

Classification of Quasi-Finite Irreducible Modules over Affine Virasoro Algebras

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Abstract. We give a complete classification of quasi-finite irreducible modules over affine Virasoro algebras. It is shown that they are all highest weight modules, lowest weight modules, or loop modules.

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Key Words: Virasoro algebra, affine Lie algebra, affine Virasoro algebra, quasi-finite module.

1. Introduction

Let \mathfrak{g} be a finite-dimensional simple Lie algebra equipped with the Killing form $(\ , \)$. Associated to the pair $(\mathfrak{g}, (\ , \))$ we have the untwisted affine Lie algebra $\hat{\mathfrak{g}}$. Denote by Vir the Virasoro algebra, which is a central extension of Lie algebra of complex vector fields on the circle. The affine Virasoro algebra $\tilde{\mathfrak{g}} = \text{Vir} \ltimes \hat{\mathfrak{g}}$ is the semi-direct sum of the affine Lie algebra $\hat{\mathfrak{g}}$ and the Virasoro algebra Vir . It is known that in literature $\tilde{\mathfrak{g}}$ is also named the conformal current algebra [4, 13], the entire gauge algebra [7]. The physical context in which the affine Virasoro algebra appears is a two-dimensional conformal field theory on the circle with an internal symmetry algebra [22]. In particular, the even part of the $N = 3$ superconformal algebra is the affine Virasoro algebra with $\mathfrak{g} = \mathfrak{sl}_2$. Kac [14] classified all unitary irreducible highest weight modules over $\tilde{\mathfrak{g}}$ via the Segal-Sugawara construction. Fock space representations of $\tilde{\mathfrak{g}}$ were further studied in [2, 6, 8, 14, 17, 18, 19, 23]. Later Cheng and Kac [4] classified all finite simple conformal modules over $\tilde{\mathfrak{g}}$. Eswara Rao and Jiang [5] (see also [12]) obtained a complete classification of integrable irreducible weight modules with finite-dimensional weight spaces over $\tilde{\mathfrak{g}}$. It is shown that they are all integrable highest weight modules, integrable lowest weight modules, or loop modules.

A module over $\tilde{\mathfrak{g}}$ is called quasi-finite if it has a weight space decomposition with respect to the action of $d_0 \in \text{Vir}$ and all weight spaces are finite-dimensional. It is clear that every integrable irreducible weight module over $\tilde{\mathfrak{g}}$ is always quasi-finite. However, it is worth noting that the converse is not true, i.e., a quasi-finite irreducible module is not integrable in general. Then it is natural to consider classification of all quasi-finite irreducible modules over the affine Virasoro algebra. Suppose that all

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real parts of eigenvalues of d_0 are bounded from above, Billig [2] showed that a quasi-finite irreducible module over $\tilde{\mathfrak{g}}$ must be a highest weight module with nonzero level. Without the assumption of eigenvalues of d_0 , using certain complicated induction, Gao, Hu and the first author of this paper obtained a complete classification of quasi-finite irreducible modules of the affine Virasoro algebra in the case when $\mathfrak{g} = \mathfrak{sl}_2$ [11]. Their proof depends heavily on the classification of irreducible modules over the twisted Heisenberg-Virasoro algebra due to Lu and Zhao [20]. It seems that there is no easy way to extend their approach to the general case.

In this paper, using a new approach, we classify quasi-finite irreducible modules over the affine Virasoro algebras for general case. Our main result can be stated as follows.

Main Theorem (Theorem 3.5). *Every quasi-finite irreducible module over the affine Virasoro algebra is a highest weight module, a lowest weight module, or a loop module.*

Our proof is based on the classification of irreducible weight modules over the Virasoro algebra with finite-dimensional weight spaces [21], and the classification of irreducible integrable weight modules with finite-dimensional weight spaces over the affine Virasoro algebra [5]. Our approach can be applied to study quasi-finite irreducible modules over full toroidal algebras [2] and their twisted versions [9].

