfrak{g}^\psi]$ we see that $[\mathfrak{g}, \mathfrak{g}^\psi, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}^\psi]$. Since $[a_1, a_2^\psi, a_3^\psi] = [a_1, a_2, a_3^\psi] \in [\mathfrak{g}, \mathfrak{g}^\psi]$ we see moreover that $[\mathfrak{g}, \mathfrak{g}^\psi, \mathfrak{g}^\psi] \subseteq [\mathfrak{g}, \mathfrak{g}^\psi]$. ■

The following result implies Theorem B.

Proposition 4.7. *Consider the ideal $[\mathfrak{g}, \mathfrak{g}^\psi]$ of $\nu(\mathfrak{g})$. Then $\mathfrak{g} \otimes \mathfrak{g} \cong [\mathfrak{g}, \mathfrak{g}^\psi]$ and $V(\mathfrak{g}) \simeq \nu(\mathfrak{g})$.*

Proof. (1): We define a linear map $\tau : V(\mathfrak{g}) \to \nu(\mathfrak{g})$, which is the identity on $\mathfrak{g} \cup \mathfrak{g}^\psi$ and $\tau(a \otimes b) = [a, b^\psi]$. We claim that it is well-defined and is a homomorphism of Lie algebras. Thus we have to check that the defining relators of $\mathfrak{g} \otimes \mathfrak{g}$ are mapped to zero in $\nu(\mathfrak{g})$ and that $\tau([u_1, u_2]) = [\tau(u_1), \tau(u_2)]$ where $u_1, u_2 \in \mathfrak{g} \cup \mathfrak{g}^\psi \cup \{a \otimes b \mid a, b \in \mathfrak{g}\}$. But observe that (i), (ii) and (iii) in Definition 4.2 follow by the bilinearity of the Lie bracket. The other items are a consequence of the relators of $\nu(\mathfrak{g})$ and the Jacobi identity: let $a, a', b, b' \in \mathfrak{g}$, then

$$[[a, a'], b^\psi] = [a, [a', b^\psi]] + [a', [b^\psi, a]] = [a, [(a')^\psi, b^\psi]] - [a', [a^\psi, b^\psi]] \text{ in } \nu(\mathfrak{g})$$

yields (iv). For (v) we do something similar:

$$[a, [b^\psi, (b')^\psi]] = -[b^\psi, [(b')^\psi, a]] - [(b')^\psi, [a, b^\psi]] = [[b', a], b^\psi] - [[b, a], (b')^\psi] \text{ in } \nu(\mathfrak{g}).$$

Note that (vi) is a consequence of

$$[[a, b^\psi], [a', (b')^\psi]] = [[a', (b')^\psi], [b^\psi, a]] = [[(a')^\psi, (b')^\psi], [b^\psi, a]] = \tag{9}$$
$$-[[b^\psi, a], [(a')^\psi, (b')^\psi]] = -[[b, a], [(a')^\psi, (b')^\psi]] = -[[b, a], [a', b']^\psi] \text{ in } \nu(\mathfrak{g}).$$

Finally we show that τ commutes with the Lie bracket. Note that

$$\tau([a_1 \otimes b_1, a_2 \otimes b_2]) = \tau([a_1, b_1] \otimes [a_2, b_2]) = [[a_1, b_1], [a_2^\psi, b_2^\psi]]$$

and by (9)

$$[\tau(a_1\otimes b_1),\tau(a_2\otimes b_2)]=[[a_1,b_1^\psi],[a_2,b_2^\psi]]=[[a_1,b_1],[a_2^\psi,b_2^\psi]].$$

For $c\in\mathfrak{g}$ we have

$$\tau([a\otimes b,c])=\tau([a,b]\otimes c)=[a,b,c^\psi]=[a,b^\psi,c]=[\tau(a\otimes b),\tau(c)]$$

and

$$\tau([a\otimes b,c^\psi])=\tau([a,b]\otimes c)=[a,b,c^\psi]=[a,b^\psi,c^\psi]=[\tau(a\otimes b),\tau(c^\psi)].$$

Note that the definition of $\mathfrak{g}\otimes\mathfrak{g}$ implies that $\tau(\mathfrak{g}\otimes\mathfrak{g})=[\mathfrak{g},\mathfrak{g}^\psi]$ and that τ is surjective.

(2): We construct an inverse of the map τ i.e. a homomorphism of Lie algebras

$$\mu:\nu(\mathfrak{g})\to V(\mathfrak{g})$$

which is the identity on $\mathfrak{g}\cup\mathfrak{g}^\psi$. Thus $\mu([a,b^\psi])=a\otimes b$. We have to check that the defining relations of $\nu(\mathfrak{g})$ are sent to 0 under μ. Indeed for $a_1,a_2,a_3\in\mathfrak{g}$ we have

$$\mu([a_1,a_2^\psi,a_3])=[\mu([a_1,a_2^\psi]),\mu(a_3)]=[a_1\otimes a_2,a_3]=[a_1,a_2]\otimes a_3,$$

$$\mu([a_1,a_2,a_3^\psi])=[\mu([a_1,a_2]),\mu(a_3^\psi)]=[[a_1,a_2],a_3^\psi]=[a_1,a_2]\otimes a_3$$

and

$$\mu([a_1,a_2^\psi,a_3^\psi])=[\mu([a_1,a_2^\psi]),\mu(a_3^\psi)]=[a_1\otimes a_2,a_3^\psi]=[a_1,a_2]\otimes a_3.$$ ■

Recall that in $\mathcal{X}(\mathfrak{g})$ we have $[D,L]=0$ provided $\operatorname{char}(K)\neq 2$ ([27, Lemma 3.2]). Therefore, for any $a,b,c\in\mathfrak{g}$,

$$[[a,b^\psi],c-c^\psi]=0,\quad \text{thus}\quad [a,b^\psi,c]=[a,b^\psi,c^\psi].$$

