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Conformal Vector Fields on Lie Groups of Dimension 4 with Signature of (2,2)

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Abstract. We classify Lie groups of dimension 4 with signature of (2,2) which admit non-Killing left-invariant conformal vector fields.

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1. Introduction

Let $(M, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian manifold. A *conformal vector field* X on M is a vector field satisfying

$$\mathfrak{L}_X\langle\cdot,\cdot\rangle = 2\rho\langle\cdot,\cdot\rangle,$$

where $\mathfrak{L}_X\langle\cdot,\cdot\rangle$ is the Lie derivative and ρ is a smooth function. It connects with the topological structure of the pseudo-Riemannian manifold [8, 13]. An important class of conformal vector fields are *Killing vector fields*, i.e., vector fields such that $\mathfrak{L}_X\langle\cdot,\cdot\rangle = 0$. Killing vector fields provide a close link between the geometry of a manifold M and the Lie algebra of $I(M)$, which is the set of all isometries in M (see [14]).

There are many studies on homogeneous pseudo-Riemannian manifolds of dimensions 3 and 4. In dimension 3, similar to the result for the Riemannian case in [15], Calvaruso proved that any non-symmetric three-dimensional homogeneous Lorentzian manifold is isometric to a Lorentzian Lie group [3]. The above result also holds for four dimensional Riemannian Lie groups [2]. In [6], Calvaruso and Zaeim proved that for most of the Segre types of the Ricci operator, a four-dimensional locally homogeneous Lorentzian manifold is either Ricci-parallel, or locally isometric to a four-dimensional Lorentzian Lie group. Moreover, they classified four dimensional Einstein Lorentzian Lie groups and described four dimensional Lorentzian Lie groups (see [5]). Four-dimensional homogeneous non-reductive pseudo-Riemannian manifolds, both Lorentzian and of neutral signature, were classified in [10], and an explicit description of their invariant homogeneous metrics was obtained in [4]. Besides, Einstein-like metrics and Ricci solitons were investigated on four-dimensional Pseudo-Riemannian Lie groups of signature $(2,2)$ (see [11, 12]).

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It is well-known that any left-invariant conformal vector field on a Riemannian Lie group is Killing. And, it is proved in [1] that any left-invariant conformal vector field on a unimodular pseudo-Riemannian Lie group is a Killing vector field. Also in [1], there is a non-Killing left-invariant conformal vector field on a Lorentzian Lie group of dimension 4. Recently, Z. Chen and the authors have classified all Lorentzian Lie groups of dimension 4 admitting left-invariant non-Killing conformal vector fields [9]. It is then natural to consider the case of Lie groups of dimension 4 with signature of (2,2).

In this paper, we classify Lie groups of dimension 4 with signature of (2,2) admitting left-invariant non-Killing conformal vector fields.

Theorem 1.1. *Let G be a four dimensional Lie group with signature of $(2,2)$ with the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ admits a non-Killing left-invariant conformal vector field, and dim$[\mathfrak{g},\mathfrak{g}]=2$, then there is a basis $\{e_1,e_2,e_3,e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g},\mathfrak{g}]$ is $\{e_1,e_2\}$, the metric associated with the basis $\{e_1,e_2,e_3,e_4\}$ is defined by*

$$\mathtt{A}=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{1}$$

and the non-zero brackets are one of the following cases:

Case 1: $[e_1,e_3]=ae_1+be_2$, $[e_1,e_4]=(a\lambda_1+\lambda_2)e_1+b\lambda_1e_2$, $[e_2,e_3]=me_1+ne_2$, $[e_2,e_4]=m\lambda_1e_1+(\lambda_2+n\lambda_1)e_2$, *where* $a,b,m,n,\lambda_1\in\mathbb{R},\lambda_2\neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $-\lambda_1e_3+e_4$.

Case 2: $[e_1,e_3]=\lambda_1e_1$, $[e_2,e_3]=\lambda_1e_2$, $[e_1,e_4]=\lambda_2e_1+\lambda_3e_2$, $[e_2,e_4]=\lambda_4e_1+\lambda_5e_2$, *where* $\lambda_j\in\mathbb{R},1\le j\le 5,\lambda_1\neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* e_3.

Case 3: $[e_3,e_1]=ae_1+be_2$, $[e_4,e_1]=(-\lambda_2+a\lambda_1)e_1+b\lambda_1e_2$, $[e_3,e_2]=me_1+ne_2$, $[e_4,e_2]=m\lambda_1e_1+(n\lambda_1-\lambda_2)e_2$, $[e_4,e_3]=\lambda_3(a+m\lambda_1)e_1+\lambda_3(b+n\lambda_1-\lambda_2)e_2$, *where* $a,b,m,n,\lambda_j\in\mathbb{R},1\le j\le 3,\lambda_2\lambda_3\neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda_3e_1+\lambda_3\lambda_1e_2-\lambda_1e_3+e_4$.

Case 4: $[e_3,e_1]=\lambda_1e_1$, $[e_4,e_1]=\lambda_3e_1+\lambda_4e_2$, $[e_3,e_2]=\lambda_1e_2$, $[e_4,e_2]=\lambda_5e_1+\lambda_6e_2$, $[e_4,e_3]=-\lambda_2(\lambda_1+\lambda_5)e_1-\lambda_2\lambda_6e_2$, *where* $\lambda_j\in\mathbb{R},1\le j\le 6,\lambda_1\lambda_2\neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda_2e_2+e_3$.

For dim$[\mathfrak{g},\mathfrak{g}]=3$, we have the following theorem.

Theorem 1.2. *Let G be a four dimensional Lie group with signature of $(2,2)$ with the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ admits a non-Killing left-invariant conformal vector field, and dim$[\mathfrak{g},\mathfrak{g}]=3$, then there is a basis $\{e_1,e_2,e_3,e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g},\mathfrak{g}]$ is $\{e_1,e_2,e_3\}$, the metric associated with the basis $\{e_1,e_2,e_3,e_4\}$ is defined by*

$$\mathtt{B}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2}$$

and the non-zero brackets are one of the following cases:

Case 1: $[e_1, e_2] = \lambda e_3$, $[e_1, e_4] = [e_2, e_4] = \lambda_1 e_1 + \lambda_1 e_2 + \lambda\lambda_2 e_3$, $[e_3, e_4] = 2\lambda_1 e_3$, *where* $\lambda\lambda_1\lambda_2 \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda_2 e_1 - \lambda_2 e_2 + e_4$.

Case 2: $[e_1, e_2] = \lambda e_3$, $[e_1, e_4] = \lambda_1 e_1 - \lambda_1 e_2 - \lambda\lambda_2 e_3$, $[e_2, e_4] = -\lambda_1 e_1 + \lambda_1 e_2 + \lambda\lambda_2 e_3$, $[e_3, e_4] = 2\lambda_1 e_3$, *where* $\lambda\lambda_1\lambda_2 \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda_2 e_1 + \lambda_2 e_2 + e_4$.