The paper is arranged as follows. In Section 2, we recall some basic definitions and results and introduce some notations. In Section 3, we classify all quasi-finite irreducible module over the affine Virasoro algebra and obtain our main theorem.

Throughout this paper, we will use $\mathbb{C}, \mathbb{Z}, \mathbb{N}, \mathbb{Z}_+$ and \mathbb{Z}^* to denote the sets of complex numbers, integers, nonnegative integers, positive integers, and nonzero integers, respectively.

2. Preliminaries

In this section we fix notations and recall some known facts on affine Virasoro algebras.

2.1. Let \mathfrak{g} denote a complex simple finite-dimensional Lie algebra, \mathfrak{h} the Cartan subalgebra, Δ the set of roots of \mathfrak{g} , and Δ_+ the set of positive roots of \mathfrak{g} . Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ has a root space decomposition of \mathfrak{g} with respect to \mathfrak{h} . Let (\cdot, \cdot) denote the Killing form on \mathfrak{g} . Let ω be an involution of \mathfrak{g} such that $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$. Let Λ^+ denote the integral dominant weights set of \mathfrak{g} .

Recall that the untwisted affine Lie algebras $\hat{\mathfrak{g}}$ is a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ defined by the following commutation relations:

$$[x(m), y(n)] = [x, y](m+n) + m(x, y)\delta_{m+n,0}\mathbf{k}, \quad (1)$$

for $m \in \mathbb{Z}, x \in \mathfrak{g}$, where $x(m) = x \otimes t^m$.

The Virasoro algebra Vir is a central extension of Lie algebra $\text{Der}[t, t^{-1}]$ defined by the following commutation relations:

$$[d_m, d_n] = (n-m)d_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} \mathbf{c}, \quad (2)$$

where $d_m = t^{m+1} \frac{d}{dt}$ for all $m, n \in \mathbb{Z}$.

Definition 2.1. ([13]) The *affine Virasoro algebra* $\tilde{\mathfrak{g}} = \text{Vir} \ltimes \hat{\mathfrak{g}}$ is the semi-direct product of the Virasoro algebra Vir and the untwisted affine Lie algebra $\hat{\mathfrak{g}}$ defined by the relations (1), (2) and

$$[d_n, x(m)] = mx(m+n), \quad [\mathbf{c}, \tilde{\mathfrak{g}}] = [\mathbf{k}, \tilde{\mathfrak{g}}] = 0. \tag{3}$$

It is clear that $\tilde{\mathfrak{g}}$ has a \mathbb{Z} -grading by the eigenvalues of the adjoint action of d_0 .

Then
$$\tilde{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \tilde{\mathfrak{g}}_n = \tilde{\mathfrak{g}}_+ \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_-,$$

where
$$\tilde{\mathfrak{g}}_{\pm} = \bigoplus_{n \in \mathbb{Z}_+} \tilde{\mathfrak{g}}_{\pm n}, \quad \tilde{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}\mathbf{c} \oplus \mathbf{k} \oplus \mathbb{C}\mathbf{d}_0.$$

2.2. Set $\bar{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{c} \oplus \mathbf{k} \oplus \mathbf{d}_0$.

Definition 2.2. A module V over $\tilde{\mathfrak{g}}$ is called a *weight module* with respect to $\bar{\mathfrak{h}}$ if $V = \bigoplus_{\lambda \in \bar{\mathfrak{h}}^*} V_{\lambda}$ where

$$V_{\lambda} = \{v \in V \mid hv = \lambda(h)v, \forall h \in \bar{\mathfrak{h}}\}.$$

Definition 2.3. If V is a $\tilde{\mathfrak{g}}$ -module on which \mathbf{c} and \mathbf{k} act as scalars c and k in \mathbb{C} , respectively, we say that V is of *level* (c, k) .