Let $R=R(\mathfrak{g})=[\mathfrak{g},L,\mathfrak{g}^\psi]$. As observed in [27] it follows from the facts that L and D are ideals in $\mathcal{X}(\mathfrak{g})$ and that $[L,D]=0$ that R is an ideal of $\mathcal{X}(\mathfrak{g})$.
In $\mathcal{X}(\mathfrak{g})/R$ we also have

$$[a,b-b^\psi,c^\psi]=0,\quad \text{thus}\quad [a,b,c^\psi]=[a,b^\psi,c^\psi].$$

This means that there is an epimorphism of Lie algebras for $a\in\mathfrak{g}\cup\mathfrak{g}^\psi$:

$$\delta:\nu(\mathfrak{g})\twoheadrightarrow\mathcal{X}(\mathfrak{g})/R,\quad a\mapsto a+R.$$

Therefore if we put $\Delta=\Delta(\mathfrak{g})=\ker\delta$ we have a short exact sequence of Lie algebras

$$0\to\Delta\to\nu(\mathfrak{g})\to\mathcal{X}(\mathfrak{g})/R\to 0 \tag{10}$$

Note that since L is generated as a Lie algebra by $\{b-b^\psi\mid b\in\mathfrak{g}\}$ [27], there is a presentation of Lie algebras (in terms of generators and relations)

$$\mathcal{X}(\mathfrak{g})/R=\langle\mathfrak{g},\mathfrak{g}^\psi\mid[a,a^\psi]=0,[a,b-b^\psi,c^\psi]=0\text{ for }a,b,c\in\mathfrak{g}\rangle. \tag{11}$$

Lemma 4.8. *Suppose* $\operatorname{char}(K)\neq 2$. *Then* $\Delta\subseteq\nu(\mathfrak{g})'\cap Z(\nu(\mathfrak{g}))$, *hence* Δ *is a quotient of the Schur multiplier* $\mathrm{H}_2(\mathcal{X}(\mathfrak{g})/R,K)$.

Proof. By the presentations (in terms of generators and relations) of Lie algebras (8) and (11) we deduce that Δ is the ideal of $\nu(\mathfrak{g})$ generated by $\{[a, a^\psi] \mid a \in \mathfrak{g}\}$. Observe that for any $a \in \mathfrak{g}$, the element $[a, a^\psi]$ lies in $Z(\nu(\mathfrak{g}))$. Indeed for $a, b \in \mathfrak{g}$ we have in $V(\mathfrak{g})$ that

$$[[a, a^\psi], b] = [a \otimes a, b] = [a, a] \otimes b = 0 \otimes b = 0.$$

Also, $[a, a^\psi] \in [G, G^\psi] \leq \nu(\mathfrak{g})'$. ■

Proof of Theorem C. (1) If $\mathrm{char}(K) \neq 2$ we can use the following arguments. The requirement about the characteristic comes from the fact that we want to use the Lie algebra $\mathcal{X}(\mathfrak{g})$ and its structure theory as developed in [27] works only when $\mathrm{char}(K) \neq 2$.

(a) There is a short exact sequence

$$0 \to W/R \to \mathcal{X}(\mathfrak{g})/R \to \mathcal{X}(\mathfrak{g})/W \to 0$$

and by Theorem 3.1 $\mathcal{X}(\mathfrak{g})/W$ is finitely presented. Moreover, by [27]

$$W/R \cong \mathrm{H}_2(\mathfrak{g}, K)$$

and the hypothesis that $\mathfrak{g}$ is finitely presented implies that $\mathrm{H}_2(\mathfrak{g}, K)$ is finite dimensional. Hence $\mathcal{X}(\mathfrak{g})/R$ is finitely presented.

As $\mathcal{X}(\mathfrak{g})/R \cong \nu(\mathfrak{g})/\Delta$, we see that Δ is finitely generated as an ideal in $\nu(\mathfrak{g})$, but since Δ is central in $\nu(\mathfrak{g})$ (Lemma 4.8) it must be finite dimensional. Then using that $\nu(\mathfrak{g})$ is an extension of the finitely dimensional ideal Δ by a finitely presented Lie algebra we deduce that also $\nu(\mathfrak{g})$ is finitely presented.

(b) By Theorem A $\mathcal{X}(\mathfrak{g})$ is nilpotent, hence $\mathcal{X}(\mathfrak{g})/R \simeq \nu(G)/\Delta$ is nilpotent. On the other hand Δ is central in $\nu(G)$, hence $\nu(\mathfrak{g})$ is nilpotent.

(c) Assume now that $\mathfrak{g}$ is free non-abelian. As $\mathcal{X}(\mathfrak{g})/W$ is a subdirect Lie sum in $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ (Lemma 3.1) we can apply [20, Theorem B] and deduce that $\mathcal{X}(\mathfrak{g})/W$ is not of type FP_3. But by the previous paragraph, $W/R \simeq \mathrm{H}_2(\mathfrak{g}, K) = 0$ and Δ is finite dimensional. Thus $\nu(\mathfrak{g})/\Delta \simeq \mathcal{X}(\mathfrak{g})/R = \mathcal{X}(\mathfrak{g})/W$ is not of type FP_3. Hence $\nu(\mathfrak{g})$ is not of type FP_3.

(2) Alternatively if we do not want to use any condition on the characteristic of K we can develop a theory for $V(\mathfrak{g})$, similar to the one developed in [27] for $\mathcal{X}(\mathfrak{g})$, that avoids restrictions on the characteristic of K.