Case 3: $[e_1, e_2] = \lambda e_3$, $[e_1, e_4] = \lambda_1 e_1 - \lambda_2 e_2$, $[e_2, e_4] = -\lambda_2 e_1 + \lambda_1 e_2$, $[e_3, e_4] = 2\lambda_1 e_3$, *where* $\lambda\lambda_1\lambda_2 \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* e_4.

Case 4: $[e_1, e_2] = -\frac{\lambda_1^2}{\lambda_2} e_3 - \lambda_1 e_2 + \lambda_1 e_1$, $[e_1, e_3] = [e_2, e_3] = \lambda_1 e_3 + \lambda_2 e_2 - \lambda_2 e_1$, $[e_4, e_1] = -[e_4, e_2] = (-\lambda_4 + \lambda_1\lambda_3) e_1 + (\lambda_4 - \lambda_1\lambda_3) e_2 - \frac{\lambda_1^2\lambda_3}{\lambda_2} e_3$, $[e_4, e_3] = 2\lambda_2\lambda_3 e_1 - 2\lambda_2\lambda_3 e_2 - (2\lambda_4 + 2\lambda_1\lambda_3) e_3$, *where* $\lambda_j \in \mathbb{R}, 1 \leq j \leq 4, \lambda_2\lambda_4 \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda_3(e_1 + e_2) + e_4$.

Case 5: $[e_1, e_2] = -\frac{\lambda_1^2}{\lambda_2} e_3 - \lambda_1 e_2 - \lambda_1 e_1$, $[e_1, e_3] = -[e_2, e_3] = \lambda_1 e_3 + \lambda_2 e_2 + \lambda_2 e_1$, $[e_4, e_1] = [e_4, e_2] = (-\lambda_4 + \lambda_1\lambda_3) e_1 + (-\lambda_4 + \lambda_1\lambda_3) e_2 + \frac{\lambda_1^2\lambda_3}{\lambda_2} e_3$, $[e_4, e_3] = -2\lambda_2\lambda_3 e_1 - 2\lambda_2\lambda_3 e_2 - (2\lambda_4 + 2\lambda_1\lambda_3) e_3$, *where* $\lambda_j \in \mathbb{R}, 1 \leq j \leq 4, \lambda_2\lambda_4 \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda_3(e_1 - e_2) + e_4$.

Case 6: $[e_1, e_2] = \lambda e_3$, $[e_1, e_4] = \lambda_1 e_1$, $[e_2, e_4] = \lambda_1 e_2, [e_3, e_4] = 2\lambda_1 e_3$, *where* $\lambda\lambda_1 \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* e_4.

Case 7: $[e_1, e_2] = \lambda e_3$, $[e_1, e_4] = \lambda_1 e_1 - \lambda_2 e_2 + (\frac{-\lambda_2\lambda_3}{\lambda_1} - \lambda_4) e_3$, $[e_2, e_4] = -\lambda_2 e_1 + \lambda_1 e_2 + \lambda_3 e_3$, $[e_3, e_4] = 2\lambda_1 e_3$. *For this case, the non-Killing conformal vector field is the constant multiple of* $e_1 + \frac{\lambda_2}{\lambda_1} e_2 + \frac{\lambda_4}{2\lambda_1} e_3 + \frac{\lambda}{\lambda_3} e_4$.

Case 8: $[e_1, e_2] = \lambda e_3$, $[e_1, e_4] = \lambda_1 e_1 - \lambda_2 e_3$, $[e_2, e_4] = \frac{\lambda\lambda_2}{\lambda_1} e_3 + \lambda_1 e_2$, $[e_3, e_4] = 2\lambda_1 e_3$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\frac{\lambda_2 e_1}{\lambda_1} + \frac{\lambda_2^2 e_3}{2\lambda_1^2} + e_4$

Case 9: $[e_1, e_4] = \alpha e_1 + \beta e_2 + \gamma e_3$, $[e_2, e_4] = \beta e_1 + \alpha e_2 + n e_3$, $[e_3, e_4] = 2\alpha e_3$, *where* $\alpha \neq 0, \beta, \gamma, n \in \mathbb{R}$. *For this case, the non-Killing conformal vector field is the constant multiple of* $-2(\gamma\alpha + n\beta) e_1 + 2(\alpha n + \gamma\beta) e_2 + (\gamma^2 - n^2) e_3 + 2(\alpha^2 - \beta^2) e_4$.

Case 10: $[e_1, e_4] = \alpha e_1 + \alpha e_2$, $[e_2, e_4] = \alpha e_1 + \alpha e_2$, $[e_3, e_4] = 2\alpha e_3$, *where* $\alpha \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda e_1 - \lambda e_2 + e_4, \lambda \in \mathbb{R}$.

Case 11: $[e_1, e_4] = \alpha e_1 + \alpha e_2 + \gamma e_3$, $[e_2, e_4] = \alpha e_1 + \alpha e_2 - \gamma e_3$, $[e_3, e_4] = 2\alpha e_3$, *where* $\alpha\gamma \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $(2\alpha^2\lambda + \gamma^2) e_1 + (-2\alpha^2\lambda + \gamma^2) e_2 - 2\alpha\gamma\lambda e_3 - 2\alpha\gamma e_4, \lambda \in \mathbb{R}$.

Case 12: $[e_1, e_4] = \alpha e_1 - \alpha e_2$, $[e_2, e_4] = -\alpha e_1 + \alpha e_2$, $[e_3, e_4] = 2\alpha e_3$, *where* $\alpha \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $\lambda e_1 + \lambda e_2 + e_4, \lambda \in \mathbb{R}$.

Case 13: $[e_1, e_4] = \alpha e_1 - \alpha e_2 + \gamma e_3, [e_2, e_4] = -\alpha e_1 + \alpha e_2 + \gamma e_3$, $[e_3, e_4] = 2\alpha e_3$, *where* $\alpha\gamma \neq 0$. *For this case, the non-Killing conformal vector field is the constant multiple of* $(-2\alpha^2\lambda - \gamma^2) e_1 + (-2\alpha^2\lambda + \gamma^2) e_2 + 2\alpha\gamma\lambda e_3 + 2\alpha\gamma e_4, \lambda \in \mathbb{R}$.

2. Preliminaries

Let G be a Lie group with the Lie algebra $\mathfrak{g}$ and let $\langle\cdot,\cdot\rangle$ be a left-invariant pseudo-Riemannian metric on G. Assume that ∇ is the Levi-Civita connection associated with $\langle\cdot,\cdot\rangle$. Then,

$$[X,Y] = \nabla_X Y - \nabla_Y X. \tag{3}$$

For a left-invariant metric $\langle\cdot,\cdot\rangle$ on G, we have

$$\langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle = 0, \tag{4}$$

for any $X, Y, Z \in \mathfrak{g}$. By (3) and (4),

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(\langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle),$$

where $X, Y, Z \in \mathfrak{g}$. Assume that $X \in \mathfrak{g}$ is a conformal vector field, i.e.

$$\mathfrak{L}_X \langle\cdot,\cdot\rangle = 2\rho\langle\cdot,\cdot\rangle. \tag{5}$$

It follows that, $0 = \mathfrak{L}_X\langle X, X\rangle = 2\rho|X|^2$.

