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Commutators of Spectral Projections of Spin Operators

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Abstract. We present a proof that the operator norm of the commutator of certain spectral projections associated with spin operators converges to $\frac{1}{2}$ in the semiclassical limit. The ranges of the projections are spanned by all eigenvectors corresponding to positive eigenvalues. The proof involves the theory of Hankel operators on the Hardy space. A discussion of several analogous results is also included, with an emphasis on the case of finite Heisenberg groups.

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Key Words: Spectral projections, commutators, quantization.

1. Introduction

Let J_x, J_y, J_z denote the generators of an irreducible, unitary, n-dimensional representation of $SU(2)$, satisfying the commutation relations

$$[J_x, J_y] = iJ_z,\ [J_y, J_z] = iJ_x,\ [J_z, J_x] = iJ_y.$$

Here, $[A, B] = AB - BA$ denotes the commutator of a pair of linear operators A, B. Consider the commutator $C_n = \left[\mathbb{1}_{(0,\infty)}(J_x), \mathbb{1}_{(0,\infty)}(J_z)\right]$, where $\mathbb{1}_{(0,\infty)}$ denotes the indicator function of $(0, \infty) \subset \mathbb{R}$. Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The main results of the present work are that

Theorem 1.1. (L. Polterovich) $\|C_{4n+2}\|_{\text{op}} = \frac{1}{2}$ *for every* $n \in \mathbb{N}_0$,

and

Theorem 1.2. $\lim_{n\to\infty} \|C_n\|_{\text{op}} = \frac{1}{2}$.

The sequence $(\|C_n\|_{\text{op}})_{n=2}^{\infty}$ is bounded from above by $\frac{1}{2}$ due to a general fact about commutators of orthogonal projections ([10]). Nonetheless, it is perhaps not evident a priori that the sequence should converge at all, let alone to the largest possible value.

As it turns out, however, analogues of Theorem 1.2 hold for several other families of pairs of spectral projections arising from non-commuting observables. A few such examples are formulated in Section 2, and a modest extension of Theorem 1.2 is included in Section 4. Ultimately, we suspect that the various results presented here

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are instances of a rather general phenomenon. We refer the reader to Section 8 for details and remarks along these lines.

The numerical simulations (originally by Y. Le Floch) of $(\|C_n\|_{\mathrm{op}})_{n=2}^{\infty}$ imply further intriguing properties.

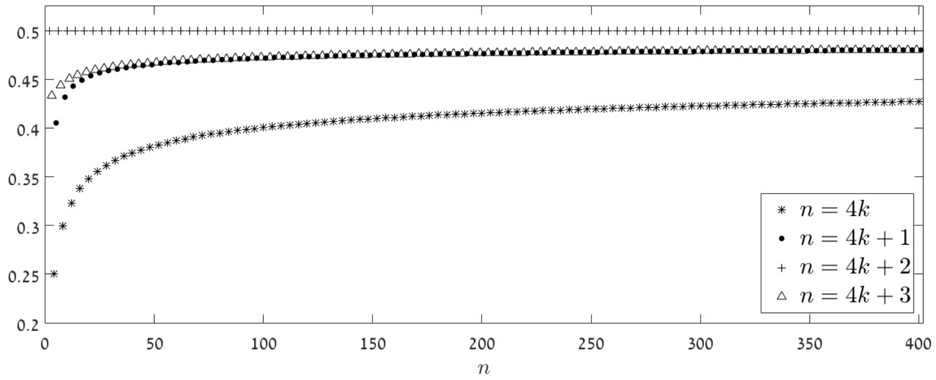

Figure 1: $\|C_n\|_{\mathrm{op}}$ as a function of n.

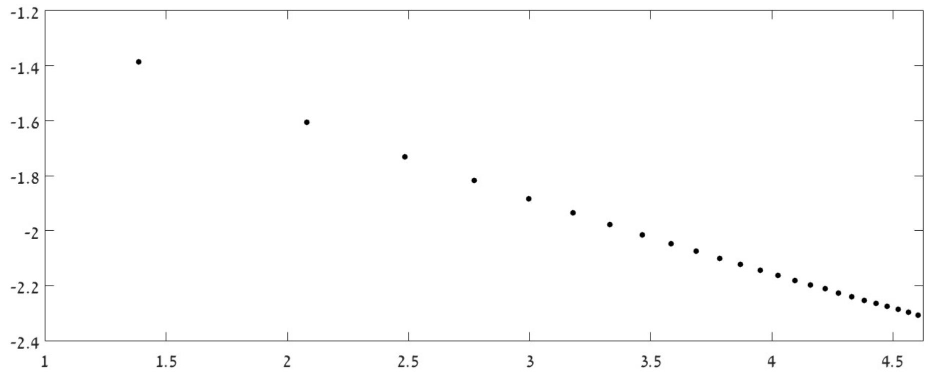

Figure 2: $\ln\left(\frac{1}{2} - \|C_{4k}\|_{\mathrm{op}}\right)$ as a function of $\ln(4k)$.

Notably, $\|C_n\|_{\mathrm{op}}$ appears to depend on the dimension of the representation modulo 4. More precisely,

Conjecture 1.3. $\frac{\|C_{4n+3}\|_{\mathrm{op}} - \|C_{4n+1}\|_{\mathrm{op}}}{\|C_{4n+p}\|_{\mathrm{op}} - \|C_{4n}\|_{\mathrm{op}}} = o(1)$ for $p = 1, 3$.

$$\frac{\|C_{4n+3}\|_{\mathrm{op}} - \|C_{4n+1}\|_{\mathrm{op}}}{\frac{1}{2} - \|C_{4n+p}\|_{\mathrm{op}}} = o(1) \text{ for } p = 0, 1, 3.$$

However, the convergence rate of the various sequences is presently unknown[1].

Finally, it holds that $\|C_n\|_{\mathrm{op}} \geq \frac{1}{4}$ for every $n \geq 2$, as depicted above. The lower bound was established as part of our initial studies of the sequence $(C_n)_{n=2}^{\infty}$, joint with Y. Le Floch and L. Polterovich.

The proof that $\|C_{4n+2}\|_{\mathrm{op}} = \frac{1}{2}$ for every $n \in \mathbb{N}_0$ is quite straightforward and requires little but symmetries. Otherwise, the lower bound is derived through specific elements of the matrix representing C_n in some orthonormal eigenbasis of J_z. The arguments involved are relatively simple when n is odd, thanks to

[1] The application of linear regression suggests a rate not faster than $O\left(n^{-\frac{1}{4}}\right)$, but otherwise does not seem to provide a conclusive answer.

symmetries again. When $n \equiv 0 \mod 4$, however, the bound is established through the use of brute-force techniques. This amounts to cumbersome calculations, since the immediate formula for the elements of C_n is complicated to estimate directly as $n \to \infty$ (and involves sums of products of certain special functions).

The proof of Theorem 1.2 also revolves around the matrix elements of C_n and their limits as $n \to \infty$. The proof appears in Section 4, relying on two main ingredients which are presented subsequently. Namely, in Section 5 we derive a concise integral formula for the elements of C_n in terms of the corresponding matrix elements of the one-parameter subgroup

$$\left\{ e^{-i\theta J_y} \middle| -2\pi \leq \theta < 2\pi \right\}.$$

Thus, we avoid the initial, complicated expressions for the elements of C_n altogether. In Section 6, we use a classical ([26], 8.21.12) asymptotic estimate for the elements (up to normalization[2]) of $e^{-i\theta J_y}$ to study the aforementioned integral formula. The estimate is valid in a restricted range of indices, which nonetheless suffices for our purpose, i.e., to calculate the limits of sufficiently many of the elements of C_n.

Ultimately, for every positive integer N and every $n > 2N+1$, we invert the rows of some sub-matrix of $-C_n$ to obtain a collection $C_{n,N} \in M_N(\mathbb{C})$, such that $C_{n,N}$ is a sub-matrix of $C_{n,N+1}$ whenever $n > 2N+3$, and furthermore, $\lim_{n\to\infty} C_{n,N}$ exists and has constant elements along the anti-diagonals.

The latter, we recall, is the defining property of (possibly infinite) Hankel matrices ([13]). More generally, any operator on some Hilbert space whose matrix relative to some orthonormal basis is a Hankel matrix may be considered as a Hankel operator. Thus, we show that

$$\left(\lim_{n\to\infty} C_{n,N} \right)_{N\geq 1}$$

is the sequence of truncated matrices of some fixed Hankel operator[3] whose norm, it turns out, equals $\frac{1}{2}$. This suffices to conclude the proof.

We note (again) that a slightly extended version of Theorem 1.2 is also included in Section 4. The proof is pretty much identical, and involves slightly modified Hankel operators.

2. Analogues of Theorem 1.2

The results of this part are proven in Section 7. Our first example involves the standard quantum model for a particle in a line (i.e., in $\mathbb{R}$). The next two examples are similar to the first, but involve the configuration spaces $\mathbb{T}$, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ rather than $\mathbb{R}$. The final example is formulated in terms of the representation theory of the group of orientation preserving Euclidean plane isometries.

Let X, Ξ denote the position and momentum operators on $L^2(\mathbb{R})$, acting on a smooth $f \in L^2(\mathbb{R})$ by

$$Xf(x) = xf(x), \ \Xi f(x) = -i\hbar f'(x).$$

[2] The estimate is formulated in terms of Jacobi polynomias, rather than Wigner d-functions, and applies in significantly more general settings. See (13) for the original result, and Conclusion 6.1 for the adaptation to our present settings.

[3] Notably, the same Hankel operator appears in all of the cases addressed in Section 7.

Theorem 2.1. *Consider the commutator* $C_\hbar^{(1)} = [\Pi_X, \Pi_\Xi]$, *where*

$$\Pi_X = \mathbb{1}_{(0,\infty)}(X),\ \Pi_\Xi = \mathbb{1}_{(0,\infty)}(\Xi).$$

Then $C_\hbar^{(1)} \equiv C^{(1)}$ *is independent of* $\hbar$, *and* $\|C^{(1)}\|_{op} = \frac{1}{2}$.

Similarly, define the operators $\Theta,\ Z$ on $L^2(S^1) \simeq L^2\left([0,2\pi), \frac{1}{2\pi}d\theta\right)$ by

$$\Theta u(\theta) = \theta u(\theta),\ Zu(\theta) = -i\frac{2\pi}{n}u'(\theta),$$

where $u \in C^\infty(S^1)$ and $n \in \mathbb{N}$. The operators $\Theta,\ Z$ may be used to construct an analogue of Weyl quantization for the cylinder T^*S^1 ([11], [20], [19]).

Theorem 2.2. *Let* $C_n^{(2)} = [\Pi_\Theta, \Pi_Z]$, *where*

$$\Pi_\Theta = \mathbb{1}_{(0,\infty)}(\cos\Theta),\ \Pi_Z = \mathbb{1}_{(0,\infty)}(\cos Z).$$

Then $\lim_{n\to\infty}\|C_n^{(2)}\|_{\mathrm{op}} = \frac{1}{2}$. *The same result holds if we replace the Heaviside function with* $\mathbb{1}_{(a,\infty)}$, *where* $a \in [0,1)$.

The previous two examples may be formulated in terms of the representation theory of the Heisenberg groups $H(\mathbb{R})$ and $H(\mathbb{T})$ (i.e., the group of unitary operators on $L^2(\mathbb{T})$ generated by translation operators and by operators of multiplication by characters). A similar result holds for representations of finite Heisenberg groups ([24, 18, 30, 27]) associated to $\mathbb{Z}_n$ as $n \to \infty$. This problem was suggested to us by D. Kazhdan, and its solution provided the model of the proof for $SU(2)$ (as well as for the rest of the examples).

Let g_1, g_2 define an irreducible unitary representation of the finite Heisenberg group $H(\mathbb{Z}_n)$ on $l^2(\mathbb{Z}_n)$ by

$$g_1(f)(x) = f(x+1),\ g_2(f)(x) = e^{\frac{2\pi xi}{n}}f(x).$$

Let Π_1, Π_2 denote the orthogonal projections on the subspaces of $l^2(\mathbb{Z}_n)$ spanned by eigenvectors of g_1, g_2 corresponding to eigenvalues with positive real part. Consider the commutator

$$C_n^{(3)} = [\Pi_1, \Pi_2].$$

The parallel of Theorem 1.2 is the following.