Given $\lambda \in \mathfrak{h}^*$, and $k, c, h \in \mathbb{C}$, we define a highest weight module over $\tilde{\mathfrak{g}}$ with highest weight (k, λ, c, h) in the space $M(k, \lambda, c, h)$, admitting a nonzero vector $\mathbf{1}$ such that

$$\begin{aligned} x\mathbf{1} &= \lambda(x)\mathbf{1}, \quad \forall x \in \mathfrak{h}; & \mathfrak{g}_{\alpha}\mathbf{1} &= 0, \quad \forall \alpha \in \Delta_+; & x(n)\mathbf{1} &= 0, \\ \mathbf{c}\mathbf{1} &= c\mathbf{1}, & \mathbf{k}\mathbf{1} &= k\mathbf{1}, & d_0\mathbf{1} &= h\mathbf{1}, & d_n\mathbf{1} &= 0, \end{aligned}$$

for $n \in \mathbb{Z}_+$. Let $L(k, \lambda, c, h)$ denote the simple quotient module of $M(k, \lambda, c, h)$. With the standard involution on $\tilde{\mathfrak{g}}$:

$$x(n) \mapsto \omega(x)(-n), \quad d_n \mapsto d_{-n}, \quad \mathbf{c} \mapsto \mathbf{c}, \quad \mathbf{k} \mapsto \mathbf{k}$$

for $x \in \mathfrak{g}, n \in \mathbb{Z}$, we can interchange the highest and the lowest weight modules over $\tilde{\mathfrak{g}}$. Similarly, the lowest weight modules over $\tilde{\mathfrak{g}}$ can be defined.

Remark 2.4. All unitary irreducible highest weight modules over $\tilde{\mathfrak{g}}$ were classified by Kac [13] via the Segal-Sugawara construction.

From [5], for $a, b \in \mathbb{C}$, the loop module $T_{a,b}(U)$ over $\tilde{\mathfrak{g}}$ is $T_{a,b}(U) = U \otimes [t, t^{-1}]$ for a \mathfrak{g} -module U with the actions defined for $m, n \in \mathbb{Z}, u \in U$ by

$$\begin{aligned} d_m(u \otimes t^n) &= (a + bm + n)u \otimes t^{m+n}, & x_m(u \otimes t^n) &= (x \cdot u) \otimes t^{m+n}, \\ \mathbf{c}(u \otimes t^n) &= 0 = \mathbf{k}(u \otimes t^n). \end{aligned}$$

Remark 2.5. Suppose that U is a finite-dimensional \mathfrak{g} -module. Then $T_{a,b}(U)$ is irreducible if and only if U is a non-trivial irreducible module or $a \notin \mathbb{Z}$ or $b \notin \{0, 1\}$. We assume that $U = L(\lambda)$ is an irreducible \mathfrak{g} -module with highest weight $\lambda \in \Lambda^+$ and use $L_{a,b}(\lambda)$ to denote the irreducible submodule or subquotient of $T_{a,b}(U)$.

Definition 2.6. A weight $\tilde{\mathfrak{g}}$ -module V respect to $\bar{\mathfrak{h}}$ is called *integrable* if the element $x(m)$ is locally nilpotent on V for all $x \in \mathfrak{g}_\alpha, \alpha \in \Delta, m \in \mathbb{Z}$.

Theorem 2.7. (Eswara Rao and Jiang [5]) *Every integrable irreducible weight module over $\tilde{\mathfrak{g}}$ is a highest weight module, a lowest weight module, or a loop module.*

Definition 2.8. A module V over $\tilde{\mathfrak{g}}$ is called *quasi-finite* if

- V has a weight space decomposition with respect to d_0 :

$$V = \bigoplus_{\mu \in \mathbb{C}} V_\mu$$

where $V_\mu = \{v \in V \mid d_0 v = \mu v\}$;

- All weight spaces V_μ are finite-dimensional.

