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# Theory of Extensions of Multiplicative Lie Algebras

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**Abstract.** We define the second cohomology of a multiplicative Lie algebra $K$ with coefficients in an abelian group $H$ with trivial multiplicative Lie algebra structure in two different cases. Consequently, we prove a natural bijective correspondence between the second cohomology and the set of equivalence classes of some special type of extensions. We also define the notion of Baer sum of extensions for multiplicative Lie algebras.

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*Key Words:* Multiplicative Lie algebras, Schreier extensions, factor system, Lie cohomology.

## 1. Introduction

In 1993, G. J. Ellis [5] conjectured that the five well known universal commutator identities generate all identities between commutators of weight $n$ and proved it for $n = 2, 3$ using homological techniques. Consequently he introduced a new algebraic concept "multiplicative Lie algebra" which is a generalization of groups as well as Lie algebras. In 2007, Donadze and Ladra [4] proved the Ellis conjecture for commutators of weight 4. In [8] F. Point and P. Wantiez studied the concept of nilpotency for multiplicative Lie algebras and proved various nilpotency results which are known for groups and Lie algebras. To see more structural properties of multiplicative Lie algebra, in [1] A. Bak et al. constructed and studied two homology theories for multiplicative Lie algebras which are already known for groups and Lie algebras. They also introduced central extensions of multiplicative Lie algebras. Furthermore, in 2015 G. Donadze et al. [2] gave the notion of non abelian tensor product of multiplicative Lie algebras and also established a relationship between the second homology and the non abelian exterior square of a multiplicative Lie algebra. In 2018, G. Donadze et al. [3] proved some algebraic structural results for the non abelian tensor product of multiplicative Lie algebras. They also obtained a six term exact sequence in the homology of multiplicative Lie algebras and proved a new version of the Stallings Theorem.

One of the problems in the theory of extensions is to determine all multiplicative Lie algebras $G$ (up to isomorphism) having $H$ as an ideal such that $G/H$ is isomorphic to $K$, for any two multiplicative Lie algebras $H$ and $K$. In other words: what are the classifications of all 2-fold extensions (up to equivalence) of $H$ by $K$? In 2019, R. Lal and S. K. Upadhyay [7] developed the Schreier extension theory for a 2-fold

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extension, in a special case, termed as "central extension" in an attempt to derive the Schur–Hopf formula for the multiplicative Lie algebras.

In this paper, we attempt to develop the Schreier extension theory for the 2-fold extension of an abelian group $H$ with trivial multiplicative Lie algebra structure by an arbitrary multiplicative Lie algebra $K$ analogous to the group theory in two different cases. We show an equivalence between the category of extensions and the category of factor systems of multiplicative Lie algebras analogous to groups [6]. We also prove a bijective correspondence between the second cohomology and the set of equivalence classes of such extensions.

The rest of the paper is organised as follows. In Section 2, we recall the basic definitions of multiplicative Lie algebra and 2-fold extensions. In Section 3, we study the 2-fold extension in the case when $H$ is contained in the center $Z(G)$ and define the second cohomology of $K$ with coefficients in $H$. We have also defined the notion of Baer sum of extensions for multiplicative Lie algebras analogous to groups [6]. In Section 4, we consider the case when $H$ is a subset of Lie center $LZ(G)$ and prove similar results as in Section 3. Moreover, there is an interesting problem to develop the Schreier extension theory for $n$-fold extensions of multiplicative Lie algebras.

## 2. Preliminaries

In this section, we give some definitions which are essential for this paper.

**Definition 2.1.** A *multiplicative Lie algebra* is a triple $(G, \cdot, *)$, where $(G, \cdot)$ is a group and $*$ is a binary operation on $G$ satisfying

(1) $x * x = 1$,

(2) $x * (y \cdot z) = (x * y) \cdot {}^{y}(x * z)$,

(3) $(x \cdot y) * z = {}^{x}(y * z) \cdot (x * z)$,

(4) $((x * y) * {}^{y}z) \cdot ((y * z) * {}^{z}x) \cdot ((z * x) * {}^{x}y) = 1$,

(5) ${}^{z}(x * y) = {}^{z}x * {}^{z}y$, where ${}^{z}x$ denotes $zxz^{-1}$,

for all $x, y, z \in G$. We call the binary operation $*$ as multiplicative Lie product.

**Example 2.2.** Let $H$ be an abelian group and $\mathrm{End}(H)$ be the set of all group endomorphisms on $H$. Then $\mathrm{End}(H)$ is an abelian group with the binary operation "$\cdot$" defined by $(F_1 \cdot F_2)(h) = F_1(h)F_2(h)$. Define a binary operation $*$ on $\mathrm{End}(H)$ by $(F_1 * F_2)(h) = F_1(F_2(h))F_2(F_1(h^{-1}))$. Then $*$ satisfies all the five properties of the multiplicative Lie product:

(1) $(F_1 * F_1)(h) = F_1(F_1(h))F_1(F_1(h^{-1})) = 0_H(h)$,

(2) $$\begin{aligned}(F_1F_2 * F_3)(h) &= F_1F_2(F_3(h))F_3(F_1F_2(h^{-1}))\\ &= F_1(F_3(h))F_3(F_1(h^{-1}))F_2(F_3(h))F_3(F_2(h^{-1})) = (F_2 * F_3)(F_1 * F_3)(h),\end{aligned}$$

(3) $$\begin{aligned}(F_1 * F_2F_3)(h) &= F_1(F_2F_3(h))F_2F_3(F_1(h^{-1}))\\ &= F_1(F_2(h))F_2(F_1(h^{-1}))F_1(F_3(h))F_3(F_1(h^{-1})) = (F_1 * F_2)(F_1 * F_3)(h),\end{aligned}$$

(4) $$\begin{aligned}&((F_1 * F_2) * {}^{F_2}F_3)((F_2 * F_3) * {}^{F_3}F_1)((F_3 * F_1) * {}^{F_1}F_2)(h)\\ &= ((F_1 * F_2) * F_3)((F_2 * F_3) * F_1)((F_3 * F_1) * F_2)(h)\end{aligned}$$

$$= F_1(F_2(F_3(h)))F_2(F_1(F_3(h^{-1})))F_3(F_1(F_2(h^{-1})))F_3(F_2(F_1(h)))$$
$$F_2(F_3(F_1(h)))F_3(F_2(F_1(h^{-1})))F_1(F_2(F_3(h^{-1})))F_1(F_3(F_2(h)))$$
$$F_3(F_1(F_2(h)))F_1(F_3(F_2(h^{-1})))F_2(F_3(F_1(h^{-1})))F_2(F_1(F_3(h)))$$
$$= 0_H(h),$$

(5) $^{F_1}(F_2 * F_3)(h) = (F_2 * F_3)(h)$,

for all $F_1, F_2, F_3 \in \text{End}(H)$ and $h \in H$, where $0_H$ is the trivial homomorphism on $H$. Therefore, $(\text{End}(H), \cdot, *)$ is a multiplicative Lie algebra.

**Definition 2.3.** Let $(G, \cdot, *)$ be a multiplicative Lie algebra.

(1) A subgroup $H$ of $G$ is said to be a *subalgebra* of $G$ if $x*y \in H$, for all $x, y \in H$.

(2) A subalgebra $H$ of $G$ is said to be an *ideal* of $G$ if it is normal subgroup of $G$ and $x * y \in H$ for all $x \in G$ and $y \in H$. It is easy to see that $G * G = \langle x * y : x, y \in G\rangle$ and $[G, G] = \langle xyx^{-1}y^{-1} : x, y \in G\rangle$ are ideals of $G$.

(3) The *Lie center* $LZ(G)$ of $G$ is defined by
$$LZ(G) = \{x \in K \mid x * y = 1 \text{ for all } y \in G\}.$$
It is an ideal of $G$.

(4) Let $(G', \circ, *')$ be an another multiplicative Lie algebra. A group homomorphism $\zeta : G \longrightarrow G'$ is called a *multiplicative Lie algebra homomorphism* if we have $\zeta(x * y) = \zeta(x) *' \zeta(y)$ for all $x, y \in G$.

**Definition 2.4.** (1) A short exact sequence
$$E(H, K) \equiv 1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$$
of multiplicative Lie algebras is called an *extension* of $H$ by $K$. A map $t : K \longrightarrow G$ is called a section of $E(H, K)$ if $\beta t = I_K$ and $t(1) = 1$.

(2) A *morphism* from an extension
$$E(H, K) \equiv 1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$$
to an extension
$$E(H', K') \equiv 1 \longrightarrow H' \xrightarrow{\alpha'} G' \xrightarrow{\beta'} K' \longrightarrow 1$$
is a triple $(\lambda, \mu, \nu)$, where $\lambda : H \longrightarrow H'$, $\mu : G \longrightarrow G'$ and $\nu : K \longrightarrow K'$ are multiplicative Lie algebra homomorphisms such that the following diagram
$$\begin{array}{ccccccccc} E(H, K) \equiv 1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 1 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ E(H', K') \equiv 1 & \longrightarrow & H' & \xrightarrow{\alpha'} & G' & \xrightarrow{\beta'} & K' & \longrightarrow & 1 \end{array}$$
is commutative.

(3) Two extensions $E_1(H, K)$ and $E_2(H, K)$ of $H$ by $K$ are *equivalent* if there is

an isomorphism $\phi : G_1 \longrightarrow G_2$ such that the following diagram

$$\begin{array}{ccccccccccc} E_1(H,K) \equiv 1 & \longrightarrow & H & \xrightarrow{i_1} & G_1 & \xrightarrow{\beta_1} & K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle I_H} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle I_K} & & \\ E_2(H,K) \equiv 1 & \longrightarrow & H & \xrightarrow{i_2} & G_2 & \xrightarrow{\beta_2} & K & \longrightarrow & 1 \end{array}$$

is commutative.

Throughout this paper, $H$ denotes an abelian group with trivial multiplicative Lie product and $K$ denotes an arbitrary multiplicative Lie algebra.

## 3. Schreier theory for center extension

One of the customary problem in the theory of multiplicative Lie algebras is to classify all multiplicative Lie algebras $G$ having $H$ as an ideal such that $G/H$ is isomorphic to $K$, for any two multiplicative Lie algebras $H$ and $K$. In analogous to groups we address this question in the Schreier theory of multiplicative Lie algebras. In [7], authors defined central extensions and developed its theory to give the notion of Schur multiplier for multiplicative Lie algebras. In this section, we discuss the theory of extensions

$$E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$$

when $H$ is contained in the center of the group $G$.

**Definition 3.1.** An extension $E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$ of an abelian group $H$ with trivial multiplicative Lie product by a multiplicative Lie algebra $K$ is called a *center extension* if $H$ is contained in the center of $G$.

Let $E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$ be a center extension of $H$ by $K$ and $t : K \longrightarrow G$ be a section of $E(H,K)$. Then it is easy to see that every element of the group $G$ can be uniquely expressed in the form $ht(x)$, for some $h \in H$ and $x \in K$.

