Quantizations on the Engel and the Cartan Groups

Marianna Chatzakou[*]

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Abstract.   This work aims to develop a global quantization in the concrete settings of two graded nilpotent Lie groups of 3-step; namely of the Engel group and the Cartan group. We provide a preliminary analysis on the structure and the representations of the aforementioned groups, and their corresponding Lie algebras. In addition, the explicit formulas for the difference operators in the two settings are derived, constituting the necessary prerequisites for the constructions of the $\Psi_{\rho,\delta}^m$ classes of symbols in both cases. In the case of the Engel group, the relation between the Kohn-Nirenberg quantization and the representations of the Engel group enables us to express operators in this setting in terms of quantization of symbols in the Euclidean space.

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1. Introduction

Since their initiation in the sixties, pseudo-differential operators have become a standard tool in the study of partial differential equations. In the Euclidean setting pseudo-differential operators are defined *globally* as the *quantization* of a smooth function called the *symbol* via the Euclidean Fourier transform. In this setting, pseudo-differential operators are linear operators usually defined by

$$a(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x,\xi) \hat{u}(\xi) \, d\xi \,, \quad u \in \mathcal{S}(\mathbb{R}^n) \,, \tag{1}$$

where $\hat{u}$ denotes the Euclidean Fourier transform of u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In (1) the pseudo-differential operator $a(x,D)$ arises as the *Kohn-Nirenberg quantization* of the symbol a.

The utility of pseudo-differential operators is due to the fact that they carry similar properties as their differential counterparts, such as the behaviour of a generalised notion of the order under composition and adjunction, as well as bounded properties between suitable Sobolev spaces. Adjunction and composition formulas are expressed in term of the corresponding symbols, and thus, the *symbolic calculus* is exactly the calculus on the symbolic side that gives rise to the calculus of operators, while also the calculus contains the parametrices of the elliptic pseudo-differential operators.

It is a natural question whether a global pseudo-differential calculus can be defined in other, more complicated, settings. To this end, in 2010 a global symbolic calculus was developed on compact Lie groups in [16]. In that work the authors defined the *global symbol* of a pseudo-differential operator via the group Fourier transform of the right-convolution kernel (on any connected manifold pseudo-differential operators can always be defined locally via local charts). They provide a definition of the classes of symbols in this setting, so that the associated (via a suitable quantization procedure) operators form an algebra of operators 'close enough' to the one in the Euclidean setting. The most pivotal part of this work was the definition of *difference operators*, which generalises the derivatives in the Fourier variable from the Euclidean case, and thus allows expressing the pseudo-differential behaviour precisely on the group.

Later on, the case of graded nilpotent Lie groups was treated in [9]. However, under this new consideration the results developed in the compact case cannot be readily extended. In particular, they dealt with the technical difficulties that arise from the fact that the dual of the group is no longer discrete and the unitary irreducible representations are infinite dimensional, and more crucially, from the fact that the Laplace-Betrami operator has to be replaced by operators associated with the group via its Lie algebra structure. On stratified Lie groups these are the sub-Laplacians, or more generally, the so-called Rockland operator, and such operators are no longer elliptic but hypoelliptic.

In their monograph, the authors study the global quantization of operators on graded Lie groups, aiming to provide an intrinsic symbolic calculus of the operators in this setting. They give an adequate definition of the difference operators in this setting, which is somehow closer to the one in the Euclidean setting, and therefore, by proving symbolic estimates on the functional calculus for the sub-Laplacian, or the Rockland operator, they provide a way to obtain the formulas for composition and adjunction. The origins of this go back to the 1970's with the works of E. Stein, G. Folland and L. Rothschild (see e.g. [12],[15]) among many others, motivated by the study of differential operators on CR or compact manifolds.

In terms of the developed calculus there, to the author knowledge, most of the versions of the global calculi of operators on homogeneous Lie groups (nilpotent Lie groups endowed with some structure of dilations) that have appeared are, except for a few notable exceptions, calculi of left-invariant operators. Since the seventies, the works on a non-invariant pseudo-differential calculus were limited to [7] where S. Dynin considers certain operators on the Heisenberg groups, and to [11], where G. Folland developed a calculus on any homogeneous group where the classes are given in terms of the kernel, i.e., they are not symbolic. Another version of a non-invariant calculus on any homogeneous group is given in [4] but this is not symbolic as well. Finally, in [18] M. Taylor describes a way that one can obtain a symbolic (non-invariant) calculus by defining a *general quantization* and the *general symbols* on any unimodular type I group, but chose to restrict his analysis mainly to invariant operators on the Heisenberg groups.

The basic example, apart from $\mathbb{R}^n$, of a nilpotent Lie group, is the Heisenberg group where the associated Fourier analysis is very well studied. In particular, most of the works that concern the non-invariant symbolic calculus on nilpotent Lie groups are restricted to the Heisenberg groups (or to manifolds having the Heisenberg group

as local model), and this is mainly due to the link between the representations of the Heisenberg groups and the Weyl quantization that enables the development of the pseudo-differential calculus on these groups with scalar-valued symbols that depend on parameters. To the best of our knowledge, the only non-invariant calculi with scalar-valued symbols on the Heisenberg groups has been developed in [2] by H. Bahouri, C. Fermanian-Kammerer and I.Gallagher, while in [9, Chapter 6] one can find the application of the general theory on graded Lie groups to this particular setting- partially these results had previously been announced in [8]. The last two works differ in the conditions on the symbol classes for small values of the parameters.

Besides the big amount of work devoted to the case of the Heisenberg group which is of 2-step, the same motivating aspects appear as well for other graded Lie groups. In particular, our interest in the two graded nilpotent Lie groups of 3-step considered in this work; namely the Engel and the Cartan group (or the generalised Dido problem), is justified by the fact that they are perhaps the most elementary nilpotent Lie groups of step higher that two not yet studied in the context of global pseudo-differential calculus.

To lay down the necessary foundation for the development of the symbol classes on a nilpotent Lie group, one needs first to study the positive Rockland operators on the group and the associated Sobolev spaces, later on to study and generalise the group Fourier transform and finally to find the concrete formulas for the difference operators on the group.

The paper is organised as follows. In Section 2, we give the necessary preliminaries for the construction of the classes of symbols as explained above. In Section 3 we explain the precise settings of our investigation for the groups we consider, including a description of their dual, and of the group Fourier transform in each setting, which, for the case of the Engel group, leads to symbols parametrized by (λ, μ)- the co-adjoint orbits. On Section 4, we find the explicit formulas for the difference operators in both settings.

2. Preliminaries

2.1. The unitary dual and the group Fourier transform. We denote by $\widehat{\mathbb{G}}$ the unitary dual of the group $\mathbb{G}$, that is, the set of all equivalence classes of irreducible, strongly continuous and unitary representations of $\mathbb{G}$. In our settings, the *Fourier transform* at $\pi \in \widehat{\mathbb{G}}$ is defined on $L^1(\mathbb{G}, dx)$ by

$$\pi(\kappa) = \hat{\kappa}(\pi) := \int_{\mathbb{G}} \kappa(x)\pi(x)^* dx \,, \tag{2}$$

where dx denotes the *Haar measure*, and $\pi^*(x)$ the adjoint operator of $\pi(x)$. This defines a linear mapping on the representation space $\mathcal{H}_\pi$, i.e., $\hat{\kappa}(\pi) : \mathcal{H}_\pi \to \mathcal{H}_\pi$.

The group Fourier transform of a vector in the Lie algebra of the group, say $X \in \mathfrak{g}$, at $\pi \in \hat{\mathbb{G}}$ is the operator $\pi(X)$ on $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi$, the subspace of smooth vectors, given via

$$\pi(X)v = \partial_{t=0}(\pi(exp_{\mathbb{G}}(tX))v \,, \tag{3}$$

where $exp_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ is the exponential mapping that identifies $\mathfrak{g}$ with $\mathbb{G}$. By setting $\pi(X^\alpha) = \pi(X)^\alpha$ we extend the group Fourier transform to the Lie algebra $\mathfrak{U}(\mathfrak{g})$.

2.2. Rockland operator and Sobolev spaces. Let us recall that a (left) *Rockland operator*, say $\mathcal{R}$, on a homogeneous Lie group $\mathbb{G}^1$ is a positive left-invariant operator on $\mathbb{G}$ that is homogeneous of degree ν [2], when for every non-trivial $\pi \in \widehat{\mathbb{G}}$ the group Fourier transform $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$. Being positive means $(\mathcal{R}f, f)_{L^2(\mathbb{G})} \geq 0$, for every f in the Schwartz space $\mathcal{S}(\mathbb{G})$, where as usual

$$(f_1, f_2)_{L^2(\mathbb{G})} = \int_{\mathbb{G}} f_1(x)\overline{f_2}(x)\, dx\,.$$

In the stratified case, we can choose $\mathcal{R} = -\mathcal{L}$, where $\mathcal{L}$ is the sub-Laplacian $\sum_{i=1}^{n_1} X_i^2$, where $\{X_i\}_{i=1}^{n_1}$ are the elements of the first stratum of $\mathfrak{g}$. In this case the homogeneous dimension of $\mathcal{R}$ is $\nu = 2$. Furthermore, by [13, Chapter 4.B] any positive Rockland operator $\mathcal{R}$, as an operator on $\mathcal{D}(\mathbb{G})$, admits a self-adjoint extension on $L^2(\mathbb{G})$.

Recall also that, for $1 \leq p \leq \infty$, let $\mathcal{R}_p$ denote the extension of $\mathcal{R}$ to $L^p(\mathbb{G})$; for a characterisation of the operator $\mathcal{R}_p$, see [8, Section 4.3.1].

Now, for a fixed Rockland operator $\mathcal{R}$ of homogeneous degree ν, following [8, Sections 4.4 and 5.1], we define the subsequent spaces, which, for our scope, will be particularly used to give a meaning to the difference operator discussed later in the Section.

Definition 2.1. If $p \in [1, \infty)$ and $s \in \mathbb{R}$, we denote by $L_s^p(\mathbb{G})$ the Sobolev space obtained by the completion of $\mathcal{S}(\mathbb{G})$ with respect to the norm

$$\|f\|_{L_s^p(\mathbb{G})} := \|(I + \mathcal{R}_p)^{\frac{s}{\nu}} f\|_{L^p(\mathbb{G})}\,, \quad f \in \mathcal{S}(\mathbb{G})\,.$$

Notice that, if $s = 0$, then $L_0^p(\mathbb{G}) = L^p(\mathbb{G})$ for $p \in [1, \infty)$, with $\|\cdot\|_{L_0^p(\mathbb{G})} = \|\cdot\|_{L^p(\mathbb{G})}$. In our setting, the group Fourier transform is an isomorphism between Banach spaces acting from $L_a^2(\mathbb{G})$ onto the $\widehat{\mathbb{G}}$-*fields of operators* denoted by $L_a^2(\mathbb{G})$ defined below:

Definition 2.2. Let $a \in \mathbb{R}$. We denote by $L_a^2(\widehat{\mathbb{G}})$ the space of fields of operators $\sigma = \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^a, \pi \in \hat{\mathbb{G}}\}$ such that

$$\{\pi(I + \mathcal{R})^{\frac{a}{\nu}} \sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi\,, \pi \in \widehat{\mathbb{G}}\} \in L^2(\hat{\mathbb{G}})\,.$$

For such a σ, we set $\|\sigma\|_{L_a^2(\widehat{\mathbb{G}})} := \|\pi(I + \mathcal{R})^{\frac{a}{\nu}} \sigma_\pi\|_{L^2(\hat{\mathbb{G}})}$.
Notice that each $\pi \in \hat{\mathbb{G}}$ is viewed as subset of $\operatorname{Rep}\mathbb{G}$.

Recall that the space $\mathcal{H}_\pi^a$ is the Sobolev space obtained by completion of $\mathcal{H}_\pi^\infty$ with respect to the norm

$$\|f\|_{\mathcal{H}_\pi^\infty} := \|\pi(I + \mathcal{R})^{\frac{a}{\nu}} f\|_{\mathcal{H}_\pi}\,.$$

The next definition is devoted to the the extension of the group Fourier transform to a suitable subset of tempered distribution as we will see later in the section. In particular the following space of fields of operators will turn out to be the image of the group Fourier transform of the latter.

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[1] A graded Lie group is naturally equipped with dilations, i.e., it is homogeneous.