Theorem 2.3. $\lim_{n\to\infty}\|C_n^{(3)}\|_{\mathrm{op}} = \frac{1}{2}$, *and the same holds if we replace the Heaviside function with* $\mathbb{1}_{(a,\infty)}$, *where* $a \in [0,1)$.

Additionally, as in the case of $SU(2)$,

Theorem 2.4. (Y. Le Floch) $\|C_{4n+2}^{(3)}\|_{\mathrm{op}} \equiv \frac{1}{2}$ *for every* $n \in \mathbb{N}_0$.

The numerical simulations (Figure 3) of the sequence $\left(\|C_n^{(3)}\|_{\mathrm{op}}\right)_{n=2}^\infty$ feature some striking similarities with the equivalent simulations (Figure 1) for $SU(2)$, including the conjectured dependence on $n \mod 4$.

Our final example may be derived as a consequence of Theorem 1.2. Let $SE(2)$ denote the group of orientation preserving Euclidean plane isometries. The asymptotic formula of Conclusion 6.1, which underlies the proof of the Theorem 1.2, provides a non-trivial relation ([7, 22, 25]) between the representations of $SU(2)$ and of $SE(2)$. This fact led us to study the analogue of C_n for the irreducible representations of $SE(2)$.

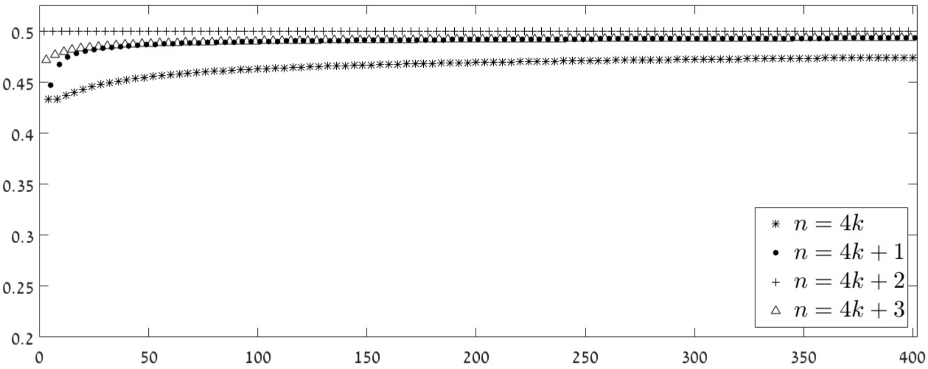

Figure 3: The norm of $C_n^{(3)}$ as a function of n for the Heisenberg groups $H(\mathbb{Z}_n)$. Note the similarity to the graph for $SU(2)$. In particular, the graph also appears to depend on $n \mod 4$.

Consider $L^2(S^1) \simeq L^2\left([0,2\pi), \frac{1}{2\pi} d\phi\right)$ as before, and fix $R > 0$. Let X_1, X_2 denote the multiplication operators $\mathcal{M}_{R\cos\phi}$, $\mathcal{M}_{R\sin\phi}$ respectively, and let Φ denote the differentiation operator $f \mapsto -if'$. Then X_1, X_2, Φ satisfy the commutation relations

$$[X_1, X_2] = 0,\ [X_2, \Phi] = iX_1,\ [\Phi, X_1] = iX_2,$$

and are well known to generate an irreducible unitary representation of $SE(2)$ on $L^2(S^1)$. In fact, every non-trivial irreducible unitary representation of $SE(2)$ is equivalent to the representation generated by X_1, X_2, Φ for some $R > 0$ ([8]).

Theorem 2.5. *Let $C_R^{(4)} = [\Pi_{X_1}, \Pi_\Phi]$, where*

$$\Pi_{X_1} = \mathbb{1}_{(0,\infty)}(X_1),\ \Pi_\Phi = \mathbb{1}_{(0,\infty)}(\Phi).$$

Then $C_R^{(4)} \equiv C^{(4)}$ is independent of R, and $\|C^{(4)}\|_{\text{op}} = \frac{1}{2}$.

3. Preliminaries on Hankel operators

Theorem 1.2 is essentially reduced to the problem of the calculation of $\|H_E\|_{\text{op}}$ for some Hankel operator H_E which we now specify. The operator H_E appears and plays roughly the same role in all of the results of Section 7 as well.

Let $\mathbb{T} \subset \mathbb{C}$ denote the unit circle, and declare the functions $z \mapsto z^p$, $p \in \mathbb{Z}$ to be an orthonormal basis of $L^2(\mathbb{T})$. Let $\Pi_\mathbb{T} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denote the Cauchy-Szegö projection on the Hardy space

$$H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) \mid \hat{f}(p) = 0 \text{ for every } p < 0\}.$$

Here, $\hat{f}(p) = \langle f, z^p \rangle_{L^2(\mathbb{T})}$ denotes the p-th Fourier coefficient of f. Finally, let $\mathcal{M}_\phi$ denote the multiplication operator defined by a function $\phi \in L^\infty(\mathbb{T})$.

Definition 3.1. The Hankel operator corresponding to the symbol $\phi \in L^\infty(\mathbb{T})$ is defined as

$$H_\phi = (\text{Id} - \Pi_\mathbb{T})\, \mathcal{M}_\phi \Pi_\mathbb{T} : H^2(\mathbb{T}) \to \left(H^2(\mathbb{T})\right)^\perp.$$

Let $[H_\phi] = (h_{k,l})_{k,l\geq 1}$ denote the matrix representing H_ϕ in the standard bases

$$\mathcal{B} = \{z^{p-1} \mid p > 0\},\ \mathcal{C} = \{z^{-p} \mid p > 0\}$$

of $H^2(\mathbb{T})$ and $H^2(\mathbb{T})^\perp$, respectively. Then

$$h_{k,l} = \langle \phi z^{l-1}, z^{-k} \rangle = \hat{\phi}(1-k-l).$$

The truncated matrices associated with a symbol $\phi \in L^\infty(\mathbb{T})$ are relevant to us as well. For an infinite matrix $A = (a_{k,l})_{k,l\geq 1}$ denote $A_N = (a_{k,l})_{1\leq k,l\leq N}$. We will require the following basic fact.

Lemma 3.2. $\lim_{N\to\infty} \|[H_\phi]_N\|_{\mathrm{op}} = \|H_\phi\|_{\mathrm{op}}$.

Finally, let $E = \{z \in \mathbb{T} \mid \Re z > 0\}$ denote the right half of the unit circle. Denote $H_E = H_{\mathbb{1}_E}$, where $\mathbb{1}_E$ is the indicator function of E. The Fourier coefficients of $\mathbb{1}_E$ are specified by

$$\hat{\mathbb{1}}_E(p) = \begin{cases} \frac{1}{2} & \text{if } p = 0, \\ \sin\left(\frac{\pi p}{2}\right)\frac{1}{\pi p} & \text{if } p \neq 0 \end{cases}. \tag{1}$$

Perhaps somewhat surprisingly, the operator H_E is closely related to the commutators C_n. In particular, the proof of Theorem 1.2 relies on the following.

Lemma 3.3. $\|H_E\|_{\mathrm{op}} = \frac{1}{2}$. *Hence by Lemma* 3.2, $\lim_{N\to\infty} \|[H_E]_N\|_{\mathrm{op}} = \frac{1}{2}$.

We present below the (simple) proof of the lemma. The inequality $\|H_E\|_{\mathrm{op}} \leq \frac{1}{2}$ follows from the contents of Section 4, though we include a separate proof using Nehari's Theorem on Hankel operators[4]. The complementary inequality $\|H_E\|_{\mathrm{op}} \geq \frac{1}{2}$ is a direct consequence of Power's Theorem on Hankel operators with piecewise continuous symbols.

We now apply two fundamental theorems on Hankel operators to obtain the proof of Lemma 3.3. For a complex sequence $a = (a_k)_{k\in\mathbb{N}_0}$, define the Hankel matrix $S_a = (a_{k+l})_{k,l\in\mathbb{N}_0} : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$.

Theorem 3.4 ([12, Theorem 2.1])**.** *S_a is bounded on $l^2(\mathbb{N}_0)$ if and only if there exists $\phi \in L^\infty(\mathbb{T})$ such that $a_k = \hat{\phi}(k)$ for every $k \geq 0$. In this case,*

$$\|S_a\|_{\mathrm{op}} = \inf\{\|\phi\|_\infty \mid \hat{\phi}(k) = a_k \textit{ for every } k \geq 0\}.$$

We recall that $[H_E] = (h_{k,l})_{k,l\geq 1} = \big(\hat{\mathbb{1}}_E(1-k-l)\big)_{k,l\geq 1}$, so the sequence associated with $[H_E]$ is

$$\big(\hat{\mathbb{1}}_E(-1-k)\big)_{k\in\mathbb{N}_0} = \left(\widehat{\bar{z}\mathbb{1}_E}(k)\right)_{k\in\mathbb{N}_0}.$$

If we define $\phi(z) = \bar{z}\big(\mathbb{1}_E(z) - \frac{1}{2}\big)$, and choose $k \geq 0$, then

$$\hat{\phi}(k) = \langle \mathbb{1}_E - \frac{1}{2}, z^{k+1} \rangle = \hat{\mathbb{1}}_E(k+1) = \hat{\mathbb{1}}_E(-1-k).$$

[4] The inequality also follows immediately from the previously mentioned general fact about commutators of orthogonal projections, applied to the projections $\Pi_{\mathbb{T}}$ and $\mathcal{M}_{\mathbb{1}_E}$.

Therefore

Corollary 3.5. $\|H_E\|_{\text{op}} \leq \|\phi\|_\infty = \frac{1}{2}$.

To obtain the complementary inequality, assume that $\phi \in L^\infty(\mathbb{T})$ has well defined one-sided limits at every point of $\mathbb{T}$. For $\alpha \in \mathbb{T}$, define the jump of ϕ at α as

$$\phi_\alpha = \frac{1}{2} \lim_{t \to 0^+} \left(\phi\left(\alpha e^{it}\right) - \phi\left(\alpha e^{-it}\right) \right).$$

Then

Theorem 3.6 ([17]). *The essential spectrum of the Hankel operator H_ϕ is given by*

$$\sigma_{ess}(H_\phi) = [0, i\phi_1] \cup [0, i\phi_{-1}] \cup \left(\cup_{\alpha \in \mathbb{T} \setminus \{\pm 1\}} \left[-\sqrt{-\phi_\alpha \phi_{\bar{\alpha}}}, \sqrt{-\phi_\alpha \phi_{\bar{\alpha}}} \right] \right).$$

The inequality $\|H_E\|_{\text{op}} \geq \frac{1}{2}$ now follows, since

Corollary 3.7. *The essential spectrum of H_E equals $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Note that $\sigma_{ess}(H_E)$ is a (closed) subset of the spectrum of H_E, hence $\|H_E\|_{\text{op}} \geq \frac{1}{2}$.*

4. The proof of Theorem 1.2

We begin with a few preliminary notations and definitions ([3], [28]). Recall that the spectrum of J_x, J_y, J_z equals the set $\{j, j-1, ..., -j\}$, where $j = \frac{1}{2}(n-1)$ is the spin number associated with the representation. Let

$$\mathcal{E}_{z,j} = \{e_m \mid m = j, j-1, ..., -j\}$$

denote an orthonormal eigenbasis of J_z, such that $J_z e_m = m e_m$. The matrices representing $\mathbb{1}_{(0,\infty)}(J_x)$, $\mathbb{1}_{(0,\infty)}(J_z)$ in $\mathcal{E}_{z,j}$ may be written as

$$P_{x,j} = \left(P_{x,j,m',m}\right)_{|m'|,|m| \leq j} = \begin{pmatrix} P_{1,x,j} & P_{2,x,j} \\ P_{2,x,j}^* & P_{3,x,j} \end{pmatrix}, \; P_{z,j} = \begin{pmatrix} \tilde{I}_j & 0 \\ 0 & 0 \end{pmatrix}.$$

Here, $\tilde{I}_j$ is the identity matrix of size $\left\lfloor j + \frac{1}{2} \right\rfloor$. The matrix of C_n is given by

$$[P_{x,j}, P_{z,j}] = \begin{pmatrix} 0 & -P_{2,x,j} \\ P_{2,x,j}^* & 0 \end{pmatrix},$$

hence $\|C_n\|_{\text{op}} = \|P_{2,x,j}\|_{\text{op}}$. In our notations,

$$P_{2,x,j} = \left(P_{x,j,m',m}\right)_{j \geq m' > 0 \geq m \geq -j}.$$

We turn our attention to the "central elements" of $P_{x,j}$, that is, to the sequences $\left(P_{x,j+k,m',m}\right)_{k \in \mathbb{N}}$ with j, m', m fixed such that $m', m \in \{j, j-1, ..., -j\}$. According to Conclusion 6.5,

$$\lim_{k \to \infty} P_{x,j+k,m',m} = \hat{\mathbb{1}}_E(m - m'),$$

where we recall that $\mathbb{1}_E$ is the indicator function of the right half of the unit circle in $\mathbb{C}$, as well as the symbol of the Hankel operator H_E of Lemma 3.3.

