

On Computing Linearizing Coordinates from the Symmetry Algebra

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Abstract. A characterization of the symmetry algebra of the N th-order ordinary differential equations (ODEs) with maximal symmetry and all third-order linearizable ODEs is given. This is used to show that such an algebra \mathfrak{g} determines – up to a point transformation – only one linear equation whose symmetry algebra is \mathfrak{g} and an algorithmic procedure is given to find the linearizing coordinates. This procedure is applied to several examples from the literature.

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1. Introduction

The Lie theory of transformation groups with applications to differential equations, both linear and nonlinear, was initiated by Sophus Lie [19, 20]. The determining equations for the symmetry algebra of the equations constitute a linear homogeneous overdetermined system of differential equations whose solutions do not rely on the solutions of the original equation if the equation is nonlinear [28]. This fact can be used to linearize nonlinear equations by using the structure of the symmetry algebra as described below. The main results of this paper are given as Theorems 3.4, 4.1 and 4.2 of Sections 3 and 4, respectively. These results use in an essential way the following results of Mahomed-Leach and Krause-Michel on linearizable ODEs.

Theorem 1.1. [23, 16] (a) *A necessary and sufficient condition for an N th order ODE ($N \geq 3$) to be linearizable by a point transformation is the existence of an N -dimensional abelian subalgebra of the symmetry algebra of the ODE.*

(b) *An N th-order ODE with $N \geq 3$ is linearizable by a point transformation if the dimension m of its symmetry algebra is $N + 4$, in which case the equation is equivalent to $y^{(N)} = 0$. Moreover, if the equation is linearizable and $m \neq N + 4$, then m must be $N + 1$ or $N + 2$.*

The case $N = 2$ was completely settled by Lie [19]. He showed that a second-order ODE is linearizable if and only if its symmetry algebra is eight dimensional; in this case the symmetry algebra is $\mathfrak{sl}(3, \mathbb{R})$. Lie [19] also determined the symmetry algebra of $y^{(N)} = 0$ for $N \geq 3$.

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Regarding third-order linearizable equations, such an equation can be transformed by a point transformation to one of the three canonical forms given by $y''' = 0$, $y''' - (\lambda + \mu)y'' + \lambda\mu y' = 0$ ($\lambda, \mu \neq 0$) or $y''' - (\phi'''/\phi'')y'' = 0$, where $\phi'' \neq 0$; see [7, 16].

The linearization problem for partial differential equations (PDEs) was settled by Kumei and Bluman in [17], where they give necessary and sufficient conditions for a pde to be linearizable and linearizing transformation is also determined.

In summary, the main contributions of this paper are: (1) an invariant characterization of the symmetry algebra of the equation $y^{(N)} = 0$ ($N > 2$) and of all third-order linearizable ODEs; (2) algorithms for finding linearizing coordinates from a knowledge of the symmetry algebra of the equation. We have used elementary representation theory to achieve this.

A few words about the linearizability problem for ODEs are in order. Almost everything about symmetry analysis of ODEs is implicit in Lie [20]. However, it is desirable to have algorithms that can be formulated more transparently and that can be implemented using programs like GAP, Maple and Sage.

The algebraic linearization and invariant criteria for scalar second-order ordinary differential equations (ODEs) are known from the classical works of Lie [18] and Tressé [30]. These are given in terms of coefficients of the equation. Grissom et al. [9] used the Cartan method to obtain these results. Moreover, a geometric method of projection was utilized in [24] to re-derive the Lie-Tressé linearization conditions. The reader is referred to Bluman and Kumei [5], Ibragimov [11] and Olver [26] for modern expositions of symmetries and invariants.

For scalar N th-order ($N > 2$) ODEs, the necessary and sufficient conditions for linearization by point transformations are given in [23] and [16]. There are three canonical forms for such ODEs. In the case of third-order ODEs, the canonical forms are listed in [12]. The maximal algebra for third order ODEs is of dimension seven and this is the symmetry algebra for the ODE $y''' = 0$. The remaining cases are for symmetry algebras of dimensions four and five, respectively.

Chern [6] provided impetus for the Cartan equivalence method for linearization of certain classes of third-order ODEs by means of contact transformations. In recent times, Neut and Petitot [25] obtained more general criteria for equivalence.