[2] A linear operator $T : \mathcal{D}(\mathbb{G}) \to \mathcal{D}'(\mathbb{G})$ is homogeneous of degree $\nu \in \mathbb{C}$ when
$T(f \circ D_r) = r^\nu (Tf) \circ D_r$ for any $f \in \mathcal{D}(\mathbb{G})\,, r > 0$.

Definition 2.3. Let $\alpha, \beta \in \mathbb{R}$. We denote by $L_{\alpha,\beta}^\infty(\hat{\mathbb{G}})$ the space of fields of operators $\sigma = \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\beta, \pi \in \hat{\mathbb{G}}\}$ such that for some $C > 0$

$$||\sigma\hat{\phi}||_{L_\beta^2(\hat{\mathbb{G}})} \leq C||\phi||_{L_\alpha^2(G)}, \quad \phi \in \mathcal{S}(\mathbb{G}), \tag{4}$$

and for such a field σ, the associated norm $||\sigma||_{L_{\alpha,\beta}^\infty(\hat{\mathbb{G}})}$ is given by the infimum over the constants $C > 0$ such that (4) holds true.

Remark 2.4. We notice that any two norms of the above spaces corresponding to different choices of Rockland operators are equivalent, see [9, Theorem 4.4.20, Proposition 5.1.7 and Lemma 5.1.7].

2.3. Quantization and symbol classes. As recalled in the introduction there is a natural quantization introduced by [18] which is valid on any unimodular type-I group due to the Plancherel formula and produces operators $\mathcal{D}(\mathbb{G}) \to \mathcal{D}'(\mathbb{G})$, where by $\mathcal{D}(\mathbb{G})$ we denote the space of smooth and compactly supported functions on $\mathbb{G}$. In particular, the *quantization*, i.e., the mapping $\sigma \mapsto Op(\sigma)$ is analogous to the Kohn-Nirenberg quantization in the Euclidean setting and associates an operator $Op(\sigma)$ to a *symbol* σ in the following way: For any $f \in \mathcal{D}(\mathbb{G})$ and $x \in \mathbb{G}$, the operator

$$Op(\sigma)\phi(x) = \int_{\hat{G}} \text{Tr}\left(\pi(x)\sigma(x,\pi)\hat{\phi}(x)\right) d\mu(\pi), \tag{5}$$

where μ is the Plancherel measure on $\hat{\mathbb{G}}$, is well-defined and continuous. Recall that a *symbol* σ is a field of operators $\{\sigma(x,\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, (x,\pi) \in \mathbb{G} \times \hat{\mathbb{G}}\}$, satisfying for each $x \in \mathbb{G}$

$$\sigma(x,\cdot) := \{\sigma(x,\pi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \hat{\mathbb{G}}\} \in L_{a,b}^\infty(\hat{\mathbb{G}}), \tag{6}$$

for some $a, b \in \mathbb{R}$, where $L_{a,b}^\infty(\hat{\mathbb{G}})$ is as in Definition 2.3.

Concrete examples of symbols that do not depend on $x \in \mathbb{G}$ are operators of the form $\pi(X)^\alpha$, $\alpha \in \mathbb{N}_0^n$.

In order to motivate the work presented in this paper, we focus our attention in the symbol classes $S_{\rho,\delta}^m = S_{\rho,\delta}^m(\mathbb{G})$ for (ρ, δ) with $0 \leq \delta \leq \rho \leq 1$ and $m \in \mathbb{R} \cup \{-\infty\}$, introduced in [9, Section 5.2] in the context of a graded Lie group $\mathbb{G}$. An application of the quantization process described in (5) then, yields the corresponding operator classes

$$\Psi_{\rho,\delta}^m = Op(S_{\rho,\delta}^m).$$

More accurately:

Definition 2.5. $(S_{\rho,\delta}^m(\mathbb{G})$ symbol classes). Let $m, \rho, \delta \in \mathbb{R}$ with $0 \leq \delta \leq \rho \leq 1$, and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. A symbol in the sense of (6), is called a symbol of order m and of type (ρ, δ) if, for each $\alpha, \beta \in \mathbb{N}_0^n$ and $\gamma \in \mathbb{R}$, we have

$$\sup_{x \in \mathbb{G}} ||X_x^\beta \Delta^\alpha \sigma(x,\cdot)||_{L_{\gamma,\rho[\alpha]-m-\delta[\beta]+\gamma}^\infty(\hat{\mathbb{G}})} < \infty^3,$$

where Δ^α are the difference operators in $\mathbb{G}$, discussed extensively later in the section.

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[3]For non-zero multi-indexes $\alpha = (\alpha_1, \cdots, \alpha_N), \beta = (\beta_1, \cdots, \beta_N)$ and $\nu_1, \cdots, \nu_N$ the dilations' weights of $\mathbb{G}$, $[\alpha] := \nu_1\alpha_1 + \cdots + \nu_N\alpha_N$ and $[\beta] := \nu_1\beta_1 + \cdots \nu_N\beta_N$.

Symbols in the symbol classes $S_{\rho,\delta}^m(\mathbb{G})$ via the quantization (5) give rise to the operators $\Psi_{\rho,\delta}^m := Op(S_{\rho,\delta}^m)$. Before giving the main properties of the operators in $\Psi_{\rho,\delta}^m(\mathbb{G})$, let us state a few remarks:

Remark 2.6. For the purposes of Definition 2.5, we assume that the Rockland operator $\mathcal{R}$ is fixed. However, the appearing class $S_{\rho,\delta}^m(\mathbb{G})$ exists independently of the choice of $\mathcal{R}$ as Remark 2.4 suggests.

Remark 2.7. For the case where $\mathbb{G} = (\mathbb{R}^n, +)$, the symbol classes $S_{\rho,\delta}^m(\mathbb{G})$ coincide with the usual Hörmander classes $S_{\rho,\delta}^m(\mathbb{R}^n)$ as in [14], where the difference operator Δ^α becomes the usual derivatives with respect to the Fourier variable; see Section 2 Paragraph 2.4.

Remark 2.8. For m as in Definition 2.5, we can allow $m = -\infty$. This can be justified if we set $S^{-\infty} := \bigcap_{m \in \mathbb{R}} S_{\rho,\delta}^m$ to denote the class of *smoothing symbols*. In this case the associated operator class is denoted by $\Psi^{-\infty}$.

The class of symbols $S_{\rho,\delta}^m$ fulfills the desired properties of a symbolic calculus. Briefly, the following properties hold:

- $\bigcup_{m \in \mathbb{R}} \Psi_{\rho,\delta}^m$ forms an algebra of operators.

- If T is in $\Psi_{\rho,\delta}^m$, then its formal adjoint T^* is also in $\Psi_{\rho,\delta}^m$.

- If $T \in \Psi_{\rho,\delta}^m$, then T extends to a continuous operator from $L_s^2(\mathbb{G})$ to $L_{s-m}^2(\mathbb{G})$, for any $s \in \mathbb{R}$.

- If $T \in \Psi_{\rho,\delta}^m$ is an elliptic (or more generally a hypoelliptic) [4] operator in this setting, then there exists $T^{-1} \in \Psi_{\rho,\delta}^{-m}$ such that $TT^{-1} - I \in \Psi^{-\infty}$.

One can refer to ([9, Theorem 5.5.3]),([9, Theorem 5.5.12]), ([9, Corollary 5.7.2]) and ([9, Theorem 5.8.7]), respectively, for proofs of the above properties.

2.4. Difference operators. The difference operators in the setting of a graded Lie group are defined as acting on the spaces $\mathcal{K}_{a,b}(\mathbb{G})$.

Recall that by $\mathcal{K}_{a,b}(\mathbb{G})$ we denote the subspace of tempered distributions $\kappa \in \mathcal{S}'(\mathbb{G})$ such that the operator

$$\mathcal{S}(\mathbb{G}) \ni \phi \mapsto \phi * \kappa,$$

extends to an operator in $\mathscr{L}(L_a^2(\mathbb{G}), L_b^2(\mathbb{G}))$, where the latter stands for the set of linear bounded operators from $L_a^2(\mathbb{G})$ to $L_b^2(\mathbb{G})$. In view of [9, Proposition 5.1.24], one can extend the definition of the group Fourier transform to the above space.

Definition 2.9. (The group Fourier transform on $\mathcal{K}_{a,b}(\mathbb{G})$) The group Fourier transform of a tempered distribution $\kappa \in \mathcal{K}_{a,b}(\mathbb{G})$ is the field of operators

$$\sigma := \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^b, \pi \in \hat{\mathbb{G}}\} \in L_{a,b}^\infty(\hat{\mathbb{G}}),$$

where for $\pi \in \hat{\mathbb{G}}$ the operator $\sigma_\pi := \pi(\kappa) = \hat{\kappa}(\pi)$ is such that

$$\sigma_\pi \hat{\phi}(\pi) = \widehat{\phi * \kappa}(\pi), \quad \phi \in \mathcal{S}(\mathbb{G}).$$

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[4]A linear differential operator T on a manifold M is called hypoelliptic, if for any $u \in \mathcal{D}'(M)$, the condition $Lu \in C^\infty(N)$ with $N \subset M$ implies that $u \in C^\infty(N)$ for any $N \subset M$.

Following ([9, Section 5.2.1]), the difference operators in the setting of a graded Lie group are defined as:

Definition 2.10. (Difference operators) For any $q \in C^\infty(\mathbb{G})$, the difference operator associated to q, is the operator Δ_q acting by

$$\Delta_q(\hat{f}) := \widehat{qf}(\pi) \equiv \pi(qf)\,,$$

on any $f \in \mathcal{D}'(\mathbb{G})$ such that $f \in \mathcal{K}_{a,b}(\mathbb{G})$ and $qf \in \mathcal{K}_{a',b'}(\mathbb{G})$ for some $a, b, a', b' \in \mathbb{R}$.

Note that the definition of the difference operators has to be given as each Δ_q is acting on the fields of operators parametrized by $\widehat{\mathbb{G}}$. Even in the simplest case where $\mathbb{G} = (\mathbb{R}^n, +)$, the difference operators associated to the inverse of the coordinate function $q_i(x) = x_i$ include derivatives in the dual variable. Indeed in this case, the dual space $\widehat{\mathbb{R}}^n$ is isomorphic to $\mathbb{R}^n$, and we have

$$\Delta_{(q_i)^{-1}} \mathcal{F}_{\mathbb{R}^n} \phi(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} (-x_i)\phi(x)\, dx = \left(\frac{1}{i}\frac{\partial}{\partial \xi_i}\right) \mathcal{F}_{\mathbb{R}^n}\phi(\xi)\,, \ \phi \in \mathcal{S}(\mathbb{R}^n)\,.$$

More generally, for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, and $q^\alpha(x) = (q_1^{\alpha_1}(x), \cdots, q_n^{\alpha_n}(x))$, the difference operators $\Delta_{(q^\alpha)^{-1}}$ coincide with the operators $D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial \xi}\right)^\alpha$. The latter means essentially that the difference operators generalise the notion of the derivative with respect to the Fourier variable.

We note that the difference operators Δ^α appearing in Definition 2.5 of the $S^m_{\rho,\delta}(\mathbb{G})$ classes are associated with the Taylor polynomials q_α, see [9, Proposition 5.2.3], of a function in each setting. Although, one can easily check that Definition 2.5 could be simplified if one uses instead the operators Δ_{x^α}, Fischer and Ruzhansky in their monograph [9] decided to use the operators Δ^α, since are those that naturally appear in the asymptotic formula for composition and in the formula for the adjoint.

The difference operators in the setting of Engel and Cartan groups whose concrete formula we find in Section 4 are exactly those of the form Δ_{x_i}. Knowing their explicit formulas is enough to describe symbols in the classes $S^m_{\rho,\delta}$ in these settings, since one can show that for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, and for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$

$$(\Delta_{x_1})^{\alpha_1} \cdots (\Delta_{x_n})^{\alpha_n} = \Delta_{x^\alpha}\,.$$

3. Preliminaries on the groups

In this section we expose the main results necessary for our analysis on the groups. As our major source mainly for the description of the Lie algebras and the corresponding Lie groups, and wherever else stated, we have used [3, Section 3].