The group operation "$\cdot$" (for details see [7]) in $G$ is given by

$$ht(x) \cdot kt(y) = hkf^t(x,y)t(xy), \tag{1}$$

where $f^t : K \times K \to H$ is a map satisfying

$$f^t(1,x) = f^t(x,1) = 1 \qquad \text{and} \qquad f^t(x,y)f^t(xy,z) = f^t(y,z)f^t(x,yz). \tag{2}$$

Further, $\beta(t(x) * h) = \beta(t(x)) * \beta(h) = x * 1 = 1$, $t(x) * h \in \ker(\beta) = H$. Thus for all $x \in K$, there is a map $\Gamma^t_x : H \to H$ defined by $\Gamma^t_x(h) = t(x) * h$. Also for $h, k \in H$,

$$\Gamma^t_x(hk) = t(x) * hk = (t(x) * h)^h (t(x) * k) = (t(x) * h)(t(x) * k)$$
$$\Longrightarrow \Gamma^t_x(hk) = \Gamma^t_x(h)\Gamma^t_x(k).$$

This shows that for all $x \in K$, $\Gamma^t_x$ is a group endomorphism on $H$.

Now the multiplicative Lie product "$*$" in $G$ is given as follows.

Since $\beta(t(x)*t(y)) = \beta(t(x))*\beta(t(y)) = x*y = \beta(t(x*y))$, we have in consequence $(t(x)*t(y))(t(x*y))^{-1} \in H$. Hence there exists a map $h^t : K \times K \to H$ such that

$$t(x)*t(y) = h^t(x,y)t(x*y).$$

Since $t(x)*t(x) = t(x)*1 = 1*t(x) = 1$, we have

$$h^t(x,x) = h^t(x,1) = h^t(1,x) = 1. \tag{3}$$

Further, $h*kt(y) = (h*k)^k(h*t(y)) = h*t(y) = \Gamma^t_y(h^{-1})$ and

$$t(x)*kt(y) = (t(x)*k)^k(t(x)*t(y)) = \Gamma^t_x(k)h^t(x,y)t(x*y).$$

Therefore $ht(x)*kt(y) = {}^h(t(x)*kt(y))(h*kt(y)) = (t(x)*(kt(y)))(h*(kt(y))) = \Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)t(x*y)$ (since $H \subseteq Z(G)$). Thus the multiplicative Lie product "$*$" in $G$ is given by

$$ht(x)*kt(y) = \Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)t(x*y). \tag{4}$$

**Lemma 3.2.** *The map $\Gamma^t_E : K \longrightarrow \mathrm{End}(H)$ defined by $\Gamma^t_E(x) = \Gamma^t_x$ is a multiplicative Lie algebra homomorphism.*

**Proof.** Let $x, y \in K$ and $h \in H$. Then

$$\Gamma^t_{xy}(h) = t(xy)*h = (f^t(x,y)^{-1}t(x)t(y))*h = {}^{f^t(x,y)^{-1}}((t(x)t(y))*h)(f^t(x,y)^{-1}*h)$$
$$= (t(x)t(y))*h = {}^{t(x)}(t(y)*h)(t(x)*h) = (t(y)*h)(t(x)*h) = \Gamma^t_x(h)\Gamma^t_y(h))$$
$$\implies \Gamma^t_{xy} = \Gamma^t_x\Gamma^t_y \Rightarrow \Gamma^t_E(xy) = \Gamma^t_E(x)\Gamma^t_E(y).$$

Thus $\Gamma^t_E$ is a group homomorphism. Now

$$\Gamma^t_{x*y}(h) = t(x*y)*h = (h^t(x,y)^{-1}(t(x)*t(y)))*h$$
$$= {}^{h^t(x,y)^{-1}}((t(x)*t(y))*h)(h^t(x,y)^{-1}*h) = (t(x)*t(y))*h.$$

By the Jacobi identity,

$$((t(x)*t(y)) *^{t(y)} h)((t(y)*h) *^h t(x))((h*t(x)) *^{t(x)} t(y)) = 1$$
$$\implies ((t(x)*t(y))*h)(\Gamma^t_y(h)*t(x))(\Gamma^t_x(h^{-1})*(f^t(x,y)t(xy)t(x)^{-1}) = 1$$
$$\implies (\Gamma^t_{x*y}(h)\Gamma^t_x(\Gamma^t_y(h^{-1})))(\Gamma^t_x(h^{-1})*(f^t(x,y)t(xy)f^t(x^{-1},x)^{-1}t(x^{-1}))) = 1$$
$$\implies \Gamma^t_{x*y}(h)\Gamma^t_x(\Gamma^t_y(h^{-1}))(\Gamma^t_x(h^{-1})*(f^t(x,y)f^t(x^{-1},x)^{-1}f^t(xy,x^{-1})t(xyx^{-1}))) = 1$$
$$\implies \Gamma^t_{x*y}(h)\Gamma^t_x(\Gamma^t_y(h^{-1}))\Gamma^t_{xyx^{-1}}(\Gamma^t_x(h)) = 1$$
$$\implies \Gamma^t_{x*y}(h) = \Gamma^t_x(\Gamma^t_y(h))\Gamma^t_y(\Gamma^t_x(h^{-1})) = (\Gamma^t_x*\Gamma^t_y)(h)$$
$$\implies \Gamma^t_{x*y} = \Gamma^t_x*\Gamma^t_y \Rightarrow \Gamma^t_E(x*y) = \Gamma^t_E(x)*\Gamma^t_E(y).$$

Therefore $\Gamma^t_E$ is a multiplicative Lie algebra homomorphism. ∎

Now we will see the properties of the function $h^t$ by using Equations (1), (2), and (4).

Consider the expression

$$\begin{aligned} ht(x) * (kt(y) \cdot lt(z)) &= ht(x) * (klf^t(y,z)t(yz)) \\ &= \Gamma^t_x(klf^t(y,z))\Gamma^t_{yz}(h^{-1})h^t(x,yz))t(x * yz). \end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned} &ht(x) * (kt(y) \cdot lt(z)) = (ht(x) * kt(y)) \cdot {}^{kt(y)}(ht(x) * lt(z)) \\ &= (\Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)t(x*y))(t(y)\Gamma^t_x(l)\Gamma^t_z(h^{-1})h^t(x,z)t(x*z)t(y)^{-1}) \\ &= (\Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)t(x*y))(t(y)\Gamma^t_x(l)\Gamma^t_z(h^{-1})h^t(x,z)t(x*z) \\ &\qquad \cdot f^t(y^{-1},y)^{-1}t(y^{-1})) \\ &= (\Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)t(x*y))(\Gamma^t_x(l)\Gamma^t_z(h^{-1})h^t(x,z)f^t(y,x*z)t(y(x*z)) \\ &\qquad \cdot f^t(y^{-1},y)^{-1}t(y^{-1})) \\ &= (\Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)t(x*y))(\Gamma^t_x(l)\Gamma^t_z(h^{-1})h^t(x,z)f^t(y,x*z)f^t(y^{-1},y)^{-1} \\ &\qquad \cdot f^t(y(x*z),y^{-1})t({}^y(x*z))). \end{aligned}$$

Finally, we have,

$$\begin{aligned} ht(x) * (kt(y) \cdot lt(z)) &= \Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)\Gamma^t_x(l)\Gamma^t_z(h^{-1})h^t(x,z)f^t(y,x*z) \\ &\cdot f^t(y^{-1},y)^{-1}f^t(y(x*z),y^{-1})f^t(x*y,{}^y(x*z))t((x*y)^y(x*z))). \end{aligned} \tag{6}$$

Now equating equation (5) and (6), we have

$$\begin{aligned} f^t(y^{-1},y(x*z))^{-1}f^t(y(x*z),y^{-1})f^t(x*y,{}^y(x*z))h^t(x,y)h^t(x,z) \\ = \Gamma^t_x(f^t(y,z))h^t(x,yz). \end{aligned} \tag{7}$$

Similarly, consider

$$\begin{aligned} (ht(x) \cdot kt(y)) * lt(z) &= (hkf^t(x,y)t(xy)) * lt(z) \\ &= \Gamma^t_{xy}(l)\Gamma^t_z(h^{-1}k^{-1}f^t(x,y)^{-1})h^t(xy,z)t(xy*z). \end{aligned} \tag{8}$$

On the other hand

$$\begin{aligned} &(ht(x) \cdot kt(y)) * lt(z) = {}^{ht(x)}\, (kt(y) * lt(z)) \cdot (ht(x) * lt(z)) \\ &= \Gamma^t_y(l)\Gamma^t_z(k^{-1})h^t(y,z)f^t(x,y*z)f^t(x^{-1},x)^{-1}f^t(x(y*z),x^{-1}) \\ &\qquad \cdot t({}^x(y*z))\Gamma^t_x(l)\Gamma^t_z(h^{-1})h^t(x,z)t(x*z) \\ &= \Gamma^t_{xy}(l)\Gamma^t_z(h^{-1}k^{-1})h^t(y,z)h^t(x,z)f^t(x,y*z)f^t(x^{-1},x)^{-1} \\ &\qquad \cdot f^t(x(y*z),x^{-1})f^t({}^x(y*z),x*z)t({}^x(y*z)(x*z)). \end{aligned} \tag{9}$$

Now equating Equations (8) and (9), we have

$$\begin{aligned} f^t(x^{-1},x(y*z))^{-1}f^t(x(y*z),x^{-1})f^t({}^x(y*z),x*z)h^t(x,z)h^t(y,z) \\ = \Gamma^t_z(f^t(x,y)^{-1})h^t(xy,z). \end{aligned} \tag{10}$$

Now consider the expressions

$$\begin{aligned}
&{}^{lt(z)}(ht(x)*kt(y)) = t(z)\Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)t(x*y)f^t(z^{-1},z)^{-1}t(z^{-1})\\
&= \Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)f^t(z,x*y)t(z(x*y))f^t(z^{-1},z)^{-1}t(z^{-1})\\
&= \Gamma^t_x(k)\Gamma^t_y(h^{-1})h^t(x,y)f^t(z,x*y)f^t(z^{-1},z)^{-1}f^t(z(x*y),z^{-1})t({}^z(x*y)). \qquad (11)
\end{aligned}$$

Also we have

$$\begin{aligned}
&{}^{lt(z)}(ht(x)) * {}^{lt(z)}(kt(y))\\
&= (t(z)(ht(x)f^t(z^{-1},z)^{-1}t(z^{-1})) * (t(z)(kt(y))f^t(z^{-1},z)^{-1}t(z^{-1}))\\
&= (hf^t(z,x)f^t(z^{-1},z)^{-1}f^t(zx,z^{-1})t({}^zx)) * (kf^t(z,y)f^t(z^{-1},z)^{-1}f^t(zy,z^{-1})t({}^zy))\\
&= \Gamma^t_x(kf^t(z,y)f^t(z^{-1},z)^{-1}f^t(zy,z^{-1}))(\Gamma^t_y(h^{-1}f^t(z,x)^{-1}f^t(z^{-1},z)f^t(zx,z^{-1})^{-1})\\
&\qquad \cdot h^t({}^zx,{}^zy)t({}^zx*{}^zy). \qquad (12)
\end{aligned}$$

