Journal of Lie Theory
Volume **31** (2021) 493–516

Hamiltonian Systems on Co-Adjoint Lie Groupoids

Ghorbanali Haghighatdoost and Rezvaneh Ayoubi

Communicated by M. Schlichenmaier

Abstract. Our purpose is to introduce by means of co-adjoint representation of a Lie groupoid on its isotropy Lie algebroid a class of Lie groupoids. In other words, we show that the orbits of the co-adjoint representation on the isotropy Lie algebroid of a Lie groupoid are Lie groupoid. We will call this type of Lie groupoid, co-adjoint Lie groupoid. Also, we try to construct and define Hamiltonian systems on the co-adjoint Lie groupoids. By considering the trivial Lie groupoid as an example, we show that our construction can be considered as a generalization of the construction of the Lie groups to the Lie groupoids. Finally we present the types I and II of Hamilton-Jacobi theorem of the Hamiltonian system corresponding to the co-adjoint Lie algebroid.

Mathematics Subject Classification: 18B40, 53D17, 70H08, 70H20.

Key Words: Lie groupoids, Lie algebroids, Hamiltonian system, Hamilton-Jacobi equation.

1. Introduction

As we know, the theory of Lie groupoids is a generalized theory of Lie groups. Unfortunately, for the Lie groupoid there is not a normal adjoint and co-adjoint representation. So, the authors, solve this by different methods [8].

Also, it is well-known that the co-adjoint orbits of a Lie group are symplectic manifolsd and therefore they are natural candidates for phase space of Hamiltonian systems. Hamiltonian systems on the phase space of this type endowed with the standard Lie-Poisson bracket. The natural methods for describing this type of systems are provided by the theory of Lie groups and Lie algebras[2]. For studying the structure of Hamiltonian systems on co-adjoint orbits one can see [13].

In this work, we introduce a new class of Lie groupoid by using adjoint and co-adjoint action, naturally, analogue to Lie groups and Lie algebras. By considering some examples, exactly, the well-known trivial Lie groupoid, one can see that our theory of Hamiltonian systems on co-adjoint Lie groupoid is the generalization of Hamiltonian systems on Lie groups. On the other hand, in [5] authors defined Poisson structure on dual of a Lie algebroid of a Lie groupoid. In this work we show that this Poisson structure on dual of co-adjoint Lie algebroid for trivial Lie groupoid give us the well-known Kirillov-Kostant bracket on the dual space of the tangent space of co-adjoint orbit of a Lie group. So, as a result for every Hamiltonian section on dual of co-adjoint Lie algebroid we obtain the interesting correspondence between Hamiltonian vector field on dual space of trivial Lie algebroid and Hamiltonian vector field on dual space of tangent space of co-adjoint orbit of the Lie group.

ISSN 0949–5932 / $2.50

The Hamilton-Jacobi theory was developed in 1866 by Jacobi. Then Abraham and Marsden took this issue from a geometric perspective [1] and using what they put forward, Wang was able to introduce two types of Hamilton-Jacobi theory of Hamiltonian system on the cotangent bundle [14]. Also, de León et al. were able to apply the version of Abraham and Marsden to Lie algebroid [5]. Finally, in our previous article [6], we express the types I and II of the Hamiltonian-Jacobi theory of the Hamiltonian system on Lie algebroid.

In the last section of this article, using the concepts stated in [6], we present the types I and II of Hamilton-Jacobi theorem of the Hamiltonian system corresponding to co-adjoint Lie algebroid.

2. Some basic concepts and definitions

2.1. Review of Lie groupoid and Lie algebroid conceptions

Let us have a survey on some definitions about Lie groupoids [9] and related conceptions that we can consider in this paper (for review see [3] and [4] and [10]).

Definition 2.1. A *groupoid* consists of two sets G and M together with structural mappings $\alpha, \beta, 1, \iota$ and m, where α, β are mappings from G to M, and are called source and target mappings, respectively. $1 : M \longrightarrow G$ is the *unit mapping*, $\iota : G \longrightarrow G$ is the *inverse mapping* and

$$m : G_2 = \{(g, h) \in G \times G \mid \alpha(g) = \beta(h)\} \longrightarrow G$$

is called the *multiplication mapping*.
A groupoid G over M will be denoted by $G \rightrightarrows M$.
A *Lie groupoid* is a groupoid $G \rightrightarrows M$ for which G and M are smooth manifolds, $\alpha, \beta, 1, \iota$ and m, are differentiable mappings and besides, α, β are differentiable submersions.

Definition 2.2. Let $G \rightrightarrows M$ be a Lie groupoid. A smooth map $\sigma : M \longrightarrow G$ is called a *bisection* of G if it is right inverse to $\alpha : G \longrightarrow M$, $(\alpha \circ \sigma = id_M)$ and $\beta \circ \sigma : M \longrightarrow M$ is a diffeomorphism.
As mentioned in [9], for each bisection σ, the left translation corresponding to σ is defined as $L_\sigma : G \longrightarrow G,\ g \longmapsto \sigma(\beta g)g$, and the right translation corresponding to σ is defined by $R_\sigma : G \longrightarrow G,\ g \longmapsto g\sigma\big((\beta \circ \sigma)^{-1}(\alpha g)\big)$.

Definition 2.3. Let $G \rightrightarrows M$ be a Lie groupoid and $U \subseteq M$ be an open subset of M. A map $\sigma : U \longrightarrow M$ is called a *local bisection* of G if it is right inverse to α and $\beta \circ \sigma : U \longrightarrow (\beta \circ \sigma)(U)$ is diffeomorphism, where $\beta \circ \sigma(U)$ is an open set in M.

Proposition 2.4. *Let G be a Lie groupoid over M. For $g \in G$, there is a local bisection σ such that $\sigma(\alpha g) = g$.*

Proof. see [9]. ∎

The definition of a groupoid's left action on the smooth mapping is presented below, which there is a similar concept of a right smooth action.

Definition 2.5. A *smooth left action* of a Lie groupoid G on a smooth map $J\colon N \longrightarrow M$ is a smooth map $\theta\colon G\ {}_{\alpha}\times_{J} N \longrightarrow N$ which satisfies the following properties:

1. For every $(g,n) \in G\ {}_{\alpha}\times_{J} N, \quad J(g.n) = \beta(g)$,
2. For every $n \in N, \quad 1_{J(n)}.n = n$,
3. For every $(g,g') \in G_2$ and $n \in J^{-1}(\alpha(g')), \quad g.(g'.n) = (gg').n$

(where $g.n := \theta(g,n)$ and $\theta(g)(n) := \theta(g,n)$).

Definition 2.6. A *Lie algebroid* A over a manifold M is a vector bundle $\tau\colon A \to M$ which is equipped with the following data:

1. A Lie bracket $[|\ ,\ |]$ on the space of smooth section of τ, for which that space is a Lie algebra $[|\ ,\ |]\colon \Gamma(\tau)\times\Gamma(\tau) \longrightarrow \Gamma(\tau),\ (X,Y) \longmapsto [|X,Y|]$.
2. A vector bundle map $\rho\colon A \longrightarrow TM$ called the *anchor map* such that we have for $X,Y \in \Gamma(\tau)$ and $f \in C^\infty(M)$,
$$[|X,fY|] = f[|X,Y|] + \rho(X)(f)Y.$$

Definition 2.7. Let $G \rightrightarrows M$ be a Lie groupoid with structure mappings $\alpha, \beta, 1, \iota$ and m. The *Lie algebroid* corresponding to the Lie groupoid $G \rightrightarrows M$ is defined as

$$AG := kerT\alpha|_{1_p}$$

where $\alpha\colon G \longrightarrow M$ is the source mapping of the Lie groupoid $G \rightrightarrows M$, and $T\alpha\colon TG \longrightarrow TM$ is the tangent mapping of α. AG has Lie algebroid structure as follows: If we consider the Lie algebroid AG as a vector bundle $\tau\colon AG \longrightarrow M$, it is well-known that there exists a bijection between the space of sections $\Gamma(\tau)$ and the set of left (right) invariant vector fields on G (see [9]).

If X be a section of $\tau\colon AG \longrightarrow M$, right invariant and left invariant vector fields corresponding to X are, respectively, defined by

$$\overrightarrow{X}(g) = TR_g(X(\beta(g))) \quad \text{and} \quad \overleftarrow{X}(g) = -T(L_g)T(\iota)(X(\alpha(g)),$$

where L_g and R_g are left translation and right translation corresponding to $g \in G$.

By using the above information, Lie algebroid structure $([|\ ,\ |], \rho)$ on AG can be introduced as follows:

1. The anchor map $\rho\colon AG \longrightarrow TM,\ \rho(X)(x) = T_{1(p)}\beta(X(p))$, where $X \in \Gamma(\tau)$, $p \in M$.
2. Lie bracket: $$\Gamma(AG)\times\Gamma(AG) \longrightarrow \Gamma(AG)$$ $$\overrightarrow{[|X,Y|]} := [\overrightarrow{X}, \overrightarrow{Y}], \quad (\overleftarrow{[|X,Y|]} := -[\overleftarrow{X}, \overleftarrow{Y}])$$ where $X,Y \in \Gamma(\tau)$ and $[\ ,\ ]$ is standard Lie bracket of vector fields.

Similar to the action of the Lie groupoid on smooth mapping (Definition 2.5), the definition of action of a Lie algebroid on a smooth mapping will be as follows:

Definition 2.8. An action of a Lie algebroid $(A, M, \pi, \rho, [| \ , \ |])$ on the map $J\colon N \longrightarrow M$ is a map $\theta\colon \Gamma(A) \longrightarrow \mathfrak{X}(N)$ which for all $f \in C^\infty(M)$ and $X, Y \in \Gamma^\infty(A)$, satisfies the following properties:

1. $\theta(X+Y) = \theta(X) + \theta(Y)$
2. $\theta(fX) = J^* f \theta(X)$
3. $\theta([|X,Y|]) = [\theta(X), \theta(Y)]$
4. $TJ(\theta(X)) = \rho(X)$

where $J^*\colon C^\infty(M) \longrightarrow C^\infty(N)$ such that $J^* f = f \circ J \in C^\infty(N)$ is the pullback of f by J.

Remark 2.9. Let θ be the action of a Lie groupoid G on smooth map $J\colon N \longrightarrow M$ which was introduced into definition 2.5. As mentioned in [3], every action of a Lie groupoid G on $J\colon N \longrightarrow M$ induce an action θ' of a Lie algebroid $A(G)$ on $J\colon N \longrightarrow M$ as follows:

$$\theta'(X)(n) := \frac{d}{dt}|_{t=0}\ Exp(tX)_{J(n)}.n.$$

Definition 2.10. Let G be a Lie groupoid over M. Then G is a *regular Lie groupoid* if the anchor $(\beta, \alpha)\colon G \longrightarrow M;\ g \longmapsto \big((\beta(g), \alpha(g)\big)$ is a mapping of constant rank.

Remark 2.11. In the following, throughout the article, we assume that G be a regular Lie groupoid over M.

2.2. The adjoint and co-adjoint actions

In this subsection, we will briefly discuss some concepts to see how adjoint and co-adjoint actions are constructed and defined, which one can refer to [3] for more details.

Let $G \rightrightarrows M$ be a Lie groupoid over M. The isotropy group of $G \rightrightarrows M$ is defined by $I_p = \alpha^{-1}(p) \cap \beta^{-1}(p)$ where $p \in M$ is an arbitrary element of M. It is well-known that I_p is a Lie group where its composition law and inverse map are the restrictions of multiplication map m and inverse map ι to I_p, respectively. The union of all isotropy groups I_p when p rounds over in M construct a groupoid over M, i.e. $I_G = (\cup I_p)_{p \in M}$ is a groupoid over M. Note that the isotropy groupoid of Lie groupoid is not a smooth manifold in general, see example A.10 in [12].