Grebot [8] considered the linearization of third-order ODEs by fibre preserving transformations which are restricted invertible transformations. Ibragimov and Meleshko [13] re-looked at linearization for such ODEs. They invoked a direct method. This works for point and contact transformations.

Moreover, the Cartan equivalence method is used in [1] where the authors arrived at an invariant characterization of scalar third-order ODEs for different dimensions of the symmetry Lie algebras.

The book of Schwarz [29] contains very useful information on algorithmic Lie theory as well as a large number of differential equations of higher orders of physical interest. The paper [27] of Oudshoorn and Van Der Put gives a new algorithm for computing the Lie symmetries for linear ODEs.

It is often difficult to find the linearizing coordinates with the approaches mentioned. The algorithmic approach proposed in this paper – combined with the use of software – is quite efficient as shown by Examples 4.4–4.7.

2. Review of real representations of $\mathfrak{sl}(2, \mathbb{R})$

The algebra $\mathfrak{sl}(2, \mathbb{R})$ plays a special role in finite dimensional Lie algebras of local vector fields in the plane. The basic reason is that it is the only algebra that can occur as a proper Levi complement in such algebras: see Section 3 for an explanation of this fact. Thus the structure of the radical can be understood by using real representation theory of $\mathfrak{sl}(2, \mathbb{R})$.

The real finite dimensional representation theory of $\mathfrak{sl}(2, \mathbb{R})$ is identical to that of complex finite dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$; indeed the real representation theory for any real semisimple algebra with a split Cartan subalgebra is identical to that of its complexification [2].

Let X, Y, H be the standard basis of $\mathfrak{sl}(2, \mathbb{R})$ with $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$. If V is a real representation space of $\mathfrak{sl}(2, \mathbb{R})$, then the dimension of the null space of X in V gives the number of irreducible components. The element H operates as a real diagonalizable transformation in the null space of X with nonnegative integer eigenvalues. If v_1, \dots, v_r form a basis of eigenvectors of H in the null space of X , with eigenvalues d_1, d_2, \dots, d_r then the irreducible components are $\langle v_1, Yv_1, \dots, Y^{d_1}v_1 \rangle, \dots, \langle v_r, Yv_r, \dots, Y^{d_r}v_r \rangle$: the vectors v_1, \dots, v_r are the high weight vectors of the representation: see Hilgert-Neeb [10], and Knapp [15] for further details.

3. Characterization of the symmetry algebra of the equation

$$y^{(N)} = 0 \text{ for } N \geq 3$$

In this section we want to show that the structure of the symmetry algebra of $y^{(N)} = 0$ ($N \geq 3$), determines the equation and give an algorithmic procedure to find linearizing coordinates for any ODE of order N whose symmetry algebra has the same Levi decomposition as that of the symmetry algebra of the equation $y^{(N)} = 0$. The Levi decomposition can be obtained by using standard programs in Computer Algebra Systems while the radical can be decomposed into its irreducible components using representation theory of $\mathfrak{sl}(2, \mathbb{R})$ – as explained in Section 2.

From the classification of Lie [19] and González-Lopez-Kamran-Olver [22], it is clear that only $\mathfrak{sl}(2, \mathbb{R})$ can appear as a proper Levi complement. This can also be deduced from the following result. Before proving this result, let us note that by a proper Levi complement we mean a Levi complement which is neither the null subalgebra nor the whole algebra. Moreover the rank of a Lie algebra of vector fields is the dimension of its generic orbit; note that there is an open set on which all the orbits have a fixed dimension r and the orbits in its complement have dimensions lower than r . We refer to this number r as a rank of the Lie algebra of vector fields [3].

Proposition 3.1. *Let L be a semisimple Lie algebra of vector fields on an open set U of \mathbb{R}^K . Assume that L has a Cartan subalgebra C with all real eigenvalues in the adjoint representation. Let $B \supset C$ be a Borel subalgebra of L and \mathfrak{N} the nilpotent radical of B . If \mathfrak{N} has an abelian subalgebra of rank K , then L cannot occur as a proper Levi complement of any finite dimensional subalgebra of local vector fields on \mathbb{R}^K .*

Proof. Because of the rank condition, one can introduce coordinates in which the abelian subalgebra of the nilpotent radical is spanned by ∂_{x_i} , $i = 1, \dots, K$.