If $\langle\cdot,\cdot\rangle$ is a left-invariant Riemannian metric, then $\rho = 0$ or $X \equiv 0$. That is, X is Killing or X is trivial. For this reason, we focus on a left-invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$.

Lemma 2.1 ([1]). *Let notations be as above. If $X \in \mathfrak{g}$ is a non-Killing conformal vector field, then X is a lightlike vector field, i.e., $\langle X, X\rangle = 0$.*

Lemma 2.2 ([1]). *Let G be an unimodular pseudo-Riemannian Lie group. Then any left-invariant conformal vector field on G is a Killing vector field.*

For non-unimodular pseudo-Riemannian Lie groups, we have the following result.

Lemma 2.3 ([1]). *Let G be a non-unimodular pseudo-Riemannian Lie group. If G admits a non-Killing left-invariant conformal vector field, then*

$$\dim C(\mathfrak{g}) \le \min(p,q), \quad \dim[\mathfrak{g},\mathfrak{g}] \ge \dim\mathfrak{g} - \min(p,q).$$

Here $\mathfrak{g}$ is the Lie algebra of G and (p,q) is the signature of the pseudo-Riemannian metric.

Also, there are non-Killing left-invariant conformal vector fields on non-unimodular Lorentzian Lie groups [1, 7]. Let G be a non-unimodular Lie group of dimension 4 with the Lie algebra $\mathfrak{g}$ and let $<,>$ be a left-invariant metric of signature (2,2) on G. Assume that $\mathfrak{g}$ admits a non-Killing conformal vector field X.

Lemma 2.4. *Let notations be as above. Then the restriction of $<,>$ on $[\mathfrak{g},\mathfrak{g}]$ is degenerate.*

Proof. From Lemma 2.3, $\dim[\mathfrak{g},\mathfrak{g}] \ge \dim\mathfrak{g} - \min(p,q)$, since the dimension of $\mathfrak{g}$ is 4, and signature is (2,2), then $\dim[\mathfrak{g},\mathfrak{g}] \ge 2$. When $\dim[\mathfrak{g},\mathfrak{g}]$=4, $\mathfrak{g}$ is unimodular, by Lemma 2.2, there does not exist any non-Killing conformal vector field. Thus, $\dim[\mathfrak{g},\mathfrak{g}]$=2 or 3. To see the restriction of $<,>$ on $[\mathfrak{g},\mathfrak{g}]$ is degenerate, we use method of *proof by contradiction.*

Assume that the restriction of $<,>$ on $[\mathfrak{g},\mathfrak{g}]$ is non-degenerate, first, we claim that the restriction of $<,>$ on orthogonal complement $[\mathfrak{g},\mathfrak{g}]^{\perp}$ of $[\mathfrak{g},\mathfrak{g}]$ is also non-degenerate.
To see this, if $w \in [\mathfrak{g},\mathfrak{g}] \cap [\mathfrak{g},\mathfrak{g}]^{\perp}$, then $< w,w >= 0$, and we get $w = 0$. So $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus [\mathfrak{g},\mathfrak{g}]^{\perp}$. If $x \in [\mathfrak{g},\mathfrak{g}]^{\perp}$, $< x,y >= 0$, for any $y \in [\mathfrak{g},\mathfrak{g}]^{\perp}$, then $< x,z >= 0$, for any $z \in \mathfrak{g}$. Since the metric $<,>$ is non-degenerate on the Lie algebra $\mathfrak{g}$, we have $x = 0$. Thus, the restriction of $<,>$ on the orthogonal complement $[\mathfrak{g},\mathfrak{g}]^{\perp}$ of $[\mathfrak{g},\mathfrak{g}]$ is also non-degenerate. Notice $\dim[\mathfrak{g},\mathfrak{g}]$=2 or 3, so $\dim[\mathfrak{g},\mathfrak{g}]^{\perp}$ = 2 or 1. Then we can find one vector e_1 which satisfies $< e_1, e_1 > \neq 0$, here $e_1 \in [\mathfrak{g},\mathfrak{g}]^{\perp}$.
On the other hand, by definition of conformal vector field X, we have

$$\mathfrak{L}_X < e_1, e_1 >= - < [X, e_1], e_1 > - < e_1, [X, e_1] >= 2\rho < e_1, e_1 > .$$

Notice e_1 is from the space $[\mathfrak{g},\mathfrak{g}]^{\perp}$, we have $\rho < e_1, e_1 >= 0$. However, $\rho \neq 0$, (otherwise, X is Killing vector field), so we get $< e_1, e_1 > = 0$. It is a contradiction. Thus, the restriction of $<,>$ on $[\mathfrak{g},\mathfrak{g}]$ is degenerate. ■

Lemma 2.5. *Any 4-dimensional Lie group G with signature of (2,2) admitting a non-Killing left invariant conformal vector field is solvable.*

Proof. Otherwise, the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ is $s \times r$. Here s is the 3-dimensional simple Lie subalgebra and r is the 1-dimensional radical of $\mathfrak{g}$. Then $\mathfrak{g}^{\mathbb{C}}$ is unimodular. That is, $\mathfrak{g}$ is unimodular. Then the lemma follows from Lemma 2.2. ■

3. Proof of Theorem 1.1

Let G be a non-unimodular Lie group of dimension 4 with the Lie algebra $\mathfrak{g}$ and let $<,>$ be a left-invariant metric of signature (2,2) on G. Assume that $\mathfrak{g}$ admits a non-Killing conformal vector field X. By Lemma 2.3, $\dim[\mathfrak{g},\mathfrak{g}] \geq \dim \mathfrak{g} - 2$. If $\dim[\mathfrak{g},\mathfrak{g}] = \dim \mathfrak{g}$, it implies $\mathfrak{g}$ is unimodular. By Lemma 2.2, $\dim[\mathfrak{g},\mathfrak{g}] = 2$ or 3.

Lemma 3.1. *Let the notations as above, if $\dim[g,g] = 2$, then there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g},\mathfrak{g}]$ is $\{e_1, e_2\}$ and the metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by* $\mathtt{A}$ (*see equation* (1)).