Remark 2.9. If V is a quasi-finite irreducible module over $\tilde{\mathfrak{g}}$, then there exists $\mu \in \mathbb{C}$ such that $\text{Supp}(V) \subset \mu + \mathbb{Z}$.

Definition 2.10. A quasi-finite module V over $\tilde{\mathfrak{g}}$ is called *uniformly bounded* if there exists a positive integer N such that $\dim V_\mu < N$ for all $\mu \in \mathbb{C}$.

Remark 2.11. It is clear that integrable irreducible highest weight modules are quasi-finite. The loop module $L_{a,b}(\lambda)$ over $\tilde{\mathfrak{g}}$ is also quasi-finite and uniformly bounded.

3. Classification of quasi-finite irreducible modules

In this section, we give a complete classification of quasi-finite irreducible modules over an affine Virasoro algebra.

Next we always assume that V is a quasi-finite irreducible module over $\tilde{\mathfrak{g}}$. From Remark 2.9, we can suppose that

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

Lemma 3.1. V has a weight space decomposition with respect to $\bar{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}L_0 \oplus \mathbb{C}\mathbf{c} \oplus \mathbb{C}\mathbf{k}$.

Proof. For $n \in \mathbb{Z}$, V_n is a finite-dimensional \mathfrak{g} -module. It follows that V is equipped with a weight space decomposition with respect to \mathfrak{h} :

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \text{ where } V_\lambda = \{v \in V \mid hv = \lambda(h)v, \forall h \in \mathfrak{h}\}.$$

Then
$$V = \bigoplus_{n \in \mathbb{Z}, \lambda \in \mathfrak{h}^*} V_{n,\lambda}, \text{ where } V_{n,\lambda} = V_n \cap V_\lambda. \quad \blacksquare$$

Lemma 3.2. Suppose that $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is a uniformly bounded quasi-finite irreducible module over $\tilde{\mathfrak{g}}$. Then the actions of central elements \mathbf{c} and \mathbf{k} on V are trivial, i.e., $\mathbf{c}V = \mathbf{k}V = 0$.

Proof. By Schur’s Lemma, we may assume that \mathfrak{c} and \mathfrak{k} act on V by scalars c and ℓ , respectively. It follows that V has a weight decomposition with respect to $\bar{\mathfrak{h}}$. Note that V is a uniformly bounded L_0 -weight module over the Virasoro algebra Vir . By [16], $c = 0$. Moreover, V is also a uniformly bounded \mathfrak{h} -weight module over the affine algebra $\hat{\mathfrak{g}}$. By [3], we have $\ell = 0$. ■

Next we shall show that every uniformly bounded irreducible $\tilde{\mathfrak{g}}$ -module is a loop module.

Proposition 3.3. *Let V be a uniformly bounded irreducible $\tilde{\mathfrak{g}}$ -module. Then $V \cong L_{a,b}(\lambda)$ for some $\lambda \in \Lambda^+$ and $a, b \in \mathbb{C}$.*

Proof. As the \mathfrak{g} -modules, V is completely reducible by the Weyl’s theorem. Since $V = \bigoplus_{k \in \mathbb{Z}} V_k$ is uniformly bounded, there exists a positive integer N such that $\dim V_k < N$ for all $k \in \mathbb{Z}$. We claim that V has a weight space decomposition corresponding to a finite set of dominant weights.

In fact, let $\lambda = a_1\Lambda_1 + \dots + a_r\Lambda_r \in \Lambda^+$ and $V(\lambda)$ be the irreducible \mathfrak{g} -module with highest weight λ . By the Weyl character formula, $\dim V(\lambda)$ is increasing about a_1, \dots, a_r . Then for any positive integer m , up to isomorphisms there are only finitely many irreducible \mathfrak{g} -modules $V(\lambda)$ whose dimension less than m . This implies that $e_\alpha(n)$ (also $f_\alpha(n)$) is nilpotent on V for $\alpha \in \Delta$ and $n \in \mathbb{Z}$. Together with Lemma 3.2, V is an irreducible integrable $\tilde{\mathfrak{g}}$ -module with uniformly bounded weight spaces. From Theorem 2.7, V is isomorphic to the irreducible loop module $L_{a,b}(\lambda)$ for some $\lambda \in \Lambda^+, a, b \in \mathbb{C}$. ■

Next we shall prove a result about the behavior of dimensions of weight spaces of quasi-finite irreducible modules over $\tilde{\mathfrak{g}}$.