By equating (11) and (12), we obtain

$$\begin{aligned}
&\Gamma^t_x(f^t(z^{-1},zy)^{-1}f^t(zy,z^{-1}))\Gamma^t_y(f^t(z^{-1},zx)f^t(zx,z^{-1})^{-1})h^t({}^zx,{}^zy)\\
&\qquad = f^t(z^{-1},z(x*y))^{-1}f^t(z(x*y),z^{-1})h^t(x,y). \qquad (13)
\end{aligned}$$

Now consider the expressions,

$$\begin{aligned}
&((ht(x)*kt(y)) * {}^{kt(y)}lt(z))((kt(y)*lt(z)) * {}^{lt(z)}ht(x))\\
&\quad \cdot ((lt(z)*ht(x)) * {}^{ht(x)}kt(y)) = 1\\
&\implies (\Gamma^t_{x*y}(lf^t(y,z)f^t(y^{-1},y)^{-1}f^t(yz,y^{-1}))\Gamma^t_z(\Gamma^t_x(k^{-1})\Gamma^t_y(h)h^t(x,y)^{-1})h^t(x*y,{}^yz)\\
&\quad \cdot t((x*y)*{}^yz))(\Gamma^t_{y*z}(hf^t(z,x)f^t(z^{-1},z)^{-1}f^t(zx,z^{-1}))\Gamma^t_x(\Gamma^t_y(l^{-1})\Gamma^t_z(k)h^t(y,z)^{-1})\\
&\quad \cdot h^t(y*z,{}^zx)t((y*z)*{}^zx))(\Gamma^t_{z*x}(kf^t(x,y)f^t(x^{-1},x)^{-1}f^t(xy,x^{-1}))\\
&\quad \cdot \Gamma^t_y(\Gamma^t_z(h^{-1})\Gamma^t_x(l)h^t(z,x)^{-1})h^t(z*x,{}^xy)t((z*x)*{}^xy)) = 1\\
&\implies \Gamma^t_{x*y}(lf^t(y,z)f^t(y^{-1},y)^{-1}f^t(yz,y^{-1}))\Gamma^t_z(\Gamma^t_x(k^{-1})\Gamma^t_y(h)h^t(x,y)^{-1})\\
&\quad \cdot \Gamma^t_{y*z}(hf^t(z,x)f^t(z^{-1},z)^{-1}f^t(zx,z^{-1}))\Gamma^t_x(\Gamma^t_y(l^{-1})\Gamma^t_z(k)h^t(y,z)^{-1})\Gamma^t_{z*x}(kf^t(x,y)\\
&\quad \cdot f^t(x^{-1},x)^{-1}f^t(xy,x^{-1}))\Gamma^t_y(\Gamma^t_z(h^{-1})\Gamma^t_x(l)h^t(z,x)^{-1})h^t(x*y,{}^yz)h^t(y*z,{}^zx)\\
&\quad \cdot f^t((x*y)*{}^yz,(y*z)*{}^zx)h^t(z*x,{}^xy)f^t(((x*y)*{}^yz)((y*z)*{}^zx),(z*x)*{}^xy) = 1\\
&\implies \Gamma^t_{x*y}(lf^t(y,z)f^t(y^{-1},y)^{-1}f^t(yz,y^{-1}))\Gamma^t_{y*z}(hf^t(z,x)f^t(z^{-1},z)^{-1}f^t(zx,z^{-1}))\\
&\quad \cdot \Gamma^t_{z*x}(kf^t(x,y)f^t(x^{-1},x)^{-1}f^t(xy,x^{-1}))\Gamma^t_{x*z}(k)\Gamma^t_{z*y}(h)\Gamma^t_{y*x}(l)\Gamma^t_z(h^t(x,y)^{-1})\\
&\quad \cdot \Gamma^t_x(h^t(y,z)^{-1}))\Gamma^t_y(h^t(z,x)^{-1}))h^t(x*y,{}^yz)h^t(y*z,{}^zx)\\
&\quad \cdot f^t((x*y)*{}^yz,(y*z)*{}^zx)h^t(z*x,{}^xy)\\
&\quad \cdot f^t(((x*y)*{}^yz)((y*z)*{}^zx),(z*x)*{}^xy) = 1.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\Gamma^t_{x*y}(f^t(y^{-1},yz)^{-1}f^t(yz,y^{-1}))\Gamma^t_{y*z}(f^t(z^{-1},zx)^{-1}\\
&\quad \cdot f^t(zx,z^{-1}))\Gamma^t_{z*x}(f^t(x^{-1},xy)^{-1}f^t(xy,x^{-1}))\Gamma^t_z(h^t(x,y)^{-1})\\
&\quad \cdot \Gamma^t_x(h^t(y,z)^{-1})\Gamma^t_y(h^t(z,x)^{-1})h^t((x*y),{}^yz)h^t((y*z),{}^zx)h^t((z*x),{}^xy)\\
&\quad \cdot f^t((x*y)*{}^yz,(y*z)*{}^zx)f^t(((z*x)*{}^xy)^{-1},((z*x)*{}^xy)) = 1. \qquad (14)
\end{aligned}$$

**Definition 3.3.** Let $H$ be an abelian group with trivial multiplicative Lie product and $K$ be an arbitrary multiplicative Lie algebra. Then a *center factor system* is a quintuple $(K, H, f, h, \Gamma)$, where $\Gamma : K \longrightarrow \mathrm{End}(H)$ is a multiplicative Lie algebra homomorphism and $f,\ h$ are maps from $K \times K$ to $H$ satisfying the conditions like equations $(2), (3), (7), (10), (13)$ and $(14)$.

So, we can say that for every center extension $E(H, K)$ with a choice of a section $t$, we have a center factor system $\mathrm{CFac}(E, t) = (K, H, f^t, h^t, \Gamma^t_E)$, called as center factor system given by the center extension $E(H, K)$. Now, we prove the following proposition:

**Proposition 3.4.** *Every center extension $E(H, K)$ with a choice of a section $t$ determines a center factor system $\mathrm{CFac}(E, t) = (K, H, f^t, h^t, \Gamma^t_E)$ and conversely for a given center factor system $(K, H, f, h, \Gamma)$, there exists a center extension $E(H, K)$ of $H$ by $K$ with a section $t$ such that $(K, H, f, h, \Gamma) = \mathrm{CFac}(E, t)$.*

**Proof.** From the above discussions and the Definition 3.3, it can be seen that every center extension $E(H, K)$ of $H$ by $K$ with a given section $t$ determines a center factor system $\mathrm{CFac}(E, t) \equiv (K, H, \Gamma^t, f^t, h^t)$.
Conversely, let $G = H \times K$. Define $\cdot$ and $*$ operations on $G$ as

$$(a, x) \cdot (b, y) = (abf(x, y), xy) \quad \text{and} \quad (a, x) * (b, y) = (\Gamma_x(b)\Gamma_y(a^{-1})h(x, y), x * y).$$

It is easy to see that $(G, \cdot, *)$ is a multiplicative Lie algebra such that

$$E(H, K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} K \longrightarrow 1$$

is a center extension of $H$ by $K$, where $i$ is the first inclusion and $p$ is the second projection. Let $t$ be a section of $E$ given by $t(x) = (1, x)$. By an easy computation, it can be seen that $\Gamma^t_E = \Gamma$, $f^t = f$, and $h^t = h$. ■

### 3.1. Equivalence between category CEXT of center extensions and category CFAC of center factor systems

Let $(\lambda, \mu, \nu)$ be a morphism from the center extension $E(H_1, K_1)$ to the center extension $E(H_2, K_2)$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} E(H_1, K_1) \equiv 1 & \longrightarrow & H_1 & \xrightarrow{i_1} & G_1 & \xrightarrow{\beta_1} & K_1 & \longrightarrow & 1 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ E(H_2, K_2) \equiv 1 & \longrightarrow & H_2 & \xrightarrow{i_2} & G_2 & \xrightarrow{\beta_2} & K_2 & \longrightarrow & 1. \end{array}$$

Let $t_1$ and $t_2$ be sections of $E(H_1, K_1)$ and $E(H_2, K_2)$ respectively. Consider the corresponding factor systems $(K_1, H_1, f^{t_1}, h^{t_1}, \Gamma^{t_1})$ and $(K_2, H_2, f^{t_2}, h^{t_2}, \Gamma^{t_2})$ of $E(H_1, K_1)$ and $E(H_2, K_2)$, respectively. Then for $x \in K_1$, $\mu(t_1(x)) \in G_2$ and $\beta_2(\mu(t_1(x))) = \nu(\beta_1(t_1(x))) = \nu(x) = \beta_2(t_2(\nu(x)))$. Thus $\mu(t_1(x))(t_2(\nu(x)))^{-1} \in H_2$. In turn, we have a unique $g(x) \in H_2$ such that $g(x) = \mu(t_1(x))(t_2(\nu(x)))^{-1}$.

This implies
$$\mu(t_1(x)) = g(x)t_2(\nu(x)). \tag{15}$$

Since $t_1(1) = 1 = t_2(1)$, it follows that
$$g(1) = 1. \tag{16}$$

Since $\mu(t_1(x)t_1(y)) = \mu(t_1(x))\mu(t_1(y))$, we have the following equation

$$\lambda(f^{t_1}(x,y))g(xy) = g(x)g(y)f^{t_2}(\nu(x),\nu(y)). \tag{17}$$

Also we have

$$\begin{aligned}\mu(t_1(x)*t_1(y)) = \mu(h^{t_1}(x,y)t_1(x*y)) &= \lambda(h^{t_1}(x,y))\mu(t_1(x*y))\\ &= \lambda(h^{t_1}(x,y))g(x*y)t_2(\nu(x)*\nu(y)).\end{aligned} \tag{18}$$

On the other hand

$$\begin{aligned}\mu(t_1(x)*t_1(y)) = \mu(t_1(x))*\mu(t_1(y)) &= (g(x)t_2(\nu(x)))*(g(y)t_2(\nu(y)))\\ &= \Gamma^{t_2}_{\nu(x)}(g(y))\Gamma^{t_2}_{\nu(y)}(g(x)^{-1})h^{t_2}(\nu(x),\nu(y))t_2(\nu(x)*\nu(y)).\end{aligned} \tag{19}$$

By comparing Equations (18) and (19), we obtain

$$\lambda(h^{t_1}(x,y))g(x*y) = \Gamma^{t_2}_{\nu(x)}(g(y))\Gamma^{t_2}_{\nu(y)}(g(x)^{-1})h^{t_2}(\nu(x),\nu(y)). \tag{20}$$

Thus a morphism $(\lambda,\mu,\nu)$ between two center extensions $E(H_1,K_1)$ and $E(H_2,K_2)$ together with choices of sections $t_1$ and $t_2$ of the corresponding extensions, induces a map $g$ from $K_1$ to $H_2$ such that the triplet $(\nu,g,\lambda)$ satisfies Equations (16), (17), and (20). It can be seen as a morphism from the factor system $(K_1,H_1,f^{t_1},h^{t_1},\Gamma^{t_1}_{E_1})$ to the factor system $(K_2,H_2,f^{t_2},h^{t_2},\Gamma^{t_2}_{E_2})$.