3.1. The Engel group. Let $\mathfrak{l}_4 = \mathrm{span}\{I_1, I_2, I_3, I_4\}$ be a 3-step nilpotent Lie algebra, whose generators satisfy the non-zero relations

$$[I_1, I_2] = I_3\,, \quad [I_1, I_3] = I_4\,,$$

while all the other commutators are zero. Thus, $\mathfrak{l}_4$ is graded, as it can be endowed with the vector space decomposition

$$\mathfrak{l}_4 = V_1 \oplus V_2 \oplus V_3, \tag{7}$$

where $V_1 = \mathrm{span}\{I_1, I_2\}$, $V_2 = \mathrm{span}\{I_3\}$, and $V_3 = \mathrm{span}\{I_4\}$, such that $[V_i, V_j] \subset V_{i+j}$.

Observe also that $\mathfrak{l}_4$ is stratified, since V_1 generates all of $\mathfrak{l}_4$. Thus, the corresponding Lie group, called the *Engel group* and denoted by $\mathcal{B}_4$ is a homogeneous Lie group, and the natural dilations on its Lie algebra are given by

$$D_r(I_1) = rI_1\,, D_r(I_2) = rI_2\,, D_r(I_3) = r^2I_3\,, \text{and } D_r(I_4) = r^3I_4\,, \quad r > 0\,. \quad (8)$$

In particular, $\mathcal{B}_4$ can be identified with the manifold $\mathbb{R}^4$ endowed with the group law:

$$(x_1, x_2, x_3, x_4) \times (y_1, y_2, y_3, y_4)$$
$$:= \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 - x_1y_2, x_4 + y_4 + \frac{1}{2}x_1^2y_2 - x_1y_3\right).$$

This identification, in turn, implies that $T_e\mathcal{B}_4 \simeq T_0\mathbb{R}^4$, where e is the identity element of $\mathcal{B}_4$ and 0 is the element $(0,0,0,0)$ in $\mathbb{R}^4$. The basis of $\mathfrak{l}^4$, now called the canonical basis, given by [3, Section 3.2], or by explicit calculations, consists of the following canonical left-invariant vector fields:

$$X_1(x) = \frac{\partial}{\partial x_1}\,, \quad X_2(x) = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_4}\,,$$
$$X_3(x) = \frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4}\,, \quad X_4(x) = \frac{\partial}{\partial x_4}\,, \quad (9)$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, that, as expected, satisfy the above relations. The right-invariant vector fields can be computed, see Appendix A, as

$$\tilde{X}_1(x) = \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\,, \quad \tilde{X}_2(x) = \frac{\partial}{\partial x_2}\,, \quad \tilde{X}_3(x) = \frac{\partial}{\partial x_3}\,, \quad \tilde{X}_4(x) = \frac{\partial}{\partial x_4}. \quad (10)$$

Additionally, we notice that the map $\exp_{\mathcal{B}_4}$ is the identity map, meaning that, for the elements of $\mathcal{B}_4$, after choosing $\{X_1, X_2, X_3, X_4\}$ as a basis for $\mathfrak{l}_4$, we have the identification

$$(x_1, x_2, x_3, x_4) = \exp_{\mathcal{B}_4}(x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4)\,, \quad (11)$$

and that we can fix the Lebesgue measure $dx_1dx_2dx_3dx_4$ on $\mathbb{R}^4$, as being the Haar measure on $\mathcal{B}_4$, see e.g. [9, Proposition 1.6.6]. Therefore, in what follows we may formulate as

$$\int_{\mathcal{B}_4} \cdots dx_1dx_2dx_3dx_4 = \int_{\mathbb{R}^4} \cdots dx_1dx_2dx_3dx_4\,. \quad (12)$$

By (11) we can transport the dilations (8) to the group side, i.e.,

$$D_r(x_1, x_2, x_3, x_4) = (rx_1, rx_2, r^2x_3, r^3x_4)\,, \quad r > 0\,,$$

whereas the homogeneous dimension of the group is $Q_{\mathcal{B}_4} = 1 + 1 + 2 + 3 = 7$. The next proposition has been proved by Dixmier in [6, p.333].

Proposition 3.1. *The dual space of $\mathcal{B}_4$ is $\widehat{\mathcal{B}_4} = \{\pi_{\lambda,\mu}|\lambda \neq 0, \mu, \lambda \in \mathbb{R}\}$. In particular, for each $(x_1, x_2, x_3, x_4) \in \mathcal{B}_4$, $\pi_{\lambda,\mu}(x_1, x_2, x_3, x_4)$ is acting on $L^2(\mathbb{R}, \mathbb{C})$ via*

$$\pi_{\lambda,\mu}(x_1, x_2, x_3, x_4)h(u) \equiv \exp\left(i\left(-\frac{\mu}{2\lambda}x_2 + \lambda x_4 - \lambda x_3 u + \frac{\lambda}{2}x_2 u^2\right)\right)h(u + x_1).$$

We assume that the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ is endowed with the standard product

$$(h_1, h_2) := \int_{\mathbb{R}} h_1(u)\overline{h_2(u)}\, du\,,$$

where du is the Lebesgue measure on $\mathbb{R}$.

Remark 3.2. For the case of the representation $\pi_{\lambda,\mu}$ given above, as well as for any representation acting on some $L^2(\mathbb{R}^m)$ of a connected, simply connected nilpotent Lie group, the space $\mathcal{H}_\pi^\infty$ is simply the Schwartz space $\mathcal{S}(\mathbb{R}^m)$, see [5, Corollary 4.1.2].

By [3, Subsection 3.2.3], or by explicit calculations using (3), the group Fourier transform of the elements of $\mathfrak{l}_4$ are the operators acting on $\mathcal{S}(\mathbb{R})$ given by

$$\pi_{\lambda,\mu}(X_1) = \frac{d}{du}\,, \quad \pi_{\lambda,\mu}(X_2) = \left(-\frac{i\mu}{2\lambda} + \frac{i}{2}\lambda u^2\right). \tag{13}$$

Further calculations show that

$$\pi_{\lambda,\mu}(X_3) = -i\lambda u\,, \quad \pi_{\lambda,\mu}(X_4) = i\lambda\,. \tag{14}$$

Due to Proposition 3.1, the group Fourier transform of a function $f \in L^1(\mathcal{B}_4)$ at $\pi_{\lambda,\mu} \in \widehat{\mathcal{B}}_4$ is, as follows from (2), given by:

$$\mathcal{F}_{\mathcal{B}_4}(f)(\pi_{\lambda,\mu}) \equiv \hat{f}(\pi_{\lambda,\mu}) \equiv \pi_{\lambda,\mu}(f) := \int_{\mathcal{B}_4} f(x)\pi_{\lambda,\mu}(x)^* dx\,, \tag{15}$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, is a linear endomorphism of $L^2(\mathbb{R})$, and, after calculations, the adjoint $\pi_{\lambda,\mu}(x)^*$ of the unitary operator $\pi_{\lambda,\mu}(x)$ is acting on $h = h(u) \in L^2(\mathbb{R})$ via

$$\pi_{\lambda,\mu}^*(x)h(u) = \exp\left(i\left(\frac{\mu}{2\lambda}x_2 - \lambda x_4 + \lambda x_3(u-x_1) - \frac{\lambda}{2}x_2(u-x_1)^2\right)\right)h(u-x_1). \tag{16}$$

More particularly one can show the following.

Proposition 3.3. *The group Fourier transform of f at $\pi_{\lambda,\mu}$ is the operator given by*

$$\mathcal{F}_{\mathcal{B}_4}(f)(\pi_{\lambda,\mu}) = Op[a_{f,\lambda,\mu}(\cdot, \cdot)]\,, \tag{17}$$

where

$$a_{f,\lambda,\mu}(v, \xi) = (2\pi)^2 \mathcal{F}_{\mathbb{R}^4}(f)\left(\xi, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda\right),$$

and $Op[a_{f,\lambda,\mu}(\cdot, \cdot)]$ denotes the Kohn-Nirenberg quantization of $a_{f,\lambda,\mu}$.

Recall that the Kohn-Nirenberg quantization of a smooth symbol a on $\mathbb{R} \times \mathbb{R}$ is the operator

$$Op(a)f(u) = (2\pi)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(u-v)\xi} a(v, \xi)f(v)\, dv\, d\xi\,,$$

for $f \in \mathcal{S}(\mathbb{R})$ and $u \in \mathbb{R}$.

Proof. Using (15) and (16), the operator $\mathcal{F}_{\mathcal{B}_4}(f)\pi_{\lambda,\mu}$ can be written as

$$\int_{\mathbb{R}^4} \left[f(x_1, x_2, x_3, x_4) \right.$$

$$\left. \cdot \exp\left(i \left(\frac{\mu}{2\lambda} x_2 - \lambda x_4 + \lambda x_3(u - x_1) - \frac{\lambda}{2} x_2(u - x_1)^2 \right) \right) h(u - x_1) \right] dx_1 \, dx_2 \, dx_3 \, dx_4$$

$$= (2\pi)^{-2} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left[\mathcal{F}_{\mathbb{R}^4}(f)(\xi, \eta, \tau, \omega) \cdot e^{ix_1\xi} \cdot e^{ix_2\eta} \cdot e^{ix_3\tau} \cdot e^{ix_4\omega} \right.$$

$$\left. \cdot \exp\left(i \left(\frac{\mu}{2\lambda} x_2 - \lambda x_4 + \lambda x_3(u - x_1) - \frac{\lambda}{2} x_2(u - x_1)^2 \right) \right) \right.$$

$$\left. \cdot h(u - x_1) \right] dx_1 \, dx_2 \, dx_3 \, dx_4 d\xi \, d\eta \, d\tau \, d\omega$$

$$= -(2\pi) \int_{\mathbb{R}} \int_{\mathbb{R}} \left[e^{ix_1\xi} \mathcal{F}_{\mathbb{R}^4}(f)(\xi, \frac{\lambda}{2}(u - x_1)^2 - \frac{\mu}{2\lambda}, \lambda(x_1 - u), \lambda)h(u - x_1) \right] dx_1 \, d\xi$$

$$= (2\pi) \int_{\mathbb{R}} \int_{\mathbb{R}} \left[e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^4}(f)(\xi, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)h(v) \right] dv \, d\xi \,,$$

where for the last inequality we have applied the change of variable $v = u - x_1$, and this proves Proposition 3.3. ∎

In the above computations, the Fourier transform $\mathcal{F}_{\mathbb{R}^m}(f)(\xi)$ has been defined via:

$$\mathcal{F}_{\mathbb{R}^m}(f)(\xi) := (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(x)e^{-ix\xi}dx \,, \quad x, \xi \in \mathbb{R}^m \,,$$

so that the Fourier inversion theorem becomes

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{i(u-v)\xi} f(v) \, dv \, d\xi = (2\pi)^m f(u) \,, \quad u, v, \xi \in \mathbb{R}^m \,.$$

Let us finally note that by (17), we see that for a symbol σ as in (6) quantized as:

$$\sigma(x, \pi_{\lambda,\mu}) \equiv \sigma(x, \lambda, \mu) = Op(a_{\kappa_x, \lambda, \mu}) \,,$$

then its symbol that is given by

$$a_{\kappa_x, \lambda, \mu}(v, \xi) = (2\pi)^2 \mathcal{F}_{\mathbb{R}^4}(\kappa_x)(\xi, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda) \,, \tag{18}$$

where $\{\kappa_x(y)\}$ is the kernel of the symbol $\sigma(x, \lambda, \mu)$, i.e.,

$$\sigma(x, \lambda, \mu) = \pi_{\lambda,\mu}(\kappa_x) \,.$$

For our notation the quantization (5) becomes

$$Op(\sigma)\phi(x) = 2^{-2}\pi^{-3} \int_{\lambda \neq 0} \int_{\mu \in \mathbb{R}} Tr(\pi_{\lambda,\mu}(x)\sigma(x, \lambda, \mu)\pi_{\lambda,\mu}(\phi)) \, d\mu \, d\lambda \,, \tag{19}$$

where $2^{-2}\pi^{-3}d\mu \, d\lambda$ is the Plancherel measure on $\widehat{\mathcal{B}}_4$ calculated in Appendix B. Summarising one can have the following equivalent formula in terms of composition of symbols on the Euclidean space.

Corollary 3.4. *For $\phi \in \mathcal{D}(\mathcal{B}_4)$, σ as above and for $x \in \mathcal{B}_4$ we can write*

$$Op(\sigma)\phi(x) = 2^{-2}\pi^{-3}\int_{\lambda \neq 0}\int_{\mu \in \mathbb{R}} Tr(Op(a_{\kappa_x,\lambda,\mu})Op(a_{\phi(x\cdot),\lambda,\mu}))\,d\mu\,d\lambda, \qquad (20)$$

where $2^{-2}\pi^{-3}d\mu d\lambda$ is the Plancherel measure on $\widehat{\mathbb{G}}$.