Proof. If $\dim[\mathfrak{g},\mathfrak{g}] = 2$, by Lemma 2.4, the restriction of the metric $<,>$ on the derived algebra $[\mathfrak{g},\mathfrak{g}]$ is degenerate, then the restriction of metric $<,>$ on the basis $\{e_1, e_2\}$ of $[\mathfrak{g},\mathfrak{g}]$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the signature of the left invariant metric $<,>$ on the four dimensional Lie group G is (2,2), the Lie algebra $\mathfrak{g}$ can be viewed as a direct sum of two Lorentzian subspaces. Thus, it is easy to see that there exists a basis e_1, e_2, e_3, e_4 of $\mathfrak{g}$ such that the basis of $[\mathfrak{g},\mathfrak{g}]$ is $\{e_1, e_2\}$ and the metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by

$$\mathtt{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{or} \quad \mathtt{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6}$$

(In fact, when the metric is $\mathtt{C}$, $\{e_1, e_4\}$ and $\{e_2, e_3\}$ are two Lorentzian subspaces. When the metric is $\mathtt{A}$, $\{e_1, e_3\}$ and $\{e_2, e_4\}$ are two Lorentzian subspaces.)

Assume that the metric is $\mathtt{C}$ and suppose that with respect to the basis $\{e_1, e_2, e_3, e_4\}$, the matric of adX is

$$\mathtt{D} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

since

$$\mathfrak{L}_X < e_i, e_j >= - < [X, e_i], e_j > - < e_i, [X, e_j] >= 2\rho < e_i, e_j > . \tag{8}$$

By some calculations, we obtain: $a_{11} = 0$, $a_{12} = 0$, $a_{14} = 0$, $a_{21} = -a_{13}$, $a_{22} = 0$, $a_{23} = 0$, $a_{24} = 0$, $\rho = 0$. So it is a contradiction to the assumption that $\mathfrak{g}$ admits a non-killing conformal vector field. Thus, if $\dim[\mathfrak{g}, \mathfrak{g}]$=2, there exists a basis e_1, e_2, e_3, e_4 of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] = span\{e_1, e_2\}$ and associated with this basis, the metric is defined by $\mathtt{A}$. ■

Now we can give the proof of the first main result of this paper.

Proof of Theorem 1.1. If $\dim[\mathfrak{g},\mathfrak{g}]$=2, by Lemma 3.1, there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g}, \mathfrak{g}]$ is $\{e_1, e_2\}$ and the metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by $\mathtt{A}$, while the matrix of adX is $\mathtt{D}$.

By Lemma 3.1, we can obtain: $a_{12} = 0$, $a_{13} = 0$, $a_{21} = 0$, $a_{22} = a_{11}$, $a_{23} = -a_{14}$, $a_{24} = 0$, $\rho = -\frac{a_{11}}{2}$. If we denote a_{11} by λ, a_{14} by η, then the matrix of adX with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is:

$$\begin{pmatrix} \lambda & 0 & 0 & \eta \\ 0 & \lambda & -\eta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the nilpotent Lie group of dimension 2 is abelian, by Lemma 2.5, it is easy to see $[e_1, e_2] = 0$. Set $X = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$, we have

$$[X, e_1] = x_3[e_3, e_1] + x_4[e_4, e_1] = \lambda e_1, \tag{9}$$
$$[X, e_2] = x_3[e_3, e_2] + x_4[e_4, e_2] = \lambda e_2, \tag{10}$$
$$[X, e_3] = x_1[e_1, e_3] + x_2[e_2, e_3] + x_4[e_4, e_3] = -\eta e_2, \tag{11}$$
$$[X, e_4] = x_1[e_1, e_4] + x_2[e_2, e_4] + x_3[e_3, e_4] = \eta e_1. \tag{12}$$

The eigenvalue of adX on $[\mathfrak{g}, \mathfrak{g}]$ is λ, and by the Jacobi identity, we obtain

$$adX([e_1, e_3]) = \lambda[e_1, e_3], \quad adX([e_1, e_4]) = \lambda[e_1, e_4], \tag{13}$$
$$adX([e_2, e_3]) = \lambda[e_2, e_3], \quad adX([e_2, e_4]) = \lambda[e_2, e_4], \tag{14}$$
$$adX([e_3, e_4]) = \eta[e_4, e_2] + \eta[e_3, e_1] = \lambda[e_3, e_4]. \tag{15}$$

From the definition of conformal vector field, we have

$$- < [X, e_i], X >= 2\rho < e_i, X >, \quad i = 1, 2, 3, 4. \tag{16}$$

Notice $< X, X >= 0$, we obtain:

$$\eta x_4 = 2\rho x_1, \quad -\eta x_3 = 2\rho x_2, \quad x_1x_3 + x_2x_4 = 0.$$

Case A: If $\eta = 0$, it is easy to see: $x_1 = x_2 = 0$, $[e_3, e_4] = 0$. Set

$$[e_1, e_3] = ae_1 + be_2, \quad [e_1, e_4] = ce_1 + de_2,$$
$$[e_2, e_3] = me_1 + ne_2, \quad [e_2, e_4] = we_1 + ve_2,$$

here $a, b, c, d, m, n, w, v \in \mathbb{R}$. From

$$x_3[e_3, e_1] + x_4[e_4, e_1] = \lambda e_1, \quad \text{and} \quad x_3[e_3, e_2] + x_4[e_4, e_2] = \lambda e_2,$$

we obtain:
$$\begin{cases} -ax_3 - cx_4 &= \lambda, \\ -bx_3 - dx_4 &= 0, \\ -mx_3 - wx_4 &= 0, \\ -nx_3 - vx_4 &= \lambda. \end{cases}$$

When $x_4 \neq 0$, the solution is $\frac{mx_3}{x_4}$ and $v = \frac{-\lambda - nx_3}{x_4}$.
And we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = ae_1 + be_2, \quad [e_1, e_4] = \frac{-ax_3 - \lambda}{x_4} e_1 + \frac{-bx_3}{x_4} e_2,$$
$$[e_3, e_4] = 0, \quad [e_2, e_3] = me_1 + ne_2, \quad [e_2, e_4] = \frac{-mx_3}{x_4} e_1 + \frac{-nx_3 - \lambda}{x_4} e_2.$$

Replacing $-\frac{x_3}{x_4}$ and $-\frac{\lambda}{x_4}$ by λ_1 and λ_2, we have Case 1 of Theorem 1.1.

When $x_4 = 0$, the solution is $a = n = -\frac{\lambda}{x_3}$ and $b = m = 0$. And we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{\lambda}{x_3} e_1, \quad [e_1, e_4] = ce_1 + de_2,$$
$$[e_3, e_4] = 0, \quad [e_2, e_3] = -\frac{\lambda}{x_3} e_2, \quad [e_2, e_4] = we_1 + ve_2.$$

Replacing $-\frac{\lambda}{x_3}, c, d, w, v$ by $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, we have the Case 2 of Theorem 1.1.