Let $(\lambda_1,\mu_1,\nu_1)$ be a morphism from the center extension

$$E(H_1,K_1) \equiv 1 \longrightarrow H_1 \xrightarrow{i_1} G_1 \xrightarrow{\beta_1} K_1 \longrightarrow 1$$

to the center extension

$$E(H_2,K_2) \equiv 1 \longrightarrow H_2 \xrightarrow{i_2} G_2 \xrightarrow{\beta_2} K_2 \longrightarrow 1$$

and $(\lambda_2,\mu_2,\nu_2)$ be another morphism from the center extension $E(H_2,K_2)$ to the center extension

$$E(H_3,K_3) \equiv 1 \longrightarrow H_3 \xrightarrow{i_3} G_3 \xrightarrow{\beta_3} K_3 \longrightarrow 1.$$

Let $t_1$, $t_2$, and $t_3$ be corresponding choices of sections. Then

$$\mu_1(t_1(x)) = g_1(x)t_2(\nu_1(x)) \text{ and } \mu_2(t_2(x)) = g_2(x)t_3(\nu_2(x)),$$

where $g_1 : K_1 \longrightarrow H_2$ and $g_2 : K_2 \longrightarrow H_3$ are uniquely determined maps same as $g$ in Equation (15). In turn, we have,

$$\begin{aligned}\mu_2(\mu_1(t_1(x))) = \mu_2((g_1(x)))\mu_2(t_2(\nu_1(x))) &= \mu_2(g_1(x)))g_2(\nu_1(x))t_3(\nu_2(\nu_1(x)))\\ &= \lambda_2(g_1(x))g_2(\nu_1(x))t_3(\nu_2(\nu_1(x))) = g_3(x)t_3(\nu_2(\nu_1(x))),\end{aligned}$$

where $g_3(x) = \lambda_2(g_1(x))g_2(\nu_1(x))$, for each $x \in K_1$. In consequence the composition $(\lambda_2\circ\lambda_1,\ \mu_2\circ\mu_1,\nu_2\circ\nu_1)$ induces the triple $(\nu_2\circ\nu_1, g_3, \lambda_2\circ\lambda_1)$.

Now we introduce the category **CFAC** whose objects are center factor system, and a morphism from $(K_1,H_1,f^1,h^1,\Gamma^1)$ to $(K_2,H_2,f^2,h^2,\Gamma^2)$ is a triple $(\nu,g,\lambda)$, where

$\nu : K_1 \longrightarrow K_2$, $\lambda : H_1 \longrightarrow H_2$ are multiplicative Lie algebra homomorphism, and $g : K_1 \longrightarrow H_2$ is a map such that

(1) $g(1) = 1$,

(2) $\lambda(f^1(x,y))g(xy) = g(x)g(y)f^2(\nu(x),\nu(y))$,

(3) $\lambda(h^1(x,y))g(x*y) = \Gamma^2_{\nu(x)}(g(y))\Gamma^2_{\nu(y)}(g(x)^{-1})h^2(\nu(x),\nu(y))$.

The composition of morphisms $(\nu_1, g_1, \lambda_1)\colon (K_1, H_1, f^1, h^1, \Gamma^1) \to (K_2, H_2, f^2, h^2, \Gamma^2)$ and $(\nu_2, g_2, \lambda_2)\colon (K_2, H_2, f^2, h^2, \Gamma^2) \to (K_3, H_3, f^3, h^3, \Gamma^3)$ is naturally the triple $(\nu_2 \circ \nu_1, g_3, \lambda_2 \circ \lambda_1)$, where $g_3$ is given by $g_3(x) = g_2(\nu_1(x))\lambda_2(g_1(x))$ for each $x \in K_1$. So, finally from the above discussion, we have the following theorem:

**Theorem 3.5.** *There is an equivalence between the category* **CEXT** *of center extensions to the category* **CFAC** *of center factor systems.*

Now we extend the Lemma 3.2 to illustrate the equivalence classes of center extensions.

**Lemma 3.6.** *Let $E(H,K)$ be a center extension and $t$ be a section of $E(H,K)$. Then the map $\Gamma^t_E$ is independent of choice of section as well as choice of the representative $E(H,K)$ of the equivalence class.*

**Proof.** Let $s$ be an another section of $E(H,K)$ and $\Gamma^s_E : K \longrightarrow \mathrm{End}(H)$ be the corresponding multiplicative Lie algebra homomorphism. Since $s$ and $t$ are two sections of $E(H,K)$, there exists a map $g : K \to H$ with $g(1) = 1$ such that $s(x) = g(x)t(x)$ for all $x \in K$. Therefore for $x \in K$ we have

$$\Gamma^s_x(h) = s(x) * h = (g(x)t(x)) * h = {}^{g(x)}(t(x) * h)(g(x) * h) = t(x) * h = \Gamma^t_x(h).$$

Thus $\Gamma^s_E(x) = \Gamma^t_E(x)$ for all $x \in K$. This shows that $\Gamma^t_E$ is independent on section $t$. Now let $E'(H,K)$ be an equivalent center extension to $E(H,K)$. Let $t'$ be a section of $E'(H,K)$, $\Gamma^{t'}_{E'} : K \longrightarrow \mathrm{End}(H)$ be the corresponding multiplicative Lie algebra homomorphism and $(I_H, \phi, I_K)$ be an equivalence from $E(H,K)$ to $E'(H,K)$. Since $t$ is a section of $E(H,K)$, $\phi \circ t$ is a section of $E'(H,K)$. Therefore there exists a map $g$ from $K$ to $H$ satisfying $g(x)\phi(t(x)) = t'(x)$. Now we have

$$\begin{aligned}\Gamma^{t'}_x(h) &= t'(x) * h = (g(x)\phi(t(x))) * h = {}^{g(x)}(\phi(t(x)) * h)(g(x) * h) = \phi(t(x)) * h \\ &= \phi(t(x)) * \phi(h) = \phi(t(x) * h) = \phi(\Gamma^t_x(h)) = \Gamma^t_x(h)\end{aligned}$$

(since $\phi(h) = h$ for all $h \in H$). Thus $\Gamma^{t'}_{E'}(x) = \Gamma^t_E(x)$ for all $x \in K$. Therefore $\Gamma^t_E$ is independent of the choice of sections and the equivalent extensions. ■

Now onwards, for a center extension $E(H,K)$, we denote $\Gamma^t_E$ by $\Gamma_{[E]}$ without any ambiguity.

**Theorem 3.7.** *Let $\mathrm{CExt}(H,K)$ denote the set of all equivalence classes of center extensions of $H$ by $K$ and $\mathrm{Hom}(K, \mathrm{End}(H))$ denote the set of all multiplicative Lie algebra homomorphisms from $K$ to $\mathrm{End}(H)$. Then there is a natural surjective map $\eta : \mathrm{CExt}(H,K) \longrightarrow \mathrm{Hom}(K, \mathrm{End}(H))$ defined by $\eta([E]) = \Gamma_{[E]}$.*

**Proof.** From the Lemma 3.6, the map $\eta : \mathrm{CExt}(H,K) \to \mathrm{Hom}(K, \mathrm{End}(H))$ defined by $\eta([E]) = \Gamma_{[E]}$ is well defined. Let $\Gamma : K \longrightarrow \mathrm{End}(H)$ be a multiplicative Lie algebra homomorphism defined by $\Gamma(x) = \Gamma_x$. Let $G = H \times K$. Then $G$ is a multiplicative Lie algebra with the operations $\cdot$ and $*$, defined by $(h,x)\cdot(k,y) = (hk,xy)$ and $(h,x)*(k,y) = (\Gamma_x(k)\Gamma_y(h^{-1}), x*y)$ respectively. Consider a section $t : K \longrightarrow G$ defined by $t(x) = (1,x)$. Then $t$ is a group homomorphism. Therefore we have a center extension $E(H,K)$ of $H$ by $K$

$$E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$$

such that $f^t$ and $h^t$ are trivial maps and $\Gamma_E = \Gamma$. Thus every multiplicative Lie algebra homomorphism from $K \longrightarrow \mathrm{End}(H)$ gives a center extension of multiplicative Lie algebra of $H$ by $K$. This shows that $\eta$ is surjective. ■

**Example 3.8.** The map $\eta$ in Theorem 3.7 need not be injective. Specifically, there may exists the same multiplicative Lie algebra homomorphism $\Gamma$ from $K$ to $\mathrm{End}(H)$ for two distinct classes of extensions. Consider the center extensions:

$$E_1(\mathbb{Z},\mathbb{Z}_5) \equiv \{0\} \longrightarrow \mathbb{Z} \xrightarrow{(i_1,0)} \mathbb{Z}\bigoplus\mathbb{Z}_5 \xrightarrow{p_2} \mathbb{Z}_5 \longrightarrow \{0\}$$

and

$$E_2(\mathbb{Z},\mathbb{Z}_5) \equiv \{0\} \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\nu} \mathbb{Z}_5 \longrightarrow \{0\}$$

of the multiplicative Lie algebra $\mathbb{Z}$ by $\mathbb{Z}_5$, where $\alpha(n) = 5n$, $\forall n \in \mathbb{Z}$, and $\nu$ is the natural quotient map. Since $\mathbb{Z}$ and $\mathbb{Z}_5$ are cyclic groups, the multiplicative Lie product is trivial on these groups. Also it can be seen that $E_1(\mathbb{Z},\mathbb{Z}_5)$ and $E_2(\mathbb{Z},\mathbb{Z}_5)$ are not equivalent as center extension. Since $\mathrm{End}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, $\mathrm{Hom}(\mathbb{Z}_5,\mathbb{Z})$ is trivial. Therefore here it can be seen that $[E_1] \neq [E_2]$ but $\eta([E_1]) = \eta([E_2])$.

Now we discuss the following problem:

**Problem 3.9.** Let $H$ be an abelian group with trivial multiplicative Lie product. Classify all center extensions of $H$ by $K$ (up to equivalence) with the given multiplicative Lie algebra homomorphism $\Gamma : K \longrightarrow \mathrm{End}(H)$.

**Definition 3.10.** A multiplicative center 2-cocycle of a multiplicative Lie algebra $K$ with coefficient in an abelian group $H$ with trivial multiplicative Lie product is a triplet $(f,h,\Gamma)$, where $f \in Z^2(K,H)$ is a group 2-cocycle of $K$ with coefficient in the trivial $K$-module $H$, $\Gamma : K \longrightarrow \mathrm{End}(H)$ is a multiplicative Lie algebra homomorphism and $h : K \times K \longrightarrow H$ is a map satisfying the conditions like Equations (2), (3), (7), (10), (13), and (14), where we replace $f^t$ and $h^t$ with $f$ and $h$, respectively.

**Remark 3.11.** Let $E(H,K)$ be a center extension with a choice of section $t$. Then from above discussion, it is clear that we have a multiplicative center 2-cocycle $(f^t,h^t,\Gamma_E)$.