Let L_0 be the ideal of $V(\mathfrak{g})$ generated by $\{a - a^\psi \mid a \in \mathfrak{g}\}$. Then $V(\mathfrak{g}) = L_0 \leftthreetimes \mathfrak{g}$. Consider the diagonal map

$$\rho_0 : V(\mathfrak{g}) \to (V(\mathfrak{g})/L_0) \oplus (V(\mathfrak{g})/\mathfrak{g} \otimes \mathfrak{g}) \simeq \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$$

i.e. it sends $v \in V(\mathfrak{g})$ to $(v + L_0, v + \mathfrak{g} \otimes \mathfrak{g})$. By Theorem 2.4

$$\mathrm{Im}(\rho_0) \simeq V(\mathfrak{g})/\ker(\rho_0) \text{ is finitely presented.}$$

Note that $\ker(\rho_0) = L_0 \cap (\mathfrak{g} \otimes \mathfrak{g})$. Then $m = \sum_i a_i \otimes b_i \in \ker(\rho_0)$ if and only if the image of m in the canonical projection $V(\mathfrak{g}) \to V(\mathfrak{g})/L_0$ is trivial, but the last canonical projection identifies b_i^ψ with b_i and $m = \sum_i [a_i, b_i^\psi]$. Thus $m \in \ker(\rho_0)$ if and only if $\sum_i [a_i, b_i] = 0$ in $\mathfrak{g}$. Note that by the definition of non-abelian tensor product of Lie algebras in [14]

$$\mathfrak{g} \wedge \mathfrak{g} = (\mathfrak{g} \otimes \mathfrak{g})/\Delta_0,$$

where Δ_0 is the central Lie subalgebra of $V(\mathfrak{g})$ generated by $\{a \otimes a \mid a \in \mathfrak{g}\}$ and by [14]

$$H_2(\mathfrak{g}, K) \simeq \ker(\mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}),$$

where the map $\mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ sends $a \otimes b$ to $[a, b]$. This shows that there is a short exact sequence of Lie algebras

$$0 \to \Delta_0 \to \ker(\rho_0) \to H_2(\mathfrak{g}, K) \to 0.$$

Hence there is a short exact sequence of Lie algebras

$$0 \to H_2(\mathfrak{g}, K) \to V(\mathfrak{g})/\Delta_0 \to Im(\rho_0) \to 0.$$

At this point we can continue as in the previous proof:

(a), (c) Suppose $\mathfrak{g}$ is finitely presented. Then the Schur multiplier $H_2(\mathfrak{g}, K)$ is finite dimensional, hence $V(\mathfrak{g})/\Delta_0$ is finitely presented, hence Δ_0 is finitely generated as an ideal in $V(\mathfrak{g})$, hence is finite dimensional.

If $\mathfrak{g}$ is finitely generated, free, non-abelian Lie algebra, $H_2(\mathfrak{g}, K) = 0$, $V(\mathfrak{g})/\Delta_0 \subseteq \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ and by [20, Theorem B] $V(\mathfrak{g})/\Delta_0$ is not of type FP_3.

(b) Suppose that $\mathfrak{g}$ is nilpotent. By [30] the non-abelian tensor square $\mathfrak{g} \otimes \mathfrak{g}$ is nilpotent and by the definition of the bracket in $V(\mathfrak{g})$ we have that $V(\mathfrak{g})/(\mathfrak{g} \otimes \mathfrak{g}) \simeq \mathfrak{g} \oplus \mathfrak{g}^{\psi}$ acts nilpotently on $\mathfrak{g} \otimes \mathfrak{g}$.

5. The Lie algebra $\mathcal{E}(L)$

In this section we consider the Lie algebra version of the group $\mathcal{E}(G)$ defined by Lima and Sidki in [26]. Let $\mathfrak{g}$ be a Lie algebra and consider the following ideals of $\mathfrak{g} * \mathfrak{g}^{\psi}$, where $\psi : \mathfrak{g} \to \mathfrak{g}^{\psi}$ is an isomorphism

$$L_0 = \langle\langle \{a - a^{\psi} \mid a \in \mathfrak{g}\}\rangle\rangle, \quad D_0 = \langle\langle \{[a, b^{\psi}] \mid a, b \in \mathfrak{g}\}\rangle\rangle.$$

We define the following Lie algebra: $\mathcal{E}(\mathfrak{g}) = \langle \mathfrak{g}, \mathfrak{g}^{\psi} \mid [L_0, D_0] = 0 \rangle$.

We denote by $\mathcal{L}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$ the images of L_0 and D_0 respectively under the canonical epimorphism $\mathfrak{g} * \mathfrak{g}^{\psi} \to \mathcal{E}(\mathfrak{g})$. In other words,

$\mathcal{L}(\mathfrak{g}) =$ ideal of $\mathcal{E}(\mathfrak{g})$ generated by $\{a - a^{\psi} \mid a \in \mathfrak{g}\}$ = kernel of the map $\widehat{\alpha}$ below,

$\mathcal{D}(\mathfrak{g}) =$ ideal of $\mathcal{E}(\mathfrak{g})$ generated by $\{[a, b^{\psi}]\} \mid a, b \in \mathfrak{g}\}$ = kernel of the map $\widehat{\beta}$ below.

We also set $\mathcal{W}(\mathfrak{g}) =: \mathcal{L}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})$ = kernel of the map $\widehat{\rho}$ below.

The homomorphisms of Lie algebras $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\rho}$ are defined for $x \in \mathfrak{g}$ by

$$\widehat{\alpha} : \mathcal{E}(\mathfrak{g}) \to \mathfrak{g}, \quad x \mapsto x, \quad x^{\psi} \mapsto x,$$

$$\widehat{\beta} : \mathcal{E}(\mathfrak{g}) \to \mathfrak{g} \oplus \mathfrak{g}, \quad x \mapsto (x, 0), \quad x^{\psi} \mapsto (0, x),$$

and $$\widehat{\rho} : \mathcal{E}(\mathfrak{g}) \to \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}, \quad x \mapsto (x, x, 0), \quad x^{\psi} \mapsto (0, x, x).$$

Note that $Im(\widehat{\rho}) = \{(a_1, a_2, a_3) \in \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \mid a_1 - a_2 + a_3 \in [\mathfrak{g}, \mathfrak{g}]\}$.