Proposition 3.4. *Let V be a quasi-finite irreducible weight module over $\tilde{\mathfrak{g}}$. If V is neither a highest nor a lowest module, then V is uniformly bounded.*

Proof. Suppose that $V = \sum_{k \in \mathbb{Z}} V_k$ is a quasi-finite irreducible $\tilde{\mathfrak{g}}$ -module without highest and lowest weights. We shall prove that for any $n \in \mathbb{Z}_+, k \in -\mathbb{Z}_+$,

$$d_n|_{V_k} \oplus d_{n+1}|_{V_k} \oplus \bigoplus_{i=1}^d x_i(n+i+1)|_{V_k} : V_k \rightarrow V_{k+n} \oplus V_{k+n+1} \oplus \bigoplus_{i=1}^d V_{k+n+i+1}$$

is injective, where $\{x_i \mid i = 1, 2, \dots, d\}$ is a basis of \mathfrak{g} . In particular, by taking $k = -n$, we obtain that $V_k, k < 0$ is uniformly bounded.

In fact, suppose that there exists some $v_0 \in V_k$ such that

$$d_n v_0 = d_{n+1} v_0 = x_i(n+i+1)v_0 = 0$$

for $i = 1, 2, \dots, d$. Note that when $\ell \gg 0$, we have

$$\ell = n_1 n + n_2(n+1)$$

for some $n_1, n_2 \in \mathbb{N}$, from this and the relations in the definition, one can easily deduce that $d_\ell, x_i(\ell), i = 1, 2, \dots, d$, can be generated by $d_n, d_{n+1}, x_i(n+i+1)$ for $i = 1, 2, \dots, d$.

Then there exists $N_0 > 0$ such that $d_\ell v_0 = x_i(\ell)v_0 = 0$ for all $\ell \geq N_0$ and $i = 1, 2, \dots, d$. This implies that

$$\tilde{\mathfrak{g}}_{[N_0, +\infty)} v_0 = 0,$$

where $\tilde{\mathfrak{g}}_{[N_0, +\infty)} = \bigoplus_{i \geq N_0} \tilde{\mathfrak{g}}_i$. Clearly, $V = U(\tilde{\mathfrak{g}})v_0$ ($v_0 \in V_k$) and $V_k = U(\tilde{\mathfrak{g}})_0 v_0$. For any $v \in V_k$, there exists $u \in U(\tilde{\mathfrak{g}})_0$ such that $v = uv_0$.

Suppose that

$$u = \sum a_{-i_1, \dots, -i_m, i_{m+1}, \dots, i_n} X_{-i_1} \cdots X_{-i_m} X_{i_{m+1}} \cdots X_{i_n},$$

where $X_k = d_k$ or $x_i(k)$ for some $i = 1, 2, \dots, d$ and $i_1, \dots, i_n > 0$. Define the degree of u by

$$\deg(u) := \max\{i_1 + \cdots + i_m \mid a_{-i_1, \dots, -i_m, i_{m+1}, \dots, i_n} \neq 0\}.$$

It is well-defined since the sum is finite. Choose a basis $\{v_1, v_2, \dots, v_{d_k}\}$ of the finite-dimensional space V_k , and set

$$n_k := \max\{\deg(v_i) \mid 1 \leq i \leq d_k\}.$$

It follows that $\tilde{\mathfrak{g}}_{[N, +\infty)} V_k = 0$, where $N = N_0 + n_k$.