Therefore since the radical is a sum of irreducibles for L , it must have a high weight vector in the radical – if the radical is non-zero. This means that there is a vector field v in the radical which commutes with ∂_{x_i} , $i = 1, \dots, K$ and therefore it must belong to the nilpotent radical of L . Therefore the radical must be 0. ■

Remark 3.2. If we have a three-dimensional algebra of vector fields X, Y and Z with relations $[X, Y] = Z$ then we can always introduce coordinates so that one of the algebras $\langle X, Z \rangle$ or $\langle Y, Z \rangle$ is $\langle \partial_{x_1}, \partial_{x_2} \rangle$. The algebra $[X, Y] = Z$ is isomorphic to the nilpotent radical of a Borel subalgebra of $\mathfrak{sl}(3, \mathbb{R})$. Moreover using the main result of [4] the nil radical of the algebra of vector fields in the plane isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ is also of rank 2. Thus using Prop 3.1 we see why only $\mathfrak{sl}(2, \mathbb{R})$ can appear as a proper Levi complement in subalgebras of vector fields in the real plane.

Remark 3.3. The condition of a Cartan subalgebra being split can be relaxed if the semisimple algebra consists of real analytic vector field – by working in the complexification of real analytic vector fields on U .

Recall that the Levi decomposition of the symmetry algebra of $y^{(N)} = 0$ ($N \geq 3$) is

$$\langle \partial_x, \frac{x^2}{2}\partial_x + \frac{N-1}{2}xy\partial_y, x\partial_x + \frac{N-1}{2}y\partial_y \rangle \oplus \langle \partial_y, x\partial_y, \dots, x^{N-1}\partial_y, y\partial_y \rangle. \quad (1)$$

We note that the radical is a sum of an N -dimensional irreducible representation and a 1-dimensional representation of the Levi complement with highest weight vectors ∂_y and $y\partial_y$ – relative to the Borel subalgebra $\langle \partial_x, x\partial_x + \frac{N-1}{2}y\partial_y \rangle$. The radical is of rank 1 and the nilradical of the Borel subalgebra and the highest weight vector of the N -dimensional component give an abelian algebra of rank 2.

In the following proposition, we show that these conditions characterize the symmetry algebra of $y^{(N)} = 0$ ($N \geq 3$).

Theorem 3.4. *Let A be an algebra of local vector fields in the plane with Levi decomposition $A = S \oplus R$, where S has a basis X, Y, H with $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$ and the radical R is a sum of an irreducible N dimensional representation ($N \geq 2$) and a 1-dimensional representation.*

Let V be a highest weight vector relative to the subalgebra $\langle H, X \rangle$ for the N -dimensional component. Assume that R is of rank 1 and the subalgebra $\langle X, V \rangle$ is of rank 2. Then the canonical coordinates for $\langle X, V \rangle$ – after an affine change of coordinates – give coordinates in which the algebra coincides with the algebra of the equation $y^{(N)} = 0$, for $N \geq 3$ – and this is the only hypersurface of the form $y^{(N)} = f(x, y, y', \dots, y^{(N-1)})$ which is invariant under A .

Proof. (a) Canonical form of the algebra: Choose coordinates (x, y) so that $X = \partial_x, V = \partial_y$. Let $Y = a\partial_x + b\partial_y$. Thus, $H = a_x\partial_x + b_x\partial_y$. Moreover, $[H, V] = (N-1)V$. By assumption on rank of the radical $[Y, V] = \lambda(x, y)V$. This gives $a_y = 0$. Using $[H, \partial_x] = 2\partial_x$ and $[H, \partial_y] = (N-1)\partial_y$, we see that $a''(x) = -2, b_{xx} = 0, b_{xy} = -d$, where $d = (N-1)$. Hence, $a(x) = -x^2 + \alpha x + \beta$, $b(x, y) = -dxy + \gamma x + H(y)$ and $Y = (-x^2 + \alpha x + \beta)\partial_x + (-dxy + \gamma x + H(y))\partial_y$. Therefore, $H = [X, Y] = -2(x - \alpha/2)\partial_x - d(y - \gamma/d)\partial_y$. Put $\tilde{x} = x - \alpha/2, \tilde{y} = y - \gamma/d$.