Case B: If $\eta \neq 0$, it is easy to see: $x_3 = -\frac{2\rho x_2}{\eta} = \frac{\lambda}{\eta} x_2$, $x_4 = \frac{2\rho x_1}{\eta} = -\frac{\lambda}{\eta} x_1$. Set

$$[e_3, e_1] = \vec{a}, \quad [e_4, e_1] = \vec{b}, \quad [e_3, e_2] = \vec{c}, \quad [e_4, e_2] = \vec{d}, \quad [e_4, e_3] = \vec{q}.$$

Here $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{q}$ belong to the subspace which is spanned by e_1 and e_2.
From (9), (10), (11), (12) and (15), we have

$$\begin{cases} \vec{a}x_3 + \vec{b}x_4 &= \lambda e_1, \\ \vec{c}x_3 + \vec{d}x_4 &= \lambda e_2, \\ \vec{d}x_1 - \vec{c}x_2 &= -\eta e_2, \\ \vec{a}x_2 - \vec{b}x_1 &= \eta e_1. \end{cases}$$

Further, we set

$$[e_3, e_1] = ae_1 + be_2, \quad [e_4, e_1] = ce_1 + de_2, \quad [e_3, e_2] = me_1 + ne_2, \quad [e_4, e_2] = ge_1 + he_2,$$

where $a, b, c, d, m, n, g, h \in \mathbb{R}$.

We obtain:

$$\begin{cases} ax_3 + cx_4 &= \lambda, \\ bx_3 + dx_4 &= 0, \\ mx_3 + gx_4 &= 0, \\ nx_3 + hx_4 &= \lambda, \\ gx_1 - mx_2 &= 0, \\ hx_1 - nx_2 &= -\eta, \\ ax_2 - cx_1 &= \eta, \\ bx_2 - dx_1 &= 0. \end{cases}$$

Note that $x_1 = \frac{-\eta x_4}{\lambda}$, $x_2 = \frac{\eta x_3}{\lambda}$ when $x_4 \neq 0$. The solution of the above equations is

$$c = \frac{\lambda - ax_3}{x_4}, \quad d = -\frac{bx_3}{x_4}, \quad g = -\frac{mx_3}{x_4}, \quad h = \frac{\lambda - nx_3}{x_4}.$$

Replacing $-\frac{x_3}{x_4}, -\frac{\lambda}{x_4}, -\frac{\eta}{\lambda}$ by $\lambda_1, \lambda_2, \lambda_3$, we have

$$\begin{aligned} &[e_1, e_2] = 0, \quad [e_3, e_1] = ae_1 + be_2, \quad [e_4, e_1] = (-\lambda_2 + a\lambda_1)e_1 + b\lambda_1 e_2, \\ &[e_3, e_2] = me_1 + ne_2, \quad [e_4, e_2] = m\lambda_1 e_1 + (n\lambda_1 - \lambda_2)e_2, \\ &[e_4, e_3] = \lambda_3(a + m\lambda_1)e_1 + \lambda_3(b + n\lambda_1 - \lambda_2)e_2. \end{aligned}$$

So we have the Case 3 of Theorem 1.1.

When $x_4 = 0$, the solution of the above equations is

$$b = e = 0, \quad a = f = \frac{\lambda}{x_3}.$$

We denote $\frac{\lambda}{x_3}, \frac{\eta}{\lambda}, c, d, g, h$ by $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$, we have

$$\begin{aligned} &[e_1, e_2] = 0, \quad [e_3, e_1] = \lambda_1 e_1, \quad [e_4, e_1] = \lambda_3 e_1 + \lambda_4 e_2, \\ &[e_3, e_2] = \lambda_1 e_2, \quad [e_4, e_2] = \lambda_5 e_1 + \lambda_6 e_2, \\ &[e_4, e_3] = -\lambda_2(\lambda_1 + \lambda_5)e_1 - \lambda_2\lambda_6 e_2. \end{aligned}$$

So we have the Case 4 of Theorem 1.1. This completes the proof. ■

4. Proof of Theorem 1.2

If $\dim[\mathfrak{g}, \mathfrak{g}] = 3$, from Lemma 2.4, the restriction of the metric on the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is degenerate. And it is easy to see that there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g}, \mathfrak{g}]$ is $\{e_1, e_2, e_3\}$ and metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{17}$$

Suppose that the matrix of adX with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_{ij} \in \mathbb{R}, \ 1 \leq i \leq 3, \ 1 \leq j \leq 4.$$

By (8), it is easy to see that the matrix of adX is

$$\begin{pmatrix} -\rho & a_{12} & 0 & a_{14} \\ a_{12} & -\rho & 0 & a_{24} \\ a_{14} & -a_{24} & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By a suitable change of bases, we can get a new basis $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}, \tilde{e_4}\}$ of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] = span\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$, the matrix of the metric is unchanged associated with the new basis, while the matrix of adX is

$$\begin{pmatrix} -\rho & a_{12} & 0 & a_{14} \\ a_{12} & -\rho & 0 & 0 \\ a_{14} & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

We still denote $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}, \tilde{e_4}\}$ by $\{e_1, e_2, e_3, e_4\}$.

Proof of Theorem 1.2. If $dim[\mathfrak{g}, \mathfrak{g}] = 3$, by Lemma 2.5, we know that $\mathfrak{g}$ is solvable. Therefore, $[\mathfrak{g}, \mathfrak{g}]$ is a 3-dimensional nilpotent Lie algebra. Firstly, we consider the case when $[\mathfrak{g}, \mathfrak{g}]$ is not abelian. That is, $[\mathfrak{g}, \mathfrak{g}]$ is a 3-dimensional Heisenberg Lie algebra. Let x, y, z be a basis of $[\mathfrak{g}, \mathfrak{g}]$ satisfying $[x, y] = z$. Then we have $adX(z) = adX[x, y] = kz$. That is, the center z of $[\mathfrak{g}, \mathfrak{g}]$ is an eigenvector of adX.

Case 1: $a_{14} = 0, a_{12} \neq 0$.
In this case, the matrix of adX with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is

$$\begin{pmatrix} -\rho & a_{12} & 0 & 0 \\ a_{12} & -\rho & 0 & 0 \\ 0 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the eigenvalues of adX on $[\mathfrak{g}, \mathfrak{g}]$ are $-a_{12} - \rho$ and $a_{12} - \rho, -2\rho$, and the corresponding eigenvectors are $-e_1 + e_2, e_1 + e_2, e_3$. We have

$$adX([e_1, e_2]) = [[X, e_1], e_2] + [e_1, [X, e_2]] = [-\rho e_1, e_2] + [e_1, -\rho e_2] = -2\rho[e_1, e_2],$$

and

$$adX([e_1, e_3]) = -3\rho[e_1, e_3] + a_{12}[e_2, e_3], \ adX([e_2, e_3]) = -3\rho[e_2, e_3] + a_{12}[e_1, e_3].$$