Lemma 2.12. *Let G be a regular Lie groupoid, then its associated isotropy groupoid is Lie groupoid.*

Proof. see [12] ∎

We denote the associated isotropy Lie groupoid to Lie groupoid $G \rightrightarrows M$ by I_G and the Lie algebroid associated to isotropy Lie groupoid by AI_G and call it isotropy Lie algebroid.

Consider Lie groupoid $G \rightrightarrows M$ and its associated isotropy Lie groupoid I_G. G acts smoothly from the left on $J : I_G \longrightarrow M$ by conjugation, it means $C : G \times I_G \longrightarrow I_G$, $C(g)(g') := g g' g^{-1}$ is an action of G on I_G which we call it conjugation action.

On the other hand, the conjugation action induces an action of a Lie groupoid G on $AI_G \longrightarrow M$. We call this action *adjoint action* of G on AI_G which can be defined as follows:

$$Ad : G \times AI_G \longrightarrow AI_G, \quad Ad_g X := \frac{d}{dt}\Big|_{t=0} C(g)Exp(tX)$$

where $p \in M$, $g \in G_p = \alpha^{-1}(p)$ (the α-fibers over p) and $X \in (AI_G)_p$.

According to Remark 2.9, the action Ad induces an adjoint action of AG on $AI_G \to M$ as follows:

$$ad : AG \times AI_G \to AI_G, \quad ad_X Y = ad(X)(Y) := \frac{d}{dt}\Big|_{t=0} Ad(Exp(tX))Y$$

where $X \in (AG)_p$, $Y \in (AI_G)_p$ and $p \in M$.

One can easily prove that for every $X \in \Gamma(AG)$ and $Y \in \Gamma(AI_G)$

$$ad_X(Y) = [|X, Y|].$$

Another action of G on dual bundle A^*I_G which is called *co-adjoint action* of G, is defined as follows:

$$Ad^* \colon G \times A^*I_G \longrightarrow A^*I_G, \quad Ad^*_g \xi(X) := \xi(Ad_{g^{-1}}X).$$

In other words $\langle Ad^*_g \xi, X\rangle = \langle \xi, Ad_{g^{-1}}X\rangle$ where $g \in G_p$, $\xi \in (A^*I_G)_p$.

Again, as previously stated (remark 2.9), the action Ad^* induces so-called co-adjoint action of a Lie algebroid AG on A^*I_G which is defined by:

$$ad^* \colon AG \times A^*I_G \longrightarrow A^*I_G, \quad ad^*_X \xi(Y) := \xi(ad_{-X}(Y)) = \xi([|Y, X|])$$

or $\langle ad^*_X \xi, Y\rangle = \langle \xi, ad(-X)Y\rangle$ where $\xi \in (A^*I_G)_p$.

3. Co-adjoint orbits

In this section, we define the orbits of co-adjoint action of Lie groupoid using the actions which we discussed in the previous section and show that these orbits of co-adjoint action of Lie groupoid are themselves a Lie groupoid.

Definition 3.1. Let $G \rightrightarrows M$ be a Lie groupoid. We define the orbit of co-adjoint action of a Lie groupoid G as follows:

$$O(\xi) = \{Ad^*_g \xi \mid g \in G\}$$

where ξ is an element of $(A^*I_G)_p$. We call $O(\xi)$ *co-adjoint orbit* of the Lie groupoid G.

3.1. Co-adjoint Lie groupoid

In the later, we try to equip the groupoid structure on the co-adjoint orbit $O(\xi)$ over M.

Let $G \rightrightarrows M$ be a Lie groupoid with source, target, multiplication, unit and inverse mapping α, β, m, 1, ι, respectively. We claim that $O(\xi)$ is a Lie groupoid over M. So, we introduce the structural mappings of $O(\xi)$ as follows:

1. source mapping: $\alpha' : O(\xi) \longrightarrow M;\ Ad^*_g\xi \longmapsto \alpha(g)$,
2. target mapping: $\beta' : O(\xi) \longrightarrow M;\ Ad^*_g\xi \longmapsto \beta(g)$,
3. multiplication mapping: $m' \colon (O(\xi))_2 \longrightarrow O(\xi)$;

$$(Ad^*_g\xi, Ad^*_h\xi) \longmapsto Ad^*_{m(g,h)}\xi = Ad^*_{gh}\xi,$$

4. unit mapping: $1' : M \longrightarrow O(\xi);\ p \longmapsto Ad^*_{1_p}\xi$,
5. inverse mapping: $\iota' \colon O(\xi) \longrightarrow O(\xi);\ Ad^*_g\xi \longmapsto Ad^*_{g^{-1}}\xi$.

We call $\alpha', \beta', m', 1'$ and ι', source, target, multiplication, unit and inverse mapping, respectively, for Lie groupoid $O(\xi)$.

Theorem 3.2. *The orbits of co-adjoint action of any Lie groupoid, are Lie groupoids.*

Proof. Let G be a Lie groupoid over smooth manifold M. Consider co-adjoint action Ad^* of G which is introduced in subsection 2.2. For an arbitrary element ξ in $(A^*I_G)_p$, consider the orbit $O(\xi)$ with the structure mappings $\alpha', \beta', m', 1'$ and ι'. In order to $O(\xi)$ be a groupoid over M with $\alpha', \beta', m', 1'$ and ι', one must check the following conditions:

1. $\alpha'(g'h') = \alpha'(h'), \qquad \beta'(g'h') = \beta'(g') \qquad$ for all $g', h' \in (\mathcal{G}_\xi)_2$.
2. $(g'h')j' = g'(h'j') \qquad$ for all $g', h', j' \in \mathcal{G}_\xi$ $s.t$ $\alpha'(g') = \beta'(h'), \quad \alpha'(h') = \beta'(j')$.
3. $\alpha'(1'_p) = \beta'(1'_p) = p \qquad$ for all $p \in M$.
4. $g'(1'_{\alpha'(g')}) = g', \quad$ *and* $\quad (1_{\beta'(g')})g' = g' \qquad$ for all $g' \in \mathcal{G}_\xi$.
5. each $g' \in \mathcal{G}_\xi$ has a two-sided inverse ${g'}^{-1}$ such that

$$\alpha'({g'}^{-1}) = \beta'(g'), \qquad \beta'({g'}^{-1}) = \alpha'(g') \quad \text{and} \quad {g'}^{-1}g' = 1'_{\alpha'(g')}, \qquad g'{g'}^{-1} = 1'_{\beta'(g')}$$

Here, we only check parts 1 and 5 for example. The other relations can be checked similarly.

1. By definition of the source and target mappings, for every $g', h' \in \mathcal{G}_{\xi_2}$, we have

$$\alpha'(g'h') = \alpha'(Ad^*_{gh}\xi) = \alpha(gh) = \alpha(h) = \alpha'(h'),$$

and

$$\beta'(g'h') = \beta'(Ad^*_{gh}\xi) = \beta(gh) = \beta(g) = \beta'(g').$$

2. Let $g' = Ad^*_g\xi \in \mathcal{G}_\xi$, where $g \in G$. Based on well known properties of the Lie groupoid $G \rightrightarrows M$, we have

$$\alpha'(Ad^*_{g^{-1}}\xi) = \alpha(g^{-1}) = \beta(g) = \beta'(Ad^*_{g^{-1}}\xi)$$

and

$$\beta'(Ad^*_{g^{-1}}\xi) = \beta(g^{-1}) = \alpha(g) = \alpha'(Ad^*_{g^{-1}}\xi).$$

Also

$$Ad^*_{g^{-1}}\xi \,.\, Ad^*_g\xi = Ad^*_{g^{-1}.g}\xi = Ad^*_{1_{\alpha(g)}}\xi = 1'_{\alpha'(Ad^*_g\xi)}$$

and

$$Ad^*_g\xi \,.\, Ad^*_{g^{-1}}\xi = Ad^*_{g.g^{-1}}\xi = Ad^*_{1_{\beta(g)}}\xi = 1'_{\beta'(Ad^*_g\xi)}.$$

As we know, associated to the smooth groupoid action G on $A^*I_G \longrightarrow M$ is an action Lie groupoid $G \ltimes A^*I_G$ over A^*I_G whose orbits are the same as the orbits of the action and the set of orbits is a submanifold of A^*I_G (see Theorem 1.5.11 in [9] for more details). So $O(\xi)$ is smooth submanifold of A^*I_G.

In addition, since $G \rightrightarrows M$ is a Lie groupoid, its structural mappings have the properties described in Definition 2.1. So, according to definition of mappings $\alpha', \beta', m', 1'$ and ι', they inherit these properties from structural mappings of Lie groupoid $G \rightrightarrows M$. Therefore, $O(\xi) \rightrightarrows M$ is a Lie groupoid. ■

As a result, we obtain $O(\xi)$ has a Lie groupoid structure on M, with structural mappings $\alpha', \beta', m', 1'$ and ι', and we will call this Lie groupoid *co-adjoint Lie groupoid* and denote it by $\mathcal{G}_\xi$.

In the following, using the definition of bisection and some related content, we will define bisection and related concepts for the co-adjoint Lie groupoid.

At first, we recall following theorem that is proven in [9] and actually introduces the tangent mapping of multiplication mapping m. We will use this theorem to establish a key lemma, which is an important tool for purposes of our work in this paper.

Theorem 3.3. *Consider the Lie groupoid $G \rightrightarrows M$ and its multiplication mapping m. Let σ, τ be any (local) bisections of G that $\sigma(\alpha g) = g$ and $\tau(\alpha h) = h$. Assuming that $X \in T_gG$, $Y \in T_hG$ and $T\alpha(X) = T\beta(Y) = W$. Then*

$$X.Y = Tm(X,Y) = TL_\sigma(Y) + TR_\tau(X) - TL_\sigma TR_\tau(T(1)(W)).$$

Definition 3.4. Let $G \rightrightarrows M$ be a Lie groupoid. We define a bisection of the co-adjoint Lie groupoid $\mathcal{G}_\xi$ corresponding to a bisection σ of G as follows:

$$\sigma' \colon M \longrightarrow \mathcal{G}_\xi, \ p \longmapsto Ad^*_{\sigma(p)}\xi$$

One can easily check that σ' is a right inverse to source map α', that is, σ' is a bisection corresponding to σ.

Now, similar to Definition 2.2, in the next definition we present left and right translations corresponding to bisection σ' of the co-adjoint Lie groupoid $\mathcal{G}_\xi$ as follows:

Definition 3.5. Let σ' be a bisection of the co-adjoint Lie groupoid $\mathcal{G}_\xi \rightrightarrows M$. We define the left and right translations corresponding to σ', respectively, as follows:

$$L_{\sigma'} \colon \mathcal{G}_\xi \longrightarrow \mathcal{G}_\xi$$

$$Ad^*_g\xi \longmapsto \sigma'(\beta'(Ad^*_g\xi))Ad^*_g\xi = \sigma'(\beta(g))Ad^*_g\xi = Ad^*_{\sigma(\beta(g))}\xi \ . \ Ad^*_g\xi = Ad^*_{\sigma(\beta(g))g}\ \xi,$$

and
$$R_{\sigma'} : \mathcal{G}_\xi \longrightarrow \mathcal{G}_\xi, \quad Ad^*_g\xi \longmapsto Ad^*_g\xi\sigma'((\beta' \circ \sigma')^{-1}\alpha'(Ad^*_g\xi))$$
$$= Ad^*_g\xi((\beta' \circ \sigma')^{-1}(\alpha(g))) = \sigma'((\beta \circ \sigma)^{-1}\alpha(g))$$
$$= Ad^*_g\xi \ . \ Ad^*_{\sigma((\beta\circ\sigma)^{-1}\alpha(g))}\xi = Ad^*_{g.\sigma((\beta\circ\sigma)^{-1}\alpha(g))}\xi.$$

Their tangent mappings are obtained by:

$$TL_{\sigma'} \colon T\mathcal{G}_\xi \longrightarrow T\mathcal{G}_\xi$$
$$ad^*_X\xi \longmapsto T\sigma'(T\beta'(ad^*_X\xi))\ ad^*_X\xi = T\sigma'(T\beta(X))ad^*_X\xi$$
$$= ad^*_{T\sigma(T\beta(X))}\xi \ . \ ad^*_X\xi = ad^*_{Tm(T\sigma(T\beta(X)),X)}\xi$$

and
$$TR_{\sigma'} : T\mathcal{G}_\xi \longrightarrow T\mathcal{G}_\xi$$
$$ad^*_X\xi \longmapsto ad^*_X\xi \ . \ T\sigma'((T\beta \circ T\sigma)^{-1}(T\alpha(X)))$$
$$= ad^*_X\xi \ . \ ad^*_{T\sigma((T\beta\circ T\sigma)^{-1}\alpha(X))}\xi = ad^*_{Tm(X,T\sigma((T\beta\circ T\sigma)^{-1}T\alpha(X)))}\xi.$$

Now, we describe and prove a useful lemma that is the result of Theorem 3.3.