In these coordinates, $H = -2\tilde{x}\partial_{\tilde{x}} - d\tilde{y}\partial_{\tilde{y}}$. Dropping the tildes for notational convenience, we have $X = \partial_x$, $Y = (-x^2 + \alpha^2/4 + \beta)\partial_x + (-dxy - d\alpha y/2 + F(y))\partial_y$. Finally, $[H, Y] = -2Y$, requires $\alpha^2/4 + \beta = 0$ and $-yF'(y) + F(y) = \alpha y - (2/d)F(y)$. The solutions of this equation are $F(y) = \frac{d\alpha}{2}y + ky^{1+2/d}$. Therefore,

$$Y = -x^2\partial_x + (-dxy + ky^{1+2/d})\partial_y.$$

Lastly, we use the fact that the radical also has a 1-dimensional representation of the algebra $\langle X, Y, H \rangle$. As the radical is of rank 1, the highest weight vector of the trivial representation is necessarily $y\partial_y$. Now $[Y, y\partial_y] = [ky^{1+2/d}\partial_y, y\partial_y] = 0$ can only hold if $k = 0$. Thus in the canonical coordinates of the algebra $\langle X, V \rangle$ and by a suitable translation, the algebra has the canonical form

$$\langle \partial_x, -(x^2\partial_x + (N-1)xy\partial_y), -(2x\partial_x + (N-1)y\partial_y) \rangle \oplus \langle \partial_y, x\partial_y, \dots, x^{(N-1)}\partial_y, y\partial_y \rangle \quad (2)$$

(b) Invariant hypersurfaces: To determine invariant hypersurfaces of the form $y^{(N)} = f(x, y, y', \dots, y^{(N-1)})$, we first determine invariant hypersurfaces under the maximal solvable subalgebra $B = \langle \partial_x, \partial_y, x\partial_y, \dots, x^{(N-1)}\partial_y, y\partial_y \rangle$. We need formulas for the N -th order prolongations of these operators. They can be computed as in Section 5 of [4], following Lie [19].

Denoting the N -th order prolongation of a vector field X by $X^{(N)}$, we have

$$\partial_x^{(N)} = \partial_x, \quad \partial_y^{(N)} = \partial_y \quad \text{and} \quad (x^a\partial_y)^{(N)} =$$

$$= x^a\partial_y + ax^{a-1}\partial_{y'} + a(a-1)x^{a-2}\partial_{y''} + \dots + (a(a-1)(a-2)\dots 2)x\partial_{y^{(a-1)}} + (a!)\partial_{y^{(a)}}$$

for $a \leq N$. Therefore, $(x^a\partial_y)^{(N)}$ is congruent to $(a!)\partial_{y^{(a)}} \pmod{\partial_y, \dots, \partial_{y^{(a-1)}}}$. Invariance of the hypersurface $y^{(N)} = f(x, y, y', \dots, y^{(N-1)})$ under $\partial_x^{(N)}$, $\partial_y^{(N)}$ and $(x^a\partial_y)^{(N)}$ (for $a \leq N$) shows that the equation must be $y^{(N)} = K$. Applying the N -th order prolongation of $y\partial_y$ – namely $y\partial_y + y'\partial_{y'} + \dots + y^{(N)}\partial_{y^{(N)}}$, we see that $y^{(N)} = 0$ on the hypersurface $y^{(N)} = K$. Thus K must be 0.

When $N \geq 3$, the symmetry algebra of $y^{(N)} = 0$ is the algebra A , while for $N = 2$, the symmetry algebra of the equation contains A as a subalgebra. The algebra A is a parabolic subalgebra in the sense that it contains a maximal solvable subalgebra of the symmetry algebra of $y'' = 0$. ■

The above proposition gives an algorithm for linearizing an ODE with maximal symmetry which is illustrated with the help of an example at the end of next section.

4. Linearizing coordinates of third-order ODEs

We will consider in this section the cases where the symmetry algebra is solvable, because in the previous section an algorithm to find linearizing coordinates for any equation equivalent to $y^{(N)} = 0$ ($N \geq 3$) has already been given. The canonical forms for equations with solvable symmetry algebras are: $y''' - (\phi'''/\phi'')y'' = 0$, where $\phi'' \neq 0$ and $y''' - (\lambda + \mu)y'' + \lambda\mu y' = 0$ ($\lambda, \mu \neq 0$). The corresponding symmetry algebras are $\mathfrak{g}_1 = \langle \partial_y, x\partial_y, y\partial_y, \phi(x)\partial_y \rangle$ and $\mathfrak{g}_2 = \langle \partial_x, \partial_y, y\partial_y, e^{\lambda x}\partial_y, e^{\mu x}\partial_y \rangle$, respectively: this is for the case when λ, μ are real and distinct, the remaining cases are given in Remark 4.3 below.