Proof. By using the property of the Fourier transform

$$\hat{\phi}(\pi_{\lambda,\mu})\pi_{\lambda,\mu}(x) = \mathcal{F}_{\mathcal{B}_4}(\phi(x\cdot))(\pi_{\lambda,\mu}),$$

as well as the properties of the trace, and formula (19), the equivalent formula (20) follows immediately. ∎

3.2. The Cartan group. Let $\mathfrak{l}_5 = \mathrm{span}\{I_1, I_2, I_3, I_4, I_5\}$ be the 3-step nilpotent Lie algebra, whose generators satisfy the non-zero relations

$$[I_1, I_2] = I_3\,, [I_1, I_3] = I_4\,, [I_2, I_3] = I_5\,,$$

while all the other commutators are zero. Thus, $\mathfrak{l}_5$ is graded, as it admits a vector space decomposition of the form

$$\mathfrak{l}_5 = V_1 \oplus V_2 \oplus V_3 \qquad (21)$$

where $V_1 = \mathrm{span}\{I_1, I_2\}$, $V_2 = \mathrm{span}\{I_3\}$, and $V_3 = \mathrm{span}\{I_4, I_5\}$, such that we have $[V_i, V_j] \subset V_{i+j}$. Now, observe that $\mathfrak{l}_5$ is stratified, since V_1 generates all of $\mathfrak{l}_5$.

The corresponding Lie group, called the *Cartan group* and denoted by $\mathcal{B}_5$ is a homogeneous Lie group, and the natural dilations on $\mathfrak{l}_5$ are given by

$$D_r(I_1) = rI_1\,, \ D_r(I_2) = rI_2\,, \ D_r(I_3) = r^2 I_3\,, \ D_r(I_4) = r^3 I_4\,, \ \text{and } D_r(I_4) = r^3 I_5\,,$$

where $r > 0$. We identify $\mathcal{B}_5$ with the manifold $\mathbb{R}^5$ endowed with the group law

$$(x_1, x_2, x_3, x_4, x_5) \times (y_1, y_2, y_3, y_4, y_5)$$
$$:= \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - x_1 y_2, x_4 + y_4 + \frac{x_1^2 y_2}{2} - x_1 y_3,$$
$$x_5 + y_5 + \frac{x_1 y_2^2}{2} - x_2 y_3 + x_1 x_2 y_2\Big).$$

Therefore, $T_e\mathcal{B}_5 \simeq T_0\mathbb{R}^5$, where e is the identity element of $\mathcal{B}_5$ and 0 is the element $(0, 0, 0, 0, 0)$ in $\mathbb{R}^5$. The canonical basis of $\mathfrak{l}^5$, given by [3, Section 3.3], or by explicit calculations, consists of the canonical left-invariant vector fields

$$X_1(x) = \frac{\partial}{\partial x_1}\,, \quad X_2(x) = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_4} + x_1 x_2\frac{\partial}{\partial x_5}\,,$$
$$X_3(x) = \frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_5}\,, \quad X_4(x) = \frac{\partial}{\partial x_4}\,, \quad X_5(x) = \frac{\partial}{\partial x_5}\,, \qquad (22)$$

where $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, that satisfy, as one expects, the above relations. If we choose the set $\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4, \tilde{X}_5\}$ of right-invariant vector fields as a basis of $\mathfrak{l}_5$, then the later acts on the space $C^\infty(\mathcal{B}_5)$, see Appendix A, by

$$\tilde{X}_1(x) = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} + \frac{x_2^2}{2} \frac{\partial}{\partial x_5}, \quad \tilde{X}_2(x) = \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_5}, \quad (23)$$

and $$\tilde{X}_3(x) = \frac{\partial}{\partial x_3}, \quad \tilde{X}_4(x) = \frac{\partial}{\partial x_4}, \quad \text{and} \quad \tilde{X}_5(x) = \frac{\partial}{\partial x_5}. \quad (24)$$

The map $\exp_{\mathcal{B}_5}$ is the natural diffeomorphism from $\mathfrak{l}_5$ onto $\mathcal{B}_5$ as in (11) for the case of $\mathcal{B}_4$, yielding analogous results, see (12), concerning the integration on $\mathcal{B}_5$. Finally, transporting the dilations above to the group side, we get

$$D_r(x_1, x_2, x_3, x_4, x_5) = (r x_1, r x_2, r^2 x_3, r^3 x_4, r^3 x_5).$$

In this case the homogeneous dimension is $Q_{\mathcal{B}_5} = 1 + 1 + 2 + 3 + 3 = 10$. Dixmier in [6, p.338] showed the following.

Proposition 3.5.   *The dual space of $\mathcal{B}_5$ is $\widehat{\mathcal{B}}_5 = \{\pi_{\lambda,\mu,\nu} | \lambda^2 + \mu^2 \neq 0, \nu, \lambda, \mu \in \mathbb{R}\}$. In particular, for each $(x_1, x_2, x_3, x_4, x_5) \in \mathcal{B}_5$, $\pi_{\lambda,\mu,\nu}(x_1, x_2, x_3, x_4, x_5)$ is acting on $L^2(\mathbb{R}, \mathbb{C})$ via*

$$\pi_{\lambda,\mu,\nu}(x_1, x_2, x_3, x_4, x_5) h(u) \equiv \exp(i A^{\lambda,\mu,\nu}_{x_1,x_2,x_3,x_4,x_5}(u)) h\left(u + \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right),$$

with

$$A^{\lambda,\mu,\nu}_{x_1,x_2,x_3,x_4,x_5}(u) = -\frac{1}{2} \frac{\nu}{\lambda^2 + \mu^2}(\mu x_1 - \lambda x_2) + \lambda x_4 + \mu x_5$$

$$-\frac{1}{6} \frac{\mu}{\lambda^2 + \mu^2}(\lambda^2 x_1^3 + 3\lambda\mu x_1^2 x_2 + 3\mu^2 x_1 x_2^2 - \lambda\mu x_2^3)$$

$$+ \mu^2 x_1 x_2 u + \lambda\mu(x_1^2 - x_2^2)u + \frac{1}{2}(\lambda^2 + \mu^2)(\mu x_1 - \lambda x_2)u^2.$$

As before, the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ is endowed with the standard product. Working as we did previously for the case of $\mathcal{B}_4$, the group Fourier transform at $\pi_{\lambda,\mu,\nu}$ acts on the elements X_1, X_2 of the canonical basis of $\mathcal{L}_5$ via

$$\pi_{\lambda,\mu,\nu}(X_1) = \left(-\frac{i}{2} \frac{\nu\mu}{\lambda^2 + \mu^2} - \frac{i}{2}(\lambda^2 + \mu^2)\mu u^2 + \frac{\lambda}{\lambda^2 + \mu^2} \frac{d}{du}\right),$$

$$\pi_{\lambda,\mu,\nu}(X_2) = \left(\frac{i}{2} \frac{\lambda\nu}{\lambda^2 + \mu^2} + \frac{i}{2}(\lambda^2 + \mu^2)\lambda u^2 + \frac{\mu}{\lambda^2 + \mu^2} \frac{d}{du}\right), \quad (25)$$

see [3, Subsection 3.3.2], or after explicit calculations using (3). Further calculations show that

$$\pi_{\lambda,\mu,\nu}(X_3) = i(\lambda^2 + \mu^2)u, \quad \pi_{\lambda,\mu,\nu}(X_4) = i\lambda, \quad \pi_{\lambda,\mu,\nu}(X_5) = i\mu. \quad (26)$$

Due to Proposition 3.5 and (2), the group Fourier transform of a function $f \in L^1(\mathcal{B}_5)$ at $\pi_{\lambda,\mu,\nu} \in \widehat{\mathcal{B}}_5$, is given by

$$\mathcal{F}_{\mathcal{B}_5}(f)(\pi_{\lambda,\mu,\nu}) \equiv \hat{f}(\pi_{\lambda,\mu,\nu}) \equiv \pi_{\lambda,\mu,\nu}(f) := \int_{\mathcal{B}_5} f(x) \pi_{\lambda,\mu,\nu}(x)^* dx, \quad (27)$$

where $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, is a linear endomorphism on $L^2(\mathbb{R})$, where, after calculations, the adjoint $\pi_{\lambda,\mu,\nu}(x)^*$ of the unitary operator $\pi_{\lambda,\mu,\nu}(x)$ is acting on $h \in L^2(\mathbb{R})$ via

$$\pi_{\lambda,\mu,\nu}(x)^* h(u) = \exp\left\{i\left(A^{\lambda,\mu,\nu}_{x_1,x_2,x_3,x_4,x_5}\right)(u)^*\right\} h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right),$$

with

$$\left(A^{\lambda,\mu,\nu}_{x_1,x_2,x_3,x_4,x_5}\right)(u)^* = \frac{1}{2}\frac{\nu}{\lambda^2 + \mu^2}(\mu x_1 - \lambda x_2) - \lambda x_4 - \mu x_5$$

$$+ \frac{\mu}{6(\lambda^2 + \mu^2)}\left(\lambda^2 x_1^3 + 3\lambda\mu x_1^2 x_2 + 3\mu^2 x_1 x_2^2 - \lambda\mu x_2^3\right)$$

$$- \mu^2 x_1 x_2\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) - \lambda\mu(x_1^2 - x_2^2)\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right)$$

$$- \frac{1}{2}(\lambda^2 + \mu^2)(\mu x_1 - \lambda x_2)\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right)^2.$$

We note that, unlike the case of the Engel group $\mathcal{B}_4$, the group Fourier transform in this case of the Cartan group $\mathcal{B}_5$, cannot be entirely expressed via a pseudo-differential form.

4. On the Construction of $\Psi^m_{\rho,\delta}$

For the description of the classes $S^m_{\rho,\delta}(\mathbb{G})$ of symbols for any graded Lie group $\mathbb{G}$, the main obstacle one has to overcome is to find explicit expression for the difference operators in each particular setting.

In this section, we find the expressions for the difference operators of the form Δ_{x_i} in the setting of the Engel and Cartan groups. Knowing the form of these operators we get explicit sufficient conditions under which a function belongs in some class of symbols $S^m_{\rho,\delta}$ for reasons explained above in Section 2 Paragraph 2.4.

Let us note that [9, Remark 5.2.18] implies that any left-invariant differential operator belongs to $\cup_{m\in\mathbb{R}}\Psi^m_{\rho,\delta}(\mathbb{G})$, for any graded Lie group $\mathbb{G}$. However, this in not the case for every right-invariant operators.

Proposition 4.1. *The right-invariant vector fields on some graded Lie group are not necessarily in some Ψ^m.*

Let us first make a useful observation:

Lemma 4.2. *If $A \in \Psi^m$ is an operator on $\mathbb{G}$ of homogeneous degree ν_A, then m must be greater or equal that ν_A.*

Proof. Recall, see [9, Corolly 5.7.2], that if $A \in \Psi^m$ then

$$\forall s \in \mathbb{R} \quad \exists C > 0 \quad \forall f \in \mathcal{S}(\mathbb{G}) \quad ||Af||_{L^2_{s-m}} \leq C||f||_{L^2_s}. \tag{28}$$

Assuming a fixed Rockland operator $\mathcal{R}$ of homogeneous degree ν, we can write

$$||f||_{L^2_s} = ||f||_{L^2} + ||\mathcal{R}^{\frac{s}{\nu}}f||_{L^2}, \tag{29}$$

for all s; see [9, Theorem 4.4.3].