Lemma 3.6. *Let $G \rightrightarrows M$ be a Lie groupoid and m be its multiplication mapping. Suppose that $X \in T_gG$, $Y \in T_hG$ where $T\alpha(X) = T\beta(Y) = W$, and σ, τ are bisections of G that satisfy the conditions of Theorem 3.3. For bisections σ', τ' of $\mathcal{G}_\xi$ and for arbitrary members $\eta_1 = ad^*_X\xi$, $\eta_2 = ad^*_Y\xi$ such that $T\alpha'(\eta_1) = T\beta'(\eta_2)$, we have*
$$Tm'(\eta_1, \eta_2) = ad^*_{Tm(X,Y)}\xi,$$
where m' is the multiplication mapping of the co-adjoint Lie gruopoid $\mathcal{G}_\xi \rightrightarrows M$.

Proof. Note that for every $\eta_1 = ad^*_X\xi$, $\eta_2 = ad^*_Y\xi$ such that $T\alpha'(\eta_1) = T\beta'(\eta_2)$, we have $T\alpha'(\eta_1) = T\alpha'(ad^*_X\xi) = T\alpha(X) = W$.
Thus, according to $T\alpha(X) = T\beta(Y)$, one can obtain $T\alpha'(\eta_1) = T\beta'(\eta_2) = W$.
On the other hand, according to Theorem 3.3
$$Tm'(\eta_1, \eta_2) = TL_{\sigma'}(\eta_2) + TR_{\tau'}(\eta_1) - TL_{\sigma'}TR_{\tau'}(T(1')(W))$$
By doing simple calculations, we obtain
$$TL_{\sigma'}(\eta_2) = ad^*_{Tm(T\sigma(T\beta(Y)),Y)}\xi = ad^*_{Tm(T\sigma(T\alpha(X)),X)}\xi = ad^*_{Tm(X,Y)}\xi,$$
$$TR_{\tau'}(\eta_1) = ad^*_{Tm(X,T\tau((T\beta\circ T\tau)^{-1}T\alpha(X)))}\xi$$
$$= ad^*_{Tm(X,T\tau((T\tau^{-1}\circ T\beta^{-1})T\beta(Y)))}\xi = ad^*_{Tm(X,Y)}\xi,$$
$$TL_{\sigma'}TR_{\tau'}(T(1')(W)) = TL_{\sigma'}TR_{\tau'}(ad^*_{T1_W}\xi) = TL_{\sigma'}(ad^*_{Tm(1_W,T\tau((T\beta\circ T\tau)^{-1}T\alpha(1_W)))}\xi$$
$$= TL_{\sigma'}(ad^*_{Tm(1_W,T\tau((T\tau^{-1}\circ T\beta^{-1})T\beta(Y)))}\xi = TL_{\sigma'}(ad^*_Y\xi)$$
$$= ad^*_{Tm(T\sigma(T\beta(Y)),Y)}\xi = ad^*_{Tm(X,Y)}\xi$$
By embedding the calculations, we obtain
$$Tm'(\eta_1, \eta_2) = ad^*_{Tm(X,Y)}\xi.$$
■

3.2. Co-adjoint Lie algebroid

In this section, using the definition of a Lie algebroid associated to the Lie groupoid, we define the Lie algebroid associated to co-adjoint Lie groupoid, and show how its associated Lie bracket and the anchor map will be defined.

Let $\mathcal{G}_\xi := O(\xi) \rightrightarrows M$ be co-adjoint Lie groupoid. We define its associated Lie algebroid as follows:

Consider the source mapping $\alpha' : O(\xi) \longrightarrow M; \quad Ad^*_g\xi \longmapsto \alpha(g)$, so
$$T\alpha' : TO(\xi) \longrightarrow TM, \quad ad^*_X\xi \longmapsto T\alpha(X).$$
On the other hand, as a result, analogous to what is stated in [2], we have the following lemma:

Lemma 3.7. *Let $G \rightrightarrows M$ be a Lie groupoid and $g \in G$. Consider its associated co-adjoint Lie groupoid $\mathcal{G}_\xi := O(\xi) \rightrightarrows M$. The tangent space of $O(\xi)$ at any $\xi \in (A^* I_G)_p$ is equal to*

$$T_\xi O(\xi) = \{ad^*_X \xi \mid X \quad \textit{runs over the tangent space}\, T_g G\}.$$

According to Definition 2.7, we have $A\mathcal{G}_\xi = kerT\alpha'|_{Ad^*_{1_p}\xi} = T_\xi O(\xi)|_{Ad^*_{1_p}\xi}$.
It turns out that the Lie algebroid of the co-adjoint Lie groupoid is the set of vectors, tangent to the $\alpha'-$fibres restricted to the units of $O(\xi)$. We call this Lie algebroid *co-adjoint Lie algebroid* associated with the co-adjoint Lie groupoid.

Definition 3.8. Let $g' = Ad^*_g \xi$, $h' = Ad^*_h \xi$ be two elements of $\mathcal{G}_\xi$ where $h, g \in G$. The *left translation* corresponding to g' is

$$L'_{g'} \colon \mathcal{G}^p_\xi \longrightarrow \mathcal{G}^q_\xi, \quad h' \longmapsto L'_{g'}(h') = Ad^*_{gh}\xi$$

and the *right translation* corresponding to g' is

$$R'_{g'} \colon \mathcal{G}_{\xi_q} \longrightarrow \mathcal{G}_{\xi_p}, \quad h' \longmapsto R'_{g'}(h') = Ad^*_{hg}\xi.$$

where $p, q \in M$ and $\mathcal{G}^p_\xi, \mathcal{G}_{\xi_p}$ are β-fiber and α-fiber, respectively.

Definition 3.9. Let $G \rightrightarrows M$ be a Lie groupoid, AG its associated Lie algebroid and $\overrightarrow{X}$ (resp., $\overleftarrow{X}$) be a right invariant (resp., left invariant) vector field on G corresponding to $X \in \Gamma(AG)$. Consider the co-adjoint Lie groupoid $\mathcal{G}_\xi$ and its associated Lie algebroid $A\mathcal{G}_\xi$. We define the section X' of the vector bundle $\tau\prime \colon A\mathcal{G}_\xi \longrightarrow M$ as follows:

$$X' \colon M \longrightarrow A\mathcal{G}_\xi, \quad x \longmapsto ad^*_{X(x)}\xi.$$

The right invariant vector field corresponding to X' on $\mathcal{G}_\xi$ is

$$\begin{aligned}\overrightarrow{X'}(g') &= TR'_{g'}(X'(\beta'(g'))) = TR'_{g'}(X'(\beta(g)))\\ &= ad^*_{TR_g(X(\beta(g)))}\xi = ad^*_{\overrightarrow{X}(g)}\xi\end{aligned}$$

where $g \in G$, $g' = Ad^*_g \xi \in O(\xi)$, and

$$\begin{aligned}\overleftarrow{X'}(g') &= -T(L'_{g'})T(\iota')(X'(\alpha'(g')) = -T(L'_{g'})T(\iota')(X'(\alpha(g)))\\ &= ad^*_{-T(L_g)T(\iota)(X(\alpha(g)))}\xi = ad^*_{\overleftarrow{X}(g)}\xi.\end{aligned}$$

Now, similar to definition 2.7, we define the Lie algebroid structure $([| \ , \ |]', \rho')$ on $A\mathcal{G}_\xi$ as follows:

1. The anchor map: $\rho' : A\mathcal{G}_\xi \longrightarrow TM$

   $$\rho'(X')(x) = T_{1'(x)}\beta'(X'(x)) = T\beta'(ad^*_{X(x)}\xi) = T\beta(X(x)) = \rho(X(x))$$

   where $X \in \Gamma(\tau)$ and $x \in M$.

2. Lie bracket: $\overrightarrow{[|X', Y'|]'} = [\overrightarrow{X'}, \overrightarrow{Y'}]$, $(\overleftarrow{[|X', Y'|]'} = -[\overleftarrow{X'}, \overleftarrow{Y'}])$
   where $X', Y' \in \Gamma(\tau')$ and and $[\ ,\ ]$ is the standard Lie bracket of vector fields.

Let $X' = ad^*_X\xi$ and $Y' = ad^*_Y\xi$, so we have

$$\overrightarrow{[|ad^*_X\xi, ad^*_Y\xi|]'} = [\overrightarrow{ad^*_X\xi}, \overrightarrow{ad^*_Y\xi}].$$

Note that the right-hand bracket is the bracket on the vector fields. Using the definition of the right invariant vector fields on the Lie groupoid $O(\xi)$, for $g \in G$ we get

$$\begin{aligned}
[\overrightarrow{ad^*_X\xi}\ ,\ \overrightarrow{ad^*_Y\xi}] &= [ad^*_{X(\beta(g))}\xi, ad^*_{Y(\beta(g))}\xi] \\
&= ad^*_{X(\beta(g))}\xi\ .\ ad^*_{Y(\beta(g))}\xi\ -\ ad^*_{Y(\beta(g))}\xi\ .\ ad^*_{X(\beta(g))}\xi \\
&= ad^*_{Tm(X(\beta(g),Y(\beta(g)))}\xi\ -\ ad^*_{Tm(Y(\beta(g),X(\beta(g)))}\xi \\
&= ad^*_{\overrightarrow{X}\ .\ \overrightarrow{Y}}\xi\ -\ ad^*_{\overrightarrow{Y}\ .\ \overrightarrow{X}}\xi = ad^*_{\overrightarrow{X}\ .\ \overrightarrow{Y}\ -\ \overrightarrow{Y}\ .\ \overrightarrow{X}}\xi = ad^*_{[\overrightarrow{X}\ ,\ \overrightarrow{Y}]}\xi.
\end{aligned}$$

Based on the calculation above, we can deduce the following commonly used lemma:

Lemma 3.10. *Consider the Lie algebroids $(AG, [|\ ,\ |], \rho)$ and $(A\mathcal{G}_\xi, [|\ ,\ |]', \rho')$ and let $X, Y \in \Gamma(AG)$ and $X', Y' \in \Gamma(A\mathcal{G}_\xi)$, then*

$$[|\overrightarrow{X', Y'}|]' = ad^*_{\overrightarrow{[|X,Y|]}}\xi.$$

3.3. Poisson structure

As stated in [5] and [11], for every Lie algebroid $(\tau : A \longrightarrow M, \rho, [|\ ,\ |])$, its dual A^* has linear Poisson structure. In this section we use this structure for the dual of a Lie algebroid AG, the associated Lie algebroid to the Lie groupoid $G \rightrightarrows M$, and define a linear Poisson structure for the dual of the co-adjoint Lie algebroid.