For simplicity we write $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{W}$. These ideals behave in a similar way as the ideals L, D and W in the $\mathcal{X}(\mathfrak{g})$-construction for $\mathrm{char}(K) \neq 2$. In this section we do not assume any condition on the caracteristic of K unless otherwise stated.

Lemma 5.1. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) *$\mathcal{W}$ is abelian and central in $\langle \mathcal{L}, \mathcal{D} \rangle = \mathcal{L} + \mathcal{D}$;*

(ii) *The quotient $\mathcal{E}(\mathfrak{g})/\mathcal{W}$ is a subdirect sum in $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ that maps surjectively on pairs;*

(iii) *If $\mathfrak{g}$ is finitely presented then so is $\mathcal{E}(\mathfrak{g})/\mathcal{W}$;*

(iv) *If $\mathrm{char}(K) \neq 2$ then $\mathcal{E}(\mathfrak{g})/\mathcal{W} \simeq \mathcal{X}(\mathfrak{g})/W$ and there is an epimorphsm of Lie algebras $\mathcal{E}(\mathfrak{g}) \to \mathcal{X}(\mathfrak{g})$ that is the identity on $\mathfrak{g} \cup \mathfrak{g}^{\psi}$.*

Proof. (i) Follows directly from $[\mathcal{L}, \mathcal{D}] = 0$;

(ii) $\mathcal{E}(\mathfrak{g})/\mathcal{W} = \mathcal{E}(\mathfrak{g})/\ker(\widehat{\rho}) \simeq \mathrm{Im}(\widehat{\rho})$ is a subdirect sum of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ that maps surjectively on pairs;

(iii) Follows directly from Theorem 2.4 and (ii);

(iv) The defining relations of $\mathcal{E}(\mathfrak{g})$ hold in $\mathcal{X}(\mathfrak{g})$ by [27]. By [27] $\mathcal{X}(\mathfrak{g})/W$ is isomorphic to the same subdirect sum of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ as $\mathrm{Im}(\rho)$. Actually it is the same construction of a diagonal map $\mathcal{X}(\mathfrak{g}) \to (\mathcal{X}(\mathfrak{g})/L) \oplus (\mathcal{X}(\mathfrak{g})/D) \simeq \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. ■

We recall here that $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ and that $\mathrm{Aug}(U(\mathfrak{g})) = \mathfrak{g}U(\mathfrak{g})$ is the augmentation ideal. The bracket in $\mathrm{Aug}(U(\mathfrak{g})) \leftthreetimes \mathfrak{g}$ is defined as follows : for $u_1, u_2 \in \mathrm{Aug}(U(\mathfrak{g}))$, $x_1, x_2 \in \mathfrak{g}$ we have

$$(u_1, x_1), (u_2, x_2) \in \mathrm{Aug}(U(\mathfrak{g})) \leftthreetimes \mathfrak{g} \quad \text{with} \quad [(u_1, x_1), (u_2, x_2)] = (x_1 u_2 - x_2 u_1, [x_1, x_2]).$$

Our next result is also a Lie algebra version of a group theoretical result in [31].

Lemma 5.2. *Let $\mathfrak{g}$ be a Lie algebra and L_0 be the ideal in the free product Lie algebra $\mathfrak{g} * \mathfrak{g}^{\psi}$ generated by $a - a^{\psi}$ for $a \in \mathfrak{g}$. Then the homomorphism of Lie algebras*

$$\theta : \mathfrak{g} * \mathfrak{g}^{\psi} \to \mathrm{Aug}(U(\mathfrak{g})) \leftthreetimes \mathfrak{g},$$

given by $\theta(a) = (0, a)$ and $\theta(a^{\psi}) = (-a, a)$ for $a \in \mathfrak{g}$, induces an isomorphism

$$\theta_0 : \mathfrak{g} * \mathfrak{g}^{\psi}/[L_0, L_0] \to \mathrm{Aug}(U(\mathfrak{g})) \leftthreetimes \mathfrak{g}.$$

Proof. Note first that $\theta([L_0, L_0]) = 0$. Indeed $\theta(a - a^{\psi}) = (a, 0) \in Aug(U(\mathfrak{g}))$. Hence $\theta([L_0, L_0]) = [\theta(L_0), \theta(L_0)] \subseteq [Aug(U(\mathfrak{g})), Aug(U(\mathfrak{g}))] = 0$.

A technical proof that the induced map θ_0 is an isomorphism can be obtained using the ideas in the proof of [27, Lemma 4.2]. Here we give a different proof using relation modules of Lie algebras. Note first that we have a split extension of Lie algebras

$$0 \to L_0 \to \mathfrak{g} * \mathfrak{g}^{\psi} \to \mathfrak{g} \to 0$$

such that the projection map $\mathfrak{g} * \mathfrak{g}^{\psi} \to \mathfrak{g}$ factors through θ, so the result will follow if we prove that $L_0/L_0' \cong \mathrm{Aug}(U(\mathfrak{g}))$ via θ_0.

Let X be a generating set for $\mathfrak{g}$, $F(X)$ the free Lie algebra on X and $I(X)$ an ideal of $F(X)$ such that

$$F(X)/I(X) \simeq \mathfrak{g}.$$

Note that if $\hat{L}_0$ is the ideal in $\mathfrak{g}*\mathfrak{g}^\psi$ generated by $x-x^\psi$ for $x \in X$, then $\mathfrak{g}*\mathfrak{g}^\psi/\hat{L}_0$ is a Lie algebra where we can identify x with x^ψ i.e. it is isomorphic to $\mathfrak{g}$, so $L_0 = \hat{L}_0$ is generated as an ideal of $\mathfrak{g} * \mathfrak{g}^\psi$ by $x - x^\psi$ for $x \in X$.