Thus, evidently, for fixed $N \in \mathbb{N}$, the bottom left $N \times N$ corner of $P_{2,x,j}$ converges in $M_N(\mathbb{C})$ as $n \to \infty$. More precisely,

Corollary 4.1. *Let $N \in \mathbb{N}$, and $j > N$. Let $C_{n,N} = (c_{n,k,l})_{k,l=1,\dots,N}$, where*

$$c_{n,k,l} = \begin{cases} P_{x,j,k,1-l} & \text{if } n \in 2\mathbb{N}+1 \\ P_{x,j,-\frac{1}{2}+k,\frac{1}{2}-l} & \text{if } n \in 2\mathbb{N} \end{cases}$$

Then $\lim_{n\to\infty} C_{n,N} = [H_E]_N$.

The proof of Theorem 1.2 easily follows now, since $\lim_{N\to\infty} \|[H_E]_N\|_{\mathrm{op}} = \frac{1}{2}$ by Lemma 3.3, and clearly

$$\frac{1}{2} \geq \liminf_n \|C_n\|_{\mathrm{op}} \geq \liminf_n \|C_{n,N}\|_{\mathrm{op}} = \|[H_E]_N\|_{\mathrm{op}}. \tag{2}$$

Letting $N \to \infty$, we obtain the desired result.

Let us now outline the proof of the following extension of Theorem 1.2.

Theorem 4.2. *Fix $a \in [0,1)$ and $b \in (0,1]$, and let*

$$C_{n,a,b} = \left[\mathbb{1}_{\left(a\left(j+\frac{1}{2}\right),\infty\right)}(J_x), \mathbb{1}_{\left(0,b\left(j+\frac{1}{2}\right)\right]}(J_z)\right].$$

Then $\lim_{n\to\infty} \|C_{n,a,b}\|_{\mathrm{op}} = \frac{1}{2}$.

This can be extended further, for instance by replacing $\mathbb{1}_{\left(a\left(j+\frac{1}{2}\right),\infty\right)}(J_x)$ with spectral projections corresponding to (not necessarily open) intervals whose end-points are $a_1\left(j+\frac{1}{2}\right)$, $a_2\left(j+\frac{1}{2}\right)$, where $0 < a_1 < a_2 \leq \infty$. However, once $0 < a_1$ is chosen, the projection arising from J_z must correspond to an interval of the form above (or $(0,\infty)$), due to the limitations of the asymptotic estimate underlying the results of Section 6.

The proof of Theorem 4.2 is rather identical to that of Theorem 1.2, except that we apply the more general Conclusion 6.6, instead of Conclusion 6.5. However, we note that the conjectured modulo 4 dependence on the dimension n seems to be (more or less) a unique feature of the case $a = b = 0$ (see Figures 6, 7, for example).

Let $P_{x,a,j}$, $Q_{z,b,j}$ denote the matrices representing the spectral projections in $\mathcal{E}_{z,j}$. Then

$$P_{x,a,j} = (P_{x,a,j,m',m})_{|m'|,|m|\leq j} = \begin{pmatrix} P_{1,x,a,j} & P_{2,x,a,j} \\ P^*_{2,x,a,j} & P_{3,x,a,j} \end{pmatrix}, \; Q_{z,b,j} = \begin{pmatrix} \tilde{I}_{b_j} & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\tilde{I}_{b_j} = \begin{pmatrix} 0 & 0 \\ 0 & I_{b_j} \end{pmatrix},$$

and $b_j = \left\lfloor b\left(j+\frac{1}{2}\right)\right\rfloor$, so that $\lim_{j\to\infty} b_j = \infty$. Hence, $C_{n,a,b}$ is represented by the matrix

$$[C_{n,a,b}] = \begin{pmatrix} \left[P_{1,x,a,j}, \tilde{I}_{b_j}\right] & -\tilde{I}_{b_j} P_{2,x,a,j} \\ P^*_{2,x,a,j}\tilde{I}_{b_j} & 0 \end{pmatrix}.$$

We may proceed to define a sequence of sub-matrices of $[C_{n,a,b}]$ as before, since $b_j \xrightarrow{j\to\infty} \infty$. According to Conclusion 6.6,

$$\lim_{k\to\infty} P_{a,x,j+k,m',m} = \hat{\mathbb{1}}_{E_a}(m-m'),$$

where $E_a = \{z \in \mathbb{T} \mid \Re z > a\}$. It follows that the bottom left $N \times N$ corner of $\tilde{I}_{b_j} P_{2,x,a,j}$ converges in $M_N(\mathbb{C})$ to the truncated matrix of the Hankel operator $H_{\mathbb{1}_{E_a}}$, whose norm equals $\frac{1}{2}$ by the same arguments that were applied to H_E.

5. Matrix elements of spectral projections

In this section, we establish a concise integral formula for the coefficients of $P_{x,j} = (P_{x,j,m',m})_{|m'|,|m|\le j}$, which is the matrix representing the spectral projection $\mathbb{1}_{(0,\infty)}(J_x)$ in the basis $\mathcal{E}_{z,j}$. The latter, we recall, is an eigenbasis of J_z. The eventual formula that we obtain for $P_{x,j,m',m}$ treats the cases $m-m' \in 2\mathbb{Z}$ and $m-m' \in 2\mathbb{Z}+1$ separately.

The Wigner small d-matrix $d^j(\theta) = \left(d^j_{m',m}(\theta)\right)_{|m'|,|m|\le j}$ is the matrix of $e^{-i\theta J_y}$ in the basis $\mathcal{E}_{z,j}$. It is fundamental in the representation theory of $SU(2)$. The Wigner d-functions $d^j_{m',m}(\theta)$ are real valued 4π-periodic trigonometric polynomials, commonly specified by the formula

$$d^j_{m',m}(\theta) = \sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}}\cdot$$

$$\sum_s (-1)^{m'-m+s} \binom{j+m'}{j+m-s} \binom{j-m'}{s} \left(\cos\frac{\theta}{2}\right)^{2j+m-m'-2s} \left(\sin\frac{\theta}{2}\right)^{m'-m+2s}.$$

The parity of $d^j_{m',m}$ with respect to m', m and θ is specified by ([28], 4.4)

$$d^j_{m',m}(-\theta) = (-1)^{m'-m} d^j_{m',m}(\theta) = d^j_{m,m'}(\theta) = d^j_{-m',-m}(\theta). \tag{3}$$

Another useful relation is

$$d^j_{m',m}(\theta+\pi) = (-1)^{j-m} d^j_{m',-m}(\theta). \tag{4}$$

Finally, we will rely on the Fourier expansion of $d^j_{m',m}$, specified by ([3], 3.78, [6])

$$d^j_{m',m}(\theta) = e^{i\frac{\pi}{2}(m-m')} \sum_{\mu=-j}^{j} d^j_{m,\mu}\left(\frac{\pi}{2}\right) d^j_{m',\mu}\left(\frac{\pi}{2}\right) e^{-i\mu\theta}. \tag{5}$$

We may rotate one spin operator to another, and in particular, J_x and J_z are related by the formula $e^{i\frac{\pi}{2}J_y} J_x e^{-i\frac{\pi}{2}J_y} = J_z$. This means that the vectors

$$f_m = e^{-i\frac{\pi}{2}J_y} e_m, \; m = j, j-1, ..., -j$$

form an orthonormal eigenbasis of J_x, with $J_x f_m = m f_m$. Note that

$$P_{x,j,m',m} = \langle \mathbb{1}_{(0,\infty)}(J_x) e_m, e_{m'} \rangle = \sum_{\mu>0} \langle e_m, f_\mu \rangle \langle f_\mu, e_{m'} \rangle.$$

Consequently,
$$P_{x,j,m',m} = \sum_{\mu>0} d^j_{m,\mu}\left(\frac{\pi}{2}\right) d^j_{m',\mu}\left(\frac{\pi}{2}\right).$$

Comparing the last expression with (5), we deduce that $P_{x,j,m',m}$ equals the sum of negative Fourier coefficients of $d^j_{m',m}$, up to multiplication by $e^{i\frac{\pi}{2}(m'-m)}$. Equivalently (see Section 3),

$$P_{x,j,m',m} = e^{i\frac{\pi}{2}(m'-m)} \left(\mathrm{Id} - \Pi_{\mathbb{T}}\right) (d^j_{m',m})(0).$$

$\Pi_{\mathbb{T}}$ acts on $L^2(\mathbb{T})$ by $z^p \mapsto \mathbb{1}_{[0,\infty)}(p) z^p$. A closely related operator is the periodic Hilbert transform $\mathcal{H}_{\mathbb{T}}$, which acts by $z^p \mapsto -i\mathrm{sgn}(p) z^p$.

Thus, another equivalent formula is

$$P_{x,j,m',m} = \frac{1}{2} e^{i\frac{\pi}{2}(m'-m)} \left(d^j_{m',m}(0) - \left\langle d^j_{m',m}, 1 \right\rangle_{L^2(\mathbb{T})} - i\mathcal{H}_{\mathbb{T}} \left(d^j_{m',m} \right)(0) \right). \tag{6}$$

Furthermore, by (5), the zeroth Fourier coefficient of $d^j_{m',m}$ is specified by

$$\langle d^j_{m',m}, 1 \rangle_{L^2(\mathbb{T})} = \begin{cases} e^{i\frac{\pi}{2}(m-m')} d^j_{m,0}\left(\frac{\pi}{2}\right) d^j_{m',0}\left(\frac{\pi}{2}\right) & \text{if } j \in \mathbb{N}, \\ 0 & \text{if } j \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}. \end{cases} \tag{7}$$

$\mathcal{H}_{\mathbb{T}}$ maps even functions[5] to odd functions and vice versa. In light of the parity properties of (3), we finally obtain the following.

Corollary 5.1. *The matrix elements of* $\mathbb{1}_{(0,\infty)}(J_x)$ *in the eigenbasis* $\mathcal{E}_{z,j}$ *of* J_z *are given by*

$$P_{x,j,m',m} = \begin{cases} \frac{1}{2} e^{i\frac{\pi}{2}(m'-m)} \left(\delta_{m',m} - \langle d^j_{m',m}, 1 \rangle_{L^2(\mathbb{T})} \right) & \textit{if } m' - m \in 2\mathbb{Z}, \\ -\frac{i}{2} e^{i\frac{\pi}{2}(m'-m)} \mathcal{H}_{\mathbb{T}}(d^j_{m',m})(0) & \textit{if } m' - m \in 2\mathbb{Z}+1. \end{cases}$$

Here $m', m \in \{j, j-1, ..., -j\}$ *and* $\delta_{m',m}$ *is Kronecker's delta.*

The periodic Hilbert transform admits the representation

$$\mathcal{H}_{\mathbb{T}} f(\theta_0) = \frac{1}{4\pi} \lim_{\varepsilon \to 0^+} \int_{\varepsilon \le |\theta| \le 2\pi} f(\theta) \cot\left(\frac{\theta_0 - \theta}{4} \right) d\theta,$$

where $f \in L^2(\mathbb{T})$ is a 4π-periodic function. For $f = d^j_{m',m}$ with $m - m' \in 2\mathbb{Z}+1$, we obtain the formula

$$\mathcal{H}_{\mathbb{T}}(d^j_{m',m})(0) = -\frac{1}{2\pi} \int_0^{2\pi} d^j_{m',m}(\theta) \cot\left(\frac{\theta}{4} \right) d\theta, \tag{8}$$

which will be studied in the next section.