Let $(AG, M, \rho, [|\ ,\ |])$ be associated Lie algebroid to Lie groupoid $G \rightrightarrows M$. So, the dual bundle A^*G to AG, has a linear Poisson structure. First we describe it in more details.

Remark 3.11. Let X be a section of $\tau\colon AG \longrightarrow M$. The linear function $\hat{X}$ is defined on A^*G as follows: $\hat{X}\colon A^*G \longrightarrow \mathbb{R}$, $\hat{X}(\theta) = \theta(X(\tau^*(\theta)))$, here $\theta \in A^*G$ and $\tau^* : A^*G \longrightarrow M$ is the dual bundle of $\tau\colon AG \longrightarrow M$.

The linear Poisson structure on A^*G, which is indicated by $\{.,.\}_{A^*G}$, is characterized by the following conditions:

$$\{.,.\}_{A^*G}\colon C^\infty(A^*G) \times C^\infty(A^*G) \longrightarrow C^\infty(A^*G)$$

$$\{\hat{X}, \hat{Y}\}_{A^*G} = -\widehat{[|X,Y|]},\quad \{f\circ\tau^*, \hat{X}\}_{A^*G} = (\rho(X)(f))\circ\tau^*,\quad \{f\circ\tau^*, g\circ\tau^*\}_{A^*G} = 0.$$

where $\tau^*\colon A^*\mathcal{G}_\xi \longrightarrow M$, $f, g \in C^\infty(M)$ and $f\circ\tau^*, g\circ\tau^* \in C^\infty(A^*G)$.

Also, the linear Poisson bivector on A^*G is defined by $\Pi_{A^*G}(d\varphi, d\psi) = \{\varphi, \psi\}_{A^*G}$ where $\varphi, \psi \in C^\infty(A^*G)$.

Let $H : A^*G \longrightarrow \mathbb{R}$ be a smooth function on A^*G. The Hamiltonian vector field $\mathcal{X}^{\Pi_{A^*G}}_H$ of H is defined by

$$\mathcal{X}^{\Pi_{A^*G}}_H(F) = \{F, H\}_{A^*G} = \Pi_{A^*G}(dF, dH) \tag{1}$$

where $F \in C^\infty(A^*G)$. ■

Now, consider $(A\mathcal{G}_\xi, \rho', [| \ , \ |]')$ as the associated Lie algebroid to the co-adjoint Lie groupoid $\mathcal{G}_\xi \rightrightarrows M$. Using what was previously described in the linear Poisson structure on A^*G, in the following we will show that $A^*\mathcal{G}_\xi$, the dual of $A\mathcal{G}_\xi$, has linear Poisson structure.

As we know (see Definition 3.9), every section of $\tau' : A\mathcal{G} \to M$ can be written by $X' = ad^*_X\xi$, where X is a section of $\tau : AG \to M$.

According to the remark 3.11, for a section $X' = ad^*_X\xi$ of $\Gamma(\tau')$ we consider the associated linear function $\hat{X'}$ on $A^*\mathcal{G}_\xi$ as follows:

$$\hat{X'} \colon A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}, \quad \hat{X'}(\delta) = \delta(X'(\tau'^*(\delta)))$$

where $\delta \in A^*\mathcal{G}_\xi$ and $\tau'^* : A^*\mathcal{G}_\xi \longrightarrow M$ is dual bundle of $\tau' \colon A\mathcal{G}_\xi \longrightarrow M$.

In other words, the above formula indicates that

$$\hat{X'} = \widehat{ad^*_X\xi}.$$

Also the linear Poisson structure on $A^*\mathcal{G}_\xi$ can be considered

$$\{.,.\}_{A^*\mathcal{G}_\xi} \colon C^\infty(A^*\mathcal{G}_\xi) \times C^\infty(A^*\mathcal{G}_\xi) \longrightarrow C^\infty(A^*\mathcal{G}_\xi)$$

$$\{\hat{X'}, \hat{Y'}\}_{A^*\mathcal{G}_\xi} = -\widehat{[|X', Y'|]'} = -\widehat{ad^*_{[|X,Y|]}\xi}$$

It is easy to check that this Poisson structure on $A^*\mathcal{G}_\xi$ satisfies in the characterized conditions which are mentioned in Remark 3.11.

For every Hamiltonian $H' \colon A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}$ the Hamiltonian vector field $\mathcal{X}_{H'}^{\Pi_{A^*\mathcal{G}_\xi}}$ on $A^*\mathcal{G}_\xi$ will be considered as equation (1).

3.4. Hamilton-Jacobi equations

Hamiltonian-Jacobi theorem is one of the most important parts of classical mechanics and has been studied extensively in last years. The geometric view of this theorem was expressed by Abraham and Marsden in [1]. Also, using these results expressed by Abraham and Marsden, Wang presented two types of geometric Hamilton-Jacobi theorem for a Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic form and dynamical vector field (see [14] for more details).

In our previous article [6], we developed these results to Lie algebroids, i.e. we introduced types I and II of for the Hamilton-Jacobi equation for Hamiltonian systems on Lie algebroids by using symplectic section and Hamiltonian vector field. Moreover, as we explained in [6], if we apply these theorems to the particular case when E is the standard Lie algebroid TM then we directly deduce a well-known theorems stated in [14].

Here, after introducing some items for the co-adjoint Lie algebroid $A\mathcal{G}_\xi$, we will present the types I and II of Hamilton-Jacobi theorem for Hamiltonian systems with this kind of Lie algebroids.

Definition 3.12. Consider a vector bundle $\tau : AG \longrightarrow M$ and suppose that $X : M \longrightarrow AG$ be a section of AG. Let $\tau^* : A^*G \longrightarrow M$ be dual of vector bundle τ. We define section $\gamma = X^* \in \Gamma(\tau^*)$ as follows:

$$\gamma(x)(Y(x)) = \langle X^*(x), Y(x) \rangle,$$

where $X^*(x) = \big(X(x)\big)^*$ is the dual section corresponding to $X \in \Gamma(\tau)$, $x \in M$ and $Y \in \Gamma(\tau)$.

Now consider the vector bundle $\tau' : A\mathcal{G}_\xi \longrightarrow M$ and section $X' = ad^*_X\xi \in \Gamma(\tau')$, where $X \in \Gamma(\tau)$. We define a section γ' of vector bundle $\tau'^* : A^*\mathcal{G}_\xi \longrightarrow M$ (the dual bundle of τ') as follows:

$$\gamma'(x) := \big(ad^*_{X(x)}\xi\big)^*.$$

Definition 3.13. Let $X : M \longrightarrow AG$ be a section of vector bundle $\tau : AG \longrightarrow M$ and $\tau^* : A^*G \longrightarrow M$ be dual of vector bundle τ.

Suppose that $X^* \in \Gamma(\tau^*)$ be the dual section associated to $X \in \Gamma(\tau)$ and $TX^* : TM \longrightarrow TA^*G$ is tangent map to X^*, we define a vector field $\chi := TX^*$ on A^*G as follows:

$$\chi(X(x)) = TX^*(x).$$

Moreover, consider the vector bundle $\tau' : A\mathcal{G}_\xi \longrightarrow M$ and $X' = ad^*_X\xi \in \Gamma(\tau')$, where $X \in \Gamma(\tau)$. Let X'^* be the dual section corresponding to $X' \in \Gamma(\tau')$ and $TX'^* : TM \longrightarrow TA^*\mathcal{G}_\xi$ be tangent map to X'^*.

We define a vector field $\chi' := TX'^*$ on $A^*\mathcal{G}_\xi$ as follows: $\chi'(X(x)) = T\big(ad^*_{X(x)}\big)^*$.

Definition 3.14. For every Hamiltonian function H on A^*G, we define a Hamiltonian function H' on $A^*\mathcal{G}_\xi$ as follows:

$$H' : A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}, \quad \gamma'(x) \longmapsto H'\Big(\big(ad^*_{X(x)}\xi\big)^*\Big) = H(\gamma(x)),$$

where $\gamma(x) = X^*(x)$.

As is well-known in the literature the prolongation of every Lie algebroid over the vector bundle projection of the dual bundle is a Lie algebroid over dual of the Lie algebroid, for more details one can see for example [5]. In the next three definitions follows [5], we define the prolongation for the co-adjoint Lie algebroid and introduce its Liouville section and canonical symplectic section. Also we construct the Hamiltonian section for the prolongation of the co-adjoint Lie algebroid.

Definition 3.15. Let $(A\mathcal{G}_\xi, [| \ , \ |]', \rho')$ be a co-adjoint Lie algebroid and let $\tau'^* : A^*\mathcal{G}_\xi \longrightarrow M$ be the dual of vector bundle $\tau' : A\mathcal{G}_\xi \longrightarrow M$. Define the prolongation $\mathcal{T}^{\tau'^*}A\mathcal{G}_\xi$ of $A\mathcal{G}_\xi$ over τ'^* as follows:

$$\mathcal{T}^{\tau'^*}A\mathcal{G}_\xi = \left\{ (a', \vartheta') \in A\mathcal{G}_\xi \times TA^*\mathcal{G}_\xi \;\middle|\; \begin{array}{l} a' = ad^*_{X(x)}\xi, \ X \in \Gamma(AG), \\ \vartheta' = T(ad^*_{X(x)}\xi)^*, \ \rho'(a') = T\tau'^*(\vartheta') \end{array} \right\}$$

which is subset of $A\mathcal{G}_\xi \times TA^*\mathcal{G}_\xi$ and $T\tau'^* : TA^*\mathcal{G}_\xi \longrightarrow TM$ is tangent map to $\tau'^* : A^*\mathcal{G}_\xi \longrightarrow M$. $\mathcal{T}^{\tau'^*}A\mathcal{G}_\xi$ is Lie algebroid over $A^*\mathcal{G}_\xi$ with following structures.

In other words, the vector bundle $\tilde{\tau}' : \mathcal{T}^{\tau^*}A\mathcal{G}_\xi \longrightarrow A^*\mathcal{G}_\xi$ has Lie algebroid structure $(\tilde{\rho}', \widetilde{[| \ , \ |]'})$ such that:

(1) The anchor $\tilde{\rho}' : \mathcal{T}^{\tau'^*}A\mathcal{G}_\xi \longrightarrow TA^*\mathcal{G}_\xi$ is projection onto the second factor,

$$\tilde{\rho}'(a', \vartheta') = \vartheta'. \tag{2}$$

(2) The bracket is given in terms of projectable sections (X'_1, χ'_1) and (X'_2, χ'_2)

$$[|\widetilde{(X'_1, \chi'_1), (X'_2, \chi'_2)}|]' = \big([|X'_1, X'_2|]', [\chi'_1, \chi'_2]\big) = \big(ad^*_{[|X_1, X_2|]}, [\chi'_1, \chi'_2]\big)$$

where $X'_i = ad^*_{X_i}\xi \in \Gamma(A\mathcal{G}_\xi)$ and $\chi'_i = T(ad^*_{X_i}\xi)^* \in \mathfrak{X}(A^*\mathcal{G}_\xi)$.