Conversely, if we have a multiplicative center 2-cocycle $(f,h,\Gamma)$ of multiplicative Lie algebra $K$ with coefficient in the abelian group $H$ with trivial multiplicative Lie product, then from Proposition 3.4, it can be seen that there exists a multiplicative Lie algebra $G$ and a center extension $E(H,K)$ with a choice of section $t$ such that $f^t = f$, $\Gamma_E = \Gamma$ and $h^t = h$. ■

Let $E(H,K)$ be a center extension with a choice of a section $t$. Then we have a multiplicative center 2-cocycle $(f^t,h^t,\Gamma_E)$. Now, let $s$ be an another section of $E$.

Then there exists a map $g : K \to H$ with $g(1) = 1$ such that $s(x) = g(x)t(x)$ for all $x \in K$. So

$$hs(x) \cdot ks(y) = hkf^s(x,y)s(xy) = hkf^s(x,y)g(xy)t(xy). \tag{21}$$

On the other hand

$$hs(x) \cdot ks(y) = (hg(x)t(x)) \cdot (kg(y)t(y)) = hkg(x)g(y)f^t(x,y)t(xy). \tag{22}$$

By comparing equations (21) and (22), we have

$$f^s(x,y) = g(x)g(xy)^{-1}g(y)f^t(x,y).$$

Consider the expression

$$\begin{aligned} hs(x) * ks(y) &= \Gamma_x(k)\Gamma_y(h^{-1})h^s(x,y)s(x*y) \\ &= \Gamma_x(k)\Gamma_y(h^{-1})h^s(x,y)g(x*y)t(x*y). \end{aligned} \tag{23}$$

On the other hand

$$\begin{aligned} hs(x) * ks(y) &= (hg(x)t(x)) * (kg(y)t(x)) \\ &= \Gamma_x(kg(y))\Gamma_y(h^{-1}g(x)^{-1})h^t(x,y)t(x*y). \end{aligned} \tag{24}$$

By comparing Equations (23) and (24), we have

$$h^s(x,y) = \Gamma_x(g(y))\Gamma_y(g(x)^{-1})g(x*y)^{-1}h^t(x,y).$$

This motivates us for the following definition:

**Definition 3.12.** Two multiplicative center 2-cocycle $(f^s, h^s, \Gamma)$ and $(f^t, h^t, \Gamma)$ are said to be *equivalent* if there exists an identity preserving map $g : K \longrightarrow H$ satisfying:

(1) $f^s(x,y) = g(x)g(y)g(xy)^{-1}f^t(x,y)$,

(2) $h^s(x,y) = \Gamma_x(g(y))\Gamma_y(g(x)^{-1})g(x*y)^{-1}h^t(x,y)$.

The set $Z^2_{ML(\Gamma)}(K,H)$ of all multiplicative center 2-cocycles of $K$ with coefficient in $H$ is easily seen to be an abelian group with respect to coordinate wise operation given by $(f,h,\Gamma) \cdot (f^{'},h^{'},\Gamma) = (ff^{'},hh^{'},\Gamma)$. Given any identity preserving map $g : K \longrightarrow H$, the triple $(\delta g, g^*, \Gamma)$ is a member of $Z^2_{ML(\Gamma)}(K,H)$, where $\delta g$ and $g^*$ are maps from $K \times K$ to $H$ given by $\delta g((x,y)) = g(y)g(xy)^{-1}g(x)$ and $g^*(x,y) = \Gamma_x(g(y))\Gamma_y(g(x)^{-1})g(x*y)^{-1}$ respectively.

Let $\mathrm{MAP}(K,H)$ denote the group of identity preserving maps from $K$ to $H$. So we have a homomorphism $\chi : \mathrm{MAP}(K,H) \longrightarrow Z^2_{ML(\Gamma)}(K,H)$ given by $\chi(g) = (\delta g, g^*, \Gamma)$. The image of $\chi$ is called the group of multiplicative center 2-coboundaries of $K$ with coefficient in $H$ and it is denoted by $B^2_{ML(\Gamma)}(K,H)$. The quotient group $Z^2_{ML(\Gamma)}(K,H)/B^2_{ML(\Gamma)}(K,H)$ is called the second center cohomology of $K$ with coefficients in $H$ and it is denoted by $H^2_{ML(\Gamma)}(K,H)$. In turn, we get the following exact sequence of abelian groups:

$$1 \longrightarrow \mathrm{Hom}(K,H) \xrightarrow{i} \mathrm{MAP}(K,H) \xrightarrow{\chi} Z^2_{ML(\Gamma)}(K,H) \xrightarrow{\nu} H^2_{ML(\Gamma)}(K,H) \longrightarrow 1,$$

where $\nu$ is a quotient map.

**Theorem 3.13.** *Let $H$ be an abelian group with trivial multiplicative Lie product, $K$ be a multiplicative Lie algebra and $\Gamma$ be a multiplicative Lie algebra homomorphism from $K$ to* $\mathrm{End}(H)$. *Then there is a natural bijective correspondence between the set* $\mathrm{CExt}_\Gamma(H,K)$ *of equivalence classes of center extensions of $H$ by $K$ with given $\Gamma$ and the second center cohomology* $H^2_{ML(\Gamma)}(K,H)$.

**Proof.** Let $E(H,K)$ and $E'(H,K)$ be two equivalent center extensions of $H$ by $K$ with multiplicative Lie algebra homomorphism $\Gamma$ and sections $t$, $t'$ respectively. Let $(f^t,h^t,\Gamma)$ and $(f^{t'},h^{t'},\Gamma)$ be the multiplicative center 2-cocycles corresponding to center extensions $E(H,K)$ and $E'(H,K)$ respectively. Then there exists a map $g$ from $K$ to $H$ with $g(1)=1$ such that $f^{t'}(x,y)=g(x)g(xy)^{-1}g(y)f^t(x,y)$ and $h^{t'}(x,y)=\Gamma_x(g(y))\Gamma_y(g(x)^{-1})g(x*y)h^t(x,y)$.
This shows that $(f^t,h^t,\Gamma)B^2_{ML(\Gamma)}(K,H)=(f^{t'},h^{t'},\Gamma)B^2_{ML(\Gamma)}(K,H)$. Therefore we have a map $\Phi$ from $\mathrm{CExt}_\Gamma(H,K)$ to $H^2_{ML(\Gamma)}(K,H)$ defined by

$$\Phi([E])=(f^t,h^t,\Gamma)B^2_{ML(\Gamma)}(K,H),$$

where $t$ is a section of $E(H,K)$. Let $(f,h,\Gamma)\in Z^2_{ML(\Gamma)}(K,H)$. Then by the Remark (3.11), we have a center extension $E(H,K)$ of $H$ by $K$ and section $t$ such that $f^t=f$, $h^t=h$ and $\Gamma_E=\Gamma$. This shows that $\Phi$ is surjective.
Let $E(H,K)$ and $E'(H,K)$ be two center extensions of $H$ by $K$, with sections $t$ and $t'$ respectively such that $\Phi([E])=\Phi([E'])$; then

$$(f^t,h^t,\Gamma)B^2_{ML(\Gamma)}(K,H)=(f^{t'},h^{t'},\Gamma)B^2_{ML(\Gamma)}(K,H).$$

Hence again there exists a map $g$ from $K$ to $H$ with $g(1)=1$ satisfying $f^{t'}(x,y)=g(x)g(xy)^{-1}g(y)f^t(x,y)$ and $h^{t'}(x,y)=\Gamma_x(g(y))\Gamma_y(g(x)^{-1})g(x*y)h^t(x,y)$. It follows that $(f^t,h^t,\Gamma)$ and $(f^{t'},h^{t'},\Gamma)$ are equivalent. Therefore the center factor systems $(K,H,f^t,h^t,\Gamma)$ and $(K,H,f^{t'},h^{t'},\Gamma)$ are equivalent and hence $E(H,K)$ and $E'(H,K)$ are equivalent. Thus $\Phi$ is injective. ∎

**Remark 3.14.** Let $(G_1,\circ_1,*_1)$ and $(G_2,\circ_2,*_2)$ be two multiplicative Lie algebras. Then $G=G_1\times G_2$ is also a multiplicative Lie algebra with operations $\cdot$ and $*$ given by

$$(g_1,g_2)\cdot(g_1',g_2')=(g_1\circ_1 g_1',g_2\circ_2 g_2') \quad \text{and} \quad (g_1,g_2)*(g_1',g_2')=(g_1 *_1 g_1',g_2 *_2 g_2').$$

**3.2. Baer sum on the class of center extensions.** Now we will define Baer sum on the class of center extensions of multiplicative Lie algebras analogues to the class of group extensions. Let

$$E_1(H,K)\equiv 1\longrightarrow H\xrightarrow{i_1}G\xrightarrow{\beta_1}K\longrightarrow 1$$

and

$$E_2(H,K)\equiv 1\longrightarrow H\xrightarrow{i_2}G'\xrightarrow{\beta_2}K\longrightarrow 1$$

be two center extensions of $H$ by $K$ with the corresponding fixed multiplicative Lie algebra homomorphism $\Gamma: K\longrightarrow \mathrm{End}(H)$. Consider

$$L=\{(g_1,g_2): g_1\in G,\ g_2\in G',\ \beta_1(g_1)=\beta_2(g_2)\}$$

and $D=\{(h,h^{-1}): h\in H\}$. Then it is easy to verify that $L$ is a subalgebra of $G_1\times G_2$ and $D$ is an ideal of $L$.

Also we get a center extension

$$E_1(H,K) \uplus E_2(H,K) \equiv 1 \longrightarrow H \xrightarrow{\eta} \bar{G} \xrightarrow{\bar{\chi}} K \longrightarrow 1,$$

where $\bar{G} = L/D$, $\eta(h) = (h,1)D$ and $\bar{\chi}((g_1,g_2)D) = \beta_1(g_1)$ are multiplicative Lie algebra homomorphisms (for the group structure details, see pages 388–389 of [6]). Further, let $t_1$ be a section of $E_1(H,K)$ and $t_2$ be a section of $E_2(H,K)$ with corresponding multiplicative center 2-cocycles $(f^{t_1}, h^{t_1}, \Gamma)$ and $(f^{t_2}, h^{t_2}, \Gamma)$, respectively. Then we have a section $t_1 + t_2$ of $E_1(H,K) \uplus E_2(H,K)$ given by $(t_1+t_2)(x) = (t_1(x), t_2(x))D$.