Next, we extend (6) to projections of the form $\mathbb{1}_{\left(a\left(j+\frac{1}{2}\right),\infty\right)}(J_x)$, where $0 \le a < 1$. More generally, the method may be used to obtain similar formulas for the elements of spectral projections corresponding to (not necessarily open) intervals with end-points at $a_1\left(j+\frac{1}{2}\right)$, $a_2\left(j+\frac{1}{2}\right)$, where $a_1 < a_2$ (hence also for projections corresponding to combinations of such intervals). The matrix elements in the present case are given by

$$\begin{aligned} P_{x,a,j,m',m} = \left\langle \mathbb{1}_{\left(a\left(j+\frac{1}{2}\right),\infty\right)}(J_x) e_m, e_{m'} \right\rangle &= \sum_{\mu > a\left(j+\frac{1}{2}\right)} \langle e_m, f_\mu \rangle \langle f_\mu, e_{m'} \rangle \\ &= \sum_{\mu > a\left(j+\frac{1}{2}\right)} d^j_{m,\mu}\left(\frac{\pi}{2}\right) d^j_{m',\mu}\left(\frac{\pi}{2}\right). \end{aligned}$$

This formula is the analogue of (5), and it also admits an interpretation through the Fourier expansions of the Wigner d-functions.

[5] i.e., functions $f \in L^2(\mathbb{T})$ with $\hat{f}(p) = \hat{f}(-p)$ for every $p \in \mathbb{Z}$.

Indeed, denote the Fourier coefficients of $d^j_{m',m}$ by $\hat{d}^j_{m',m}(p)$. Then

$$\hat{d}^j_{m',m}(p) = \begin{cases} e^{-i\frac{\pi}{2}(m'-m)} d^j_{m,-\frac{p}{2}}\left(\frac{\pi}{2}\right) d^j_{m',-\frac{p}{2}}\left(\frac{\pi}{2}\right) & \text{if } p = 2j, 2j-2, ..., -2j, \\ 0 & \text{otherwise,} \end{cases}$$

therefore
$$P_{x,a,j,m',m} = e^{i\frac{\pi}{2}(m'-m)} \sum_{p<\lceil -a(2j+1)\rceil} \hat{d}^j_{m',m}(p).$$

Thus, $P_{x,a,j,m',m}$ equals the sum of Fourier coefficients of $d^j_{m',m}$ corresponding to indices lesser than $\lceil -a(2j+1)\rceil$. Equivalently, we may shift the Fourier expansion of $d^j_{m',m}$ using the relation $\widehat{z^{-p}f}(l) = \hat{f}(l+p)$ to obtain

$$P_{x,a,j,m',m} = e^{i\frac{\pi}{2}(m'-m)} \left(\mathrm{Id} - \Pi_{\mathbb{T}}\right)\left(g^j_{m',m}\right)(0),$$

where $g^j_{m',m}(\theta) = e^{-ia_j\frac{\theta}{2}} d^j_{m',m}(\theta)$ and $a_j = \lceil -a(2j+1)\rceil$. As in (6), we translate the former expression to

$$P_{x,a,j,m',m} = \frac{1}{2} e^{i\frac{\pi}{2}(m'-m)} \left(\delta_{m',m} - \langle g^j_{m',m}, 1\rangle_{L^2(\mathbb{T})} - i\mathcal{H}_{\mathbb{T}}(g^j_{m',m})(0)\right). \tag{9}$$

In the final part of the next section, we will study $\lim_{k\to\infty} P_{a,x,j+k,m',m}$.

6. Limits of central matrix elements

In this section, we use an asymptotic approximation of Wigner d-functions by Bessel functions of the first kind in order to compute $\lim_{k\to\infty}\langle d^{j_k}_{m',m}, 1\rangle_{L^2(\mathbb{T})}$ and $\lim_{k\to\infty}\mathcal{H}_{\mathbb{T}}\left(d^{j_k}_{m',m}\right)(0)$, where j, m', m are fixed, $k \in \mathbb{N}$ and $j_k = j+k$. Since $P_{x,j,m',m}$ are the elements of a symmetric matrix, we further assume that $m - m' \geq 0$. The values of the limits $\lim_{k\to\infty} P_{x,j_k,m',m}$ will then follow from Conclusion 5.1.

The relevant asymptotic relation between Bessel functions and Wigner d-functions follows from a formula for the latter in terms of Jacobi polynomials.

Let $p \in \mathbb{N}_0$. The Bessel function of the first kind J_p may be specified by ([1])

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!}\left(\frac{x}{2}\right)^{2k+p} = \frac{1}{\pi}\int_0^{\pi} \cos(pt - x\sin t)dt. \tag{10}$$

We note, for later use, that for $x \in \mathbb{R}$, it holds that ([1], 9.1.7, 9.2.1)

$$J_p(x) = \mathcal{O}(x^p),\ J_p(x) = \mathcal{O}\left(x^{-\frac{1}{2}}\right) \tag{11}$$

as $x \to 0$ and as $x \to +\infty$, respectively. The Bessel functions associated with negative integers are specified by ([1], 9.1.5)

$$J_{-p}(x) = (-1)^p J_p(x) = J_p(-x). \tag{12}$$

The Jacobi polynomials $P_k^{(\alpha,\beta)}$ are a class of classical orthogonal polynomials specified by ([26], 4.3.1)

$$P_k^{(\alpha,\beta)}(x) = \frac{(-1)^k}{2^k k!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}\left[(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^k\right].$$

They are orthogonal on the interval $[-1,1]$ with respect to the weight function $W^{(\alpha,\beta)}(x)=(1-x)^\alpha(1+x)^\beta$. The results of the present section are based on the following classical estimate. Denote

$$Q^{(\alpha,\beta)}(\theta)=\left(\sin\frac{\theta}{2}\right)^\alpha\left(\cos\frac{\theta}{2}\right)^\beta$$

and assume that $\alpha>-1,\beta\in\mathbb{R}$. Then ([26], 8.21.12)

$$Q^{(\alpha,\beta)}(\theta)P_k^{(\alpha,\beta)}(\cos\theta)=\frac{(k+\alpha)!}{r^\alpha k!}\sqrt{\frac{\theta}{\sin\theta}}J_k(r\theta)+E_k^{(\alpha,\beta)}(\theta), \tag{13}$$

where $r=k+\frac{\alpha+\beta+1}{2}$ and $E_k^{(\alpha,\beta)}(\theta)=\sqrt{\theta}\mathcal{O}(k^{-\frac{3}{2}})$ in intervals of the form $[0,\pi-\delta]$. Wigner d-functions are related to Jacobi polynomials by ([3], 3.72)

$$d^j_{m',m}(\theta)=\sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}}Q^{(m-m',m+m')}(\theta)P_{j-m}^{(m-m',m+m')}(\cos\theta).$$

Thus, choosing $k=j-m$, $\alpha=m-m'$, $\beta=m+m'$, we obtain a powerful asymptotic approximation[6] of $d^j_{m',m}$.

Corollary 6.1. *Fix $m,m'\in\mathbb{N}$ or $m,m'\in\frac{1}{2}\mathbb{N}\setminus\mathbb{N}$ such that $m-m'\geq-1$. Then*

$$d^j_{m',m}(\theta)=C_{j,m',m}\sqrt{\frac{\theta}{\sin\theta}}J_{m-m'}\left(\frac{2j+1}{2}\theta\right)+E_{j-m}^{(m-m',m+m')}(\theta),$$

where $E_{j-m}^{(m-m',m+m')}=\sqrt{\theta}\mathcal{O}\left(j^{-\frac{3}{2}}\right)$ in intervals of the form $[0,\pi-\delta]$, and

$$C_{j,m',m}=\sqrt{\frac{(j-m')!(j+m)!}{(j-m)!(j+m')!}}\frac{1}{\left(j+\frac{1}{2}\right)^{m-m'}}$$

satisfies $\lim_{j\to\infty}C_{j,m',m}=1$.

The asymptotic approximation may be extended to $d^j_{m',m}$ with $m-m'<0$ using the parity relations (3), and to further intervals using (4).

Conclusion 6.1 together with the fact that $\lim_{x\to\infty}J_{m-m'}(x)=0$ by (11) imply that $\lim_{j\to\infty}d^j_{m',m}(\theta)=0$ for $m-m'\geq0$ and $\theta\in(0,\pi)$ fixed. This remains true when $m-m'<0$, as may be shown using the parity properties of $d^j_{m',m}$. In particular, if $j\in\mathbb{N}$ and m' is fixed, then $\lim_{k\to\infty}d^{j+k}_{m',0}\left(\frac{\pi}{2}\right)=0$. Thus, in light of (7), we find that

$$\lim_{k\to\infty}\langle d^{j+k}_{m',m},1\rangle_{L^2(\mathbb{T})}=0. \tag{14}$$

The latter also follows from the next lemma, which will be used in the analysis of $\mathcal{H}_{\mathbb{T}}(d^j_{m',m})(0)$. As before, denote $j_k=j+k$.

Lemma 6.2. *Let $f\in L^\infty(\mathbb{T})$. Then $\lim_{k\to\infty}\langle d^{j_k}_{m',m},f\rangle_{L^2(\mathbb{T})}=0$.*

[6] We refer the reader to [22] for a survey of the asymptotic properties of Wigner d-functions.

Proof. Recall that $d^j_{m',m}(\theta)$ is an element of a unitary matrix, so $|d^j_{m',m}(\theta)| \le 1$ for every j, m', m and θ. Moreover, $\lim_{k\to\infty} d^{j_k}_{m',m}(\theta) = 0$ for $\theta \in (0,\pi)$. By the dominated convergence theorem, it follows that

$$\lim_{k\to\infty}\int_0^{\pi} f(\theta) d^{j_k}_{m',m}(\theta) d\theta = 0$$

for every $f \in L^\infty(\mathbb{T})$. The symmetries (4), (3) of $d^j_{m',m}(\theta)$ imply, similarly, that the integrals over the intervals $[\pi, 2\pi]$ and $[-2\pi, 0]$ converge to 0. ■

The lemma is not immediately applicable to $\mathcal{H}_{\mathbb{T}}(d^j_{m',m})(0)$, since by (8),

$$\mathcal{H}_{\mathbb{T}}(d^j_{m',m})(0) = -\frac{1}{2\pi}\int_0^{2\pi} d^j_{m',m}(\theta)\cot\left(\frac{\theta}{4}\right) d\theta,$$

and $\cot\left(\frac{\theta}{4}\right)$ is unbounded. However, it allows us to truncate this integral to an interval of the form $[0,\delta]$, where $\delta > 0$ is arbitrarily small.