Definition 3.16. We may introduce the *Liouville section* $\Theta' \in \Gamma((\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)^*)$ and canonical symplectic section $\Omega' \in \Gamma(\Lambda^2(\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)^*)$ as follows:

$$\Theta'(a'^*)(a', \vartheta') = a'^*(a') = (ad^*_{X(x)}\xi)^*(ad^*_{X(x)}\xi).$$

and

$$\Omega' = -d\Theta'.$$

Definition 3.17. Let $H' : A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}$ be a Hamiltonian function on $A^*\mathcal{G}_\xi$ which is defined in Definition 3.14. Suppose that $\mu_H = (X_H, TX^*_H) \in \Gamma(\mathcal{T}^{\tau^*} AG)$ be the Hamiltonian section corresponding to the Hamiltonian function $H : A^*G \longrightarrow \mathbb{R}$. Then, there exists a unique section $\mu'_{H'} \in \Gamma(\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)$ which satisfying

$$i_{\mu'_{H'}}\Omega' = dH'$$

where $dH' \in \Gamma((\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)^*)$ and $\Omega' \in \Gamma(\Lambda^2(\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)^*)$ is canonical symplectic section. We define $\mu'_{H'}$ as follows: $\mu'_{H'} = \big(ad^*_{X_H}\xi, T(ad^*_{X_H}\xi)^*\big)$.

Now, using the content above, we present a proposition, which is in fact the case of the Hamilton-Jacobi theorem for the co-adjoint Lie algebroid $(A\mathcal{G}_\xi, [| \ , \ |]', \rho')$ and it can easily be proved by using theorem 7 of [6] or theorem 3.16 of [5] which are Lie algebroid version of geometric Hamilton-Jacobi theorem on cotangent bundles (theorem 5.2.4 in [1]).

Proposition 3.18. *Let $(A\mathcal{G}_\xi, [| \ , \ |]', \rho')$ be co-adjoint Lie algebroid and assume that $H' \colon A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}$ is a Hamiltonian function. Consider the Lie algebroid $(\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi, \tilde{\rho}', \widetilde{[| \ , \ |]'})$ and the Hamiltonian section $\mu'_{H'} \in \Gamma(\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)$. Let $\gamma' \in \Gamma(A^*\mathcal{G}_\xi)$ be a 1-cocycle, i.e. γ' is a section of $A^*\mathcal{G}_\xi$ such that $d\gamma' = 0$, and denote the section $\zeta' = pr_1 \circ \mu'_{H'} \circ \gamma'$ by $\zeta' \in \Gamma(A\mathcal{G}_\xi)$. Then the following statements are equivalent:*

1. *For every curve $t \longrightarrow \sigma(t)$ in M that applies to the property $\rho'(\zeta')(\sigma(t)) = \dot{\sigma}(t)$ for all t, the curve $t \longrightarrow \gamma'(\sigma(t))$ on $A^*\mathcal{G}_\xi$ applies to the Hamilton equation.*
2. *γ' applies to the Hamilton-Jacobi equation, $d(H' \circ \gamma') = 0$.*

Definition 3.19. The section γ' is called to be cocycle with respect to projection $pr'_1 \colon \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi \longrightarrow A\mathcal{G}_\xi$ if for any $\kappa_1, \kappa_2 \in \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi$ we have that

$$d\gamma'(pr'_1(\kappa_1), pr'_1(\kappa_2)) = 0.$$

Definition 3.20. Let $\tau'^* \colon A^*\mathcal{G}_\xi \to M$ be the dual of the vector bundle $\tau \colon A\mathcal{G}_\xi \to M$, $\Omega' \in \Gamma(\Lambda^2(\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)^*)$ the canonical symplectic section and $H' \colon A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}$ a Hamiltonian function. We call the triple $(A^*\mathcal{G}_\xi, \Omega', H')$ the *Hamiltonian system* on the co-adjoint Lie algebroid.

Now using the Definition 3.20 and simulation of what is explained in [6], we can easily present types I and II Hamilton-Jacobi equation for Hamiltonian system $({}^*\mathcal{G}_\xi, \Omega', H')$ which is a Hamiltonian system associated to co-adjoint Lie algebroid $(A\mathcal{G}_\xi, [| \, , \, |]', \rho')$. We state these under two propositions without proof, they can be easily proved using the types I and II of Hamiltonian-Jacobi equations of [6].

Proposition 3.21. *Consider the Hamiltonian system* $(A^*\mathcal{G}_\xi, \Omega', H')$. *Assume* $\gamma' \in \Gamma(A^*\mathcal{G}_\xi)$ *and* $\zeta' = pr'_1 \circ \mu'_{H'} \circ \gamma'$, *where* $\mu'_{H'}$ *is the corresponding Hamiltonian section. Suppose that the section* $\gamma' \in \Gamma(A^*\mathcal{G}_\xi)$ *is a cocycle with respect to* $pr'_1 \colon \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi \longrightarrow A^*\mathcal{G}_\xi$, *then* γ' *is a solution of the equation* $(\phi'_{\gamma'}, \gamma') \circ \zeta' = \mu'_{H'} \circ \gamma'$, *which is called type I of Hamilton-Jacobi equation for Hamiltonian system* $(A^*\mathcal{G}_\xi, \Omega', H')$.

$$\begin{array}{ccccc}
A^*\mathcal{G}_\xi & \xrightarrow{\tau'^*} & M & \xrightarrow{\gamma'} & A^*\mathcal{G}_\xi \\
 & & \downarrow{\scriptstyle \mu^{\gamma'}_{H'}} & & \downarrow{\scriptstyle \mu'_{H'}} \\
\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi & \xleftarrow[(\phi'_{\gamma'},\gamma')]{} & A\mathcal{G}_\xi & \xleftarrow[pr'_1]{} & \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi
\end{array}$$

Proposition 3.22. *Consider the Hamiltonian system* $(A^*\mathcal{G}_\xi, \Omega', H')$. *Let* $\gamma' \in \Gamma(A^*\mathcal{G}_\xi)$ *and* $\tilde{\mathcal{T}}\lambda' := (\mathcal{T}\lambda' \circ \gamma') \circ pr'_1 = (\phi'_{\gamma'}, \gamma') \circ pr'_1 \colon \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi \longrightarrow \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi$, *and also for every symplectic morphism* $\varepsilon' \colon A^*\mathcal{G}_\xi \longrightarrow A^*\mathcal{G}_\xi$, *define* $\zeta'^{\varepsilon'} = pr'_1 \circ \mu'_{H'} \circ \varepsilon'$, *where* $\mu'_{H'}$ *is the corresponding Hamiltonian section. Then* ε' *is a solution of the equation* $\mathcal{T}\varepsilon' \circ \mu'_{H' \circ \varepsilon'} = \tilde{\mathcal{T}}\lambda' \circ \mu'_{H'} \circ \varepsilon'$, *if and only if it is a solution of the equation* $(\phi'_{\gamma'}, \gamma') \circ \mu^{\varepsilon'}_{H'} = \mu'_{H'} \circ \varepsilon'$, *where* $\mu'_{H' \circ \varepsilon'} \in \Gamma(\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi)$ *is Hamiltonian section of the function* $H' \circ \varepsilon' \colon A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}$. *The equation* $(\phi'_{\gamma'}, \gamma') \circ \mu^{\varepsilon'}_{H'} = \mu'_{H'} \circ \varepsilon'$, *is called type II of Hamilton-Jacobi equation for Hamiltonian system* $(A^*\mathcal{G}_\xi, \Omega', H')$.

$$\begin{array}{ccccccc}
A^*\mathcal{G}_\xi & \longrightarrow & A^*\mathcal{G}_\xi & \xrightarrow{\tau'^*} & M & \xrightarrow{\gamma'} & A^*\mathcal{G}_\xi \\
 & & \downarrow{\scriptstyle \mu'_{H'\circ\varepsilon'}} & \searrow{\scriptstyle \mu^{\varepsilon'}_{H'}} & & & \downarrow{\scriptstyle \mu'_{H'}} \\
\mathcal{T}^{\tau'^*} A\mathcal{G}_\xi & \xleftarrow[\mathcal{T}\varepsilon']{} & \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi & \xleftarrow[(\phi'_{\gamma'},\gamma')]{} & A\mathcal{G}_\xi & \xleftarrow[pr'_1]{} & \mathcal{T}^{\tau'^*} A\mathcal{G}_\xi
\end{array}$$

4. Example

In this section we consider an example, namely the trivial Lie groupoid (we refer the reader to [9]). By describing the example and realizing our theory for the trivial Lie groupoid one can see that the theory of Hamiltonian systems on the co-adjoint Lie groupoid can be consider as a generalization of the theory of Hamiltonian systems on Lie groups.

Now let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Also, let M be a smooth manifold and consider $\Upsilon := M \times G \times M$. As mentioned in [9], Υ has a Lie groupoid structure over M, called *trivial Lie groupoid*, with the following mappings:

1. source mapping: $\alpha : \Upsilon \longrightarrow M, \ (x, a, y) \longmapsto y$,
2. target mapping: $\beta \colon \Upsilon \longrightarrow M, \ (x, a, y) \longmapsto x$,

3. multiplication mapping: $m\colon (\Upsilon)_2 \longrightarrow \Upsilon,\ ((x,a,y),(y,b,z)) \longmapsto (x,ab,z)$,
4. unit mapping: $1 : M \longrightarrow \Upsilon,\ x \longmapsto (x,1,x)$,
5. inverse mapping: $\iota : \Upsilon \longrightarrow \Upsilon,\ (x,a,y) \longmapsto (y,a^{-1},x)$.

The Lie algebroid associated to the trivial Lie groupoid is $A\Upsilon = TM \oplus (M \times \mathfrak{g})$ ([9]). It is easy to see that isotropy Lie groupoid corresponding to the trivial Lie groupoid is $I_\Upsilon = M \times G$ and its associated Lie algebroid is $AI_\Upsilon = M \times \mathfrak{g}$.

If $g = (x,a,p)$ is an element of Υ and $g' = (p,b,p) \in I_\Upsilon$, as mentioned in subsection 2.2, the action of the Lie groupoid Υ on $J : I_\Upsilon \longrightarrow M$ is as follows:

$$C_{(x,a,p)}(p,b,p) = (x,a,p)(p,b,p)(p,a^{-1},x) = (x,aba^{-1},x) = (x,C_a(b),x)$$

where $x = \beta(g)$ and $C_a\colon G \longrightarrow G,\quad b \longmapsto aba^{-1}$ is the conjugation action of the Lie group G by $a \in G$.

Before describing adjoint and co-adjoint actions on the Lie groupoid Υ, we first consider it necessary to focus on the content of the exponential map. Consider trivial Lie groupoid $\Upsilon = M \times G \times M$ and its associated Lie algebroid $A\Upsilon = TM \oplus (M \times \mathfrak{g})$.

According to the exponential mapping defined for Lie groupoids and Lie algebroids, $Exp\colon A\Upsilon \longrightarrow \Upsilon,\ \tilde{X} \longmapsto Exp(\tilde{X})$, and it results that $Exp(\tilde{X}) \in \Upsilon$. Suppose that $\tilde{X} = X \oplus U$ where $X \in TM$ and $U \in M \times \mathfrak{g}$. Let $U = (y,V,y)$ where $y \in M$ and $V \in \mathfrak{g}$. Since $\tilde{X} \in A\Upsilon$, there is a smooth curve $\gamma\colon I \subseteq \mathbb{R} \longrightarrow \Upsilon,\ \gamma(t) = Exp(t\tilde{X})$ such that $\frac{d\gamma(t)}{dt}|_{t=0} = (X,U)$. Let $x(t) = \beta(\gamma(t))$, so

$$\gamma(t) = \big(x(t), Exp(tU)\big) = \big(x(t), exp(tV), y\big). \tag{3}$$

Note that here $exp\colon \mathfrak{g} \longrightarrow G$ is the usual exponential map on Lie groups.
Now we write the definition of the adjoint representation for Υ. As mentioned before,

$$Ad : \Upsilon \times AI_\Upsilon \longrightarrow AI_\Upsilon$$

$$(g,X) \longmapsto Ad_g X = \frac{d}{dt}\Big|_{t=0} C_g\ Exp(tX)$$

Let $X \in (AI_\Upsilon)_p$, from equation (3), $Exp(tX) = (p, exp(tV), p)$ where $p \in M$ and $V \in \mathfrak{g}$, we have

$$Ad_g X = \frac{d}{dt}\Big|_{t=0} C_g\ Exp(tX) = \frac{d}{dt}\Big|_{t=0} C_g\ (p, exp(tV), p)$$

Now, if $g = (x,a,p) \in \Upsilon_p$, then

$$\begin{aligned} C_g(p, exp(tV), p) &= g(p, exp(tV), p)g^{-1} = (x,a,p)(p,exp(tV),p)(p,a^{-1},x) \\ &= (x, a(exp(tV))a^{-1}, x) = (x, C_a\ exp(tV), x) \end{aligned}$$

So for every $g = (x,a,p) \in \Upsilon_p$, we have

$$C_g(p, exp(tV), p) = (x, C_a\ exp(tV), x)$$

which implies
$$Ad_g X = (x, Ad_a V, x), \tag{4}$$

where $x = \beta(g),\ Ad_a = TC_a$.