Now, inequality (28) for $s = 0$ applied to the function $f \circ D_r$, $r > 0$, using (29), and allowing $r \to \infty$, yields $\nu_A \leq m$, and this completes the proof. ∎

Proof.   It is enough to prove our statement in the case of the Engel group. Recall the canonical base $\{X_1, X_2, X_3, X_4\}$, with X_i's as in (9), of its Lie algebra. Then, using (10), we get that $X_1 - \tilde{X}_1 = x_2 \partial_{x_3} + x_3 \partial_{x_4}$. Now by [10, Corollary 4.13] for the particular case of $\mathcal{B}_4$ we have,

$$L_1^2(\mathcal{B}_4) := \{f \in L^2(\mathcal{B}_4) : X_1 f, X_2 f \in L^2(\mathcal{B}_4)\}. \tag{30}$$

Let $\phi, \chi \in C^\infty(\mathbb{R})$, such that $\phi, \phi' \in L^2(\mathbb{R})$, and $\chi : \mathbb{R} \to [0,1]$ be compactly supported. Then, define $f \in C^\infty(\mathcal{B}_4)$ via

$$f(x_1, x_2, x_3, x_4) := \chi(x_1)\chi(x_3)\chi(x_4)\phi(x_2).$$

Then, it can be easily checked that $f, X_1 f$ and $X_2 f$ are square integrable, and so, using (30), $f \in L_1^2(\mathcal{B}_4)$. However, since we have $x_2 \partial_{x_3} f \notin L^2(\mathcal{B}_4)$, then also $(X_1 - \tilde{X}_1)f \notin L^2(\mathcal{B}_4)$, and in particular $\tilde{X}_1 f \notin L^2(\mathcal{B}_4)$. Assume now that $\tilde{X}_1$ is in some Ψ^m, $m \geq 1$. Then, an application of (28) yields that $\tilde{X}_1$ maps $L_1^2(\mathcal{B}_4)$ to some $L_{1-m}^2(\mathcal{B}_4)$, $m \geq 1$, and hence to $L^2(\mathcal{B}_4)$, continuously. Thus, the chosen f shows that the right invariant operator $\tilde{X}_1$ is not in $\cup_{m \in \mathbb{R}} \Psi_{\rho,\delta}^m(\mathcal{B}_4)$. ∎

Finally, we note that in the process of finding the formulas for the difference operators in what follows, instead of making direct calculations, we adopt other, more efficient, techniques. In some of those, we make use of the subsequent properties without mentioning:

- The linearity of the group Fourier transform.

- For $\kappa \in \mathcal{D}(\mathbb{G})$ and for $X \in \mathfrak{g}$, viewed as a left invariant vector field we have

$$\pi(X\kappa) = \pi(X)\pi(\kappa),$$

  while viewing X as a right invariant vector field $\tilde{X}$ we have

$$\pi(\tilde{X}\kappa) = \pi(\kappa)\pi(X),$$

  see e.g. [9, Proposition 1.7.6(iv)].

4.1. The case of the Engel group.   The difference operators on a graded Lie group, have been defined as acting on the space of tempered distributions $\mathcal{K}_{a,b}$, see Definition 2.10. Thus, it makes sense to clarify this notion in the particular setting of $\mathcal{B}_4$:

Let κ be in some $\mathcal{K}_{a,b}(\mathcal{B}_4)$, and let $\pi_{\lambda,\mu} \in \widehat{\mathcal{B}}_4$, where $\widehat{\mathcal{B}}_4$ is as in Proposition 3.1. Then, the group Fourier transform of κ is by Definition 2, and by Remark 3.2, the field of operators

$$\{\pi_{\lambda,\mu}(\kappa) : \mathcal{S}(\mathbb{R}) \to \mathcal{H}_{\pi_{\lambda,\mu}}^b, \pi_{\lambda,\mu} \in \widehat{\mathcal{B}}_4\},$$

in $L_{a,b}^\infty(\widehat{\mathcal{B}}_4)$, where $\pi_{\lambda,\mu}(\kappa)$ acts on $\mathcal{S}(\mathbb{R})$ as in (15).

Proposition 4.3. *Let $\kappa \in \mathcal{K}_{a,b}(\mathcal{B}_4)$ be a tempered distribution, such that $x_1\kappa$ is in some $\mathcal{K}_{a',b'}(\mathcal{B}_4)$, or $x_2\kappa$ is in some $\mathcal{K}_{a',b'}(\mathcal{B}_4)$, so that the difference operator Δ_{x_1} or Δ_{x_2}, respectively, make sense. Then the formulas for the above difference operators, acting on the space $\mathcal{S}(\mathbb{R})$, are given by:*

$$\Delta_{x_1}\hat{\kappa}(\pi_{\lambda,\mu}) = \frac{i}{\lambda}\left(\pi_{\lambda,\mu}(X_3)\pi_{\lambda,\mu}(\kappa) - \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3)\right),$$

where $\pi_{\lambda,\mu}(X_3) = -i\lambda u$, and $\Delta_{x_2}\hat{\kappa}(\pi_{\lambda,\mu}) = \frac{2\lambda}{i}\partial_\mu\pi_{\lambda,\mu}(\kappa)$.

Proof. Observe that by (9) and (10) we get

$$\tilde{X}_3 - X_3 = x_1\frac{\partial}{\partial x_4} = \frac{\partial}{\partial x_4}x_1. \tag{31}$$

Observe also that for any suitable κ, using (14) we have

$$\pi_{\lambda,\mu}\left(\frac{\partial}{\partial x_4}\kappa\right) = \pi_{\lambda,\mu}(X_4\kappa) = \pi_{\lambda,\mu}(X_4)\pi_{\lambda,\mu}(\kappa) = i\lambda\pi_{\lambda,\mu}(\kappa), \tag{32}$$

since $X_4 = \frac{\partial}{\partial x_4}$. Then, by the above,

$$\begin{aligned}
\pi_{\lambda,\mu}(x_1\kappa) &= \frac{1}{i\lambda}\pi_{\lambda,\mu}(X_4 x_1\kappa) = \frac{1}{i\lambda}\pi_{\lambda,\mu}((\tilde{X}_3 - X_3)\kappa)\\
&= \frac{1}{i\lambda}(\pi_{\lambda,\mu}(\tilde{X}_3\kappa) - \pi_{\lambda,\mu}(X_3\kappa))\\
&= \frac{1}{i\lambda}(\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3) - \pi_{\lambda,\mu}(X_3)\pi_{\lambda,\mu}(\kappa))\\
&= \frac{i}{\lambda}(\pi_{\lambda,\mu}(X_3)\pi_{\lambda,\mu}(\kappa) - \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3)).
\end{aligned}$$

Now, for the difference operator corresponding to x_2, we differentiate the group Fourier transform of κ given in (15), at $h \in \mathcal{S}(\mathbb{R})$ with respect to μ and get

$$\begin{aligned}
\partial_\mu\{\pi_{\lambda,\mu}(\kappa)h(u)\} &= \partial_\mu\left\{\int_{\mathbb{R}^4}\kappa(x)\exp\left(i\left(\frac{\mu}{2\lambda}x_2 - \lambda x_4\right)\right)\right.\\
&\quad \left.\cdot\exp\left(i\left(\lambda x_3(u-x_1) - \frac{\lambda}{2}x_2(u-x_1)^2\right)\right)h(u-x_1)dx\right\}\\
&= \int_{\mathbb{R}^4}\kappa(x)\exp\left(i\left(\frac{\mu}{2\lambda}x_2 - \lambda x_4\right)\right)\\
&\quad \cdot\exp\left(i\left(\lambda x_3(u-x_1) - \frac{\lambda}{2}x_2(u-x_1)^2\right)\right)h(u-x_1)\left(\frac{i}{2\lambda}x_2\right)dx,
\end{aligned}$$

or in terms of difference operators,

$$\partial_\mu\pi_{\lambda,\mu}(\kappa) = \pi_{\lambda,\mu}\left(\frac{i}{2\lambda}x_2\kappa\right) = \frac{i}{2\lambda}\Delta_{x_2}\pi_{\lambda,\mu}(\kappa).$$

The proof is complete. ∎

Remark 4.4. As we mention earlier in the discussion that follows after Definition 2.10 the difference operators are not necessarily local, in the sense that, as the last proposition shows, they might for example be expressed by using derivatives in $\mu \in \mathbb{R}$, and therefore they should act on the field of operators $\{\pi_{\lambda,\mu}(\kappa), \lambda \neq 0, \mu \in \mathbb{R}\}$.

This is also the case for the difference operators $\Delta_{x_3|\pi_{\lambda,\mu}}$ and $\Delta_{x_4|\pi_{\lambda,\mu}}$ in the setting of $\mathcal{B}_4$, see Propositions 4.5 and 4.6, whereas in the setting of $\mathcal{B}_5$ any difference operator of the form $\Delta_{x_i|\pi_{\lambda,\mu,\nu}}, i = 1, \cdots, 4$ is not local, see Propositions 4.7, 4.8, 4.9 and 4.10.

Proposition 4.5. Let $\kappa \in \mathcal{K}_{a,b}(\mathcal{B}_4)$ be such that $x_3\kappa$ is also in some $\mathcal{K}_{a',b'}(\mathcal{B}_4)$. Then the difference operator that corresponds to x_3 is given by

$$\Delta_{x_3}\hat{\kappa}(\pi_{\lambda,\mu}) = \frac{i}{\lambda}\left(\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3) + \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_1) - \pi_{\lambda,\mu}(X_1)\pi_{\lambda,\mu}(\kappa)\right),$$

where $\Delta_{x_2|\pi_{\lambda,\mu}}$ is given in Proposition 4.3, and $\pi_{\lambda,\mu}(X_1), \pi_{\lambda,\mu}(X_3)$, are as in (13) and (14).

Proof. Formulas (9) and (10) yield

$$X_1 - \tilde{X}_1 - x_2\tilde{X}_3 = x_3\frac{\partial}{\partial x_4} = \frac{\partial}{\partial x_4}x_3.$$

Let κ be a suitable distribution. Then, using (32), and as $\frac{\partial}{\partial x_4} = X_4$, we have

$$\begin{aligned}
\pi_{\lambda,\mu}(x_3\kappa) &= \frac{1}{i\lambda}\pi_{\lambda,\mu}(X_4 x_3\kappa) = \frac{1}{i\lambda}\pi_{\lambda,\mu}\left((X_1 - \tilde{X}_1 - x_2\tilde{X}_3)\kappa\right)\\
&= \frac{1}{i\lambda}\left(\pi_{\lambda,\mu}(X_1\kappa) - \pi_{\lambda,\mu}(\tilde{X}_1\kappa) - \pi_{\lambda,\mu}(x_2\tilde{X}_3\kappa)\right)\\
&= \frac{1}{i\lambda}\left(\pi_{\lambda,\mu}(X_1)\pi_{\lambda,\mu}(\kappa) - \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_1) - \pi_{\lambda,\mu}(x_2\kappa)\pi_{\lambda,\mu}(X_3)\right)\\
&= \frac{1}{i\lambda}\left(\pi_{\lambda,\mu}(X_1)\pi_{\lambda,\mu}(\kappa) - \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_1) - \Delta_{x_2}\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3)\right)\\
&= \frac{i}{\lambda}\left(\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_3) + \pi_{\lambda,\mu}(\kappa)\pi_{\lambda,\mu}(X_1) - \pi_{\lambda,\mu}(X_1)\pi_{\lambda,\mu}(\kappa)\right),
\end{aligned}$$

completing the proof. ∎

Proposition 4.6. Let $\kappa \in \mathcal{K}_{a,b}(\mathcal{B}_4)$ be such that $x_4\kappa$ is also in some $\mathcal{K}_{a',b'}(\mathcal{B}_4)$. Then, for all $h \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}
(\Delta_{x_4}\pi_{\lambda,\mu}(\kappa))h(u) &= i\partial_\lambda\{\pi_{\lambda,\mu}(\kappa)h(u)\} - \left(\frac{\mu}{2\lambda^2} + \frac{u^2}{2}\right)\left\{\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)h(u)\right\}\\
&\quad + u\left\{\Delta_{x_3}\pi_{\lambda,\mu}(\kappa)h(u)\right\} - \left\{\Delta_{x_3}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\right\}\\
&\quad + u\left\{\Delta_{x_2}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\right\} - \frac{1}{2}\left\{\Delta_{x_2}\Delta_{x_1}^2\pi_{\lambda,\mu}(\kappa)h(u)\right\},
\end{aligned}$$

where the difference operators $\Delta_{x_i|\pi_{\lambda,\mu}}$, $i = 1, 2, 3$, are given in Propositions 4.3 and 4.5.

Proof. For κ as in the statement and for $h \in \mathcal{S}(\mathbb{R})$, we differentiate with respect to λ the group Fourier transform of κ, given in (15), at h.