Set $[e_1, e_2] = \lambda z$, $[e_1, e_3] = \mu z$, $[e_2, e_3] = \tau z$. Here $\lambda, \mu, \tau \in \mathbb{R}$, and z is the center of $[\mathfrak{g}, \mathfrak{g}]$. Then

$$adX(\lambda z) = -2\rho\lambda z, \ adX(\mu z) = (-3\rho\mu + a_{12}\tau)z, \ adX(\tau z) = (-3\rho\tau + a_{12}\mu)z.$$

Case 1a: If $\mu\tau = \mathbf{0}$, since $a_{12} \neq 0$, we can get $\mu = 0, \tau = 0$. Then, e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$. And we assume $[e_i, e_j] = \lambda e_3$, where $\lambda \neq 0, i, j = 1, 2, 3$. Set $X = \alpha e_1 + \beta e_2 + \gamma e_3 + \theta e_4$,

$$[X, e_1] = -\lambda\beta e_3 + \theta[e_4, e_1] = -\rho e_1 + a_{12} e_2,$$

it is easy to see $\theta \neq 0$, so $[e_1, e_4] = \dfrac{\rho e_1 - a_{12}e_2 - \lambda\beta e_3}{\theta}$.
Similarly, we have $[e_2, e_4] = \dfrac{-a_{12}e_1 + \rho e_2 + \alpha\lambda e_3}{\theta}$ and $[e_3, e_4] = \dfrac{2\rho e_3}{\theta}$.
By (8) and definition of conformal vector field, substitute e_j with X, we have

$$- < [X, e_i], X >= 2\rho < e_i, X >, \quad i = 1, 2, 3, 4. \tag{19}$$

Thus, we have $\theta \neq 0$, $\gamma = 0$, $a_{12}\beta = \alpha\rho$, $\beta\rho = a_{12}\alpha$. Hence, if $\rho \neq \pm a_{12}$, we have $\alpha = 0, \beta = 0, \gamma = 0, \theta \neq 0$. If $\rho = \pm a_{12}$, we have $\gamma = 0$, $\theta \neq 0$, $\beta = \pm\alpha$.
Hence, we conclude: $a_{12} = -\rho$, $\beta = -\alpha$, or $a_{12} = \rho$, $\beta = \alpha$, or $\alpha = \beta = 0$.
For $a_{12} = -\rho, \beta = -\alpha$, we have

$$[e_1, e_2] = \lambda e_3, \quad [e_3, e_4] = \frac{2\rho e_3}{\theta}, \quad [e_1, e_4] = [e_2, e_4] = \frac{\rho e_1 + \rho e_2 + \lambda\alpha e_3}{\theta}.$$

Set $\frac{\rho}{\theta} = \lambda_1, \frac{\alpha}{\theta} = \lambda_2$, and we have Case 1 of Theorem 1.2.
For $a_{12} = \rho$, $\beta = \alpha$, we have

$$[e_1, e_2] = \lambda e_3, \quad [e_3, e_4] = \frac{2\rho e_3}{\theta}, \quad [e_1, e_4] = -[e_2, e_4] = \frac{\rho e_1 - \rho e_2 - \lambda\alpha e_3}{\theta}.$$

Set $\frac{\rho}{\theta} = \lambda_1$, $\frac{\alpha}{\theta} = \lambda_2$, and we have Case 2 of Theorem 1.2.
For $\alpha = \beta = 0$, we have

$$[e_1, e_2] = \lambda e_3, \quad [e_3, e_4] = \frac{2\rho e_3}{\theta}, \quad [e_1, e_4] = \frac{\rho e_1 - a_{12}e_2}{\theta}, \quad [e_2, e_4] = \frac{-a_{12}e_1 + \rho e_2}{\theta},$$

Set $\frac{\rho}{\theta} = \lambda_1, \frac{a_{12}}{\theta} = \lambda_2$, we have the Case 3 of Theorem 1.2.

Case 1b: If $\mu\tau \neq \mathbf{0}$, we have $adX(z) = (-3\rho + \frac{a_{12}}{\mu}\tau)z$, $adX(z) = (-3\rho + \frac{a_{12}}{\tau}\mu)z$, then $\mu = \pm\tau$.

When $\mu = \tau$, then $adX(z) = (-3\rho + a_{12})z$, since the eigenvalues of adX on $[\mathfrak{g}, \mathfrak{g}]$ are $-a_{12} - \rho, a_{12} - \rho, -2\rho$, then it is easy to see that $a_{12} = \rho$, and z lies in the subspace $Span\{-e_1 + e_2, e_3\}$. Set $z = k_1e_3 + k_2e_2 - k_2e_1, k_2 \neq 0$, we have

$$[e_1, e_2] = \lambda(k_1e_3 + k_2e_2 - k_2e_1), \quad [e_1, e_3] = [e_2, e_3] = \mu(k_1e_3 + k_2e_2 - k_2e_1).$$

By $[e_i, z] = 0, i = 1, 2, 3$, we get: $\lambda = -\frac{k_1\mu}{k_2}$. Since $X = \alpha e_1 + \beta e_2 + \gamma e_3 + \theta e_4$, by (19), we have $\gamma = 0, \theta \neq 0, \beta = \alpha$. Since $adX(e_1) = -\rho e_1 + a_{12}e_2$, we have

$$[e_4, e_1] = \frac{1}{\theta}[(-\rho + \alpha k_1\mu)e_1 + (\rho - \alpha k_1\mu)e_2 - \frac{k_1^2\mu\alpha}{k_2}e_3].$$

Similarly, we obtain:

$$[e_4, e_2] = \frac{1}{\theta}[(\rho - \alpha k_1\mu)e_1 - (\rho - \alpha k_1\mu)e_2 + \frac{k_1^2\mu\alpha}{k_2}e_3],$$

$$[e_4, e_3] = \frac{1}{\theta}[2\alpha\mu k_2e_1 - 2\alpha\mu k_2e_2 - (2\rho + 2\alpha k_1\mu)e_3].$$

Replacing $k_1\mu, k_2\mu, \frac{\alpha}{\theta}, \frac{\rho}{\theta}$ by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we have

$$\begin{aligned}
&[e_1, e_2] = -\frac{\lambda_1^2}{\lambda_2}e_3 - \lambda_1 e_2 + \lambda_1 e_1,\\
&[e_1, e_3] = [e_2, e_3] = \lambda_1 e_3 + \lambda_2 e_2 - \lambda_2 e_1,\\
&[e_4, e_1] = -[e_4, e_2] = (-\lambda_4 + \lambda_1\lambda_3)e_1 + (\lambda_4 - \lambda_1\lambda_3)e_2 - \frac{\lambda_1^2\lambda_3}{\lambda_2}e_3,\\
&[e_4, e_3] = 2\lambda_2\lambda_3 e_1 - 2\lambda_2\lambda_3 e_2 - (2\lambda_4 + 2\lambda_1\lambda_3)e_3.
\end{aligned}$$

So we have Case 4 of Theorem 1.2.