Now we construct the adjoint representation of the Lie algebroid $A\Upsilon$ on AI_Υ :

$$ad : A\Upsilon \times AI_\Upsilon \longrightarrow AI_\Upsilon$$

$$(\tilde{X}, Y) \longmapsto ad_{\tilde{X}} Y := \frac{d}{dt}\Big|_{t=0} Ad_{Exp(t\tilde{X})} Y = \frac{d}{dt}\Big|_{t=0} Ad_{(x(t), exp(tV), p)} Y.$$

where $Exp(\tilde{X}) = g \in \Upsilon_p$ and $V \in \mathfrak{g}$.
If $Y = (p, W, p) \in (AI_\Upsilon)_p$, then using equation (4),

$$ad_{\tilde{X}} Y := (x, \frac{d}{dt}\Big|_{t=0} Ad_{exp(tV)} W, x)$$

So $ad_{\tilde{X}} Y = (x, ad_V W, x)$. Furthermore, if $X = (x, V, x)$, $Y = (y, W, y) \in M \times \mathfrak{g}$, then

$$[|X, Y|] = [V, W],$$

where $[|.,.|]$ is the bracket of the Lie algebroid AI_Υ and $[.,.]$ is the bracket of the Lie algebra $\mathfrak{g}$.

Remark 4.1. This fact which we obtain above also is known from the definition of Lie algebroid for the trivial Lie groupoid and Lie algebroid for its isotropy Lie groupoid. In other words, the bracket of the isotropy Lie algebroid of the trivial Lie groupoid is same with the bracket of the Lie algebra $\mathfrak{g}$.

Later, we try to construct the co-adjoint representation of Υ on A^*I_Υ and consider it as follows:

$$Ad^* : \Upsilon \times A^*I_\Upsilon \longrightarrow A^*I_\Upsilon$$

$$Ad^*_g \xi(X) = \xi(Ad_{g^{-1}}(X)) = \xi(p, Ad_{a^{-1}}(V), p)$$

where $p = \beta(g^{-1})$, $X = (x, V, x) \in M \times \mathfrak{g}$.
In order to $AI_\Upsilon = M \times \mathfrak{g}$, so $A^*I_\Upsilon = M \times \mathfrak{g}^*$. Therefore, for every section ξ of $(A^*I_\Upsilon)_p$, there exists $\xi' \in \mathfrak{g}^*$ and $p \in M$, such that $\xi = (p, \xi', p)$. So

$$Ad^*_g \xi(X) = Ad^*_a \xi'(V)$$

By doing the same calculations, we conclude that the co-adjoint representation of the Lie algebroid $A^*\Upsilon$ on A^*I_Υ is defined as follows:

$$ad^* : A\Upsilon \times A^*I_\Upsilon \longrightarrow A^*I_\Upsilon$$

$$ad^*_{\tilde{X}} \xi(Y) = \xi(ad_{(-\tilde{X})} Y) = \xi(x, ad_{(-V)} W, x) = \xi'(ad_{(-V)} W)$$

where $\xi = (p, \xi', p) \in (A^*I_\Upsilon)_p$ and $Y = (p, W, p) \in (AI_\Upsilon)_p$. Similar to what we did above, we obtain

$$ad^*_{\tilde{X}} \xi(Y) = ad^*_V \xi'(W).$$

Consider $\xi = (p, \xi', p) \in (A^*I_\Upsilon)_p$, its orbit under the co-adjoint action of Υ is as follows:

$$O(\xi) = \{Ad^*_g \xi \mid g \in \Upsilon\} = \{(p, Ad^*_a \xi', p) \mid \ a \in G, \ p \in M\}.$$

Therefore, we have the following relation between $O(\xi)$-the orbit of co-adjoint action of the trivial Lie groupoid Υ at ξ and $O(\xi')$-the orbit of co-adjoint action of the Lie group G at ξ':

$$O(\xi) = M \times O(\xi').$$

It is easy to see that $O(\xi) = M \times O(\xi')$ has Lie groupoid structure on M with the following mappings:

1. source mapping: $\alpha' : M \times O(\xi') \longrightarrow M; \qquad Ad^*_g\xi = (p, Ad^*_a\xi', p) \longmapsto \alpha(g)$,
2. target mapping: $\beta' : M \times O(\xi') \longrightarrow M; \qquad Ad^*_g\xi = (p, Ad^*_a\xi', p) \longmapsto \beta(g)$,
3. multiplication mapping: $m' \colon (M \times O(\xi'))_2 \longrightarrow M \times O(\xi')$;
$$\big(Ad^*_g\xi, Ad^*_h\xi\big) = \big((p, Ad^*_a\xi', p), (q, Ad^*_b\xi', q)\big) \longmapsto (q, Ad^*_{ab}\xi', q) = Ad^*_{m(g,h)}\xi,$$
4. unit mapping: $1' : M \longrightarrow M \times O(\xi'); \qquad p \longmapsto (p, Ad^*_e\xi', p) = Ad^*_{1_p}\xi$, where e is the identity element of the Lie group G.
5. inverse mapping: $\iota' \colon M \times O(\xi') \longrightarrow M \times O(\xi')$;
$$Ad^*_g\xi = (p, Ad^*_a\xi', p) \longmapsto (p, Ad^*_{a^{-1}}\xi', p) = Ad^*_{g^{-1}}\xi.$$

From now on, $\mathcal{G}_\xi$ denotes the co-adjoint orbit $O(\xi) = M \times O(\xi')$. According to definition 2.7, it can be easily concluded that the Lie algebroid associated to $\mathcal{G}_\xi$ is as follows:

$$A\mathcal{G}_\xi = M \times T_{\xi'}O(\xi') = \{(x, ad^*_V\xi') \mid x \in M \text{ and } V \in \mathfrak{g}\}.$$

It is worthy of note that ξ' is an element of $\mathfrak{g}^*$ such that $\xi = (x, \xi', x) \in (A^*I_\Upsilon)_x$. Consider the vector bundle $\tau : M \times T_{\xi'}O(\xi') \longrightarrow M$. We equip it with the following Lie algebroid structures:

1. The anchor: Using the explanations of Section 3.2, we can easily conclude that
$$\rho' : M \times T_{\xi'}O(\xi') \longrightarrow TM$$
$$\rho'(x, ad^*_V\xi')(p) = T_{1'(p)}\beta'((x, ad^*_V\xi')(p)) = X(p) \tag{5}$$
where $X \in \Gamma(TM) = \mathfrak{X}(M)$ is equal to $\dot{p}(0)$, $p(t) = \beta(\gamma(t)) \in M$, $\gamma(t) = (p(t), Ad^*_a\xi', p(t)) \in \mathcal{G}_\xi$, $\frac{d}{dt}|_{t=0}(\gamma(t)) = (x, ad^*_V\xi') \in A\mathcal{G}_\xi$ and $p(0) = p \in M$.

2. The space of sections of the vector bundle $\tau \colon M \times T_{\xi'}O(\xi')) \longrightarrow M$ may be identified with the space of sections of $TO(\xi')$. Under this identification, the Lie bracket on the space of sections $\Gamma(M \times T_{\xi'}O(\xi'))$ is
$$[|ad^*_V\xi', ad^*_W\xi'|]' = ad^*_{[V,W]}\xi'.$$
for every $ad^*_V\xi'$ and $ad^*_W\xi'$ in $\Gamma(M \times T_{\xi'}O(\xi'))$.

The compatibility condition of the Lie bracket and the anchor map, as is easy to check and we leave it to the reader to verify.

Now consider the trivial Lie groupoid $\Upsilon = M \times G \times M$ on M and its associated Lie algebroid $A\Upsilon = TM \oplus (M \times \mathfrak{g})$. Let $\Sigma = X \oplus V$ be a section of $A\Upsilon$. As we mentioned before, the orbit of co-adjoint action of trivial groupoid is $M \times O(\xi')$ and its associated Lie algebroid is $M \times T_{\xi'}O(\xi')$.

As we know, if we assume that $\hat{V}', \hat{W}' \in C^{\infty}(T^*_{\xi'}O(\xi'))$, then $T^*_{\xi'}O(\xi')$ carries the Kirillov-Kostant bracket as follows:

$$\{\hat{V}', \hat{W}'\}(\lambda) = \langle \lambda, [V', W'] \rangle.$$

Now, consider vector bundle $\tau\colon M \times T_{\xi'}O(\xi') \longrightarrow M$, where is projection over the first factor, and its dual $\tau^*\colon M \times T^*_{\xi'}O(\xi') \longrightarrow M$. Let $\Sigma' := V' = ad^*_V \xi' \in \Gamma(\tau)$ and $\delta = (p, \lambda) \in M \times T^*_{\xi'}O(\xi')$, so the linear function $\hat{\Sigma}'$ on $M \times T^*_{\xi'}O(\xi')$ will be as follows:

$$\hat{\Sigma}'\colon M \times T^*_{\xi'}O(\xi') \longrightarrow \mathbb{R}$$

$$\hat{\Sigma}'(\delta) = \delta(\Sigma'(\tau^*(\delta))) = \langle \lambda, V' \rangle$$

In other words, we have $\hat{\Sigma}' = (p, \hat{V}')$.

Now, we try to clear relation between $C^{\infty}(M \times T^*_{\xi'}O(\xi'))$ and $C^{\infty}(T^*_{\xi'}O(\xi'))$. Let $V' \in C^{\infty}(T^*_{\xi'}O(\xi'))$ be a vector field on $O(\xi')$. We define

$$\hat{V}' : T^*_{\xi'}O(\xi')) \longrightarrow \mathbb{R}, \quad \hat{V}'(\lambda) = \langle \lambda, V' \rangle = \lambda(V')$$

$\hat{V}'$ is linear function on $T^*_{\xi'}O(\xi')$.