The algebra \mathfrak{g}_1 is of rank 1, its commutator is abelian and $\mathfrak{g}_1/\mathfrak{g}'_1$ operates on \mathfrak{g}_1 with eigenvalue -1 of multiplicity 3 while \mathfrak{g}'_2 has an abelian complement, say A , of

rank 2, and the weights of A on \mathfrak{g}'_2 , for an ordered basis of A , are $(0, -1)$, $(\lambda, -1)$, $(\mu, -1)$. These conditions characterize the algebras and the differential equations intrinsically.

Theorem 4.1. *Let \mathfrak{g} be a four-dimensional algebra of vector fields in the plane which is of rank 1 whose commutator is abelian and such that $\mathfrak{g}/\mathfrak{g}'$ operates on \mathfrak{g}' with a non-zero eigenvalue of multiplicity 3. Then, one can algorithmically introduce coordinates (x, y) so that $\mathfrak{g}' = \langle \partial_y, x\partial_y, \phi(x)\partial_y \rangle$ and $\mathfrak{g} = \langle y\partial_y, \partial_y, x\partial_y, \phi(x)\partial_y \rangle$. The generic orbit of \mathfrak{g} in the extended space (x, y, y', y'', y''') is four-dimensional and such points are in the complement of the set defined by $y''\phi''' - \phi''y''' = 0$. In these coordinates, this is the only hypersurface invariant under \mathfrak{g} .*

Proof. By assumption $\mathfrak{g}/\mathfrak{g}'$ operates on \mathfrak{g}' with a nonzero eigenvalue of multiplicity 3. By scaling, we can assume that this eigenvalue is -1 . Let the eigenvectors be f_1, f_2, f_3 . The whole algebra is of rank 1 and the commutator is abelian. Therefore, we can introduce coordinates (x, y) so that $f_1 = \partial_y, f_2 = x\partial_y, f_3 = \phi(x)\partial_y$, with $1, x, \phi(x)$ linearly independent. Let q be a representative of $\mathfrak{g}/\mathfrak{g}'$, say $q = f(x, y)\partial_y$. Then, $[q, \partial_y] = -\partial_y$ implies $\frac{\partial f}{\partial y} = 1$. Hence, $f = y + g(x)$. Let $\tilde{x} = x, \tilde{y} = y + g(x)$. In these coordinates, $\partial_{\tilde{x}} = \partial_x - g'(x), \partial_{\tilde{y}} = \partial_y$. Thus \mathfrak{g} has the form stated in the statement of the proposition. For notational convenience, we write \mathfrak{g} in its canonical form as $\mathfrak{g} = \langle y\partial_y, \partial_y, x\partial_y, \phi(x)\partial_y \rangle$, where $\phi'' \neq 0$.

Now the generic orbits of a Lie algebra of analytic vector fields L on an open set of \mathbb{R}^N are the same as that of the algebra of vector fields obtained from the matrix of coefficients of L by putting it in reduced row echelon form [21] (p.102, Proposition 4), [3]. Thus the rank of the coefficient matrix gives the dimension of the generic orbit.

We want to describe such orbits in the extended space (x, y, y', y'', y''') .

The third prolongation of \mathfrak{g} can be computed as in Section 5 of [2], following Lie [19] (pp. 261–274). As the third-order prolongation of $f(x)\partial_y$ is

$$f(x)\partial_y + f'(x)\partial_{y'} + f''(x)\partial_{y''} + f'''(x)\partial_{y'''}$$

and the third prolongation of $y\partial_y$ is $y\partial_y + y'\partial_{y'} + y''\partial_{y''} + y'''\partial_{y'''}$, the rank of the vector fields coming from the given basis of \mathfrak{g} in the third prolongation is the rank of

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & y' & y'' & y''' \\ \phi & \phi' & \phi'' & \phi''' \end{bmatrix} \quad (3)$$

Thus the generic orbit is four-dimensional on the complement of the hypersurface defined by $y''\phi''' - y'''\phi'' = 0$. Therefore, bringing the algebra into its canonical form determines the equation $y''' = (\phi'''/\phi'')y''$.