Corollary 6.3. *Fix $0 < \delta < 1$. Then $\lim_{k\to\infty}\left[\mathcal{H}_{\mathbb{T}}(d^{j_k}_{m',m})(0) - I_{j_k,\delta}\right] = 0$, where*

$$I_{j,\delta} = -\frac{1}{2\pi}\int_0^{\delta} d^j_{m',m}(\theta)\cot\left(\frac{\theta}{4}\right) d\theta.$$

At this point, we wish to use the asymptotic formula of Conclusion 6.1. The error satisfies $E^{(m-m',m+m')}_{j-m}(\theta) = \sqrt{\theta}\mathcal{O}\left(j^{-\frac{3}{2}}\right)$, and the function $\sqrt{\theta}\cot\frac{\theta}{4}$ is integrable,

hence $$\lim_{k\to\infty}\int_0^{\delta} E^{(m-m',m+m')}_{j_k-m}(\theta)\cot\left(\frac{\theta}{4}\right) d\theta = 0.$$

Therefore, we have obtained that

$$\lim_{k\to\infty}\mathcal{H}_{\mathbb{T}}(d^{j_k}_{m',m})(0) = -\frac{1}{2\pi}\lim_{k\to\infty}\int_0^{\delta}\sqrt{\frac{\theta}{\sin\theta}} J_{m-m'}\left(\frac{2j_k+1}{2}\theta\right)\cot\left(\frac{\theta}{4}\right) d\theta. \qquad (15)$$

Let $\mathcal{H}_{\mathbb{R}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the standard Hilbert transform, specified by

$$\mathcal{H}_{\mathbb{R}} f(x) = -\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{\infty}\frac{f(x+t) - f(x-t)}{t} dt.$$

As in the case of $\mathcal{H}_{\mathbb{T}}$, if $f \in L^2(\mathbb{R})$ is even then $\mathcal{H}_{\mathbb{R}}(f)$ is odd, and vice versa. Finally, we are ready to prove the main result of this section.

Proposition 6.4. *Let $j_k = j + k$ as above, with $j \in \frac{1}{2}\mathbb{N}$ fixed and $k \in \mathbb{N}$, and fix $m', m \in \{j, j-1, ..., -j\}$. Then*

$$\lim_{k\to\infty}\mathcal{H}_{\mathbb{T}}\left(d^{j_k}_{m',m}\right)(0) = \mathcal{H}_{\mathbb{R}}\left(J_{m-m'}\right)(0).$$

When $m - m' \in 2\mathbb{Z}$, this simply says that $\mathcal{H}_{\mathbb{T}}\left(d^{j_k}_{m',m}\right)(0) = 0 = \mathcal{H}_{\mathbb{R}}\left(J_{m-m'}\right)(0)$.

Proof. Assume that $m-m'\in 2\mathbb{Z}+1$. By the substitution $x=\left(j+\frac{1}{2}\right)\theta$ in (15), it suffices to establish that $\lim_{k\to\infty} I_{j_k}=\mathcal{H}_{\mathbb{R}}\left(J_{m-m'}\right)(0)$, where

$$I_j=-\frac{1}{2\pi}\int_0^{\left(j+\frac{1}{2}\right)\delta}\sqrt{\frac{\frac{x}{j+\frac{1}{2}}}{\sin\left(\frac{x}{j+\frac{1}{2}}\right)}}J_{m-m'}(x)\cot\left(\frac{x}{4\left(j+\frac{1}{2}\right)}\right)\frac{dx}{j+\frac{1}{2}}$$

with $j\in\frac{1}{2}\mathbb{N}$. To this end, denote

$$f_j^{(1)}(x)=\sqrt{\frac{\frac{x}{j+\frac{1}{2}}}{\sin\left(\frac{x}{j+\frac{1}{2}}\right)}},\ f_j^{(2)}(x)=\frac{1}{j+\frac{1}{2}}\cot\left(\frac{x}{4\left(j+\frac{1}{2}\right)}\right),$$
$$f_j=\mathbb{1}_{\left(0,\delta\left(j+\frac{1}{2}\right)\right)}f_j^{(1)}f_j^{(2)}J_{m-m'},$$

Then $\lim_{j\to\infty} f_j^{(1)}(x)=1$ and $\lim_{j\to\infty} f_j^{(2)}(x)=\frac{4}{x}$. Additionally,

$$0<f_j^{(1)}(x)<M_\delta=\max_{\theta\in[0,\delta]}\sqrt{\frac{\theta}{\sin\theta}},\ 0<f_j^{(2)}(x)\le\frac{4}{x}$$

for every $j\in\frac{1}{2}\mathbb{N},\ x\in\left[0,\delta\left(j+\frac{1}{2}\right)\right]$. Thus,

$$|f_j(x)|\le\frac{4}{x}\left|J_{m-m'}(x)\right|,\ \lim_{j\to\infty} f_j(x)=\frac{4}{x}J_{m-m'}(x)$$

for every $x\in(0,\infty)$. Finally, $\mathbb{1}_{(0,\infty)}(x)\frac{4}{x}J_{m-m'}(x)\in L^1(\mathbb{R})$, hence by the dominated convergence theorem, we deduce that

$$-\frac{1}{2\pi}\lim_{j\to\infty}\int_{\mathbb{R}} f_j(x)dx=-\frac{2}{\pi}\int_0^\infty\frac{J_{m-m'}(x)}{x}dx=\mathcal{H}_{\mathbb{R}}(J_{m-m'})(0),$$

where the last equality holds since $J_{m-m'}$ is an odd function by (12). ■

The above results, together with Conclusion 5.1, imply that

$$\lim_{k\to\infty} P_{x,j_k,m',m}=\begin{cases}\frac{1}{2} & \text{if } m-m'=0,\\ -\frac{i}{2}e^{-i\frac{\pi}{2}(m-m')}\mathcal{H}_{\mathbb{R}}(J_{m-m'})(0) & \text{if } m-m'\neq 0\end{cases},$$

where we recall that $\mathcal{H}_{\mathbb{R}}(J_p)(0)=0$ for p even. The Hilbert transform of the Bessel function J_p admits the alternative representation ([16], 15.9.1)

$$\mathcal{H}_{\mathbb{R}}(J_p)(x)=\frac{1}{\pi}\int_0^\pi\sin(x\sin t-pt)dt.$$

Thus, when $p\neq 0$, we find that $\mathcal{H}_{\mathbb{R}}(J_p)(0)=-\frac{1}{\pi p}\left(1-(-1)^p\right)$. Additionally, $ie^{-i\frac{\pi}{2}p}=\sin\left(\frac{\pi}{2}p\right)$ whenever p is odd, therefore finally

$$\lim_{k\to\infty} P_{x,j_k,m',m}=\begin{cases}\frac{1}{2} & \text{if } m-m'=0,\\ \sin\left(\frac{\pi}{2}(m-m')\right)\frac{1}{\pi(m-m')} & \text{if } m-m'\neq 0.\end{cases}$$

In light of (1), this establishes the relation between the elements of $\mathbb{1}_{(0,\infty)}(J_x)$ and the Fourier coefficients of $\mathbb{1}_E$, where we recall that E denotes the right half of the unit circle in $\mathbb{C}$.

Corollary 6.5. $\lim_{k\to\infty} P_{x,j_k,m',m} = \hat{\mathbb{1}}_E(m-m')$.

We now extend the latter as follows. Recall that we obtained a formula (9) for the matrix elements of the projection $\mathbb{1}_{\left(a\left(j+\frac{1}{2}\right),\infty\right)}(J_x)$ in the basis $\mathcal{E}_{z,j}$, with $0 \le a < 1$. Namely, for $g^j_{m',m}(\theta) = e^{-ia_j\frac{\theta}{2}} d^j_{m',m}(\theta)$, $a_j = \lceil -a(2j+1)\rceil$, we saw that

$$P_{x,a,j,m',m} = \frac{1}{2} e^{i\frac{\pi}{2}(m'-m)} \left(\delta_{m',m} - \langle g^j_{m',m}, 1\rangle_{L^2(\mathbb{T})} - i\mathcal{H}_{\mathbb{T}}\left(g^j_{m',m}\right)(0)\right).$$

Notably, $|g^j_{m',m}| = |d^j_{m',m}|$, therefore most of the arguments in the analysis of $P_{x,j,m',m} = P_{x,0,j,m',m}$ remain valid for $P_{x,a,j,m',m}$ with $a > 0$. Specifically,

$$g^j_{m',m}(0) = d^j_{m',m}(0) = \delta_{m',m}, \ \lim_{k\to\infty} \langle g^{j_k}_{m',m}, 1\rangle_{L^2(\mathbb{T})} = 0,$$

and since $\lim_{j\to\infty} \frac{a_j}{2j+1} = -a$, we can also establish that

$$\lim_{k\to\infty} \mathcal{H}_{\mathbb{T}}\left(g^{j_k}_{m',m}\right)(0) = \mathcal{H}_{\mathbb{R}}(f_{a,m-m'})(0),$$

where $f_{a,p}(x) = e^{aix} J_p(x)$. Moreover, using the parity of $\cos(ax), \sin(ax)$ and $J_p(x)$, we see that

$$\mathcal{H}_{\mathbb{R}}(f_{a,p})(0) = \begin{cases} -\frac{2i}{\pi}\int_0^\infty \frac{\sin(ax)J_p(x)}{x}dx & \text{if } p \in 2\mathbb{Z}, \\ -\frac{2}{\pi}\int_0^\infty \frac{\cos(ax)J_p(x)}{x}dx & \text{if } p \in 2\mathbb{Z}+1. \end{cases}$$

The Bessel function J_p is part of the integral kernel of the pth order Hankel transform, which provides a straightforward way to evaluate the integrals above. For $a < 1$, we have that ([2], 8.2.33, 8.7.2, 8.7.27)

$$\int_0^\infty \frac{\sin(ax)J_0(x)}{x}dx = \sin^{-1}(a)$$

and otherwise when $p \in 2\mathbb{Z}\setminus\{0\}$,

$$\int_0^\infty \frac{\sin(ax)J_p(x)}{x}dx = \frac{1}{p}\sin\left(p\sin^{-1}(a)\right).$$

Similarly, if $p \in 2\mathbb{Z}+1$,

$$\int_0^\infty \frac{\cos(ax)J_p(x)}{x}dx = \frac{1}{p}\cos\left(p\sin^{-1}(a)\right).$$

Combining the above, we obtain a generalization of Conclusion 6.5.

Corollary 6.6. *Assume that* $a = \cos\alpha = \sin\left(\frac{\pi}{2}-\alpha\right)$, *with* $\alpha \in [0, \frac{\pi}{2})$. *Then, using trigonometric identities for angle difference, we obtain that*

$$\lim_{k\to\infty} P_{x,a,j_k,m',m} = \begin{cases} \frac{\alpha}{\pi} & \text{if } m-m' = 0 \\ \frac{1}{\pi(m-m')}\sin((m-m')\alpha) & \text{if } m-m' \neq 0 \end{cases} = \hat{\mathbb{1}}_{E_a}(m-m'),$$

where $E_a = \{z \in \mathbb{T} \mid \Re z > a\}$.

7. Miscellaneous proofs

We begin with the proof of Theorem 2.1. Recall that X, Ξ act on a smooth function $f \in L^2(\mathbb{R})$ by

$$Xf(x) = xf(x), \quad \Xi f(x) = -i\hbar f'(x).$$

Let $\sigma_\hbar$ denote the rescaling $f(x) \mapsto \sqrt{\hbar} f(\hbar x)$. Then $\sigma_\hbar$ is unitary with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx \quad \text{since} \quad \langle \sigma_\hbar f, \sigma_\hbar g\rangle = \hbar \int_{-\infty}^{\infty} f(\hbar x)\bar{g}(\hbar x)dx = \langle f, g\rangle.$$

Let $\mathcal{F}$ denote the (unitary) Fourier transform on $L^2(\mathbb{R})$, acting on a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

We define the semiclassical Fourier transform $\mathcal{F}_\hbar$ using the scaling properties of $\mathcal{F}$ as

$$\mathcal{F}_\hbar = \sigma_{\hbar^{-1}}\mathcal{F} = \mathcal{F}\sigma_\hbar.$$

The observables X, Ξ are conjugated by $\mathcal{F}_\hbar$, that is,

$$\Xi = \mathcal{F}_\hbar^{-1} X \mathcal{F}_\hbar.$$

Consequently, so are the functional calculi of Ξ, X, where the latter consists of multiplication operators. Recall that

$$\Pi_X = \mathbb{1}_{(0,\infty)}(X) = \mathcal{M}_{\mathbb{1}_{(0,\infty)}}, \ \Pi_\Xi = \mathbb{1}_{(0,\infty)}(\Xi),$$

hence

Corollary 7.1. *The projections Π_X, Π_Ξ are related by*

$$\Pi_\Xi = \mathcal{F}^{-1}\sigma_\hbar \mathcal{M}_{\mathbb{1}_{(0,\infty)}} \sigma_{\hbar^{-1}}\mathcal{F} = \mathcal{F}^{-1}\mathcal{M}_{\mathbb{1}_{(0,\infty)}}\mathcal{F},$$

where $\mathcal{M}_f$ denotes the operator of multiplication by f.