Now, the Kirillov-Kostant bracket on $C^{\infty}(T^*_{\xi'}O(\xi'))$ is as follows:

$$\{.,.\}_{K.K}\colon C^{\infty}(T^*_{\xi'}O(\xi')) \times C^{\infty}(T^*_{\xi'}O(\xi')) \longrightarrow C^{\infty}(T^*_{\xi'}O(\xi'))$$

$$(\hat{V}', \hat{W}') \longmapsto \{\hat{V}', \hat{W}'\}_{K.K}, \quad \{\hat{V}', \hat{W}'\}_{K.K}(\lambda) = -\langle \lambda, [V', W'] \rangle.$$

As we know $T_{\xi'}O(\xi') = \{ad^*_V \xi' \mid V \in \mathfrak{g}\}$, so let $V' = ad^*_V \xi' \in T_{\xi'}O(\xi')$. Also by well known facts for finite dimensional vector spaces

$$T^{**}_{\xi'}O(\xi') \cong T_{\xi'}O(\xi'),$$

one can consider $V' \in T^{**}_{\xi'}O(\xi')$, i.e. $V'\colon T^*_{\xi'}O(\xi') \longrightarrow \mathbb{R}$ is linear functional, in other words, $V' \in C^{\infty}(T^*_{\xi'}O(\xi'))$. So $T_{\xi'}O(\xi') \subset C^{\infty}(T^*_{\xi'}O(\xi'))$, therefore we can take $\hat{V}' = V'$ and for every $\lambda \in T^*_{\xi'}O(\xi')$

$$\hat{V}'(\lambda) = V'(\lambda) = \langle \lambda, V' \rangle$$

or equivalently

$$\langle \lambda, V' \rangle = (ad^*_V \xi')(\lambda).$$

Now, we rewrite the bracket

$$\{.,.\}_{K.K}\colon C^{\infty}(T^*_{\xi'}O(\xi')) \times C^{\infty}(T^*_{\xi'}O(\xi')) \longrightarrow C^{\infty}(T^*_{\xi'}O(\xi'))$$

as follows:

$$\begin{aligned}\{\hat{V}', \hat{W}'\}_{K.K}(\lambda) &= -\langle \lambda, [V', W'] \rangle = -[V', W'](\lambda) \\ &= -[ad^*_V \xi', ad^*_W \xi'](\lambda) = -ad^*_{[V,W]} \xi'(\lambda),\end{aligned}$$

and as result we obtain $\{\hat{V}', \hat{W}'\}_{K.K} = -ad^*_{[V,W]} \xi'$.

So, according to Subsection 3.3, one can easily check that the first property of linear Poisson structure of functions

$$\{.,.\}_{A^*\mathcal{G}_\xi}\colon C^\infty(M\times T^*_{\xi'}O(\xi'))\times C^\infty(M\times T^*_{\xi'}O(\xi'))\longrightarrow C^\infty(M\times T^*_{\xi'}O(\xi'))$$

will be as follows:

$$\{\hat{\Sigma_1'},\hat{\Sigma_2'}\}_{A^*\mathcal{G}_\xi}(\delta)=\{\hat{V}',\hat{W}'\}_{K.K}(\lambda) \tag{6}$$

where $\hat{\Sigma_1'}=(p,\hat{V}'),\hat{\Sigma_2'}=(p,\hat{W}')$ and $\delta=(p,\lambda)\in M\times T^*_{\xi'}O(\xi')$. So we have a well known Poisson structure on $M\times T^*_{\xi'}O(\xi')$.

Also, according to equation 5, it is clear that the property (ii) of linear Poisson structure on $M\times T^*_{\xi'}O(\xi'))$ is

$$\{f\circ\tau^*,\hat{\Sigma}'\}_{A^*\mathcal{G}_\xi}(\delta)=(\rho'(\Sigma')(f))\circ\tau^*(\delta)=X(f(p))$$

where $\Sigma'=ad^*_\Sigma\xi'\in\Gamma(A\mathcal{G}_\xi)$ and $\Sigma=X\oplus V\in A\Upsilon$.

Moreover, the third feature of linear Poisson structure on $M\times T^*_{\xi'}O(\xi'))$ easily deduced based on the linear Poisson structure on $A^*\Upsilon$, i.e.

$$\{f\circ\tau^*,g\circ\tau^*\}_{A^*\mathcal{G}_\xi}=0.$$

Now, suppose that $H: M\times T^*_{\xi'}O(\xi')\longrightarrow\mathbb{R}$ be function which we define it as $H=(p,h)$ where $h: T^*_{\xi'}O(\xi')\longrightarrow\mathbb{R}$ is a Hamiltonian function. In the following, we will show that H is Hamiltonian function on $M\times T^*_{\xi'}O(\xi')$. In order to reach this result, we need to express some fundamental information which are related to Hamiltonian mechanics on cotangent bundles and Lie algebroids.

Consider $O(\xi')$ as a smooth manifold and let $T^*O(\xi')$ be its cotangent bundle. Suppose that $\delta=(p,\lambda)\in T^*O(\xi')$ and $X_\delta\in T_\delta(T^*O(\xi'))$. As we know, the Liouville form on $T^*O(\xi')$ is the 1-form θ such that

$$\theta(X_\delta)=\lambda(T\pi_{O(\xi')}(X_\delta)) \tag{7}$$

where $\pi_{O(\xi')}\colon T^*O(\xi')\longrightarrow O(\xi');\ (p,\lambda)\longmapsto p$ is canonical projection. Moreover, the 2-form

$$\omega=d\theta \tag{8}$$

is a canonical symplectic form on $T^*O(\xi')$.

Furthermore, a vector field $\mathcal{X}$, where $\mathcal{X}\in\Gamma(T(T^*O(\xi')))$, is called a *Hamiltonian vector field* if there is a $h\in C^\infty(T^*O(\xi'))$ such that $i_\mathcal{X}\omega=dh$.

Let $h\colon T^*O(\xi')\longrightarrow\mathbb{R}$ be a Hamiltonian function and $\mathcal{X}_h$ be Hamiltonian vector field associated to Hamiltonian function h. Moreover,

$$\mathcal{X}_h(f)=\Pi(df,dh)=\{f,h\}_{K.K}$$

where Π is Poisson 2-vector on $T^*O(\xi')$ and $f\in C^\infty(T^*O(\xi'))$.

Now, consider the prolongation

$$\mathcal{T}^{\tau^*}A\mathcal{G}_\xi \text{ of } A\mathcal{G}_\xi = M \times TO(\xi') \text{ over } \tau^*\colon M \times T^*O(\xi') \longrightarrow M\colon$$

$$\mathcal{T}^{\tau^*}A\mathcal{G}_\xi = \left\{ \begin{array}{c} (\delta, a, \vartheta_\delta) \in M \times T^*O(\xi') \times TO(\xi') \times T_\delta(T^*O(\xi')) \mid \\ \rho(a) = T\tau^*(\vartheta_\delta),\ \vartheta_\delta \in T_\delta A^*\mathcal{G}_\xi,\ \tau^*(\delta) = \tau(a) \end{array} \right\}$$

where $\delta = (p, \lambda) \in M \times T^*O(\xi')$ and $a = (p, V') \in M \times TO(\xi')$.

The vector bundle $\tilde{\tau}\colon \mathcal{T}^{\tau^*}A\mathcal{G}_\xi \longrightarrow A^*\mathcal{G}_\xi$ has a Lie algebroid structure $(\tilde{\rho}, \widetilde{[|\ ,\ |]})$ such that

1. The anchor $\tilde{\rho}\colon \mathcal{T}^{\tau^*}A\mathcal{G}_\xi \longrightarrow TA^*\mathcal{G}_\xi$ is projection onto the third factor,

$$\tilde{\rho}(\delta, a, \vartheta_\delta) = \vartheta_\delta.$$

2. A section $\tilde{\Sigma} \in \Gamma(\tilde{\tau})$ is projectable if there exists a section Σ' of $\tau\colon A\mathcal{G}_\xi \longrightarrow M$ and a vector field $\mathcal{X} \in \mathfrak{X}(A^*\mathcal{G}_\xi)$ which is τ-projectable to the vector field $\rho(\Sigma')$ on M, such that $\tilde{\Sigma}(\delta) = (\delta, \Sigma'(\tau\delta)), \mathcal{X}(\delta))$ for all $\delta \in A^*\mathcal{G}_\xi$. We use the notation $\tilde{\Sigma} \equiv (\Sigma', \mathcal{X})$. Then the bracket of two projectable sections $\tilde{\Sigma_1}$ and $\tilde{\Sigma_2}$ is given by

$$\widetilde{[|\tilde{\Sigma_1}, \tilde{\Sigma_2}|]}(\delta) = \big(\delta, [|\Sigma_1', \Sigma_2'|]'(\tau(\delta)), [\mathcal{X}_1, \mathcal{X}_2](\delta)\big),$$

(see [4] for more details).

Consider the Liouville section $\Theta \in \Gamma((\mathcal{T}^{\tau^*}A\mathcal{G}_\xi)^*)$ and using equation (7), we define it as follows:

$$\Theta(\delta)(a, \vartheta_\delta) = \delta(a) = (p, \lambda)(p, V') = \lambda(V') = \theta(\vartheta_\delta) \tag{9}$$

So, as a result $\Theta = (p, \theta)$, where θ is Liouville 1-form on $T^*O(\xi')$.

Furthermore, according to equations (8) and (9), the canonical symplectic section Ω will be defined as follows:

$$\Omega = -d\Theta = (p, \omega) \tag{10}$$

where ω is canonical symplectic 2-form on $T^*O(\xi')$.

Let $H\colon M \times T^*_{\xi'}O(\xi') \longrightarrow \mathbb{R}$ be a Hamiltonian function, Ω be symplectic section and $dH \in \Gamma((\mathcal{T}^{\tau^*}A\mathcal{G}_\xi)^*)$. Then, by definition, there exists the unique Hamiltonian section $\mu_H \in \Gamma(\mathcal{T}^{\tau^*}A\mathcal{G}_\xi)$ satisfying

$$i_{\mu_H}\Omega = dH.$$

In the following lemma, we will show the correspondence between Hamiltonian sections associated to Lie algebroid $M \times T_{\xi'}O(\xi')$ and tangent space $T_{\xi'}O(\xi')$.

Lemma 4.2. *Consider Hamiltonian function $H = (p, h) : M \times T^*_{\xi'}O(\xi') \longrightarrow \mathbb{R}$ where $h : T^*_{\xi'}O(\xi') \longrightarrow \mathbb{R}$ is Hamiltonian function defined on $T^*_{\xi'}O(\xi')$. Let $\mathcal{X}_h$ be Hamiltonian vector field of h, then the Hamiltonian section of H will be as follows:*

$$\mu_H = (p, \mathcal{X}_h).$$

Proof. Let $Y' = (p, Y)$ be a vector field on $M \times T^*_{\xi'}O(\xi')$, Then

$$dH(Y') = (p, dh)(p, Y) = i_{\mathcal{X}_h}\omega(Y) = \omega(\mathcal{X}_h, Y)$$
$$= (p, \omega)((p, \mathcal{X}_h)(p, Y)) = \Omega((p, \mathcal{X}_h), Y') = i_{(p,\mathcal{X}_h)}\Omega(Y').$$

So, if $\mathcal{X}_h$ be Hamiltonian vector field associated to Hamiltonian function h, then, according to what was presented above, and since the Hamiltonian vector field associated to Hamiltonian function h and Hamiltonian section associated to Hamiltonian function are unique, we conclude that the $\mu_H = (p, \mathcal{X}_h)$ is Hamiltonian section associated to Hamiltonian function H, and vice versa. ∎

Furthermore, $\tilde{\rho}(\mu_H)$ is Hamiltonian vector field of H with respect to the linear Poisson structure $\Pi_{M\times T^*_{\xi'}O(\xi')}$ on $M \times T^*_{\xi'}O(\xi')$. So, according to equation (2), we have that

$$\tilde{\rho}(\mu_H) = \mathcal{X}_H \in \mathfrak{X}(M \times T^*_{\xi'}O(\xi')).$$

We denote by $\mathcal{X}_H^{\Pi_{M\times T^*_{\xi'}O(\xi')}}$ the Hamiltonian vector field of H with respect to the linear Poisson structure $\Pi_{M\times T^*_{\xi'}O(\xi')}$ on $M \times T^*_{\xi'}O(\xi')$.

Note that by using the equations (1) and (6), we have actually proved that:

$$\mathcal{X}_H^{\Pi_{M\times T^*_{\xi'}O(\xi')}}(F) = \mathcal{X}_h(f)$$

where $F = (p, f) \in C^\infty(M \times T^*_{\xi'}O(\xi'))$ and $f \in C^\infty(T^*_{\xi'}O(\xi'))$.