This yields,

$$\partial_\lambda \{\pi_{\lambda,\mu}(\kappa)h(u)\} = \partial_\lambda \Big\{ \int_{\mathbb{R}^4} \kappa(x) \exp\Big(i\Big(\frac{\mu}{2\lambda}x_2 - \lambda x_4\Big)\Big)$$

$$\cdot \exp\Big(i\Big(\lambda x_3(u-x_1) - \frac{\lambda}{2}x_2(u-x_1)^2\Big)\Big) h(u-x_1)\, dx \Big\}$$

$$= \int_{\mathbb{R}^4} \kappa(x) \exp\Big(i\Big(\frac{\mu}{2\lambda}x_2 - \lambda x_4 + \lambda x_3(u-x_1) - \frac{\lambda}{2}x_2(u-x_1)^2\Big)\Big)$$

$$h(u-x_1)\Big\{i\Big(-\frac{\mu}{2\lambda^2}x_2 - x_4 + x_3(u-x_1) - \frac{x_2}{2}(u-x_1)^2\Big)\Big\}dx\,.$$

Rewriting the above formula in terms of difference operators we obtain

$$\partial_\lambda \{\pi_{\lambda,\mu}(\kappa)h(u)\} = i\Big[-\Big(\frac{\mu}{2\lambda^2} + \frac{u^2}{2}\Big)\Big\{\Delta_{x_2}\pi_{\lambda,\mu}(\kappa)h(u)\Big\} - \Big\{\Delta_{x_4}\pi_{\lambda,\mu}(\kappa)h(u)\Big\}$$

$$+ u\Big\{\Delta_{x_3}\pi_{\lambda,\mu}(\kappa)h\Big\}(u) - \Big\{\Delta_{x_3}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\Big\}$$

$$+ u\Big\{\Delta_{x_2}\Delta_{x_1}\pi_{\lambda,\mu}(\kappa)h(u)\Big\} - \frac{1}{2}\Big\{\Delta_{x_2}\Delta_{x_1}^2\pi_{\lambda,\mu}(\kappa)h(u)\Big\}\Big]\,,$$

completing the proof. ∎

4.2. The case of the Cartan group. Let κ be in some $\mathcal{K}_{a,b}(\mathcal{B}_5)$, and let $\pi_{\lambda,\mu,\nu} \in \widehat{\mathcal{B}}_5$, where $\widehat{\mathcal{B}}_5$ is as in Proposition 3.5. Then, the group Fourier transform of κ is by Definition 2, and by Remark 3.2, the field of operators

$$\{\pi_{\lambda,\mu,\nu}(\kappa) : \mathcal{S}(\mathbb{R}) \to \mathcal{H}^b_{\pi_{\lambda,\mu,\nu}}\,, \pi_{\lambda,\mu,\nu} \in \hat{\mathcal{B}}_5\}\,,$$

in $L^\infty_{a,b}(\widehat{\mathcal{B}}_5)$, where $\pi_{\lambda,\mu,\nu}(\kappa)$ acts on $\mathcal{S}(\mathbb{R})$ as in (27).

Proposition 4.7. *Let $\kappa \in \mathcal{K}_{a,b}(\mathcal{B}_5)$ be a tempered distribution, such that $x_1\kappa$ is in some $\mathcal{K}_{a',b'}(\mathcal{B}_5)$, or $x_2\kappa$ is in some $\mathcal{K}_{a',b'}(\mathcal{B}_5)$, so that the difference operator Δ_{x_1} or Δ_{x_2}, respectively, make sense. Then, the above difference operators act on $h \in \mathcal{S}(\mathbb{R})$ via*

$$(\Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa))h(u) = -2i\mu\frac{\partial}{\partial\nu}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$

$$+ \frac{i\lambda}{(\lambda^2+\mu^2)}\left(\pi_{\lambda,\mu,\nu}(X_3)\pi_{\lambda,\mu,\nu}(\kappa) - \pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_3)\right)h(u)\,,$$

and

$$(\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa))h(u) = 2i\lambda\frac{\partial}{\partial\nu}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$

$$+ i\mu\left(\pi_{\lambda,\mu,\nu}(X_3)\pi_{\lambda,\mu,\nu}(\kappa) - \pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_3)\right)h(u)\,,$$

where $\pi_{\lambda,\mu,\nu}(X_3)$ is given by (26).

Proof.   We proceed by finding the explicit formulas for the above operators simultaneously, by solving a system of equations. In particular, for κ as in the statement, and $h \in \mathcal{S}(\mathbb{R})$, we start by differentiating with respect to ν the group Fourier transform of κ given in (27). Then,

$$
\partial_\nu \{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} = \int_{\mathbb{R}^5} \kappa(x) \frac{\partial}{\partial \nu} \left[e^{iA^*} \cdot h \left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2} \right) \right] dx
$$

$$
= \int_{\mathbb{R}^5} \kappa(x) \cdot e^{iA^*} \cdot \frac{\partial (iA^*)}{\partial \nu} \cdot h \left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2} \right) dx
$$

$$
= \frac{i\mu}{2(\lambda^2 + \mu^2)} \int_{\mathbb{R}^5} x_1 \kappa(x) \cdot h \left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2} \right) dx
$$

$$
- \frac{i\lambda}{2(\lambda^2 + \mu^2)} \int_{\mathbb{R}^5} x_2 \kappa(x) \cdot h \left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2} \right) dx
$$

$$
= \frac{i\mu}{2(\lambda^2 + \mu^2)} \pi_{\lambda,\mu,\nu}(x_1 \kappa)h(u) - \frac{i\lambda}{2(\lambda^2 + \mu^2)} \pi_{\lambda,\mu,\nu}(x_2 \kappa)h(u). \quad (33)
$$

Now, observe that by (22) and (24), we have that

$$
\tilde{X}_3 - X_3 = x_1 X_4 + x_2 X_5 = X_4 x_1 + X_5 x_2.
$$

Then, for any suitable κ as in the statement, as $X_4 = \frac{\partial}{\partial x_4}$ and $X_5 = \frac{\partial}{\partial x_5}$, since $\pi_{\lambda,\mu,\nu}(X_4) = i\lambda$ and $\pi_{\lambda,\mu,\nu}(X_5) = i\mu$, one gets

$$
\pi_{\lambda,\mu,\nu} \left(\frac{\partial}{\partial x_4} \kappa \right) = \pi_{\lambda,\mu,\nu}(X_4 \kappa) = \pi_{\lambda,\mu,\nu}(X_4)\pi_{\lambda,\mu,\nu}(\kappa) = i\lambda \pi_{\lambda,\mu,\nu}(\kappa),
$$

and $\pi_{\lambda,\mu,\nu} \left(\frac{\partial}{\partial x_5} \kappa \right) = \pi_{\lambda,\mu,\nu}(X_5 \kappa) = \pi_{\lambda,\mu,\nu}(X_5)\pi_{\lambda,\mu,\nu}(\kappa) = i\mu \pi_{\lambda,\mu,\nu}(\kappa).$

Assume first that $\mu \neq 0$. The above observations yield

$$
\pi_{\lambda,\mu,\nu}(x_2 \kappa)h(u) = \frac{1}{i\mu} \pi_{\lambda,\mu,\nu} (X_5 x_2 \kappa) h(u)
$$

$$
= -\frac{i}{\mu} \pi_{\lambda,\mu,\nu} \left(\left(\tilde{X}_3 - X_3 - x_1 X_4 \right) \kappa \right) h(u)
$$

$$
= \frac{i}{\mu} \left(\pi_{\lambda,\mu,\nu} (X_4 x_1 \kappa) + \pi_{\lambda,\mu,\nu} (X_3 \kappa) - \pi_{\lambda,\mu,\nu} \left(\tilde{X}_3 \kappa \right) \right) h(u).
$$

Thus, since $\pi_{\lambda,\mu,\nu}(X_4) = i\lambda$, we finally get

$$
\pi_{\lambda,\mu,\nu}(x_2 \kappa)h(u) = \left[\left(-\frac{\lambda}{\mu} \right) \pi_{\lambda,\mu,\nu}(x_1 \kappa) + \frac{i}{\mu} (\pi_{\lambda,\mu,\nu}(X_3 \kappa) - \pi_{\lambda,\mu,\nu}(\tilde{X}_3 \kappa)) \right] h(u). \quad (34)
$$

Plugging (34) into (33) yields

$$
\frac{\partial}{\partial \nu} \{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} = \frac{i\mu}{2(\lambda^2 + \mu^2)} \pi_{\lambda,\mu,\nu}(x_1 \kappa)h(u)
$$

$$
- \frac{i\lambda}{2(\lambda^2 + \mu^2)} \left[\left(-\frac{\lambda}{\mu} \right) \pi_{\lambda,\mu,\nu}(x_1 \kappa) + \frac{i}{\mu} \left(\pi_{\lambda,\mu,\nu}(X_3 \kappa) - \pi_{\lambda,\mu,\nu}(\tilde{X}_3 \kappa) \right) \right] h(u)
$$

$$
= \frac{i}{2\mu} \pi_{\lambda,\mu,\nu}(x_1 \kappa)h(u) + \frac{\lambda}{2\mu(\lambda^2 + \mu^2)} \left[\pi_{\lambda,\mu,\nu}(X_3 \kappa) - \pi_{\lambda,\mu,\nu}(\tilde{X}_3 \kappa) \right] h(u).
$$

Hence, the difference operator $\Delta_{x_1|\pi_{\lambda,\mu,\nu}}$ is acting on $\mathcal{S}(\mathbb{R})$ by

$$\pi_{\lambda,\mu,\nu}(x_1\kappa)h(u) = -2i\mu\frac{\partial}{\partial\nu}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$
$$+ \frac{i\lambda}{(\lambda^2 + \mu^2)}\left(\pi_{\lambda,\mu,\nu}(X_3)\pi_{\lambda,\mu,\nu}(\kappa) - \pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_3)\right)h(u). \quad (35)$$

Lastly, if we plug (35) into (34), we see that the difference operator $\Delta_{x_2|\pi_{\lambda,\mu,\nu}}$ is acting on $\mathcal{S}(\mathbb{R})$ via

$$\pi_{\lambda,\mu,\nu}(x_2\kappa)h(u) = 2i\lambda\frac{\partial}{\partial\nu}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$
$$+i\frac{\mu}{\lambda^2 + \mu^2}\left(\pi_{\lambda,\mu,\nu}(X_3)\pi_{\lambda,\mu,\nu}(\kappa) - \pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_3)\right)h(u). \quad (36)$$

Now consider the case where $\mu = 0$. Then $\lambda \neq 0$, and by (33) we get

$$\pi_{\lambda,\mu,\nu}(x_2\kappa)h(u) = 2i\lambda\frac{\partial}{\partial\nu}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}, \quad (37)$$

and by the above observations

$$\pi_{\lambda,\mu,\nu}(x_1\kappa)h(u) = \frac{1}{i\lambda}\pi_{\lambda,\mu,\nu}\left(X_4 x_1\kappa\right)h(u)$$
$$= -\frac{i}{\lambda}\pi_{\lambda,\mu,\nu}\left(\left(\tilde{X}_3 - X_3 - X_5 x_2\right)\kappa\right)h(u)$$
$$= \frac{i}{\lambda}\left(\pi_{\lambda,\mu,\nu}\left(X_5 x_2\kappa\right) + \pi_{\lambda,\mu,\nu}\left(X_3\kappa\right) - \pi_{\lambda,\mu,\nu}\left(\tilde{X}_3\kappa\right)\right)h(u),$$

while since $\pi_{\lambda,\mu,\nu}(X_5) = i\mu = 0$ and by using (37) we finally get

$$\pi_{\lambda,\mu,\nu}(x_1\kappa)h(u) = \frac{i}{\lambda}\left(\pi_{\lambda,\mu,\nu}(X_3)\pi_{\lambda,\mu,\nu}(\kappa) - \pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_3)\right)h(u), \quad (38)$$

and this concludes the proof if one observes that (37) and (38) are particular cases of (36) and (35), respectively. $\blacksquare$

Proposition 4.8. *Let $\kappa \in \mathcal{K}_{a,b}(\mathcal{B}_5)$ be a tempered distribution, such that $x_3\kappa$ is in some $\mathcal{K}_{a',b'}(\mathcal{B}_5)$. Then, the formula for the difference operator Δ_{x_3} acting on the space $\mathcal{S}(\mathbb{R})$ is the following:*

$$\Delta_{x_3}\pi_{\lambda,\mu,\nu}(\kappa) = \frac{i}{\lambda}\left(\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_3) - \pi_{\lambda,\mu,\nu}(X_1)\pi_{\lambda,\mu,\nu}(\kappa)\right)$$
$$+ \frac{i}{\lambda}\pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_1) + \frac{\mu}{\lambda}\Delta_{x_2}^2\pi_{\lambda,\mu,\nu}(\kappa),$$

where the difference operator $\Delta_{x_2|\pi_{\lambda,\mu,\nu}}$, is as in Proposition 4.7, and the operators $\pi_{\lambda,\mu,\nu}(X_i)$, $i = 1, 3, 5$, are given in (25) and (26).