In particular, Π_Ξ is independent of $\hbar$, hence

$$C_\hbar^{(1)} = [\Pi_X, \Pi_\Xi] = C^{(1)}$$

for some fixed, bounded operator $C^{(1)}$ on $L^2(\mathbb{R})$. Next, we recall that the Hardy space on $\mathbb{R}$ is given by

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid \mathcal{F}f(\xi) = 0 \text{ for every } \xi < 0\},$$

therefore $\Pi_\Xi = \Pi_\mathbb{R}$ is the Cauchy-Szegö projection on $H^2(\mathbb{R})$, and consequently

$$C^{(1)} = \left[\mathcal{M}_{\mathbb{1}_{(0,\infty)}}, \Pi_\mathbb{R}\right].$$

Let $C(z) = \frac{z-i}{z+i}$ denote the Cayley transform (which maps $(0,\infty) \subset \mathbb{R}$ onto $\{\Im z < 0\} \subset \mathbb{T}$). The unitary operator $U_C : L^2(\mathbb{T}) \to L^2(\mathbb{R})$ specified by

$$U_C f(x) = \pi^{-\frac{1}{2}}(x+i)^{-1} f(C(x))$$

is known ([21], p. 92) to map $H^2(\mathbb{T})$ onto $H^2(\mathbb{R})$.

Lemma 7.2. *$U_C^*\Pi_X U_C = \mathcal{M}_{\mathbb{1}_{\{\Im z<0\}}}$ and $U_C^*\Pi_\Xi U_C = \Pi_\mathbb{T}$, where the latter denotes the Cauchy-Szegö projection on $H^2(\mathbb{T})$.*

Proof. Note that $U_C^*\psi(z) = 2i\frac{\sqrt{\pi}}{1-z}\psi\left(C^{-1}(z)\right)$.

For a bounded function $\psi : \mathbb{R} \to \mathbb{R}$ and $f \in L^2(\mathbb{T})$, we obtain that

$$U_C^*\mathcal{M}_\psi U_C f(z) = 2i\frac{\sqrt{\pi}}{1-z}\psi\left(C^{-1}(z)\right) \cdot (U_C f)\left(C^{-1}(z)\right) = \left(\psi \circ C^{-1}\right)(z) f(z).$$

Then,
$$\mathbb{1}_{(0,\infty)} \circ C^{-1} = \mathbb{1}_{C((0,\infty))} = \mathbb{1}_{\{\Im z<0\}},$$

which means that $U_C^*\Pi_X U_C = \mathcal{M}_{\mathbb{1}_{\{\Im z<0\}}}$.

Next, U_C is unitary and maps $H^2(\mathbb{T})$ onto $H^2(\mathbb{R})$, hence it maps $H^2(\mathbb{T})^\perp$ onto $H^2(\mathbb{R})^\perp$. It follows immediately that $U_C^*\Pi_\mathbb{R} U_C = \Pi_\mathbb{T}$. ■

In light of the previous lemma, we conclude that

$$\left\|C^{(1)}\right\|_{\text{op}} = \left\|U_C^* C^{(1)} U_C\right\|_{\text{op}} = \left\|\left[\mathcal{M}_{\mathbb{1}_{\{\Im z<0\}}}, \Pi_\mathbb{T}\right]\right\|_{\text{op}}.$$

Consider the translation operator τ on $L^2(\mathbb{T})$, specified by $f(z) \mapsto f\left(e^{i\frac{\pi}{2}}z\right)$. Then $\tau z^m = e^{i\frac{\pi m}{2}} z^m$ for every $m \in \mathbb{Z}$, therefore $\tau^*\Pi_\mathbb{T}\tau = \Pi_\mathbb{T}$. At the same time,

$$\tau^*\mathcal{M}_f\tau = \mathcal{M}_{\tau^* f}, \quad \text{therefore} \quad \tau^*\mathcal{M}_{\mathbb{1}_{\{\Im z<0\}}}\tau = \mathcal{M}_{\mathbb{1}_E},$$

where we recall that $E = \{z \in \mathbb{T} \mid \Re z > 0\}$. Finally, if we define $\Pi_\mathbb{T}^\perp = \operatorname{Id} - \Pi_\mathbb{T}$.

Then
$$\begin{aligned}[\mathcal{M}_f, \Pi_\mathbb{T}] &= \left(\Pi_\mathbb{T} + \Pi_\mathbb{T}^\perp\right)[\mathcal{M}_f, \Pi_\mathbb{T}]\left(\Pi_\mathbb{T} + \Pi_\mathbb{T}^\perp\right) \\ &= \Pi_\mathbb{T}^\perp\mathcal{M}_f\Pi_\mathbb{T} - \Pi_\mathbb{T}\mathcal{M}_f\Pi_\mathbb{T}^\perp = H_f - H_f^*,\end{aligned}$$

where H_f denotes the Hankel operator with (real-valued) symbol f.

Corollary 7.3. *$\left\|\left[\mathcal{M}_{\mathbb{1}_{\{\Im z<0\}}}, \Pi_\mathbb{T}\right]\right\|_{\text{op}} = \|[\mathcal{M}_{\mathbb{1}_E}, \Pi_\mathbb{T}]\|_{\text{op}}$, where*

$$[\mathcal{M}_{\mathbb{1}_E}, \Pi_\mathbb{T}] = H_E \oplus (-H_E)^* : H^2(\mathbb{T}) \oplus H^2(\mathbb{T})^\perp \to H^2(\mathbb{T})^\perp \oplus H^2(\mathbb{T}). \tag{16}$$

It follows from Lemma 3.3 *that* $\|\,[\Pi_X, \Pi_\Xi]\,\|_{\text{op}} = \frac{1}{2}$.

Let us now prove Theorem 2.2. Recall that we have defined the operators Θ, Z on $L^2(\mathbb{T}) \simeq L^2\left([0, 2\pi), \frac{1}{2\pi}d\theta\right)$ by

$$\Theta u(\theta) = \theta u(\theta), \; Zu(\theta) = -i\frac{2\pi}{n}u'(\theta),$$

where $u \in C^\infty(\mathbb{T})$ and $n \in \mathbb{N}$. We are interested in $C_n^{(2)} = [\Pi_\Theta, \Pi_Z]$, where

$$\Pi_\Theta = \mathbb{1}_{(0,\infty)}(\cos\Theta) = \mathcal{M}_{\mathbb{1}_E}, \quad \Pi_Z = \mathbb{1}_{(0,\infty)}(\cos Z).$$

The proof that $\lim_{n\to\infty} \|C^{(n)}\|_{\mathrm{op}} = \frac{1}{2}$ immediately reduces to Lemma 3.3, since $\{z^k \mid k \in \mathbb{Z}\}$ is an eigenbasis of Z (analogous to $\mathcal{E}_{z,j}$ for $SU(2)$), with

$$Z(z^k) = \frac{2\pi k}{n} z^k.$$

This means that $\Pi_Z(z^k) = \mathbb{1}_E\left(\lambda_{k,n}\right) z^k$, where $\lambda_{k,n} = e^{\frac{2\pi k}{n} i}$.

The matrix elements of Π_Θ are specified by $\langle \Pi_\Theta z^l, z^k \rangle = \hat{\mathbb{1}}_E(k-l)$.

Consequently, the matrix elements of $C_n^{(2)}$ are specified by

$$\begin{aligned} c_{n,k,l}^{(2)} &= \langle C_n^{(2)} z^l, z^k \rangle = \langle \Pi_\Theta \Pi_Z z^l, z^k \rangle - \langle \Pi_Z \Pi_\Theta z^l, z^k \rangle \\ &= \mathbb{1}_E\left(\lambda_{l,n}\right) \langle \mathbb{1}_E z^l, z^k \rangle - \langle \mathbb{1}_E z^l, \Pi_Z z^k \rangle = \left(\mathbb{1}_E(\lambda_{l,n}) - \mathbb{1}_E(\lambda_{k,n})\right) \hat{\mathbb{1}}_E(k-l), \end{aligned}$$

In particular, when $\frac{n}{4} < l < \frac{3n}{4}$ and $0 \le k < \frac{n}{4}$, we have $c_{n,k,l}^{(2)} = -\hat{\mathbb{1}}_E(k-l)$.

Corollary 7.4. *Fix some positive $N \in \mathbb{N}$, and assume that $n > 4N$. Define*

$$C_{n,N}^{(2)} = \left(a_{k,l}\right)_{k,l=1,\ldots,N} = \left(c^{(2)}_{n,\lceil \frac{n}{4}\rceil - k, \lceil \frac{n}{4}\rceil + l - 1}\right)_{k,l=1,\ldots,N}.$$

Then $a_{k,l} = -\hat{\mathbb{1}}_E(1-k-l)$. It follows that $-C_{n,N}^{(2)} = [H_E]_N$ is the truncated Hankel matrix associated with H_E. By Lemma 3.3 and the same argument as in (2), we deduce that $\lim_{n\to\infty} \|C_n^{(2)}\|_{\mathrm{op}} = \frac{1}{2}$.

If we replace $\mathbb{1}_{(0,\infty)}$ with $\mathbb{1}_{(a,\infty)}$, where $a \in (0,1)$, then the proof remains largely unchanged, except for the use of the Hankel operator $H_{\mathbb{1}_{E_a}}$ (as in Conclusion 6.6) instead of H_E.

Next, we consider Theorem 2.3. The proof of Theorem 2.3 may be obtained by straightforward computations. However, we will use a geometric model as follows.

We identify the standard basis of $V_n = l^2(\mathbb{Z}_n)$ with

$$\Delta_n = \left\{\delta_{\frac{2\pi k}{n}} \mid k = 0, 1, \ldots, n-1\right\} = \left\{\delta_{\frac{2\pi k}{n}} \mid k \in \mathbb{Z}\right\},$$

where $\delta_{\frac{2\pi k}{n}}$ is the Dirac measure supported in $\frac{2\pi k}{n} \in \mathbb{Z}_n \subset \mathbb{T} \simeq [0, 2\pi)$. For a vector $v = \sum_{k=0}^{n-1} v_k \delta_{\frac{2\pi k}{n}}$ and a bounded, measurable function $f : \mathbb{T} \to \mathbb{C}$ we will use the notation

$$fv = \sum_{k=0}^{n-1} f\left(\frac{2\pi k}{n}\right) v_k \delta_{\frac{2\pi k}{n}},$$

and refer to the operator $v \mapsto fv$ as the multiplication operator $\mathcal{M}_f : V_n \to V_n$.

Given $f : \mathbb{T} \to \mathbb{C}$, define the discretization

$$A_n(f) = f A_n(1) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f\left(\frac{2\pi k}{n}\right) \delta_{\frac{2\pi k}{n}}.$$

The representation of $H(\mathbb{Z}_n)$ is realized on $(V_n, \langle \cdot, \cdot \rangle_n)$, where $\langle \cdot, \cdot \rangle_n$ is specified by $\langle \delta_{\frac{2\pi k}{n}}, \delta_{\frac{2\pi l}{n}} \rangle_n = \delta_{kl}$. In these settings, g_2 is the multiplication operator $\mathcal{M}_z$, and g_1 is the operator of translation by $\frac{2\pi}{n}$.