**Claim 3.15.** *$f^{t_1+t_2} = f^{t_1} f^{t_2}$, $h^{t_1+t_2} = h^{t_1} h^{t_2}$ and $\Gamma_{E_1 \uplus E_2} = \Gamma$.*

**Proof.** Let $x, y \in K$. Then

$$\begin{aligned}
&\eta(h^{t_1+t_2}(x,y)) = ((t_1+t_2)(x) * (t_1+t_2)(y))((t_1+t_2)(x*y))^{-1} \\
&\implies (h^{t_1+t_2}(x,y),1)D = ((t_1(x),t_2(x))*(t_1(y),t_2(y)))((t_1(x*y))^{-1},(t_2(x*y))^{-1})D \\
&\quad = ((t_1(x)*t_1(y))(t_1(x*y))^{-1},(t_2(x)*t_2(y))(t_2(x*y))^{-1})D = (h^{t_1}(x,y),h^{t_2}(x,y))D \\
&\implies (h^{t_1+t_2}(x,y),1)D = (h^{t_1}(x,y),h^{t_2}(x,y))D \\
&\implies (h^{t_1+t_2}(x,y)(h^{t_1}(x,y))^{-1},(h^{t_2}(x,y))^{-1}) \in D.
\end{aligned}$$

Therefore $h^{t_1+t_2} = h^{t_1}h^{t_2}$. Similarly, we can show that $f^{t_1+t_2} = f^{t_1}f^{t_2}$. Also

$$\begin{aligned}
\eta(\Gamma_x^{t_1+t_2}(h)) &= (t_1+t_2)(x) * \eta(h) = ((t_1(x),t_2(x)) * (h,1))D = (t_1(x)*h,1)D \\
&= (\Gamma_x(h),1)D \implies (\Gamma^{t_1+t_2}(h),1)D = (\Gamma_x(h),1)D.
\end{aligned}$$

Hence $\Gamma_{E_1 \uplus E_2} = \Gamma$. ■

Therefore $\mathrm{CExt}_\Gamma(H,K)$ is an abelian group with respect to the Baer sum and the bijective map $\Phi$ defined in Theorem (3.13) from the set $\mathrm{CExt}_\Gamma(H,K)$ to $H^2_{ML(\Gamma)}(H,K)$ is an isomorphism.

**Example 3.16.** Consider the dihedral group

$$D_4 = \langle a, b \mid a^2 = 1 = b^4, aba = b^{-1} \rangle,$$

and let $H = \{1, b^2\}$ and $K = D_4/H$. Define the multiplicative Lie product $*$ on $D_4$ by $a*a = b*b = 1$ and $a*b = b$. Then $(D_4, \cdot, *)$ forms a multiplicative Lie algebra and the induced structure on $K$ is given by $aH*aH = bH*bH = H$, and $aH*bH = bH$. So, finally we have a center extension

$$1 \longrightarrow H \xrightarrow{i} D_4 \xrightarrow{\nu} K \longrightarrow 1$$

of $H$ by $K$. Let $t : K \longrightarrow D_4$ be a section of the above extension defined by $t(H) = e$, $t(aH) = a$, $t(bH) = b^3$, and $t(abH) = ab$. Then the multiplicative center 2-cocycle $(f^t, h^t, \Gamma)$ is as follows:

$$f^t(x,y) = \begin{cases} 1 & (x,y) \notin \{(aH,abH),(aH,bH),(bH,abH)\} \\ b^2 & (x,y) \in \{(aH,bH),(bH,abH),(aH,abH)\} \end{cases}$$

$$h^t(x,y) = \begin{cases} 1 & (x,y) \notin \{(bH,aH),(abH,aH),(bH,abH)\} \\ b^2 & (x,y) \in \{(bH,aH),(abH,aH),(bH,abH)\} \end{cases}$$

$$\Gamma(x) = \begin{cases} 0_H & x \in \{H, bH\} \\ I_H & x \in \{aH, abH\} \end{cases}$$

where $0_H$ and $I_H$ denotes the zero homomorphism and identity homomorphism on $H$ respectively.

## 4. Schreier Theory for Lie Center Extension

In this section, we discuss the theory of Lie center extension in a similar manner to Section 3.

**Definition 4.1.** An extension $E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$ of an abelian group $H$ with trivial multiplicative Lie product by a multiplicative Lie algebra $K$ is called a *Lie center extension* if $H$ is contained in the Lie center $LZ(G)$ of $G$.

Let $E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$ be a Lie center extension of $H$ by $K$ and $t : K \longrightarrow G$ be a section of $E(H,K)$. Then every element of the group $G$ can be uniquely expressed in the form $ht(x)$, for some $h \in H$ and $x \in K$. The group operation "$\cdot$" (for details see [6]) in $G$ is given by

$$ht(x) \cdot kt(y) = h\sigma_x^t(k) f^t(x,y) t(xy), \tag{25}$$

where $\sigma_x^t : H \longrightarrow H$ is a group automorphism defined by $\sigma_x^t(k) = t(x)kt(x)^{-1}$ and $f^t : K \times K \to H$ is a map satisfying

$$f^t(1,x) = f^t(x,1) = 1 \text{ and } f^t(x,y)f^t(xy,z) = \sigma_x^t(f^t(y,z))f^t(x,yz). \tag{26}$$

**Remark 4.2.** Since $H$ is abelian, $\mathrm{Aut}(H) = \mathrm{Out}(H)$. The map $\sigma_E^t : K \to \mathrm{Aut}(H)$ defined by $\sigma_E^t(x) = \sigma_x^t$ is independent of the section $t$ and the representative of extension $E(H,K)$ of the equivalence class (see [6, p. 378]). So further in this paper, we will use $\sigma_x$ in the place of $\sigma_x^t$. ■

Since $\beta$ is a multiplicative Lie algebra homomorphism, for $x, y \in K$ we have

$$\beta(t(x) * t(y)) = \beta(t(x)) * \beta(t(y)) = x * y = \beta t(x * y) \implies (t(x) * t(y))t(x * y)^{-1} \in H.$$

So we have a map $h^t : K \times K \to H$ such that $t(x) * t(y) = h^t(x,y)t(x * y)$, where $h^t$ satisfies

$$h^t(x,1) = h^t(1,x) = h^t(x,x) = 1. \tag{27}$$

Now consider the expression,

$$ht(x) * kt(y) = {}^k(ht(x) * t(y)) = {}^{kh}(t(x) * t(y)) = kh\sigma_{(x*y)}(h^{-1}k^{-1})h^t(x,y)t(x * y).$$

Thus the multiplicative Lie product $*$ on $G$ is defined by

$$ht(x) * kt(y) = hk\sigma_{(x*y)}(h^{-1}k^{-1})h^t(x,y)t(x * y). \tag{28}$$

Now we will see properties of the function $h^t$ by using Equations (25) and (28). Consider the expression

$$ht(x) * (kt(y) \cdot lt(z)) = ht(x) * (k\sigma_y(l)f^t(y,z)t(yz))$$

$$= hk\sigma_y(l)f^t(y,z)\sigma_{x*(yz)}(h^{-1}k^{-1}\sigma_y(l^{-1})f^t(y,z)^{-1})h^t(x,yz)t(x * yz). \tag{29}$$

On the other hand,

$$
\begin{aligned}
&ht(x) * (kt(y) \cdot lt(z)) = (ht(x) * kt(y)) \cdot {}^{kt(y)}(ht(x) * lt(z)) \\
&= (hk\sigma_{x*y}(h^{-1}k^{-1})h^t(x,y)t(x*y))(kt(y)(hl\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z))t(x*z)t(y)^{-1}k^{-1}) \\
&= (hk\sigma_{x*y}(h^{-1}k^{-1})h^t(x,y)t(x*y))(k\sigma_y(hl\sigma_{(x*z)}(h^{-1}l^{-1}) \\
&\quad \cdot h^t(x,z))t(y)t(x*z)f^t(y^{-1},y)^{-1}t(y^{-1})k^{-1}) \\
&= (hk\sigma_{x*y}(h^{-1}k^{-1})h^t(x,y)t(x*y)) \cdot (k\sigma_y(hl\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z)) \\
&\quad \cdot f^t(y,x*z)t(y(x*z))f^t(y^{-1},y)^{-1}\sigma_{y^{-1}}(k^{-1})t(y^{-1}) \\
&= hk\sigma_{x*y}(h^{-1}k^{-1})h^t(x,y)t(x*y)(k\sigma_y(hl\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z)) \\
&\quad \cdot f^t(y,x*z)\sigma_{y(x*z)}(f^t(y^{-1},y)^{-1}\sigma_{y^{-1}}(k^{-1}))f^t(y(x*z),y^{-1})t({}^y(x*z)) \\
&= hk\sigma_{x*y}(h^{-1}k^{-1})h^t(x,y)\sigma_{x*y}(k\sigma_y(hl\sigma_{x*z}(h^{-1}l^{-1})h^t(x,z))f^t(y,(x*z)) \\
&\quad \cdot \sigma_{y(x*z)}(f^t(y^{-1},y)^{-1})\sigma_{y^{-1}}(k^{-1}))f^t(y(x*z),y^{-1}))f^t(x*y,{}^y(x*z))t((x*y){}^y(x*z)) \\
&= hkh^t(x,y)\sigma_{x*y}(h^{-1}\sigma_y(hl\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z))f^t(y,x*z)\sigma_{y(x*z)}(f^t(y^{-1},y)^{-1} \\
&\quad \cdot \sigma_{y^{-1}}(k^{-1}))f^t(y(x*z),y^{-1}))f^t(x*y,{}^y(x*z))t(x*(yz)).
\end{aligned}
\tag{30}
$$

By equating Equations (29) and (30), we have

$$
\begin{aligned}
&\sigma_{x*y}(h^{-1}\sigma_y(hl\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z))f^t(y,x*z) \\
&\qquad \cdot \sigma_{y(x*z)}(f^t(y^{-1},y)^{-1}\sigma_{y^{-1}}(k^{-1}))f^t(y(x*z),y^{-1})f^t(x*y,{}^y(x*z))h^t(x,y) \\
&= \sigma_y(l)f^t(y,z)\sigma_{x*(yz)}(h^{-1}k^{-1}\sigma_y(l^{-1})f^t(y,z)^{-1})h^t(x,yz).
\end{aligned}
\tag{31}
$$

Now consider the expression

$$
\begin{aligned}
&(ht(x) \cdot kt(y)) * lt(z) \\
&= hl\sigma_x(k)f^t(x,y)\sigma_{xy*z}(h^{-1}l^{-1}\sigma_x(k^{-1})f^t(x,y)^{-1})h^t(xy,z)t((xy)*z).
\end{aligned}
\tag{32}
$$

On the other hand,

$$
\begin{aligned}
&(ht(x) \cdot kt(y)) * lt(z) = {}^{ht(x)}(kt(y) * lt(z)) \cdot (ht(x) * lt(z)) \\
&= ht(x)(kl\sigma_{(y*z)}(k^{-1}l^{-1})h^t(y,z)t(y*z))t(x)^{-1}h^{-1}(hl\sigma_{(x*z}(h^{-1}l^{-1})h^t(x,z)t(x*z)) \\
&= h\sigma_x(kl\sigma_{(y*z)}(k^{-1}l^{-1})h^t(y,z))t(x)t(y*z)t(x)^{-1}h^{-1}(hl\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z)t(x*z)) \\
&= h\sigma_x(kl\sigma_{(y*z)}(k^{-1}l^{-1})h^t(y,z))f^t(x,y*z)t(x(y*z)) \\
&\qquad \cdot t(x)^{-1}h^{-1}(hl\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z)t(x*z)) \\
&= h\sigma_x(kl\sigma_{(y*z)}(k^{-1}l^{-1})h^t(y,z))f^t(x,y*z)\sigma_{(x(y*z))}(f^t(x^{-1},x)^{-1}) \\
&\qquad \cdot f^t(x(y*z),x^{-1})t({}^x(y*z))(l\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z)t(x*z)) \\
&= h\sigma_x(kl\sigma_{(y*z)}(k^{-1}l^{-1})h^t(y,z))f^t(x,y*z)\sigma_{(x(y*z))}(f^t(x^{-1},x)^{-1}) \\
&\qquad \cdot f^t(x(y*z),x^{-1})\sigma_{({}^x(y*z))}(l\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z))t({}^x(y*z))t(x*z)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(ht(x) \cdot kt(y)) * lt(z) \\
&= h\sigma_x(kl\sigma_{(y*z)}(k^{-1}l^{-1})h^t(y,z))f^t(x,y*z)\sigma_{(x(y*z))}(f^t(x^{-1},x)^{-1})f^t(x(y*z),x^{-1}) \\
&\qquad \cdot \sigma_{{}^x(y*z)}(l\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z))f^t({}^x(y*z),(x*z))t({}^x(y*z)(x*z))).
\end{aligned}
\tag{33}
$$