Proof. Since $X_4 = \frac{\partial}{\partial x_4}$, then

$$X_4 x_3 = x_3 X_4 = x_3\frac{\partial}{\partial x_4} = x_3\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} - x_2\frac{\partial}{\partial x_3} + \frac{x_2^2}{2}\frac{\partial}{\partial x_5} - \frac{x_2^2}{2}\frac{\partial}{\partial x_5}$$
$$= X_1 - x_2\tilde{X}_3 + \frac{x_2^2}{2}X_5 - \tilde{X}_1,$$

where for the last equality we have used (22),(24), and (23).

Now, by (26) we get

$$\pi_{\lambda,\mu,\nu}(x_3\kappa) = \frac{1}{i\lambda}\pi_{\lambda,\mu,\nu}(X_4 x_3\kappa) = \frac{1}{i\lambda}\pi_{\lambda,\mu,\nu}\left(\left(X_1 - x_2\tilde{X}_3 + \frac{x_2^2}{2}X_5 - \tilde{X}_1\right)\kappa\right)$$

$$= \frac{1}{i\lambda}\left(\pi_{\lambda,\mu,\nu}(X_1\kappa) - \pi_{\lambda,\mu,\nu}(\tilde{X}_3 x_2\kappa) + \pi_{\lambda,\mu,\nu}\left(X_5\frac{x_2^2}{2}\kappa\right) - \pi_{\lambda,\mu,\nu}(\tilde{X}_1\kappa)\right)$$

$$= \frac{1}{i\lambda}\left(\pi_{\lambda,\mu,\nu}(X_1)\pi_{\lambda,\mu,\nu}(\kappa) - \Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_3)\right)$$

$$+ \frac{1}{i\lambda}\left(\frac{1}{2}\pi_{\lambda,\mu,\nu}(X_5)\Delta_{x_2}^2\pi_{\lambda,\mu,\nu}(\kappa) - \pi_{\lambda,\mu,\nu}(\kappa)\pi_{\lambda,\mu,\nu}(X_1)\right),$$

where $\pi_{\lambda,\mu,\nu}(X_5) = i\mu$, which shows the desired result. ∎

Proposition 4.9. *Let $\kappa \in \mathcal{K}_{a,b}(\mathcal{B}_5)$, such that $x_4\kappa$ is in some $\mathcal{K}_{a',b'}(\mathcal{B}_5)$. Then, we have for all $h \in \mathcal{S}(\mathbb{R})$ that*

$$(\Delta_{x_4}\pi_{\lambda,\mu,\nu}(\kappa))h(u) = i\left(\frac{\partial}{\partial\lambda}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} - \left[\frac{2\lambda\mu}{(\lambda^2+\mu^2)^2}\right]\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)\left\{\frac{d}{du}h\right\}(u)\right.$$

$$\left. + \left[\frac{\mu^2-\lambda^2}{(\lambda^2+\mu^2)^2}\right]\Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa)\left\{\frac{d}{du}h\right\}(u)\right)$$

$$+ (b_1^1 + b_1^2 u^2)\{\Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} + (b_2^1 + b_2^2 u^2)\{\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$

$$+ ub_{1,2}\{\Delta_{x_1}\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} + b_{1,1,2}\{\Delta_{x_1}^2\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$

$$+ b_{1,2,2}\{\Delta_{x_1}\Delta_{x_2}^2\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}\, b_{1,1,1}\{\Delta_{x_1}^3\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$

$$+ b_{2,2,2}\{\Delta_{x_2}^3\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}\,,$$

where the difference operators $\Delta_{x_i|\pi_{\lambda,\mu,\nu}}$, $i = 1, 2, 3$, are as in Propositions 4.7 and 4.8, and

$$b_1^1 = \frac{-\nu\lambda\mu}{(\lambda^2+\mu^2)^2}\,, \quad b_1^2 = -\lambda\mu\,, \quad b_2^1 = \frac{\nu\lambda^2 - \nu\mu^2}{2(\lambda^2+\mu^2)^2}\,,$$

$$b_2^2 = \frac{9\lambda^6 + 3\mu^6 + 15\lambda^2\mu^4 + 21\lambda^4\mu^2}{6(\lambda^2+\mu^2)^2}\,, \quad b_{1,2} = -2\lambda\,, \quad b_{1,1,2} = \frac{3\lambda^4 + 9\mu^4}{6(\lambda^2+\mu^2)^2}\,,$$

$$b_{1,2,2} = \frac{-2\lambda\mu^3}{(\lambda^2+\mu^2)^2}\,, \quad b_{1,1,1} = \frac{4\lambda\mu^3}{3(\lambda^2+\mu^2)^2}\,, \quad b_{2,2,2} = \frac{2\mu^2\lambda^2 - 2\mu^4}{3(\lambda^2+\mu^2)^2}\,.$$

Proof. Let κ be as in the statement. We start by differentiating with respect to λ the group Fourier transform of κ, given in (27), at $h \in \mathcal{S}(\mathbb{R})$. This yields,

$$\frac{\partial}{\partial\lambda}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} = \int_{\mathbb{R}^5}\kappa(x)\frac{\partial}{\partial\lambda}\left[e^{iA^*}\cdot h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2+\mu^2}\right)\right]dx$$

$$= \int_{\mathbb{R}_5}\kappa(x)\cdot e^{iA^*}\cdot\frac{\partial(iA^*)}{\partial\lambda}\cdot h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2+\mu^2}\right)dx$$

$$+ \int_{\mathbb{R}_5}\kappa(x)\cdot e^{iA^*}\left[\frac{2\lambda(\lambda x_1 + \mu x_2) - x_1(\lambda^2+\mu^2)}{(\lambda^2+\mu^2)^2}\right]\frac{d}{du}h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2+\mu^2}\right)dx$$

$$= I + II\,.$$

Dealing with integral II, we see that

$$II = \int_{\mathbb{R}_5} \kappa(x) \cdot e^{iA^*} \left\{ x_2 \frac{2\lambda\mu}{(\lambda^2 + \mu^2)^2} + x_1 \frac{-\mu^2 + \lambda^2}{(\lambda^2 + \mu^2)^2} \right\} \frac{d}{du} h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx$$

$$= \frac{2\lambda\mu}{(\lambda^2 + \mu^2)^2} \int_{\mathcal{B}_5} x_2 \kappa(x) \cdot e^{iA^*} \cdot \frac{d}{du} h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx \qquad (39)$$

$$+ \frac{-\mu^2 + \lambda^2}{(\lambda^2 + \mu^2)^2} \int_{\mathcal{B}_5} x_1 \kappa(x) \cdot e^{iA^*} \cdot \frac{d}{du} h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx$$

$$= \left(\left[\frac{2\lambda\mu}{(\lambda^2 + \mu^2)^2} \right] \Delta_{x_2} \pi_{\lambda,\mu,\nu}(\kappa) \left\{ \frac{d}{du} h \right\} + \left[\frac{-\mu^2 + \lambda^2}{(\lambda^2 + \mu^2)^2} \right] \Delta_{x_1} \pi_{\lambda,\mu,\nu}(\kappa) \left\{ \frac{d}{du} h \right\} \right)(u).$$

For the integral I, we see that since

$$\frac{\partial(iA^*)}{\partial\lambda} = i\left\{ -x_4 + (b_1^1 + b_1^2 u^2)x_1 + (b_2^1 + b_2^2 u^2)x_2 + ub_{1,2}x_1 x_2 + b_{1,1,2}x_1^2 x_2 \right.$$

$$\left. + b_{1,2,2}x_1 x_2^2 \right\} + i\left\{ b_{1,1,1}x_1^3 + b_{2,2,2}x_2^3 \right\}, \qquad (40)$$

then, reasoning as in the case of the integral II, we get

$$I = i\left[\left\{ -\Delta_{x_4}\pi_{\lambda,\mu,\nu}(\kappa) + (b_1^1 + b_1^2 u^2)\Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa) + (b_2^1 + b_2^2 u^2)\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa) \right\} h(u) \right]$$

$$+ i\left[\left\{ ub_{1,2}\Delta_{x_1}\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa) + b_{1,1,2}\Delta_{x_1}^2\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa) + b_{1,2,2}\Delta_{x_1}\Delta_{x_2}^2\pi_{\lambda,\mu,\nu}(\kappa) \right\} h(u) \right]$$

$$+ i\left[\left\{ b_{1,1,1}\Delta_{x_1}^3\pi_{\lambda,\mu,\nu}(\kappa) + b_{2,2,2}\Delta_{x_2}^3\pi_{\lambda,\mu,\nu}(\kappa) \right\} h(u) \right],$$

where the b's are as in the statement, and the proof is complete. ∎

Proposition 4.10. *Let $\kappa \in \mathcal{K}_{a,b}(\mathcal{B}_5)$, such that $x_5\kappa$ is in some $\mathcal{K}_{a',b'}(\mathcal{B}_5)$. Then, we have for all $h \in \mathcal{S}(\mathbb{R})$ that*

$$(\Delta_{x_5}\pi_{\lambda,\mu,\nu}(\kappa))h(u) = i\left(\frac{\partial}{\partial\mu}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} - \left[\frac{2\lambda\mu}{(\lambda^2 + \mu^2)^2} \right] \Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa) \left\{ \frac{d}{du} h \right\}(u) \right.$$

$$\left. + \left[\frac{\lambda^2 - \mu^2}{(\lambda^2 + \mu^2)^2} \right] \Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa) \left\{ \frac{d}{du} h \right\}(u) \right)$$

$$+ (b_1^1 + b_1^2 u^2)\{\Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} + (b_2^1 + b_2^2 u^2)\{\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$

$$+ b_{1,1,2}\{\Delta_{x_1}^2\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} + + b_{1,2,2}\{\Delta_{x_1}\Delta_{x_2}^2\pi_{\lambda,\mu,\nu}(\kappa)h(u)\}$$

$$+ b_{1,1,1}\{\Delta_{x_1}^3\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} + b_{2,2,2}\{\Delta_{x_2}^3\pi_{\lambda,\mu,\nu}(\kappa)h(u)\},$$

where the difference operators $\Delta_{x_i|\pi_{\lambda,\mu,\nu}}$, $i = 1, 2, 3$, are as in Propositions 4.7 and 4.8, and

$$b_1^1 = \frac{\nu\lambda^2 - \nu\mu^2}{2(\lambda^2 + \mu^2)^2}, \quad b_1^2 = -\frac{3\lambda^6 + 9\mu^6 + 15\lambda^4\mu^2 + 21\lambda^2\mu^4}{6(\lambda^2 + \mu^2)^2}, \quad b_2^1 = \frac{\nu\lambda\mu}{(\lambda^2 + \mu^2)^2},$$

$$b_2^2 = \lambda\mu, \quad b_{1,1,2} = \frac{2\lambda^3\mu}{(\lambda^2 + \mu^2)^2}, \quad b_{1,2,2} = \frac{\mu^4 + 3\mu^2\lambda^2}{(\lambda^2 + \mu^2)^2},$$

$$b_{1,1,1} = \frac{2\lambda^4 - 2\lambda^2\mu^2}{3(\lambda^2 + \mu^2)^2}, \quad b_{2,2,2} = \frac{-4\lambda^3\mu}{3(\lambda^2 + \mu^2)^2}.$$