Let $Z = X^\psi$ be a disjoint set from X together with a bijection $\psi : X \to Z$ sending x to x^ψ and we write z_i for x_i^ψ. Thus

$$F(Z)/I(Z) \simeq \mathfrak{g}^\psi.$$

Then $$F(X \cup Z)/S_0 \simeq \mathfrak{g} * \mathfrak{g}^\psi \text{ and } F(X \cup Z)/S \simeq \mathfrak{g},$$

where $$S_0 = \langle\langle I(X), I(Z)\rangle\rangle \text{ and } S = \langle\langle I(X), I(Z), \{x - x^\psi\}_{x\in X}\rangle\rangle$$

are ideals in $F(X \cup Z)$. Note that

$$L_0 \simeq S/S_0 \text{ and } L_0/[L_0, L_0] \simeq (S/[S,S])/\overline{S}_0,$$

where $\overline{S}_0$ is the image of S_0 in $S/[S,S]$. We need to prove that $L_0/[L_0, L_0] \simeq \mathrm{Aug}(U(\mathfrak{g}))$ via θ_0.

By Lemma 2.1 every presentation of a Lie algebra gives an exact sequence, in particular we have the following exact sequences

$$0 \to I(X)_{ab} \xrightarrow{\partial_2} \bigoplus_{x\in X} xU(\mathfrak{g}) \xrightarrow{\partial_1} U(\mathfrak{g}) \to K \to 0 \tag{12}$$

and $$0 \to I(Z)_{ab} \xrightarrow{d_2} \bigoplus_{x\in X} x^\psi U(\mathfrak{g}^\psi) \xrightarrow{d_1} U(\mathfrak{g}^\psi) \to K \to 0. \tag{13}$$

Identifying $\mathfrak{g}^\psi$ with $\mathfrak{g}$ we have

$$0 \to I(Z)_{ab} \xrightarrow{d_2} \bigoplus_{x\in X} x^\psi U(\mathfrak{g}) \xrightarrow{d_1} U(\mathfrak{g}) \to K \to 0$$

and $$0 \to S_{ab} \longrightarrow (\bigoplus_{x\in X} xU(\mathfrak{g})) \bigoplus (\bigoplus_{x\in X} x^\psi U(\mathfrak{g})) \xrightarrow{\widetilde{d_1}} U(\mathfrak{g}) \to K \to 0,$$

where $\widetilde{d_1} = (\partial_1, d_1)$. Then

$$S_{ab} = (\mathrm{Im}(\partial_2) \oplus \mathrm{Im}(d_2)) + (\bigoplus_{x\in X} (x - x^\psi)U(\mathfrak{g}))$$

and $$\overline{S}_0 = \mathrm{Im}(\partial_2) \oplus \mathrm{Im}(d_2).$$

Note that

$$\mathrm{Im}(\partial_2) \oplus \mathrm{Im}(d_2) \subseteq S_{ab} \subseteq (\bigoplus_{x\in X} xU(\mathfrak{g})) \bigoplus (\bigoplus_{x\in X} x^\psi U(\mathfrak{g})),$$

hence

$$S_{ab}/(\mathrm{Im}(\partial_2) \oplus \mathrm{Im}(d_2)) \subseteq (\bigoplus_{x\in X} xU(\mathfrak{g})/\mathrm{Im}(\partial_2)) \bigoplus (\bigoplus_{x\in X} x^\psi U(\mathfrak{g})/\mathrm{Im}(d_2)).$$

Then using the exactness of (12) and (13) we have

$$\bigoplus_{x\in X} xU(\mathfrak{g})/\operatorname{Im}(\partial_2) = \bigoplus_{x\in X} xU(\mathfrak{g})/\ker(\partial_1) \simeq \operatorname{Im}(\partial_1) \simeq \operatorname{Aug}(U(\mathfrak{g}))$$

and $$\bigoplus_{x\in X} x^{\psi}U(\mathfrak{g})/\operatorname{Im}(d_2) = \bigoplus_{x\in X} x^{\psi}U(\mathfrak{g})/\ker(d_1) \simeq \operatorname{Im}(d_1) \simeq \operatorname{Aug}(U(\mathfrak{g})).$$

Thus we have an inclusion

$$S_{ab}/(\operatorname{Im}(\partial_2)\oplus \operatorname{Im}(d_2)) \xrightarrow{\mu} (\operatorname{Aug}(U(\mathfrak{g}))) \oplus (\operatorname{Aug}(U(\mathfrak{g})))$$

and this inclusion sends $x - x^{\psi}$ to $(x, -x)$. Then

$$\begin{aligned} L_0/[L_0, L_0] &\simeq S_{ab}/(\operatorname{Im}(\partial_2)\oplus \operatorname{Im}(d_2)) \simeq \operatorname{Im}(\mu) \\ &= \{(\lambda, -\lambda) \mid \lambda \in \operatorname{Aug}(U(\mathfrak{g}))\} \simeq \operatorname{Aug}(U(\mathfrak{g})). \end{aligned}$$ ■

Lemma 5.3. *There is an isomorphism of right $U(\mathfrak{g})$-modules*

$$\mathcal{L}/\mathcal{L}' \simeq \operatorname{Aug}(U(\mathfrak{g}))/(\operatorname{Aug}(U([\mathfrak{g},\mathfrak{g}]))\operatorname{Aug}(U(\mathfrak{g})) = \mathfrak{g}U(\mathfrak{g})/[\mathfrak{g},\mathfrak{g}]\mathfrak{g}U(\mathfrak{g}),$$

where $U(\mathfrak{g})$ acts on the left hand side via the adjoint action of $\mathfrak{g}$ and on the right hand side the action is induced by the multiplication in $U(\mathfrak{g})$.