In the remainder of this work, we will consider the Hamiltonian system corresponding to the co-adjoint Lie algebroid associated to the co-adjoint action of trivial groupoid and introduce the types I and II of Hamilton-Jacobi equation for it.

Consider co-adjoint Lie algebroid $(M \times T_{\xi'}O(\xi'), [| \ , \ |]', \rho')$ associated to co-adjoint Lie groupoid $M \times O(\xi')$ corresponding to trivial Lie groupoid Υ.

Suppose that $H = (p, h)$ is the Hamiltonian function on $M \times T^*_{\xi'}O(\xi')$, where $h : T^*O(\xi') \longrightarrow \mathbb{R}$ is the Hamiltonian function defined on $T^*O(\xi')$.

Let $(M \times T^*_{\xi'}O(\xi'), \Omega, H)$ be the Hamiltonian system on the co-adjoint Lie algebroid associated to the trivial groupoid $\mathcal{G}$, where according to equation (10), $\Omega = (p, \omega)$ is the canonical symplectic section, where ω is the canonical symplectic 2-form on $T^*O(\xi')$.

Let $\gamma := (p, \varpi)\colon M \longrightarrow M \times T^*_{\xi'}O(\xi')$ be a section of τ^* such that

$$\gamma\big(\tau(p, \mathcal{B})\big) = \big(p, \varpi(\flat)\big)$$

where $\flat \in O(\xi')$, $\mathcal{B} \in T_\flat O(\xi')$, $p \in M$ and ϖ is a 1-form on $O(\xi')$.

Also, let $\zeta_H = pr_1 \circ \mu_H \circ \gamma$ be a section of $M \times T_\xi O(\xi')$, where

$$pr_1\colon \mathcal{T}^{\tau^*}A\mathcal{G}_\xi \longrightarrow M \times T_{\xi'}O(\xi')$$

is a projection, $\mu_H = (p, \mathcal{X}_h)$ is the corresponding Hamiltonian section and $\mathcal{X}_h$ is Hamiltonian vector field associated to the Hamiltonian function $h : T^*O(\xi') \longrightarrow M$. So, $\zeta_H = (p, \zeta_h)$ where $\zeta_h = T\pi_{O(\xi')} \circ \mathcal{X}_h \circ \varpi$ and $T\pi_{O(\xi')}\colon TT^*O(\xi') \longrightarrow TO(\xi')$ is cotangent bundle map.

According to Proposition 3.18, the following conditions are equivalent:

1. Let $t \longrightarrow \sigma'(t)$ and $t \longrightarrow \sigma(t)$ are two curves on $O(\xi')$, M and $\dot{\sigma}'(t)$, $\dot{\sigma}(t)$ are their tangent vector fields on $TO(\xi')$ and TM, respectively. We have

$$\begin{aligned}\dot{\sigma}(t) &= \rho'\big(p, \dot{\sigma}'(t)\big) = \rho'(\zeta_H)(\sigma(t)) \\ &= \rho'(p, \zeta_h)(\sigma(t)) = \rho'\big(p, T\pi_{O(\xi')}(\chi_h(\varpi(\sigma'(t))))\big)\end{aligned}$$

If ϖ satisfies the Hamilton-Jacobi equation, then since $\varpi \circ \sigma' : I \longrightarrow T^*O(\xi')$ is an integral curve of the Hamiltonian vector field $\mathcal{X}_h$, (see [14] for more details), then it turn out that the curve $\gamma \circ \sigma = (p, \varpi \circ \sigma')\colon I \longrightarrow M \times T^*O(\xi')$ on $A^*\mathcal{G}_\xi$ satisfies the Hamilton equations.

(2) γ satisfies the Hamilton-Jacobi equation, $d(H \circ \gamma) = 0$.

Assuming that the section γ is cocycle with respect to $pr_1\colon \mathcal{T}^{\tau^*}A\mathcal{G}_\xi \longrightarrow A\mathcal{G}_\xi$, this assumption is equivalent to the 1-form ϖ is closed with respect to

$$T\pi_{O(\xi')}\colon TT^*O(\xi') \longrightarrow TO(\xi'),$$

because for every $\kappa_1 = (p, V_1', \eta_1)$, $\kappa_2 = (p, V_2', \eta_2) \in \mathcal{T}^{\tau'^*}A\mathcal{G}_\xi$, we have that

$$\begin{aligned} 0 &= d\gamma\big(pr_1(\kappa_1), pr_1(\kappa_2)\big) \\ &= d\gamma\big(pr_1(p, V_1', \eta_1), pr_1(p, V_2', \eta_2)\big) \\ &= d\gamma\big((p, V_1'), (p, V_2')\big) \\ &= d\varpi(V_1', V_2') \\ &= d\varpi(T\pi_{O(\xi')}(\Delta_1), T\pi_{O(\xi')}(\Delta_2)) \end{aligned}$$

where $\Delta_1, \Delta_2 \in TT^*O(\xi')$.

So, according to proposition 3.21, γ is a solution of the equation $(\phi_\gamma, \gamma) \circ \zeta_H = \mu_H \circ \gamma$, which is called type I of Hamilton-Jacobi equation for Hamiltonian system $(A^*\mathcal{G}_\xi, \Omega, H)$, and

$$(\phi_\gamma, \gamma) \circ \zeta_H = \mu_H \circ \gamma = (p, \mathcal{X}_h) \circ (p, \varpi) = (p, \mathcal{X}_h \circ \varpi) = (p, T\varpi \circ \zeta_h).$$

Here we have used the fact mentioned in [14] that since ϖ is closed with respect to $T\pi_{O(\xi')}$, then ϖ is a solution of the equation $T\varpi \circ \zeta_h = \mathcal{X}_h \circ \varpi$, which is called the type I of Hamilton-Jacobi equation for the Hamiltonian system $(T^*O(\xi'), \omega, h)$.

Moreover, we define the symplectic morphism $\varepsilon\colon M \times T^*_{\xi'}O(\xi') \longrightarrow M \times T^*_{\xi'}O(\xi')$ as follows:

$$\varepsilon = (p, \varepsilon')$$

where ε' is symplectic map $\varepsilon'\colon T^*O(\xi') \longrightarrow T^*O(\xi')$ and denote by $\zeta_H^\varepsilon = pr_1 \circ \mu_H \circ \varepsilon$, where μ_H is the corresponding Hamiltonian section, so

$$\begin{aligned}\zeta_H^\varepsilon &= pr_1 \circ \mu_H \circ \varepsilon = pr_1 \circ (p, \mathcal{X}_h) \circ (p, \varepsilon') \\ &= (p, T\pi_{O(\xi')} \circ \mathcal{X}_h \circ \varepsilon') = (p, \zeta_h^{\varepsilon'}).\end{aligned}$$

Then, according to Proposition 3.22, ε is a solution of the equation

$$\mathcal{T}\varepsilon \circ \mu_{H\circ\varepsilon} = \tilde{\mathcal{T}}\lambda \circ \mu_H \circ \varepsilon,$$

if and only if it is a solution of the equation $(\phi_\gamma, \gamma) \circ \mu_H^\varepsilon = \mu_H \circ \varepsilon$, where $\mu_{H\circ\varepsilon} \in \Gamma(\mathcal{T}^{\tau^*} A\mathcal{G}_\xi)$ is Hamiltonian section of the function $H \circ \varepsilon \colon A^*\mathcal{G}_\xi \longrightarrow \mathbb{R}$.

As mentioned in [14], ε' is a solution of the equation $T\varepsilon' \circ \mathcal{X}_{h\circ\varepsilon'} = T\lambda \circ \mathcal{X}_h \circ \varepsilon'$, where $\mathcal{X}_{h\circ\varepsilon'}$ is the Hamiltonian vector field of the function $h \circ \varepsilon' \colon T^*O(\xi') \longrightarrow \mathbb{R}$, if and only if it is a solution of the equation $T\varpi \circ \zeta_h^{\varepsilon'} = \mathcal{X}_h \circ \varepsilon'$, which is called the type II of a Hamilton-Jacobi equation for the Hamiltonian system $(T^*O(\xi'), \omega, h)$. So, we have that

$$(\phi_\gamma, \gamma) \circ \zeta_H^\varepsilon = \mu_H \circ \varepsilon = \big((p, \mathcal{X}_h) \circ (p, \varepsilon')\big) = (p, \mathcal{X}_h \circ \varepsilon') = (p, T\varpi \circ \zeta_h^{\varepsilon'}).$$

The equation $(p, T\varpi\circ\zeta_h^{\varepsilon'}) = (p, \mathcal{X}_h\circ\varepsilon')$, is called type II of Hamilton-Jacobi equation for Hamiltonian system $(A^*\mathcal{G}_\xi, \Omega, H)$.

As the final note in this paper, we state that the time-dependent case of the Hamiltonian-Jacobi equations of Hamiltonian system $(A^*\mathcal{G}_\xi, \Omega', H')$ is similar to what we stated in [6], which we ignore expressing here.

Acknowledgment. The authors would like to thank the referees for carefully reading the paper and their valuable comments.

References

[1] R. Abraham, J. E. Marsden: *Foundations of Mechanics*, 2nd revised and enlarged edition, Advanced Book Program, Benjamin/Cummings Publishing Co., Reading (1978).

[2] A. V. Bolsinov, A. T. Fomenko: *Integrable Hamiltonian System: Geometry, Topology, Classification*, CRC Press, Boca Raton (2004).

[3] R. Bos: *Geometric quantization of Hamiltonian actions of Lie algebroids and Lie groupoids*, Int. J. Geom. Methods Mod. Phys. 4 (2007) 389–436.

[4] J. Cortés, M. de León, J. C. Marrero, D. M. De Diego, E. Martinez: *A survey of lagrangian mechanics and contorol on Lie algebroids and Lie groupoids*, Int. J. Geom. Methods Mod. Phys. 3 (2006) 509–558.

[5] M. de León, J. C. Marrero, E. Martínez: *Lagrangian submanifolds and dynamics on Lie algebroids*, J. Phys. A Math. Gen. 38 (2005) 241–308.

[6] Gh. Haghighatdoost, R. Ayoubi: *Generalized geometric Hamilton-Jacobi theorem on Lie algebroids*, arXiv: 1902.06969 (2019).

[7] Gh. Haghighatdoost, F. Hasani: *Lie Group and Lie Algebra 1*, Payame Noor University, Tehran (1976).

[8] H. Lang, Zh. Liu: *Coadjoint orbits of Lie groupoids*, J. Geometry Physics 129 (2018) 217–232.

[9] K. C. H. Mackenzie: *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Notes Series 213, Cambridge University Press, Cambridge (2005).

[10] C. M. Marle: *Lie, symplectic and Poisson groupoids and their Lie algebroids*, arXiv: 1402.0059 (2014).

[11] J. C. Marrero: *Hamiltonian dynamics on Lie algebroids, unimodularity and preservation of volumes*, arXiv: 0905.0123 (2009).

[12] A. Schmeding: *The Lie group of vertical bisections of a regular Lie groupoid*, Forum Math. 32 (2019) 479–489.

[13] V. V. Trofimov, A. T. Fomenko: *Dynamic systems on orbits of linear representations of Lie groups and the complete integrability of some hydrodynamic systems*, Functional Analysis Appl. 17 (1983) 23–29.

[14] H. Wang: *Hamilton–Jacobi theorems for regular reducible Hamiltonian systems on a cotangent bundle*, J. Geometry Physics 119 (2017) 82–102.

Ghorbanali Haghighatdoost (corresponding author), Rezvaneh Ayoubi, Dept. of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran;
gorbanali@azaruniv.ac.ir, rezvaneh.ayoubi@azaruniv.ac.ir.

Received July 23, 2020
and in final form November 9, 2020