In particular, $g_1\delta_{\frac{2\pi k}{n}} = \delta_{\frac{2\pi(k-1)}{n}}$, and we note that

$$g_1 A_n(f) = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f\left(\frac{2\pi k}{n}\right)\delta_{\frac{2\pi(k-1)}{n}} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f\left(\frac{2\pi(k+1)}{n}\right)\delta_{\frac{2\pi k}{n}} = A_n(\tau_n f),$$

where $\tau_n f(\theta) = f\left(\theta + \frac{2\pi}{n}\right)$. Thus,

$$g_1 A_n(z^k) = e^{2\pi\frac{k}{n}i} A_n(z^k) = \lambda_{k,n} A_n(z^k),$$

therefore $$\mathcal{E}_n = \{e_{k,n} \mid k = 0, 1, ..., n-1\} = \left\{A_n\left(z^k\right) \mid k \in \mathbb{Z}\right\}$$

is an eigenbasis of g_1, orthonormal with respect to $\langle\cdot,\cdot\rangle_n$ (as may be seen by a straightforward calculation). Δ_n is clearly an orthonormal eigenbasis of g_2.

Let $\mathcal{F}_n$ denote the (unitary) discrete Fourier transform, specified by

$$\langle\mathcal{F}_n v, \delta_{\frac{2\pi k}{n}}\rangle_n = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} v_l e^{-\frac{2\pi kl}{n}i}.$$

Then

Lemma 7.5. $g_1 = \mathcal{F}_n^{-1} g_2 \mathcal{F}_n$.

Proof. Note that $\langle\mathcal{F}_n\delta_{\frac{2\pi m}{n}}, \delta_{\frac{2\pi k}{n}}\rangle_n = \frac{1}{\sqrt{n}}e^{-2\pi\frac{km}{n}i}$, hence

$$\mathcal{F}_n\delta_{\frac{2\pi m}{n}} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\left(e^{\frac{2\pi k}{n}i}\right)^{-m}\delta_{\frac{2\pi k}{n}} = A_n(z^{-m}),$$

which means that $g_2\mathcal{F}_n\delta_{\frac{2\pi m}{n}} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\left(e^{\frac{2\pi k}{n}}\right)^{-(m-1)}\delta_{\frac{2\pi k}{n}} = \mathcal{F}_n\delta_{\frac{2\pi(m-1)}{n}}$.

We conclude that $\mathcal{F}_n^{-1}g_2\mathcal{F}_n\delta_{\frac{2\pi m}{n}} = \delta_{\frac{2\pi(m-1)}{n}} = g_1\delta_{\frac{2\pi m}{n}}$, therefore $g_1 = \mathcal{F}_n^{-1}g_2\mathcal{F}_n$. ∎

We have that $\Pi_2 = \mathbb{1}_E(\mathcal{M}_z) = \mathcal{M}_{\mathbb{1}_E}$, therefore $\Pi_1 = \mathcal{F}_n^{-1}\mathcal{M}_{\mathbb{1}_E}\mathcal{F}_n$. Since $\mathcal{E}_n$ is an eigenbasis of g_1, it holds that

$$\Pi_1 e_{k,n} = \mathbb{1}_E(\lambda_{k,n})e_{k,n}.$$

The matrix elements of Π_2 in $\mathcal{E}_n$ are given by

$$\begin{aligned}\langle\Pi_2 e_{l,n}, e_{k,n}\rangle_n &= \langle A_n\left(\mathbb{1}_E z^l\right), A_n\left(z^k\right)\rangle_n \\ &= \langle z^l A_n(\mathbb{1}_E), A_n(z^k)\rangle_n = \langle A_n(\mathbb{1}_E), A_n(z^{k-l})\rangle_n.\end{aligned}$$

Here, we have used the fact that $\mathcal{M}_{f_1}A_n(f_2) = f_1 A_n(f_2) = A_n(f_1 f_2)$ and that

$$\langle f A_n(f_1), A_n(f_2)\rangle_n = \langle A_n(f_1), \bar{f}A_n(f_2)\rangle_n.$$

The proof of Theorem 2.3 reduces to Lemma 3.3, as in all previous cases. We demonstrate this using $\mathcal{E}_n$ (though Δ_n works just as well). The matrix elements of the commutator $C_n^{(3)} = [\Pi_1, \Pi_2]$ are specified by

$$c_{n,k,l}^{(3)} = \langle C_n^{(3)} e_{l,n}, e_{k,n}\rangle_n = \left(\mathbb{1}_E(\lambda_{k,n}) - \mathbb{1}_E(\lambda_{l,n})\right)\langle\Pi_2 e_{l,n}, e_{k,n}\rangle_n.$$

In particular, when $\frac{n}{4} < l < \frac{3n}{4}$ and $0 \le k < \frac{n}{4}$,

$$c^{(3)}_{n,k,l} = \langle A_n(\mathbb{1}_E), A_n(z^{k-l})\rangle_n.$$

If $f_1, f_2 \in L^2(\mathbb{T})$ are piecewise continuous, then

$$\langle A_n(f_1), A_n(f_2)\rangle_n = \frac{1}{2\pi}\sum_{k=0}^{n-1}\left[f_1\left(\frac{2\pi k}{n}\right)\bar{f}_2\left(\frac{2\pi k}{n}\right)\frac{2\pi}{n}\right] \xrightarrow{n\to\infty} \langle f_1, f_2\rangle_{L^2(\mathbb{T})}.$$

Thus,

Corollary 7.6. *Fix some $N \in \mathbb{N}$, and assume that $n > 4N$. Define*

$$C^{(3)}_{n,N} = (b_{n,k,l})_{k,l=1,\dots,N} = \left(c^{(3)}_{n,\lceil\frac{n}{4}\rceil-k,\lceil\frac{n}{4}\rceil+l-1}\right)_{k,l=1,\dots,N}.$$

Then $\lim_{n\to\infty} b_{n,k,l} = \lim_{n\to\infty}\langle A_n(\mathbb{1}_E), A_n(z^{1-k-l})\rangle_n = \hat{\mathbb{1}}_E(1-k-l)$. It follows that

$$\lim_{n\to\infty} C^{(3)}_{n,N} = [H_E]_N$$

is the truncated Hankel matrix associated with H_E. By Lemma 3.3 and the same argument as in (2), we deduce that $\lim_{n\to\infty}\|C^{(3)}_n\|_{\mathrm{op}} = \frac{1}{2}$.

As in the case of Theorem 2.2, if we replace $\mathbb{1}_{(0,\infty)}$ with $\mathbb{1}_{(a,\infty)}$, where $a \in (0,1)$, then the proof remains largely unchanged, except for the of the Hankel operator $H_{\mathbb{1}_{E_a}}$ (as in Conclusion 6.6) instead of H_E.

Finally, the proof of Theorem 2.5 is immediate. Indeed, $\Pi_\Phi(f) = \Pi_{\mathbb{T}} f - \hat{f}(0)$, and

$$\Pi_{X_1} = \mathcal{M}_{\mathbb{1}_{(0,\infty)}(R\cos\phi)} = \mathcal{M}_{\mathbb{1}_E},$$

therefore $C^{(4)}_R = [\mathcal{M}_{\mathbb{1}_E}, \Pi_{\mathbb{T}}]$, and $\|C^{(4)}_R\|_{\mathrm{op}} = \frac{1}{2}$, as we have already seen in (16).

8. Discussion and a general conjecture

We begin with an informal interpretation of Theorem 1.2, based on a realization of the representations of $SU(2)$ through Berezin-Toeplitz quantization of the unit sphere $S^2 \subset \mathbb{R}^3$. This will lead us to formulate a conjectured, generalized version of Theorem 1.2, using the language of quantization. Subsequently, we will explore the conjectured formulation in a number of concrete examples.

In what follows, $\mathcal{L}(\mathcal{H})$ denotes the space of self-adjoint operators on a finite dimensional Hilbert space $\mathcal{H}$. Let (M,ω) denote a closed[7], quantizable[8] symplectic manifold. A Berezin-Toeplitz quantization ([4, 23, 9]) of M produces a sequence of finite dimensional complex Hilbert spaces $(\mathcal{H}_\hbar)_{\hbar\in\Lambda}$, where 0 is an accumulation point of $\Lambda \subset (0,\infty)$ and $\lim_{\hbar\to 0^+}\dim\mathcal{H}_\hbar = +\infty$, together with surjective linear maps $T_\hbar : C^\infty(M) \to \mathcal{L}(\mathcal{H}_\hbar)$, such that

[7] i.e., compact and without boundary.

[8] i.e., $\frac{\omega}{2\pi}$ represents an integral de-Rham cohomology class.

(1) $T_\hbar(1) = \mathrm{Id}_{\mathcal{H}_\hbar}$,

(2) if $f \geq 0$, then $T_\hbar(f) \geq 0$,

(3) $\|f\|_\infty - O(\hbar) \leq \|T_\hbar(f)\|_{\mathrm{op}} \leq \|f\|_\infty$,

(4) $\left\| \frac{i}{\hbar}\left[T_\hbar(f), T_\hbar(g)\right] - T_\hbar\left(\{f,g\}\right)\right\|_{\mathrm{op}} = O(\hbar)$,

(5) $\|T_\hbar\left(f^2\right) - T_\hbar(f)^2\|_{\mathrm{op}} = O(\hbar)$

for every $f, g \in C^\infty(M)$. Here $\|f\|_\infty = \max_M |f|$ is the uniform norm and $\{f,g\}$ is the Poisson bracket of f,g. The existence of a Berezin-Toeplitz quantization in these rather general settings is a non-trivial fact, though if (M,ω) is a closed Kähler manifold, then the construction itself is quite direct. Item (4) above is known as the *correspondence principle*, and it is central to our interpretation.

Let us identify S^2 with the complex projective space $\mathbb{C}P^1$ via the stereographic projection through the north pole, and let ρ denote the standard action of $SU(2)$ on $\mathbb{C}P^1$, given by

$$\rho(U)\left([z_1 : z_2]\right) = [\alpha z_1 - \bar{\beta} z_2 : \beta z_1 + \bar{\alpha} z_2],\ U = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in SU(2).$$

In addition to the properties specified above, the Berezin-Toeplitz quantization of $S^2 \simeq \mathbb{C}P^1$ is $SU(2)$-equivariant, meaning that $\mathcal{H}_\hbar$ carries an irreducible, unitary representation $\rho_\hbar$ of $SU(2)$ such that

$$T_\hbar\left(f \circ \rho(U)^{-1}\right) = \rho_\hbar(U) T_\hbar(f) \rho_\hbar(U)^*$$

for every $\hbar \in \Lambda$, $f \in C^\infty\left(\mathbb{C}P^1\right)$ and $U \in SU(2)$. Here, $\hbar^{-1} = n = \dim \mathcal{H}_\hbar$, and $\Lambda = \left\{n^{-1} \mid n \in \mathbb{N}\right\}$. The spin operators $J_x, J_y, J_z \in \mathcal{L}\left(\mathcal{H}_\hbar\right)$ are then, up to normalization, the quantum counterparts of the Cartesian coordinate functions $x, y, z : \mathbb{C}P^1 \to \mathbb{R}$. Specifically,

$$T_\hbar(x) = \frac{1}{n+1} J_x,\ T_\hbar(y) = \frac{1}{n+1} J_y,\ T_\hbar(z) = \frac{1}{n+1} J_z.$$

Since $\mathbb{1}_{(0,\infty)}$ is unaffected by positive rescalings, Theorem 1.2 means that

$$\lim_{n\to\infty} \|C_n\|_{\mathrm{op}} = \lim_{\hbar \to 0^+} \left\| \left[\mathbb{1}_{(0,\infty)}\left(T_\hbar(x)\right), \mathbb{1}_{(0,\infty)}\left(T_\hbar(z)\right)\right] \right\|_{\mathrm{op}} = \frac{1}{2}.$$

Finally, our loose interpretation of this result goes as follows. We consider the spectral projections

$$\mathbb{1}_{(0,\infty)}(J_x),\ \mathbb{1}_{(0,\infty)}(J_z)$$

as a pair of observables that are somehow related ([31, 32, 33]) to the indicator functions of the hemispheres $\{x > 0\}, \{z > 0\} \subset S^2$. Thus, we interpret Theorem 1.2 as an informal attempt to explore the correspondence principle (item (4) above) in the context of discontinuous classical observables[9]. At the moment, it is unclear whether C_n corresponds to a well-defined classical object as $n \to \infty$. Still, the behavior of $\left(C_n\right)_{n\geq 2}$ appears to be related to the intersection of the boundaries of the respective hemispheres, that is, to the points $\pm(0,1,0)$.