Further, by equating Equations (32) and (33), we have

$$
\begin{aligned}
&lf^t(x,y)\sigma_{(xy)*z}(h^{-1}l^{-1}\sigma_x(k^{-1})f^t(x,y)^{-1})h^t(xy,z)\\
&= \sigma_x(l\sigma_{(y*z)}(k^{-1}l^{-1})h^t(y,z))f^t(x,y*z)\sigma_{x(y*z)}(f^t(x^{-1},x)^{-1})\\
&\qquad \cdot f^t(x(y*z),x^{-1})\sigma_{{}^x(y*z)}(l\sigma_{(x*z)}(h^{-1}l^{-1})h^t(x,z))f^t({}^x(y*z),(x*z)).
\end{aligned} \tag{34}
$$

Now consider the expressions

$$
\begin{aligned}
&{}^{lt(z)}(ht(x)*kt(y)) = lt(z)(hk\sigma_{(x*y)}(h^{-1}k^{-1})h^t(x,y)t(x*y))t(z)^{-1}l^{-1}\\
&= l\sigma_z(hk\sigma_{(x*y)}(h^{-1}k^{-1})h^t(x,y))f^t(z,x*y)t(z(x*y))f^t(z^{-1},z)^{-1}t(z^{-1})l^{-1}\\
&= l\sigma_z(hk\sigma_{(x*y)}(h^{-1}k^{-1})h^t(x,y))f^t(z,x*y)\\
&\qquad \cdot \sigma_{(z(x*y))}(f^t(z^{-1},z)^{-1}\sigma_{z^{-1}}(l^{-1}))f^t(z(x*y),z^{-1})t({}^z(x*y)).
\end{aligned} \tag{35}
$$

On the other hand,

$$
\begin{aligned}
&{}^{lt(z)}(ht(x)) * {}^{lt(z)}(kt(y)) = (lt(z)ht(x)t(z)^{-1}l^{-1}) * (lt(z)kt(y)t(z)^{-1}l^{-1})\\
&= (l\sigma_z(h)f^t(z,x)\sigma_{zx}(f^t(z^{-1},z)^{-1})t(zx)t(z^{-1})l^{-1})\\
&\qquad * (l\sigma_z(k)f^t(z,y)\sigma_{zy}(f^t(z^{-1},z)^{-1})t(zy)t(z^{-1})l^{-1})\\
&= l^2\sigma_z(hk)f^t(z,x)f^t(z,y)\sigma_{zx}(f^t(z^{-1},z)^{-1})\sigma_{zy}(f^t(z^{-1},z)^{-1})\\
&\qquad \cdot \sigma_{{}^zx}(l^{-1})\sigma_{{}^zy}(l^{-1})f^t(zx,z^{-1})f^t(zy,z^{-1})\sigma_{({}^zx*{}^zy)}(l^{-2}\sigma_z(h^{-1}k^{-1})\\
&\qquad \cdot f^t(z,x)^{-1}f^t(z,y)^{-1}\sigma_{zx}(f^t(z^{-1},z))\sigma_{zy}(f^t(z^{-1},z))\\
&\qquad \cdot f^t(zy,z^{-1})^{-1}f^t(zx,z^{-1})^{-1}\sigma_{{}^zy}(l)\sigma_{{}^zx}(l))h^t({}^zx,{}^zy)t({}^zx*{}^zy).
\end{aligned} \tag{36}
$$

By equating Equations (35) and (36), we have

$$
\begin{aligned}
&\sigma_z(\sigma_{(x*y)}(h^{-1}k^{-1})h^t(x,y))f^t(z,x*y)\sigma_{(z(x*y))}(f^t(z^{-1},z)^{-1}\sigma_{z^{-1}}(l^{-1}))f^t(z(x*y),z^{-1})\\
&= lf^t(z,x)f^t(z,y)\sigma_{zx}(f^t(z^{-1},z)^{-1})\sigma_{zy}(f^t(z^{-1},z)^{-1})\sigma_{{}^zx}(l^{-1})\sigma_{{}^zy}(l^{-1})\\
&\quad \cdot f^t(zy,z^{-1})f^t(zx,z^{-1})\sigma_{({}^zx*{}^zy)}(l^{-2}\sigma_z(h^{-1}k^{-1})f^t(z,x)^{-1}f^t(z,y)^{-1}\sigma_{zx}(f^t(z^{-1},z))\\
&\quad \cdot \sigma_{zy}(f^t(z^{-1},z))f^t(zy,z^{-1})^{-1}f^t(zx,z^{-1})^{-1}\sigma_{{}^zy}(l)\sigma_{{}^zx}(l))h^t({}^zx,{}^zy).
\end{aligned} \tag{37}
$$

By the Jacobi identity, we have

$$
\begin{aligned}
&hk^2\sigma_{x*y}(h^{-1}k^{-1})h^t(x,y)\sigma_y(l)f^t(y,z)\sigma_{yz}(f^t(y^{-1},y)^{-1})f^t(yz,y^{-1})\sigma_{{}^yz}(k^{-1})\\
&\sigma_{(x*y)*{}^yz}(h^{-1}k^{-2}\sigma_{x*y}(hk)h^t(x,y)^{-1}\sigma_y(l^{-1})f^t(y,z)^{-1}\sigma_{yz}(f^t(y^{-1},y))f^t(yz,y^{-1})^{-1}\\
&\sigma_{{}^yz}(k))h^t(x*y,{}^yz)\sigma_{(x*y)*{}^yz}(kl^2\sigma_{y*z}(k^{-1}l^{-1})h^t(y,z)\sigma_z(h)f^t(z,x)\sigma_{zx}(f^t(z^{-1},z))^{-1}\\
&f^t(zx,z^{-1})\sigma_{{}^zx}(l^{-1})\sigma_{(y*z)*{}^zx}(k^{-1}l^{-2}\sigma_{y*z}(kl)h^t(y,z)^{-1}\sigma_z(h^{-1})f^t(z,x)^{-1}\\
&\sigma_{zx}(f^t(z^{-1},z))f^t(zx,z^{-1})\sigma_{{}^zx}(l))h^t(y*z,{}^zx))f^t((x*y)*{}^yz,(y*z)*{}^zx)\\
&\sigma_{((x*y)*{}^yz)*((y*z)*{}^zx)}(lh^2\sigma_{(z*x)}(l^{-1}h^{-1})h^t(z,x)\sigma_x(k)f^t(x,y)\sigma_{xy}(f^t(x^{-1},x)^{-1})\\
&f^t(xy,x^{-1})\sigma_{{}^xy}(h^{-1})\sigma_{((z*x)*{}^xy)}(l^{-1}h^{-2}\sigma_{z*x}(lh)h^t(z,x)^{-1}\sigma_x(k^{-1})f^t(x,y)^{-1}\\
&\sigma_{xy}(f^t(x^{-1},x))f^t(xy,x^{-1})^{-1}\sigma_{{}^xy}(h))h^t(z*x,{}^xy))\\
&f^t(((x*y)*{}^yz)*((y*z)*{}^zx),(z*x)*{}^xy) = 1.
\end{aligned} \tag{38}
$$

**Definition 4.3.** Let $H$ be an abelian group with trivial multiplicative Lie product and $K$ be an arbitrary multiplicative Lie algebra. Then a *Lie center factor system* is a quintuple $(K,H,f,h,\sigma)$, where $\sigma : K \longrightarrow \mathrm{Aut}(H)$ is a group homomorphism and $f$, $h$ are maps from $K \times K$ to $H$ satisfying the conditions like Equations (26), (27), (31), (34), (37), and (38).

So we can say that for every Lie center extension $E(H,K)$ with a choice of section $t$ we have a Lie center factor system $\mathrm{LFac}(E,t) = (K,H,f^t,h^t,\sigma)$ and called as Lie center factor system given by the Lie center extension $E(H,K)$. Now we prove the following proposition:

**Proposition 4.4.** *Let $E(H,K)$ be a Lie center extension of $H$ by $K$ with a choice of a section $t$. Then there exists a Lie center factor system* $\mathrm{LFac}(E,t) = (K,H,f^t,h^t,\sigma)$. *Conversely, for every Lie center factor system $(K,H,f,h,\sigma)$ we have a Lie center extension $E(H,K)$ of $H$ by $K$, with a section $t$ such that* $\mathrm{LFac}(E,t) = (K,H,f,h,\sigma)$.

**Proof.** From the above discussions and the Definition 4.3, it can be seen that every Lie center extension $E(H,K)$ of $H$ by $K$ with a given section $t$ determines a Lie center factor system $\mathrm{LFac}(E,t) \equiv (K,H,f^t,h^t,\sigma)$.
Conversely let $(K,H,f,h,\sigma)$ be a Lie center factor system. Take $G = H \times K$ and define the $\cdot$ and $*$ binary operations on $G$ as

$$(a,x)\cdot(b,y) = (a\sigma_x(b)f(x,y),xy) \qquad \text{and}$$
$$(a,x)*(b,y) = (ab\sigma_{x*y}(a^{-1}b^{-1})h(x,y),x*y).$$

It is easy to see that $(G,\cdot,*)$ is a multiplicative Lie algebra such that

$$E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} K \longrightarrow 1$$

is a Lie center extension of $H$ by $K$, where $i$ is the first inclusion and $p$ is the second projection. Let $t$ be a section of $E(H,K)$ given by $t(x) = (1,x)$. By an easy computation, it can be seen that $\sigma_E = \sigma$, $f^t = f$, and $h^t = h$. ∎

### 4.1. Equivalence between Category LEXT of Lie center extensions and Category LFAC of Lie center factor systems

Let $(\lambda,\mu,\nu)$ be a morphism from the Lie center extension $E_1(H,K)$ to the Lie center extension $E_2(H,K)$ of $H$ by $K$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
E(H_1,K_1) \equiv 1 & \longrightarrow & H_1 & \xrightarrow{i_1} & G_1 & \xrightarrow{\beta_1} & K_1 & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\
E(H_2,K_2) \equiv 1 & \longrightarrow & H_2 & \xrightarrow{i_2} & G_2 & \xrightarrow{\beta_2} & K_2 & \longrightarrow & 1.
\end{array}$$