When $\mu = -\tau$, then $adX(z) = (-3\rho - a_{12})$, since the eigenvalues of adX on $[\mathfrak{g}, \mathfrak{g}]$ are $-a_{12} - \rho, a_{12} - \rho, -2\rho$, then it is easy to see that $a_{12} = -\rho$, and z lies in the subspace $Span\{e_1 + e_2, e_3\}$. Set $z = k_1 e_3 + k_2 e_2 + k_2 e_1, k_2 \neq 0$, we have

$$[e_1, e_2] = \lambda(k_1 e_3 + k_2 e_2 + k_2 e_1), \quad [e_1, e_3] = -[e_2, e_3] = \mu(k_1 e_3 + k_2 e_2 + k_2 e_1).$$

By $[e_i, z] = 0, i = 1, 2, 3$, we get: $\lambda = -\frac{k_1\mu}{k_2}$. Since $X = \alpha e_1 + \beta e_2 + \gamma e_3 + \theta e_4$. By (19), we have $\gamma = 0, \theta \neq 0, \beta = -\alpha$. Notice $adX(e_1) = -\rho e_1 + a_{12}e_2$, we have

$$[e_4, e_1] = \frac{1}{\theta}\left[(-\rho + \alpha k_1\mu)e_1 + (-\rho + \alpha k_1\mu)e_2 + \frac{k_1^2\mu\alpha}{k_2}e_3\right],$$

Similarly, we obtain:

$$\begin{aligned}
&[e_4, e_2] = \frac{1}{\theta}\left[(-\rho + \alpha k_1\mu)e_1 + (-\rho + \alpha k_1\mu)e_2 + \frac{k_1^2\mu\alpha}{k_2}e_3\right],\\
&[e_4, e_3] = -\frac{2}{\theta}\left[\alpha\mu k_2 e_1 + \alpha\mu k_2 e_2 + (\rho + \alpha k_1\mu)e_3\right].
\end{aligned}$$

Replacing $k_1\mu, k_2\mu, \frac{\alpha}{\theta}, \frac{\rho}{\theta}$ by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we have

$$\begin{aligned}
&[e_1, e_2] = -\frac{\lambda_1^2}{\lambda_2}e_3 - \lambda_1 e_2 - \lambda_1 e_1,\\
&[e_1, e_3] = -[e_2, e_3] = \lambda_1 e_3 + \lambda_2 e_2 + \lambda_2 e_1,\\
&[e_4, e_1] = [e_4, e_2] = (-\lambda_4 + \lambda_1\lambda_3)e_1 + (-\lambda_4 + \lambda_1\lambda_3)e_2 + \frac{\lambda_1^2\lambda_3}{\lambda_2}e_3,\\
&[e_4, e_3] = -2\lambda_2\lambda_3 e_1 - 2\lambda_2\lambda_3 e_2 - (2\lambda_4 + 2\lambda_1\lambda_3)e_3.
\end{aligned}$$

So we have Case 5 of Theorem 1.2.

Case 2: $a_{12} = 0,\ a_{14} = 0$.

For this case, we set $X = \alpha e_1 + \beta e_2 + \gamma e_3 + \theta e_4$, by (19), we have $\alpha = \beta = \gamma = 0$. Now we assert that e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$. To see this, we suppose that z is the center of $[\mathfrak{g}, \mathfrak{g}]$. Set $[e_1, e_3] = \lambda z,\ \lambda \in \mathbb{R}$, by the Jacobi identity. Then we have

$$[X, [e_1, e_3]] = [[X, e_1], e_3] + [e_1, [X, e_3]] = -\rho[e_1, e_3] - 2\rho[e_1, e_3] = -3\rho\lambda z.$$

Since z is the eigenvector of adX, and the eigenvalues of the operator adX are -ρ, -2ρ. Thus, $[X, [e_1, e_3]] = [X, \lambda z] = k\lambda z$, where $k = -\rho$, or $k = -2\rho$. Thus, $[e_1, e_3] = 0$. And we can also obtain $[e_2, e_3] = 0$. So e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$.

In this case, the non-zero brackets are the following:

$$[e_1,e_4]=\frac{\rho e_1}{\theta},\ [e_2,e_4]=\frac{\rho e_2}{\theta},\ [e_3,e_4]=\frac{2\rho e_3}{\theta},\ [e_1,e_2]=\lambda e_3,\lambda\neq 0,$$

and the non-Killing conformal vector field $X=\theta e_4$. Replacing $\frac{\rho}{\theta}$ by λ_1, we have the Case 6 of Theorem 1.2.

Case 3: $a_{12}\neq 0,\ a_{14}\neq 0$.
It is easy to see that the eigenvectors of adX on $[g,g]$ are

$$(a_{12}+\rho)e_1+(a_{12}+\rho)e_2+a_{14}e_3 \text{ and } (-a_{12}+\rho)e_1+(a_{12}-\rho)e_2+a_{14}e_3, e_3,$$

and the corresponding eigenvalues are $a_{12}-\rho,-a_{12}-\rho,-2\rho$. Notice that

$$\begin{aligned} adX([e_1,e_2]) &= [[X,e_1],e_2]+[e_1,[X,e_2]] \\ &= [-\rho e_1+a_{14}e_3,e_2]+[e_1,-\rho e_2] \\ &= -2\rho[e_1,e_2]+a_{14}[e_3,e_2], \end{aligned}$$

and

$$adX([e_1,e_3])=-3\rho[e_1,e_3]+a_{12}[e_2,e_3],\quad adX([e_2,e_3])=-3\rho[e_2,e_3]+a_{12}[e_1,e_3].$$

We can set $[e_1,e_2]=\lambda z$, $[e_1,e_3]=\mu z$, $[e_2,e_3]=\tau z$, here $\lambda,\mu,\tau\in\mathbb{R}$, z is the center of $[\mathfrak{g},\mathfrak{g}]$. Then

$$adX(\lambda z)=-2\rho\lambda z-a_{14}\tau z,\quad adX(\mu z)=(-3\rho\mu+a_{12}\tau)z,$$
$$adX(\tau z)=(-3\rho\tau+a_{12}\mu)z.$$