We now show that this is the only equation in these coordinate of the form $y''' = f(x, y, y', y'')$ that is invariant under the third prolongation of \mathfrak{g} . Invariance under ∂_y and $x\partial_y + \partial_{y'}$ gives $y''' = F(x, y'')$. Invariance under the third order prolongation of $y\partial_y$ shows that $y''' = y''F_{y''}$ on the hypersurface $y''' = F(x, y'')$. Thus $y''F_{y''} = F(x, y'')$. Working with canonical coordinates of the field $y''\partial_{y''}$ we find that $F(x, y'') = y''G(x)$.

Finally, invariance under the third-order prolongation of $\phi(x)\partial_y$ shows that $\phi''' = \phi''G(x)$. Thus the invariant equation must be $y''' = y''(\phi'''/\phi'')$. ■

Note. This shows that expressing the symmetry algebra L of a third-order ODE whose algebra is described as in the statement of Theorem 4.1 in the coordinates (x, y) described in the proof of this proposition linearizes the equation.

Theorem 4.2. *Let \mathfrak{g} be a five-dimensional Lie algebra of vector fields in the plane whose commutator \mathfrak{g}' is abelian and of rank 1. Assume that \mathfrak{g}' has an abelian complement A . Assume also that the weights of $\mathfrak{g}/\mathfrak{g}'$ on \mathfrak{g}' with respect to an ordered basis (\bar{e}_1, \bar{e}_2) , $e_i \in A$ of $\mathfrak{g}/\mathfrak{g}'$, are $(0, k)$, (λ, k) , (μ, k) with $k, \lambda, \mu \neq 0$ and $\lambda \neq \mu$ with weight vectors e_3, e_4, e_5 . Moreover, assume that if w is the weight vector of weight $(0, k)$, then $\langle Z_A(w), w \rangle$ is of rank 2. Then in the canonical coordinates for $\langle Z_A(w), w \rangle$ the algebra $\mathfrak{g} = \langle \partial_x, x\partial_y \rangle \oplus \langle \partial_y, e^{\lambda x}\partial_y, e^{\mu x}\partial_y \rangle$. The generic \mathfrak{g} - orbit in the extended space (x, y, y', y'', y''') is five-dimensional and points with lower dimension orbits are those which satisfy the equation $y''' - (\lambda + \mu)y'' + \lambda\mu y' = 0$. Moreover, in these coordinates this is the only hypersurface invariant under \mathfrak{g} .*

Proof. As the proof is very similar to the proof of the previous proposition, we will omit some details. Using the notations in the statement of the proposition, let e_3, e_4, e_5 be vectors in \mathfrak{g}' of weights $(0, k), (\lambda, k), (\mu, k)$ for A . By assumption, the centralizer of e_3 in A , namely $\langle e_3, e_1 \rangle$, is of rank 2. Choose coordinates so that $e_1 = \partial_x, e_3 = \partial_y$. By commutativity of \mathfrak{g}' and the assumption that \mathfrak{g}' is of rank 1, we have $e_4 = f(x)\partial_y, e_5 = g(x)\partial_y$.

As e_4, e_5 are eigenvectors for ∂_x with eigenvalues λ, μ we may assume that $e_4 = e^{\lambda x}\partial_y, e_5 = e^{\mu x}\partial_y$. Now e_2 operates on \mathfrak{g}' as a nonzero scalar k . By scaling e_2 , we may assume that $k = -1$. Moreover as e_2 commutes with e_1 , it must be of the form $e_2 = a(y)\partial_x + b(y)\partial_y$. Thus $[e_2, e_3] = -e_3$, gives $a(y) = \alpha, b(y) = y + \beta$. Finally, $[e_2, e_4] = -e_4$ gives $\alpha = 0$ and $e_2 = y + \beta$. Thus in the coordinates $\tilde{x} = x, \tilde{y} = y + \beta$, the algebra \mathfrak{g} is of the form given in the statement of the proposition.

Dropping the tildas for notational convenience, we see after computing the third prolongations of the basis of \mathfrak{g} , that the coefficient matrix of the generators of \mathfrak{g} is singular at the points in the extended space (x, y, y', y'', y''') where we have $y'\lambda\mu - y''(\lambda + \mu) + y''' = 0$. Arguing as in the previous proposition, the generic orbit outside this linear subspace is five-dimensional.