Proof. Note that if we denote by J the ideal of $\mathfrak{g} * \mathfrak{g}^{\psi}$ generated by $[D_0, L_0]$, where D_0 and L_0 are the ideals defined at the beginning of this Section, then $\mathcal{L} = (L_0 + J)/J = L_0/J$ since $J \subseteq L_0$. Thus $[\mathcal{L}, \mathcal{L}] = ([L_0, L_0] + J)/J$ and

$$\mathcal{L}/[\mathcal{L}, \mathcal{L}] = L_0/([L_0, L_0] + J).$$

Then the map θ from Lemma 5.2 induces an isomorphism

$$\mathcal{L}/[\mathcal{L}, \mathcal{L}] \simeq \theta(L_0)/\theta([L_0, L_0] + J). \tag{14}$$

By the proof of Lemma 5.2 we have that

$$\theta(L_0) = \operatorname{Aug}(U(\mathfrak{g})). \tag{15}$$

On the other hand

$$\theta([x_1, x_2^{\psi}]) = [\theta(x_1), \theta(x_2^{\psi})] = [(0, x_1), (-x_2, x_2)] = (-x_1x_2, [x_1, x_2])$$

and hence $$\theta(D_0) \subseteq \operatorname{Aug}(U(\mathfrak{g})) \leftthreetimes [\mathfrak{g},\mathfrak{g}].$$

Thus, since $\theta([L_0, L_0]) = 0$, we have

$$\begin{aligned} \theta([L_0, L_0] + J) = \theta([L_0, L_0]) + \theta(J) = \theta(J) = \langle\langle \theta([D_0, L_0]) \rangle\rangle \\ \subseteq \langle\langle [\theta(D_0), \theta(L_0)] \rangle\rangle \subseteq \langle\langle [\operatorname{Aug}(U(\mathfrak{g})) \leftthreetimes [\mathfrak{g},\mathfrak{g}], \operatorname{Aug}(U(\mathfrak{g}))] \rangle\rangle \\ \subseteq [\mathfrak{g},\mathfrak{g}]\operatorname{Aug}(U(\mathfrak{g})) = \operatorname{Aug}(U([\mathfrak{g},\mathfrak{g}]))\operatorname{Aug}(U(\mathfrak{g})). \end{aligned}$$

To see that both ideals are in fact equal it suffices to observe that for $x_1, x_2, x_3 \in \mathfrak{g}$ we have

$$([x_1, x_2]x_3, 0) = [(-x_1x_2, [x_1, x_2]), (x_3, 0)] = \theta([[x_1, x_2^{\psi}], x_3 - x_3^{\psi}]) \\ \in \theta([D_0, L_0]) \subseteq \theta(J).$$

Thus
$$\theta([L_0, L_0] + J) = \mathrm{Aug}(U([\mathfrak{g}, \mathfrak{g}]))\mathrm{Aug}(U(\mathfrak{g})). \tag{16}$$

Then the proof is completed by (14), (15) and (16). ∎

Lemma 5.4. *Let $H = \langle B, t \mid [t, s] = d(s)$ for $s \in S\rangle$ be a Lie algebra HNN-extension with an abelian Lie subalgebra A. Then*

$$\dim_K(A/(B \cap A)) \leq 1.$$

Proof. Let $A = (A \cap B) \oplus A_0$ as vector spaces. Then A_0 is an abelian Lie subalgebra of H such that $A_0 \cap B = 0$. By Lemma 2.6 if L_0 is a Lie subalgebra of an HNN extension Lie algebra such that

(i) L_0 intersects trivially the associated Lie subalgebra and

(ii) L_0 intersects the base Lie subalgebra in a free Lie algebra

then L_0 is itself free. In particular for $L_0 = A_0$ we get that A_0 is a free Lie algebra and being abelian this implies that $\dim_K(A_0) \leq 1$. ∎

Proof of Theorem D. Note first that we have a semidirect product decomposition $\mathcal{E}(\mathfrak{g}) = \mathcal{L} \rtimes \mathfrak{g}$, hence since $\mathcal{E}(\mathfrak{g})$ is finitely presented then so is $\mathfrak{g}$. As we observed before, using Theorem 2.4 we can deduce that $\mathcal{E}(\mathfrak{g})/\mathcal{W}$ is also finitely presented, hence $\mathcal{W}$ is a finitely generated $U(\mathcal{E}(\mathfrak{g}))$-module via the adjoint $\mathcal{E}(\mathfrak{g})$-action. The $\mathcal{E}(\mathfrak{g})$-action factors through $\mathcal{E}(\mathfrak{g})/\langle\mathcal{D}, \mathcal{L}\rangle \simeq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] =: Q$. Thus $\mathcal{W}$ is a finitely generated $U(Q)$-module and by the Noetherianess of $U(Q)$ we deduce that

$$\mathcal{W} \cap [\mathcal{L}, \mathcal{L}] \text{ is a finitely generated } U(Q)\text{-module.} \tag{17}$$

Note that since $[\mathfrak{g}, \mathfrak{g}]$ is a finitely generated ideal in $\mathfrak{g}$ (i.e. $Q = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a finitely presented Lie algebra) and $\mathcal{E}(\mathfrak{g}) = \mathcal{L} \rtimes \mathfrak{g}$ is finitely presented we deduce that

$$(\mathcal{L}/[\mathcal{L}, [\mathfrak{g}, \mathfrak{g}]]) \rtimes (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \simeq (\mathcal{L} \rtimes \mathfrak{g})/\langle\langle[\mathfrak{g}, \mathfrak{g}]\rangle\rangle \text{ is finitely presented,} \tag{18}$$

where $\langle\langle[\mathfrak{g}, \mathfrak{g}]\rangle\rangle$ is the ideal in $\mathcal{E}(\mathfrak{g}) = \mathcal{L} \rtimes \mathfrak{g}$ generated by $[\mathfrak{g}, \mathfrak{g}]$.

Assume now that $Q \neq 0$ and consider the Lie algebra

$$H = V \rtimes Q, \quad \text{where } V = \mathcal{L}/((\mathcal{W} \cap [\mathcal{L}, \mathcal{L}]) + [\mathcal{L}, [\mathfrak{g}, \mathfrak{g}]]).$$

It follows from (17) and (18) that H is finitely presented.