Proof. For κ as in the statement, and for $h \in \mathcal{S}(\mathbb{R})$, we differentiate with respect to μ the expression (27) at h. In particular,

$$
\frac{\partial}{\partial \mu}\{\pi_{\lambda,\mu,\nu}(\kappa)h(u)\} = \int_{\mathbb{R}^5} \kappa(x)\frac{\partial}{\partial \mu}\left[e^{iA^*} \cdot h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right)\right] dx
$$

$$
= \int_{\mathbb{R}_5} \kappa(x) \cdot e^{iA^*} \cdot \frac{\partial(iA^*)}{\partial \mu} \cdot h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx
$$

$$
+ \int_{\mathbb{R}_5} \kappa(x) \cdot e^{iA^*}\left[\frac{2\mu(\lambda x_1 + \mu x_2) - x_2(\lambda^2 + \mu^2)}{(\lambda^2 + \mu^2)^2}\right]\frac{d}{du}h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx
$$

$$
= I + II.
$$

Dealing with integral II, we see that

$$
II = \int_{\mathbb{R}_5} \kappa(x) \cdot e^{iA^*}\left\{x_1\frac{2\lambda\mu}{(\lambda^2 + \mu^2)^2} + x_2\frac{\mu^2 - \lambda^2}{(\lambda^2 + \mu^2)^2}\right\}\frac{d}{du}h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx
$$

$$
= \frac{2\lambda\mu}{(\lambda^2 + \mu^2)^2}\int_{\mathcal{B}_5} x_1\kappa(x) \cdot e^{iA^*} \cdot \frac{d}{du}h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx \tag{41}
$$

$$
+ \frac{\mu^2 - \lambda^2}{(\lambda^2 + \mu^2)^2}\int_{\mathcal{B}_5} x_2\kappa(x) \cdot e^{iA^*} \cdot \frac{d}{du}h\left(u - \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}\right) dx
$$

$$
= \left(\left[\frac{2\lambda\mu}{(\lambda^2 + \mu^2)^2}\right]\Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa)\left\{\frac{d}{du}h\right\} + \left[\frac{\mu^2 - \lambda^2}{(\lambda^2 + \mu^2)^2}\right]\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)\left\{\frac{d}{du}h\right\}\right)(u).
$$

For the integral I, we see that since

$$
\frac{\partial(iA^*)}{\partial_\mu} = i\left\{-x_5 + (b_1^1 + b_1^2 u^2)x_1 + (b_2^1 + b_2^2 u^2)x_2 + b_{1,1,2}x_1^2 x_2\right\}
$$

$$
+ i\left\{b_{1,2,2}x_1 x_2^2 + b_{1,1,1}x_1^3 + b_{2,2,2}x_2^3\right\}, \tag{42}
$$

then, arguing as in the case of the integral II in terms of difference operators, we get

$$
I = i\left[\left\{-\Delta_{x_5}\pi_{\lambda,\mu,\nu}(\kappa) + (b_1^1 + b_1^2 u^2)\Delta_{x_1}\pi_{\lambda,\mu,\nu}(\kappa) + (b_2^1 + b_2^2 u^2)\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa)\right\}h\right](u)
$$

$$
+ i\left[\left\{b_{1,1,2}\Delta_{x_1}^2\Delta_{x_2}\pi_{\lambda,\mu,\nu}(\kappa) + b_{1,2,2}\Delta_{x_1}\Delta_{x_2}^2\pi_{\lambda,\mu,\nu}(\kappa)\right\}h\right](u)
$$

$$
+ i\left[\left\{b_{1,1,1}\Delta_{x_1}^3\pi_{\lambda,\mu,\nu}(\kappa) + b_{2,2,2}\Delta_{x_2}^3\pi_{\lambda,\mu,\nu}(\kappa)\right\}h\right](u).
$$

where the b's are as in the statement, and the proof is complete. ∎

A. Canonical right-invariant vector fields on the Engel and Cartan groups

In the case of the Engel group $\mathcal{B}_4$ we have: For $f \in C^\infty(\mathcal{B}_4)$, the identification (11) allows us to calculate the canonical right-invariant vector fields as follows

$$\tilde{X}_1(x)f(x) = \frac{d}{dt}\Big|_{t=0} f\left(\left(t \cdot \exp_{\mathcal{B}_4} \tilde{X}_1\right) \times x\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\left((t,0,0,0) \times (x_1, x_2, x_3, x_4)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\left(t + x_1, x_2, x_3 - tx_2, x_4 + \frac{1}{2}t^2 x_2 - tx_3\right)$$

$$= \left(\frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4}\right) f(x).$$

Analogously one has,

$$\tilde{X}_2(x)f(x) = \frac{d}{dt}\Big|_{t=0} f\left(\left(t \cdot \exp_{\mathcal{B}_4} \tilde{X}_2\right) \times x\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\left((0,1,0,0) \times (x_1, x_2, x_3, x_4)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\left(x_1, x_2 + t, x_3, x_4\right)$$

$$= \left(\frac{\partial}{\partial x_2}\right) f(x),$$

whereas,
$$\tilde{X}_3(x) = \frac{\partial}{\partial x_3}, \quad \tilde{X}_4(x) = \frac{\partial}{\partial x_4}.$$

For the case of the Cartan group $\mathcal{B}_5$ the right-invariant vector fields act on the space $C^\infty(\mathcal{B}_5)$ by

$$\tilde{X}_1(x)f(x) = \frac{d}{dt}\Big|_{t=0} f\left(\left(t \cdot \exp_{\mathcal{B}_5} \tilde{X}_1\right) \times x\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\left((t,0,0,0,0) \times (x_1, x_2, x_3, x_4, x_5)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f(t + x_1, x_2, x_3 - tx_2, x_4 + \frac{1}{2}t^2 x_2 - tx_3, x_5 + \frac{1}{2}tx_2^2)$$

$$= \left(\frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} + \frac{x_2^2}{2}\frac{\partial}{\partial x_5}\right) f(x),$$

$$\tilde{X}_2(x)f(x) = \frac{d}{dt}\Big|_{t=0} f\left(\left(t \cdot \exp_{\mathcal{B}_5} \tilde{X}_2\right) \times x\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\left((0,t,0,0,0) \times (x_1, x_2, x_3, x_4, x_5)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f(x_1, x_2 + t, x_3, x_4, x_5 - tx_3)$$

$$= \left(\frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_5}\right) f(x),$$

whereas similarly one gets,

$$\tilde{X}_3(x) = \frac{\partial}{\partial x_3}, \quad \tilde{X}_4(x) = \frac{\partial}{\partial x_4}, \quad \text{and} \quad \tilde{X}_5(x) = \frac{\partial}{\partial x_5}.$$

B. Plancherel measure on the dual of the Engel group

The first part of the abstract Plancherel theorem on unimodular, type I Lie groups, obtained by Dixmier in ([6, Section 18.8]), is the Plancherel formula, which ensures the existence of a unique positive σ-finite measure $m = m(\pi)$, $\pi \in \widehat{\mathbb{G}}$, on $\widehat{\mathbb{G}}$, called the Plancherel measure, such that for any $f \in C_c(\mathbb{G})$ we have

$$\int_{\mathbb{G}} |f(x)|^2 dx = \int_{\widehat{\mathbb{G}}} \|\hat{f}(\pi)\|_{\mathrm{HS}(\mathcal{H}_\pi)}^2 dm(\pi),$$

where $\hat{f}(\pi)$ is an endomorphism on $\mathcal{H}_\pi$. The above formula shows that the group Fourier transform is a Hilbert-Schmidt operator, and in particular that it is an isometry from the space of smooth functions with compact support $C_c(\mathbb{G})$ endowed with the $L^2(\mathbb{G})$ norm, to the Hilbert space

$$L^2(\widehat{\mathbb{G}}) := \int_{\hat{G}}^{\oplus} \mathrm{HS}(\mathcal{H}_\pi) dm(\pi),$$

introduced by Dixmier in [6, Part II, Ch. I], of fields of Hilbert-Schmidt operators that are square integrable in the above sense.

In the next proposition we obtain a concrete formula for the Plancherel formula in the setting of the Engel group, and this implies specifying the Plancherel measure on the dual of the group. The Plancherel measure in this case has been calculated by Dixmier in [6] using other methods. Here we give a straightforward proof.

Proposition B.1 (Plancherel formula in $\mathcal{B}_4$). *Let $f \in \mathcal{S}(\mathcal{B}_4)$. Then for each $\lambda \in \mathbb{R} \setminus \{0\}, \mu \in \mathbb{R}$, the operator $\hat{f}(\pi_{\lambda,\mu})$ acting on $L^2(\mathbb{R})$ is the Hilbert-Schmidt operator with integral kernel $\mathcal{K}_{f,\lambda,\mu} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$, given by*

$$\mathcal{K}_{f,\lambda,\mu}(u,v) = (2\pi)\mathcal{F}_{\mathbb{R}^3}(f)(u - v, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda),$$

and Hilbert-Schmidt norm

$$\|\hat{f}(\pi_{\lambda,\mu})\|_{HS(L^2(\mathbb{R}))} = (2\pi)^3 \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}^3}(f)(u - v, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)|^2 \, du \, dv.$$

In addition, we have

$$\int_{\mathcal{B}_4} |f(x_1, x_2, x_3, x_4)|^2 \, dx_1 \, dx_2 \, dx_3 \, dx_4 = 2^{-2}\pi^{-3} \cdot \int_{\lambda \in \mathbb{R} \setminus \{0\}} \int_{\mu \in \mathbb{R}} \|\hat{f}(\pi_{\lambda,\mu})\|_{HS(L^2(\mathbb{R}))}^2 \, d\mu \, d\lambda.$$

Furthermore, we have that the Plancherel measure $dm(\pi_{\lambda,\mu})$ on $\widehat{\mathcal{B}}_4$ is the Lebesgue measure on $\mathbb{R}^2$, namely $dm(\pi_{\lambda,\mu}) \equiv 2^{-2}\pi^{-3} \, d\lambda \, d\mu$.

Proof. By the computations as in Proposition 3.3, we can write for $h \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$,

$$\hat{f}(\pi_{\lambda,\mu})h(u) = (2\pi) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^4}(f)(\xi, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)h(v) \, d\xi \, dv$$

$$= \int_{\mathbb{R}} \mathcal{K}_{f,\lambda,\mu}(u,v)h(v) \, dv,$$

where $\mathcal{K}_{f,\lambda,\mu}(u,v)$ is the integral kernel of $\hat{f}(\pi_{\lambda,\mu})$ hence given by

$$\mathcal{K}_{f,\lambda,\mu}(u,v) = 2\pi \int_{\mathbb{R}} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^4}(f)(\xi, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)\, d\xi.$$

Using the properties of the Euclidean Fourier transform we may rewrite this as

$$\mathcal{K}_{f,\lambda,\mu}(u,v) = (2\pi)^{\frac{3}{2}} \mathcal{F}_{\mathbb{R}^3}(f)(u-v, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda),$$

where the Fourier transform above is taken with respect to the second, the third and the fourth variable of f.

The $L^2(\mathbb{R} \times \mathbb{R})$-norm of the integral kernel is

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{K}_{f,\lambda,\mu}(u,v)|^2\, du\, dv = (2\pi)^3 \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}^3}(f)(u-v, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)|^2\, du\, dv\,,$$

where since $f \in \mathcal{S}(\mathcal{B}_4)$, the last quantity is finite. Hence the operator $\hat{f}(\pi_{\lambda,\mu})$ is Hilbert-Schmidt and its Hilbert-Schmidt norm is the exactly the above quantity. This shows the first part of the statement. We now integrate each side of the last equality against $d\lambda, d\mu$ and obtain

$$\int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{K}_{f,\lambda,\mu}(u,v)|^2\, du\, dv\, d\mu\, d\lambda$$

$$= (2\pi)^3 \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}^3}(f)(u-v, \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, -\lambda v, \lambda)|^2\, du\, dv\, d\mu\, d\lambda$$

$$= (2\pi)^3 \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}^3}(f)(x_1, w_2, w_3, w_4)|^2 det(J_F(u,v,\lambda,\mu))\, dx_1\, dw_2\, dw_3\, dw_4$$

$$= 2^2\pi^3 \cdot \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_{\mathbb{R}^3}(f)(x_1, w_2, w_3, w_4)|^2\, dw_2\, dw_3\, dw_4\, dx_1\,,$$

where $det(J_F(u,v,\lambda,\mu)) = \frac{1}{2}$ is the determinant of the Jacobian matrix of the transformation $F(u,v,\lambda,\mu) = (x_1 = u-v, w_2 = \frac{\lambda}{2}v^2 - \frac{\mu}{2\lambda}, w_3 = -\lambda v, w_4 = \lambda)$. Now, using the Euclidean Plancherel formula on $\mathbb{R}^3$ in the variable (w_2, w_3, w_4) with dual variable (x_2, x_3, x_4), we have

$$\int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}} |\mathcal{K}_{f,\lambda,\mu}(u,v)|^2\, dx_1\, du\, dv\, d\mu\, d\lambda$$

$$= 2^2\pi^3 \cdot \int_{\mathbb{R}^4} |f(x_1, x_2, x_3, x_4)|^2\, dx_1\, dx_2\, dx_3\, dx_4\,,$$

and this completes the proof of Proposition B.1. ∎

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Marianna Chatzakou, Department of Mathematics, Imperial College London, UK;
    m.chatzakou16@imperial.ac.uk.