Now we assert that e_3 is the center of $[\mathfrak{g},\mathfrak{g}]$. If $\mu\tau=\mathbf{0}$, since $a_{12}\neq 0$, we can get $\mu=0,\tau=0$. Then, e_3 is the center of $[\mathfrak{g},\mathfrak{g}]$. If $\mu\tau\neq\mathbf{0}$, we have $adX(z)=(-3\rho+\frac{a_{12}}{\mu}\tau)z$, $adX(z)=(-3\rho+\frac{a_{12}}{\tau}\mu)z$, then $\mu=\pm\tau$.
When $\mu=\tau$, then $adX(z)=(-3\rho+a_{12})z$, since the eigenvalues of adX on $[\mathfrak{g},\mathfrak{g}]$ are $-a_{12}-\rho,a_{12}-\rho,-2\rho$, then it is easy to see that $a_{12}=\rho$, and e_3 is the center of $[\mathfrak{g},\mathfrak{g}]$.
When $\mu=-\tau$, then $adX(z)=(-3\rho-a_{12})$, since the eigenvalues of adX on $[\mathfrak{g},\mathfrak{g}]$ are $-a_{12}-\rho,a_{12}-\rho,-2\rho$, then it is easy to see that $a_{12}=-\rho$, and e_3 is the center of $[\mathfrak{g},\mathfrak{g}]$.
So in this case, e_3 is the center of $[\mathfrak{g},\mathfrak{g}]$. And we have the only non-zero bracket of $[e_i,e_j]$ is $[e_1,e_2]=\lambda e_3$, where $\lambda\neq 0,i,j=1,2,3$. Since $\rho\neq 0$, by computation, it easy to know $a_{12}\neq\pm\rho$. Set $X=\alpha e_1+\beta e_2+\gamma e_3+\theta e_4$, by (19), we have

$$\theta=-\frac{\alpha(a_{12}^2-\rho^2)}{a_{14}\rho},\quad \beta=\frac{\alpha a_{12}}{\rho},\quad \gamma=\frac{\alpha a_{14}}{2\rho}.$$

Since $X\neq 0$, then $\alpha\neq 0$. By by (18), we have

$$[e_1,e_4]=-\frac{a_{14}(a_{12}e_2\rho+\alpha a_{12}e_3\lambda+a_{14}e_3\rho-e_1\rho^2)}{\alpha(\rho^2-a_{12}^2)},$$
$$[e_2,e_4]=\frac{a_{14}\rho(-a_{12}e_1+e_2\rho+\alpha e_3\lambda)}{\alpha(\rho^2-a_{12}^2)},\quad [e_3,e_4]=\frac{2a_{14}e_3\rho^2}{\alpha(\rho^2-a_{12}^2)},$$

We get Case 7 of Theorem 1.2 by setting

$$\lambda_1 = \frac{\rho^2 a_{14}}{(\rho^2 - a_{12}^2)\alpha}, \quad \lambda_2 = \frac{\rho a_{14} a_{12}}{(\rho^2 - a_{12}^2)\alpha}, \quad \lambda_3 = \frac{\lambda \rho a_{14}}{(\rho^2 - a_{12}^2)}, \quad \lambda_4 = \frac{\rho a_{14}^2}{(\rho^2 - a_{12}^2)\alpha}.$$

Case 4: $a_{12} = 0, a_{14} \neq 0$.
Notice that the center of $[\mathfrak{g}, \mathfrak{g}]$ is the eigenvector adX. By the same argument as in Case 2 we know that e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$. Then we set $[e_1, e_2] = \lambda e_3$, since $X = \alpha e_1 + \beta e_2 + \gamma e_3 + \theta e_4$, by (19), we have

$$\alpha = \frac{a_{14}\theta}{\rho}, \quad \beta = 0, \quad \gamma = \frac{a_{14}^2 \theta}{2\rho^2}.$$

In this case, the non-zero brackets are

$$[e_1, e_2] = \lambda e_3, \ [e_1, e_4] = \frac{e_1\rho - a_{14}e_3}{\theta}, \quad [e_2, e_4] = \frac{a_{14}e_3\lambda}{\rho} + \frac{e_2\rho}{\theta}, \quad [e_3, e_4] = \frac{2e_3\rho}{\theta},$$

so we have Case 8 of Theorem 1.2.
Finally, we study the case when $[\mathfrak{g}, \mathfrak{g}]$ is abelian. Let the non-zero Lie brackets be

$$[e_1, e_4] = \alpha e_1 + \beta e_2 + \gamma e_3, \ [e_2, e_4] = q e_1 + m e_2 + n e_3, \ [e_3, e_4] = w e_1 + v e_2 + y e_3.$$

Suppose $X = \sum_{i=1}^4 x_i e_i$. Since $L_X < .,. >= 2\rho <,.,>$, we have

$$\begin{pmatrix} 2\alpha x_4 - 2\rho & q x_4 - \beta x_4 & w x_4 & -\alpha x_1 - q x_2 - w x_3 - \gamma x_4 \\ q x_4 - \beta x_4 & 2\rho - 2m x_4 & -v x_4 & \beta x_1 + m x_2 + v x_3 - n x_4 \\ w x_4 & -v x_4 & 0 & 2\rho - x_4 y \\ -q x_2 - w x_3 - x_1\alpha - x_4\gamma & m x_2 + v x_3 - n x_4 + x_1\beta & 2\rho - x_4 y & 2n x_2 + 2x_3 y + 2x_1\gamma \end{pmatrix} = 0$$

Notice that $\mathfrak{g}$ is non-unimodular, we have $\alpha + m + y \neq 0$, and we obtain:
$y = 2m = 2\alpha \neq 0$, $q = \beta$, $m x_4 = \rho$, $w = v = 0$. If $m \neq \pm q$,

$$x_1 = -\frac{\gamma m x_4 + n q x_4}{m^2 - q^2}, \quad x_2 = -\frac{-m n x_4 - \gamma q x_4}{m^2 - q^2}, \quad x_3 = -\frac{n^2 x_4 - \gamma^2 x_4}{2(m^2 - q^2)}.$$

Hence we have Case 9 of Theorem 1.2.

If $m = q$,we have
$$\begin{cases} -m(x_1 + x_2) &= \gamma x_4, \\ m(x_1 + x_2) &= n x_4, \\ 2n x_2 + 4m x_3 + 2x_1\gamma &= 0. \end{cases}$$

from above equations, we have $n = 0$, $x_1 = -x_2$, $x_3 = 0$, $\gamma = 0$, or $n \neq 0$,

$$x_1 = \frac{2m^2 x_3 + n^2 x_4}{2mn}, \quad x_2 = \frac{-2m^2 x_3 + n^2 x_4}{2mn}.$$

Hence we have Case 10 and Case 11 of Theorem 1.2.

If $m = -q$, we have
$$\begin{cases} -m x_1 + m x_2 &= \gamma x_4, \\ -m x_1 + m x_2 &= n x_4, \\ 2n x_2 + 4m x_3 + 2x_1\gamma &= 0. \end{cases}$$

from the above equations, we have $n = 0$, $x_1 = x_2$, $x_3 = 0$, $\gamma = 0$, or $n \neq 0$,

$$x_1 = \frac{-2m^2 x_3 - n^2 x_4}{2mn}, \quad x_2 = \frac{-2m^2 x_3 + n^2 x_4}{2mn}.$$

Hence we have Case 12 and Case 13 of Theorem 1.2. This completes the proof.

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