As we are assuming that $Q \neq 0$ we may choose an ideal Q_0 of Q of codimension one and $t \in Q$ such that Q_0 and t generate Q. Then $V \rtimes Q_0$ is an ideal of codimension one of H. By Theorem 2.7, H can be decomposed as an HNN-extension

$$H = \langle B, t \mid [t, s] = d(s) \text{ for } s \in S\rangle,$$

where $S \le B$ are finitely generated subalgebras of $V \rtimes Q_0$ and $d : S \to B$ is a derivation. Furthermore the proof of Theorem 2.7 in [32] shows that B can be generated by any finite generating set Y of the ideal $V \rtimes Q_0$ of H. In particular we can fix Y that contains a basis of Q_0 as a vector space, thus assume that $Q_0 \subseteq B$.

Let $\widetilde{\mathcal{W}}$ be the image of $\mathcal{W}$ in H. By Lemma 5.4 applied for $A = \widetilde{\mathcal{W}}$ we deduce that $\dim_K(\widetilde{\mathcal{W}}/\widetilde{\mathcal{W}} \cap B) \le 1$. If $\widetilde{\mathcal{W}} \not\subseteq B$ then $\widetilde{\mathcal{W}} = Kw_0 \oplus (\widetilde{\mathcal{W}} \cap B)$. Then since $\widetilde{W} \subseteq V \subseteq V \rtimes Q_0$, we deduce that by adding to Y the element w_0 we get a new decomposition as HNN-extension

$$H = \langle \widehat{B}, t \mid [t, s] = \widehat{d}(s) \text{ for } s \in \widehat{S} \rangle,$$

where $\widehat{S} \le \widehat{B}$ are finitely generated subalgebras of $V \rtimes Q_0$ and $\widehat{d} : \widehat{S} \to \widehat{B}$ is a derivation. Note that $Q_0 \subseteq B \subseteq \widehat{B}$, $\widetilde{\mathcal{W}} \cap B \subseteq B \subseteq \widehat{B}$ and $w_0 \in \widehat{B}$, hence

$$\widetilde{\mathcal{W}} \subseteq \widehat{B}.$$

If $\widetilde{\mathcal{W}} \subseteq B$ there is no need to change the HNN-extension i.e. $\widehat{B} = B$ and $\widehat{S} = S$.

Note that Lemma 2.5 implies that $Q_0 \le \widehat{S}$ because in other case there would be an element t_0 of Q_0 such that t_0 and t generate a free non-abelian Lie subalgebra of Q. Thus we deduce that

$$\widehat{S} = (\widehat{S} \cap V) \rtimes Q_0$$

and as $\widehat{S}$ is a finitely generated Lie algebra, $\widehat{S} \cap V$ is finitely generated as ideal of $\widehat{S}$. Then $\widehat{S} = \langle s_1, \ldots, s_m, q_1, \ldots, q_k \mid s_i \in \widehat{S} \cap V, q_j \in Q_0 \rangle$. We write $s_i \circ U(Q_0)$ for the adjoint action of $U(Q_0)$ on s_i. Then $\widehat{S} \cap V = \langle s_i \circ U(Q_0) \mid 1 \le i \le m \rangle$ and $(\widehat{S} \cap V)/[\widehat{S} \cap V, \widehat{S} \cap V]$ is finitely generated as $U(Q_0)$-module by the images of $\{s_1, \ldots, s_m\}$.

If $\widetilde{\mathcal{W}} \not\leq \widehat{S} \cap V$, then using Lemma 2.5 again we deduce that some element of $\widetilde{\mathcal{W}}$ and t generate a free Lie algebra of rank 2. But $\widetilde{\mathcal{W}}$ is an abelian ideal of H thus the Lie subalgebra generated by $\widetilde{\mathcal{W}}$ and t is metabelian and can not contain a copy of the free Lie algebra of rank 2. Therefore

$$\widetilde{\mathcal{W}} \le \widehat{S} \cap V.$$

Note that
$$\widetilde{\mathcal{W}} \cap [\widehat{S} \cap V, \widehat{S} \cap V] \subseteq \widetilde{\mathcal{W}} \cap [V, V] = 0,$$

where the last equality comes from the fact that the image of $\mathcal{W} \cap [\mathcal{L}, \mathcal{L}]$ in V is zero. Hence $\widetilde{\mathcal{W}}$ embeds in the finitely generated $U(Q_0)$-module $(\widehat{S} \cap V)/[\widehat{S} \cap V, \widehat{S} \cap V]$ as $U(Q_0)$-submodule (the last follows from the fact that $\mathcal{W}$ is an ideal in $\mathcal{E}(\mathfrak{g})$). As $U(Q_0)$ is a commutative polynomial ring, so is a Noetherian ring, we deduce that

$$\widetilde{\mathcal{W}} \text{ is finitely generated as } U(Q_0)\text{-module.}$$

Now, consider the composition of epimorphisms

$$H \overset{\gamma}{\twoheadrightarrow} \mathcal{L}/[\mathcal{L}, \mathcal{L}] \rtimes Q = (\mathrm{Aug}(U(\mathfrak{g}))/\mathrm{Aug}(U([\mathfrak{g}, \mathfrak{g}]))\mathrm{Aug}(U(\mathfrak{g}))) \rtimes Q \overset{\pi}{\twoheadrightarrow} \mathrm{Aug}(U(Q)) \rtimes Q,$$

where we have used Lemma 5.3 to identify the Lie algebras in the middle, γ is the map induced by the canonical projection $\gamma_0 : V \to \mathcal{L}/[\mathcal{L}, \mathcal{L}]$ and π is induced by the canonical projection $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Note that γ_0 is well-defined since $[\mathfrak{g},\mathfrak{g}]$ acts trivially on $\mathcal{L}/[\mathcal{L},\mathcal{L}]$, hence $[\mathcal{L},[\mathfrak{g},\mathfrak{g}]] \subseteq [\mathcal{L},\mathcal{L}]$.