It is clear that $\tilde{\mathfrak{g}}^+$ is generated by

$$S := \{d_1, d_2, x_i(1) \mid i = 1, 2, \dots, d\}.$$

For any $k \geq 1$, set

$$U_{k+1} = S V_k = \sum_{x \in S} x V_k \subset V_{k+1} + V_{k+2}.$$

Then U_{k+1} is a \mathfrak{g} -module.

If $U_{k+1} = 0$, then there exists a nonzero element $v \in V_k$ such that $Sv = 0$ and then $\tilde{\mathfrak{g}}_+ v = 0$. Since V_k is finite-dimensional, $\mathfrak{g}_+ = \{e_\alpha(0) \mid \alpha \in \Delta\}$ is nilpotent on V_k . So there exists $w \in U(\mathfrak{g}_+)v$ such that

$$\mathfrak{g}_+ w = 0. \tag{4}$$

Now we can suppose that $U_{k+1} \neq 0$. For any $l \geq N$, we have

$$\tilde{\mathfrak{g}}_l U_{k+1} = \sum_{x \in S} [\tilde{\mathfrak{g}}_l, x] V_k + \sum_{x \in S} (x \tilde{\mathfrak{g}}_l) V_k \subset (\tilde{\mathfrak{g}}_{l+1} + \tilde{\mathfrak{g}}_{l+2}) V_k + \sum_{x \in S} (x \tilde{\mathfrak{g}}_l) V_k = 0,$$

which implies that $\tilde{\mathfrak{g}}_{[N, +\infty)} U_{k+1} = 0$.

Repeating the step above, we get $U_{k+i+1} = S U_{k+i}$ and

$$\tilde{\mathfrak{g}}_{[N, +\infty)} U_{k+i} = 0, \quad \forall i \geq 0.$$

Choose $u \in U_{k+N} \neq 0$ and $u \in V_p$ ($p \geq N + k$), where U_{k+N} is also a finite-dimensional \mathfrak{g} -module. For any $z \in \tilde{\mathfrak{g}}_j, j > 0$, there exists $y \in \tilde{\mathfrak{g}}_{p-k+j}$ such that $z = [y, L_{k-p}]$. Then

$$zu = [y, L_{k-p}]u = yL_{k-p}u - L_{k-p}yu \subset yV_k - L_{k-p}yU_{k+N} = 0$$

since $p - k + j \geq N + j$. It follows that $\tilde{\mathfrak{g}}_+ u = 0$.

Similar to the proof of (4), we can find $w \in U_{k+N} \cap V_p$ such that $\tilde{\mathfrak{g}}_+ w = 0$ and $\mathfrak{g}_+ w = 0$. It is clear that the subalgebra $\bar{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{c} \oplus \mathfrak{k} \oplus \mathfrak{d}_0$ of $\tilde{\mathfrak{g}}$ is commutative. It follows that there exists a common eigenvector w' of $U(\bar{\mathfrak{h}})w$. Then w' is a highest weight vector of $\tilde{\mathfrak{g}}$. This contradicts to the assumption.

Similarly for any $n \in -\mathbb{Z}_+, k \in \mathbb{Z}_+$,

$$d_n|_{V_k} \oplus d_{n-1}|_{V_k} \oplus \bigoplus_{i=1}^d x_i(n-i-1)|_{V_k} : V_k \longrightarrow V_{k+n} \oplus V_{k+n-1} \oplus \bigoplus_{i=1}^d V_{k+n-i-1}$$

is injective. In particular, by taking $k = -n$, we have $V_k, k > 0$ is uniformly bounded. ■

As an immediate corollary of Proposition 3.3 and Proposition 3.4, we obtain main result of this paper.

Theorem 3.5. *Every quasi-finite irreducible module over the affine Virasoro algebra $\tilde{\mathfrak{g}}$ is a highest weight module, a lowest weight module, or a loop module.*

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