[9] To the best of our knowledge, a well-defined, useful (in the context of quantization) notion of Poisson bracket which is applicable to discontinuous observables does not exist.

To see this, note that $\mathcal{H}_\hbar$ may be identified with the space of homogeneous polynomials of degree $n-1$ in two complex variables, such that $\rho_\hbar$ becomes the standard irreducible unitary representation of $SU(2)$ in the latter space. Assume that $v_n \in \mathcal{H}_\hbar$ is a polynomial which realizes the norm of C_n, i.e., assume that $\|C_n v_n\| = \|C_n\|_{\mathrm{op}}$. Our numerical simulations suggest that v_n concentrates about the points $\pm(0,1,0)$ when $n \to \infty$, as illustrated in the following images.

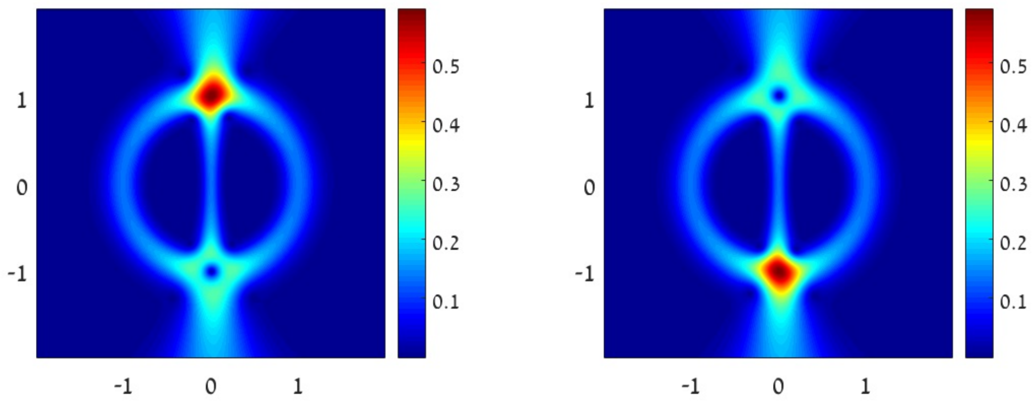

Figure 4: (originally by Y. Le Floch) The modulus of (unit) eigenvectors of C_{101} corresponding to extremal eigenvalues, realized as polynomials on $\mathbb{C}$.

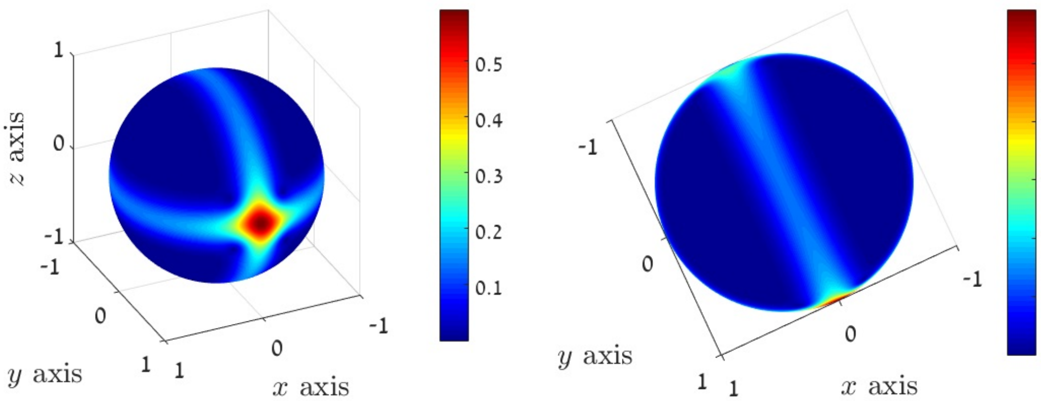

Figure 5: The image above to the left, reproduced with the eigenvector realized as a function on S^2 using the stereographic projection.

More generally, assume that $T_\hbar(f), T_\hbar(g)$ are a pair of quantum observables arising from smooth, non-commuting observables f, g on some quantizable phase space M, and let $I, J \subset \mathbb{R}$ denote some (non-trivial) intervals. As before, we view the projections

$$\Pi_{\hbar,f,I} = \mathbb{1}_I\left(T_\hbar(f)\right), \ \Pi_{\hbar,g,J} = \mathbb{1}_J\left(T_\hbar(g)\right)$$

as a pair of observables that are related to the domains $f^{-1}(I),\ g^{-1}(J) \subset M$. The numerical evidence (Figures 6, 7 in particular) and the results presented in this work appear to support the following conjecture, which was inspired by recent findings pertaining to quantization of domains in phase space ([5, 14, 15]).

Conjecture 8.1. If M is 2-dimensional, and if the intersection of the boundaries of the domains $f^{-1}(I),\ g^{-1}(J)$ is non-empty and transversal, then $\Pi_{I,f,\hbar}, \Pi_{J,g,\hbar}$ are maximally non-commuting in the semiclassical limit, i.e.,

$$\lim_{\hbar\to 0^+} \|[\Pi_{I,f,\hbar}, \Pi_{J,g,\hbar}]\|_{\mathrm{op}} = \frac{1}{2}.$$

By contrast, if the distance[10] between the boundaries of $f^{-1}(I),\ g^{-1}(J)$ is greater than some $\varepsilon > 0$, then

$$\lim_{\hbar\to 0^+} \|[\Pi_{I,f,\hbar}, \Pi_{J,g,\hbar}]\|_{\mathrm{op}} = 0.$$

In the context of S^2, Theorem 4.2 is a modest extension of our main result, and agrees with the conjecture. Similarly, consider the sequence

$$C_{n,a} = \left[\mathbb{1}_{\left(a\left(j+\frac{1}{2}\right),\infty\right)}(J_x), \mathbb{1}_{\left(a\left(j+\frac{1}{2}\right),\infty\right)}(J_z)\right],$$

where $a \in [0,1)$. A rigorous calculation of $\lim_{n\to\infty} \|C_{n,a}\|_{\mathrm{op}}$ for $a > 0$ is not possible using our current method (due to limitations in the applicability of the asymptotic estimate (13)). However, the conjecture forecasts a transition in the behavior of $(\|C_{n,a}\|_{\mathrm{op}})_{n\geq 2}$ as a crosses the value $\frac{1}{\sqrt{2}}$. According to our numerical simulations, this indeed seems to be the case. The following images are the analogues of Figure 1 above for the choices $a = 0.25,\ 0.75$ and $a = 0.7,\ 0.71$.

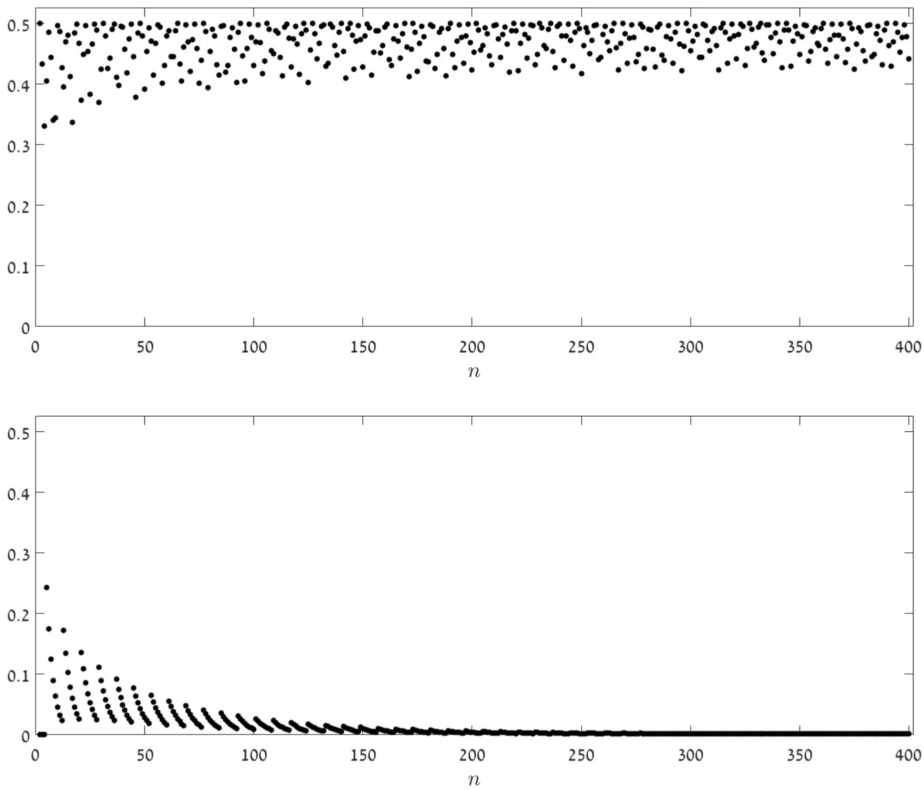

Figure 6: A plot of $\|C_{n,a}\|_{\mathrm{op}}$ as a function of n, where $a = 0.25$ (top), $a = 0.75$ (bottom).

[10] The distance with respect to any metric inducing the topology of M.

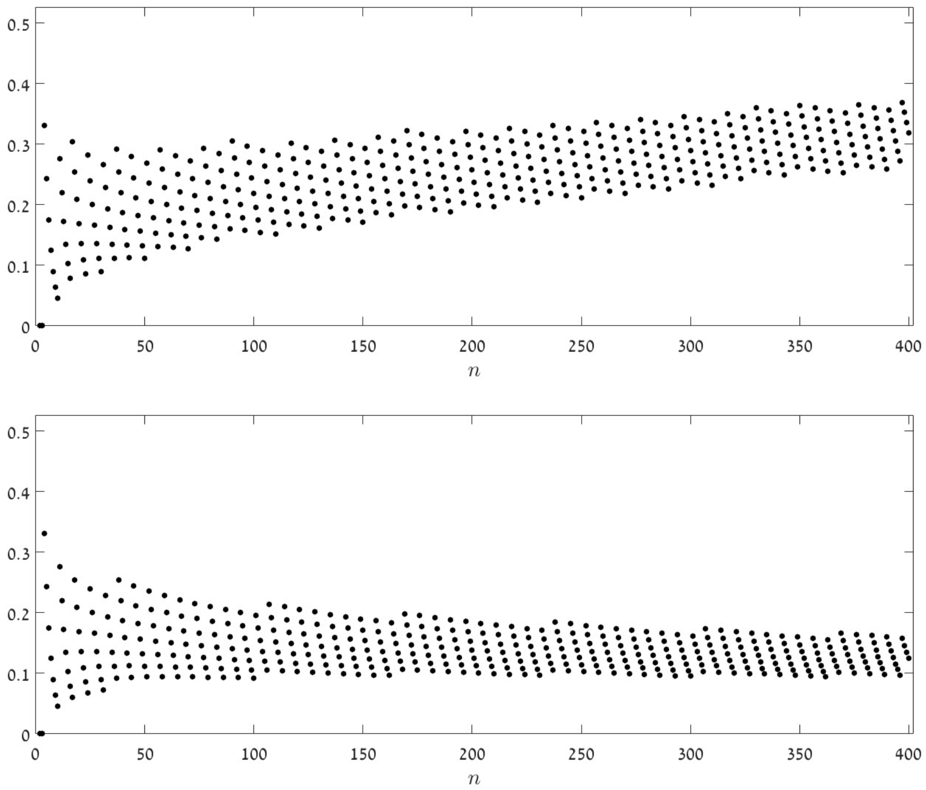

Figure 7: A plot of $\|C_{n,a}\|_{\text{op}}$ as a function of n, where $a = 0.7$ (top), $a = 0.71$ (bottom). Compare, also, with the image for $a = 0.75$.

Although we formulated the conjecture using the specific notion of Berezin-Toeplitz quantization, we expect it to hold in the context of similar or standard quantization schemes as well (Weyl quantization, in particular).

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