Let $x \in \mathfrak{g} \setminus [\mathfrak{g},\mathfrak{g}]$. Then $[x, x^\psi] \in \mathcal{W}$ and as

$$\gamma([x,x^\psi]) = [(0,x),(-x,x)] = [(0,x),(-x,0)+(0,x)] = [(0,x),(-x,0)] = (-x^2,0)$$

we deduce
$$\pi\gamma([x,x^\psi]) = (-x_1^2, 0) \in \mathrm{Aug}(U(Q)),$$

where x_1 is the image of x in $Q = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. As $\widetilde{\mathcal{W}}$ is a $U(Q)$-module finitely generated as $U(Q_0)$-module and $U(Q_0)$ is Noetherian, the $U(Q)$-submodule of $\mathrm{Aug}(U(Q))$ generated by $-x_1^2$ should also be finitely generated as $U(Q_0)$-module. But this is impossible. Indeed $U(Q)$ is a commutative polynomial ring on the variables $x_1, x_2, \dots, x_{k+1}$. Then the ideal I of $U(Q)$ generated by $-x_1^2$ is not finitely generated as $U(Q_0)$-module, since I is isomorphic to $U(Q)$ as $U(Q)$-module and $U(Q)$ is not finitely generated as $U(Q_0)$-module, since $U(Q_0)$ is a commutative polynomial ring on k variables. ∎

Lemma 5.5. *If $\mathcal{L}/[\mathcal{L},\mathcal{L}]$ is finite dimensional, $\mathfrak{g}$ is of type* FP_2 *and $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ then $\mathcal{W}$ is finite dimensional.*

Proof. Note first that $\mathcal{W}/\mathcal{W} \cap [\mathcal{L},\mathcal{L}]$ embeds in $\mathcal{L}/[\mathcal{L},\mathcal{L}]$, hence $\mathcal{W}/\mathcal{W} \cap [\mathcal{L},\mathcal{L}]$ is finite dimensional. Also, there is a stem extension

$$0 \to \mathcal{W} \cap [\mathcal{L},\mathcal{L}] \to \mathcal{L} \to \mathcal{L}/(\mathcal{W} \cap [\mathcal{L},\mathcal{L}]) \to 0$$

and as in the beginning of the proof of Theorem A the 5-term exact sequence in homology yields an epimorphism

$$\mathrm{H}_2(\mathcal{L}/\mathcal{W}, K) \to \mathcal{W} \cap [\mathcal{L},\mathcal{L}].$$

Consider the ideal in $\mathfrak{g} \oplus \mathfrak{g}$

$$S := \mathcal{L}/\mathcal{W} \simeq \widehat{\rho}(\mathcal{L}) \simeq \langle\langle \{(a,-a) \mid a \in \mathfrak{g}\} \rangle\rangle = \{(a_1,a_2) \in \mathfrak{g} \oplus \mathfrak{g} \mid a_1 + a_2 \in [\mathfrak{g},\mathfrak{g}]\}.$$

If $\mathrm{char}(K) \neq 2$ we can apply [27, Lemma 4.1] that gives that if $\mathfrak{g}$ is FP_2 and $[\mathfrak{g},\mathfrak{g}] = [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]$ then $\mathrm{H}_2(S,K)$ is finite dimensional. Since

$$S \simeq \{(a_1,a_2) \in \mathfrak{g} \oplus \mathfrak{g} \mid a_1 + a_2 \in [\mathfrak{g},\mathfrak{g}]\},$$

the proof of [27, Lemma 4.1] applies even in the case $\mathrm{char}(K) = 2$, hence in both cases $\mathrm{H}_2(S,K)$ is finite dimensional and $\mathcal{W} \cap [\mathcal{L},\mathcal{L}]$ is finite dimensional. ∎

Proof of Theorem E. The fact that (5) implies (1) follows immediately from Theorem D. Next we check that (1), (3) and (4) are all equivalent. By Lemma 5.3

$$\mathcal{L}/[\mathcal{L},\mathcal{L}] \simeq (\mathrm{Aug}U(\mathfrak{g}))/\mathrm{Aug}(U([\mathfrak{g},\mathfrak{g}]))\mathrm{Aug}(U(\mathfrak{g})). \tag{19}$$

Thus if (1) holds, i.e., if $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ we get that

$$\mathcal{L}/[\mathcal{L},\mathcal{L}] = (\mathrm{Aug}U(\mathfrak{g}))/(\mathrm{Aug}U(\mathfrak{g}))^2 \simeq \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = 0.$$

If (3) holds i.e. $\mathcal{L}/[\mathcal{L},\mathcal{L}]$ is finite dimensional, by (19) $\mathcal{L}/[\mathcal{L},\mathcal{L}]$ maps surjectivey onto $AugU(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$, we deduce that $AugU(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ is finite dimensional and so $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]=0$.

To see that (1) implies (2), by Lemma 5.5 it remains only to show that $\mathcal{L}/[\mathcal{L},\mathcal{L}]$ is finite dimensional. But we already know that (1) implies (4), actually $\mathcal{L}/[\mathcal{L},\mathcal{L}]=0$.

Finally, we check that (2) implies (5). Consider first the quotient $\mathcal{E}(\mathfrak{g})/\mathcal{W}$ which is a subdirect sum in $\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$ that maps surjectively on pairs and contains the commutator Lie subalgebra $[\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g},\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}]$. Since $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$ we deduce that $\mathcal{E}(\mathfrak{g})/\mathcal{W}\simeq\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$ which is obviously finitely presented and of type FP_m since $\mathfrak{g}$ is finitely presented and of type FP_m. As $\mathcal{W}$ is finite dimensional we deduce the same for $\mathcal{E}(\mathfrak{g})$. ∎

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