Suppose that $y''' = F(x, y, y', y'')$ is an invariant hypersurface for \mathfrak{g} . Invariance under the prolongations of ∂_y and ∂_x shows F is a function of only y', y'' . Using invariance under the third prolongation of $y\partial_y$ we see that $y''' = y'F_{y'} + y''F_{y''}$ on the set $y''' = F(y', y'')$. Thus $y'F_{y'} + y''F_{y''} = F$.

Working with canonical coordinates for the field $y'\partial_{y'} + y''\partial_{y''}$ we see that $F(y', y'') = y''G(y'/y'')$. Invariance under the third order prolongation of $e^{ax}\partial_y$ implies that $G(w)$ is a linear function of w . Thus the equation of the invariant hypersurface is $y''' = y''(\alpha x + \beta y'/y'')$. Hence invariance under $\langle e^{\lambda x}\partial_y, e^{\mu x}\partial_y \rangle$ determines the values of α, β and the equation of the invariant hypersurface is $y''' - (\lambda + \mu)y'' + \lambda\mu y' = 0$. ■

Remark 4.3. The canonical forms of the cases when λ is complex and non-real or when $\lambda = \mu$ are $\mathfrak{g}_1 = \langle \partial_x, y\partial_y \rangle \oplus \langle \partial_y, e^{ax} \cos(bx)\partial_y, e^{ax} \sin(bx)\partial_y \rangle$, where $\lambda = a + \sqrt{-1}b$ and $\mathfrak{g}_2 = \langle \partial_x, y\partial_y \rangle \oplus \langle \partial_y, e^{\lambda x}\partial_y, xe^{\lambda x}\partial_y \rangle$, respectively. The proofs of these assertions are similar to the proof given above and are therefore omitted.

To elaborate the effectiveness of the algebraic procedure constructed, we now linearize a fourth-order ODE given in [14] using Theorem 3.4 and three third-order ODEs using Theorem 4.1 and Theorem 4.2, respectively.

Example 4.4. The equation

$$y^3 y^{(4)} + 4ky^2 y' y''' + 3ky^2 y''^2 + 6k(k-1)yy''^2 y'' + k(k-1)(k-2)y'^4 = 0, \quad (4)$$

has the symmetry algebra

$$\begin{aligned} e_1 &= x\partial_x, \quad e_2 = \partial_x, \quad e_3 = y\partial_y, \quad e_4 = y^{1-a}\partial_y, \quad e_5 = xy^{1-a}\partial_y, \\ e_6 &= x^2 y^{1-a}\partial_y, \quad e_7 = x^3 y^{1-a}\partial_y, \quad e_8 = ax^2\partial_x + 3xy\partial_y, \end{aligned} \quad (5)$$

where $a = k + 1$. The Levi decomposition is

$$\mathfrak{g} = \langle e_3, e_4, e_5, e_6, e_7 \rangle \oplus \langle e_1 + (3/2a)e_3, e_2, e_8 \rangle. \quad (6)$$

In order to bring the semisimple part to the standard form of $\mathfrak{sl}(2, \mathbb{R})$, we introduce $H = -2e_1 - 3e_3/a$, $X = -2e_2$ and $Y = e_8$. Now $\langle e_2, e_4 \rangle$ form a rank 2 algebra which gives us the linearizing coordinates $\tilde{x} = x$, $\tilde{y}(\tilde{x}) = y^a/a$ that linearize the ODE (4) and bring it in the required form $\tilde{y}'''' = 0$.

Example 4.5. Consider the nonlinear ODE

$$y''' - \frac{(x+3-3(x+2))y'' - ((x+2)y' - x-3)y'^2}{x+2} = 0, \quad (7)$$

which has a solvable four-dimensional symmetry algebra of rank 1

$$e_1 = \partial_y, \quad e_2 = e^{-y}\partial_y, \quad e_3 = xe^{-y}\partial_y, \quad e_4 = e^{x-y}\partial_y, \quad (8)$$

where we invoke Theorem 4.1 to linearize it. Its derived algebra is $\langle e_2, e_3, e_4 \rangle$ of rank 1 and $\mathfrak{g}/\mathfrak{g}'$ whose representative is ∂_y , operates on \mathfrak{g}' with eigenvalues -1 of multiplicity 3. We choose coordinates so that $e_2 = \partial_{\tilde{y}}$ and $e_3 = \tilde{x}\partial_{\tilde{y}}$. Therefore, $\tilde{x} = x$, $\tilde{y}(x) = e^{y(x)}$, and ODE (7) is reduced to its canonical form given in [Omri Gat, 1992]

$$\tilde{y}''' - \frac{\tilde{x} + 3}{\tilde{x} + 2}\tilde{y}'' = 0. \quad (9)$$

Example 4.6. The nonlinear ODE

$$y''' - \frac{3y''^2}{y'} - xy'^4 = 0, \quad (10)$$

has a five-dimensional solvable Lie algebra

$$e_1 = \partial_y, \quad e_2 = x\partial_x, \quad e_3 = e^{-y}\partial_x, \quad (11)$$

$$e_4 = e^{y/2} \sin(\sqrt{3}y/2)\partial_x, \quad e_5 = e^{y/2} \cos(\sqrt{3}y/2)\partial_x, \quad (12)$$

with the relations

$$\begin{aligned} [e_1, e_3] &= -e_3, & [e_1, e_4] &= (\sqrt{3}/2)e_5 + (1/2)e_4, \\ [e_1, e_5] &= (1/2)e_5 - (\sqrt{3}/2)e_4, & [e_2, e_3] &= -e_3, \\ [e_2, e_4] &= -e_4, & [e_2, e_5] &= -e_5. \end{aligned} \quad (13)$$

Its derived algebra is

$$\mathfrak{g}' = \langle e_3, \frac{1}{2}(e_4 + \sqrt{3}e_5), \frac{1}{2}(e_5 - \sqrt{3}e_4) \rangle. \quad (14)$$

We compute the centralizer of e_3 in $\langle e_1, e_2 \rangle$ which is $\langle e_1 - e_2 \rangle$. Moreover, e_3 and $e_1 - e_2$ is of rank 2. The element e_2 has the eigenvalues -1 on \mathfrak{g}' and $e_1 - e_2$ has eigenvalues 0 and $\pm\sqrt{3}$. Thus, by using Theorem 4.2, the canonical coordinates corresponding to algebra $\langle e_3, e_1 - e_2 \rangle$, are $x = e^{-r}v(r)$, $y(x) = r$, that transforms nonlinear ODE (10) into

$$v''' - 3v'' + 3v' = 0. \quad (15)$$

Example 4.7. The nonlinear ODE

$$y'y''' = xy'^5 + y'^4 - y''y'^2 + 3y''^2, \quad (16)$$

has a five-dimensional solvable Lie algebra

$$e_1 = \partial_y, \quad e_2 = x\partial_x, \quad e_3 = \sin y\partial_x, \quad e_4 = \cos y\partial_x, \quad e_5 = e^{-y}\partial_x, \quad (17)$$

with derived algebra $\mathfrak{g}' = \langle e_3, e_4, e_5 \rangle$. The one- and two-dimensional invariant subspaces of $A = \langle e_1, e_2 \rangle$ are $\langle e_5 \rangle$ and $\langle e_3, e_4 \rangle$, respectively. By introducing the new basis $\tilde{e}_1 = e_1 - e_2$, $\tilde{e}_2 = e_2$, we find that $[\tilde{e}_1, e_3] = e_4 + e_3$, $[\tilde{e}_1, e_4] = e_4 - e_3$, $[\tilde{e}_1, e_5] = 0$, and e_2 operates on \mathfrak{g}' as a multiplication by -1 . Moreover, the centralizer of e_5 in A is \tilde{e}_1 . The eigenvalues of \tilde{e}_1 on the space spanned $\langle e_3, e_4 \rangle$ are $1 \pm i$. By using Theorem 4.2 and Remark 4.3, the canonical coordinates corresponding to algebra $\langle \tilde{e}_1, e_5 \rangle$, will give us the linearizing coordinates. The canonical coordinates are given by $x = e^{-r}v(r)$, $y(x) = r$, which transforms nonlinear ODE (16) into

$$v''' - 2v'' + 2v' = 0. \quad (18)$$

We have tested the algorithms given in this paper to several equations in literature – for example [14] – and have recovered the linearizing coordinates given there.

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