Let $t_1$ and $t_2$ be sections of $E(H_1,K_1)$ and $E(H_2,K_2)$, respectively. Consider the corresponding factor systems $(K_1,H_1,f^{t_1},h^{t_1},\sigma^1)$ and $(K_2,H_2,f^{t_2},h^{t_2},\sigma^2)$ of $E(H_1,K_1)$ and $E(H_2,K_2)$ respectively. Let $x \in K_1$. Then $\mu(t_1(x)) \in G_2$ and $\beta_2(\mu(t_1(x))) = \nu(\beta_1(t_1(x))) = \nu(x) = \beta_2(t_2(\nu(x)))$. Thus, $\mu(t_1(x))(t_2(\nu(x)))^{-1} \in H_2$. In turn, we have a unique $g(x) \in H_2$ such that

$$\mu(t_1(x)) = g(x)t_2(\nu(x)). \tag{39}$$

Since, $t_1(1) = 1 = t_2(1)$, it follows that $$g(1) = 1. \tag{40}$$

Also we get the following equations (see [6, pages 375–376]):

$$\lambda(f^{t_1}(x,y))g(xy) = g(x)\sigma^2_{\nu(x)}(g(y))f^{t_2}(\nu(x),\nu(y)) \tag{41}$$

and $$g(x)\sigma^2_{\nu(x)}(\lambda(h))g(x)^{-1} = \lambda(\sigma^1_x(h)). \tag{42}$$

Further, $$\mu(t_1(x) * t_1(y)) = \mu(h^{t_1}(x,y)t_1(x*y))$$

$$= \lambda(h^{t_1}(x,y))\mu(t_1(x*y)) = \lambda(h^{t_1}(x,y))g(x*y)t_2(\nu(x)*\nu(y)). \tag{43}$$

On the other hand,

$$\mu(t_1(x) * t_1(y)) = \mu(t_1(x)) * \mu(t_2(y)) = (g(x)t_2(\nu(x))) * (g(y)t_2(\nu(y))$$
$$= g(x)g(y)\sigma^2_{(\nu(x)*\nu(y))}(g(x)^{-1}g(y)^{-1})h^{t_2}(\nu(x),\nu(y))t_2(\nu(x)*\nu(y)). \tag{44}$$

Now comparing Equations (43) and (44), we have

$$\lambda(h^{t_1}(x,y))g(x*y) = g(x)g(y)\sigma^2_{(\nu(x)*\nu(y))}(g(x)^{-1}g(y)^{-1})h^{t_2}(\nu(x),\nu(y)). \tag{45}$$

Thus a morphism $(\lambda,\mu,\nu)$ from the Lie center extension $E_1(H,K)$ to the Lie center extension $E_2(H,K)$ together with the choices of sections $t_1$ and $t_2$ of the corresponding center extensions, induces a map $g$ from $K_1$ to $H_2$ such that the triple $(\nu,g,\sigma)$ satisfies Equations (40), (41), (42), and (45). This can be regarded as a morphism between the corresponding Lie center factor systems.

Now we introduce the category **LFAC** of Lie center factor systems whose objects are Lie center factor systems, and a morphism from $(K_1,H_1,f^1,h^1,\sigma^1)$ to $(K_2,H_2,f^2,h^2,\sigma^2)$ is a triple $(\nu,g,\lambda)$, where $\nu : K_1 \longrightarrow K_2$, $\lambda : H_1 \longrightarrow H_2$ are multiplicative Lie algebra homomorphisms, and $g : K_1 \longrightarrow H_2$ is a map such that

(1) $g(1) = 1$,

(2) $\lambda(f^1(x,y))g(xy) = g(x)\sigma^2_{\nu(x)}(g(y))f^2(\nu(x),\nu(y))$,

(3) $\lambda(\sigma^1_x(h)) = g(x)\sigma^2_{\nu(x)}(\lambda(h))g(x)^{-1}$,

(4) $\lambda(h^1(x,y))g(x*y) = g(x)g(y)\sigma^2_{(\nu(x)*\nu(y))}(g(x)^{-1}g(y)^{-1})h^2(\nu(x),\nu(y))$.

The composition of morphisms $(\nu_1,g_1,\lambda_1) : (K_1,H_1,f^1,h^1,\sigma^1) \to (K_2,H_2,f^2,h^2,\sigma^2)$ and $(\nu_2,g_2,\lambda_2) : (K_2,H_2,f^2,h^2,\sigma^2) \to (K_3,H_3,f^3,h^3,\sigma^3)$ is defined as the triple $(\nu_2\circ\nu_1, g_3, \lambda_2\circ\lambda_1)$, where $g_3$ is given by $g_3(x) = g_2(\nu_1(x))\lambda_2(g_1(x))$ for each $x \in K_1$.

So finally from the above discussion we have the following theorem:

**Theorem 4.5.** *There is an equivalence between the category* **LEXT** *of Lie center extensions and the category* **LFAC** *of Lie center factor systems.*

**Theorem 4.6.** *Let* $\mathrm{LExt}(H,K)$ *denote the set of all equivalence classes of Lie center extensions of* $H$ *by* $K$ *and* $\mathrm{Hom}(K,\mathrm{Aut}(H))$ *denote the set of all group homomorphisms from* $K$ *to* $\mathrm{Aut}(H)$. *Then there is a natural surjective map* $\eta : \mathrm{LExt}(H,K) \to \mathrm{Hom}(K,\mathrm{Aut}(H))$ *defined by* $\eta([E]) = \sigma_{[E]}$.

**Proof.** From Remark 4.2, the map $\eta : \mathrm{LExt}(H,K) \longrightarrow \mathrm{Hom}(K,\mathrm{Aut}(H))$ defined by $\eta([E]) = \sigma_{[E]}$ is well defined. Let $\sigma : K \longrightarrow \mathrm{Aut}(H)$ is a group homomorphism defined by $\sigma(x) = \sigma_x$. Let $G = H \times K$. Then $G$ is a multiplicative Lie algebra with the operations $\cdot$ and $*$, defined by $(h,x)\cdot(k,y) = (h\sigma_x(k), xy)$ and $(h,x)*(k,y) = (hk\sigma_{x*y}(h^{-1}k^{-1}), x*y)$, respectively. Consider a section $t : K \longrightarrow G$ defined by $t(x) = (1,x)$. Therefore we have a Lie center extension $E(H,K)$ of $H$ by $K$

$$E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$$

such that $f^t$ and $h^t$ are trivial maps and $\sigma_E = \sigma$. Thus every group homomorphism from $K \longrightarrow \mathrm{Aut}(H)$ gives a Lie center extension of multiplicative Lie algebra of $H$ by $K$. This shows that $\eta$ is surjective. ∎

**Example 4.7.** The map $\eta$ in Theorem 4.6 need not be injective. Specifically, there may exists the same group homomorphism $\sigma$ from $K$ to $\mathrm{Aut}(H)$ for two distinct classes of extensions. Consider the center extensions:

$$E_1(\mathbb{Z},\mathbb{Z}_5) \equiv \{0\} \longrightarrow \mathbb{Z} \xrightarrow{(i_1,0)} \mathbb{Z} \bigoplus \mathbb{Z}_5 \xrightarrow{p_2} \mathbb{Z}_5 \longrightarrow \{0\}$$

and
$$E_2(\mathbb{Z},\mathbb{Z}_5) \equiv \{0\} \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\nu} \mathbb{Z}_5 \longrightarrow \{0\}$$

of the multiplicative Lie algebra $\mathbb{Z}$ by $\mathbb{Z}_5$, where $\alpha(n) = 5n$, $\forall n \in \mathbb{Z}$, and $\nu$ is natural quotient map. Since $\mathbb{Z}$ and $\mathbb{Z}_5$ are cyclic groups, the multiplicative Lie product is trivial on these groups. Also it can be seen that $E_1(\mathbb{Z},\mathbb{Z}_5)$ and $E_2(\mathbb{Z},\mathbb{Z}_5)$ are not equivalent as Lie center extensions. Since $\mathrm{Aut}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$, $\mathrm{Hom}(\mathbb{Z}_5, \mathrm{Aut}(\mathbb{Z}))$ is trivial. Therefore here it can be seen that $[E_1] \neq [E_2]$ but $\eta([E_1]) = \eta([E_2])$.

Now we discuss the following problem:

**Problem 4.8.** Let $H$ be an abelian group with trivial multiplicative Lie product. Classify all Lie center extensions of $H$ by $K$ (up to equivalence) with the given group homomorphism $\sigma : K \longrightarrow \mathrm{Aut}(H)$.

**Definition 4.9.** A multiplicative Lie center 2-cocycle of a multiplicative Lie algebra K with coefficient in an abelian group $H$ with trivial multiplicative Lie product is a triple $(f,h,\sigma)$, where $f \in Z^2(K,H)$ is a group 2-cocycle of $K$ with coefficients in the trivial $K$-module $H$, $h : K \times K \longrightarrow H$ and $\sigma : K \longrightarrow \mathrm{Aut}(H)$ are the maps satisfying conditions like Equations (26), (27), (31), (34), (37), and (38).

**Remark 4.10.** Let $E(H,K)$ be a center extension with a choice of a section $t$. Then from the above discussion, it is clear that we have a multiplicative Lie center 2-cocycle $(f^t, h^t, \sigma_E)$.

Conversely, if we have a multiplicative Lie center 2-cocycle $(f,h,\sigma)$ of a multiplication Lie algebra $K$ with coefficient in an abelian group $H$ with trivial multiplicative Lie product, then from Proposition 4.4 it can be seen that there exists a multiplicative Lie algebra $G$ and a Lie center extension $E(H,K)$ with a choice of section $t$ such that $f^t = f$, $\sigma_E = \sigma$, and $h^t = h$. ∎

Let $E(H,K)$ be a Lie center extension with a choice of a section $t$. Then we have a multiplicative Lie center 2-cocycle $(f^t, h^t, \sigma_E)$. Now, let $s$ be an another section of $E$. Then there exists a map $g : K \to H$ with $g(1) = 1$ such that $s(x) = g(x)t(x)$ for all $x \in K$.

Then $$hs(x) \cdot ks(y) = h\sigma_x(k)f^s(x,y)s(xy) = h\sigma_x(k)f^s(x,y)g(xy)t(xy). \tag{46}$$

On the other hand, $$hs(x) \cdot ks(y) = h\sigma_x(k)s(x)s(y)$$

$$= h\sigma_x(k)g(x)t(x)g(y)t(y) = h\sigma_x(k)g(x)\sigma_x(g(y))f^t(x,y)t(xy). \tag{47}$$

Therefore by comparing Equations (46) and (47), we have

$$f^s(x,y) = g(x)g(xy)^{-1}\sigma_x(g(y))f^t(x,y). \tag{48}$$

Further, $$hs(x) * ks(y) = hk\sigma_{x*y}(h^{-1}k^{-1})h^s(x,y)s(x*y)$$

$$= hk\sigma_{x*y}(h^{-1}k^{-1})h^s(x,y)g(x*y)t(x*y). \tag{49}$$

Also we have

$$hs(x) * ks(y) = hkg(x)g(y)\sigma_{x*y}(h^{-1}k^{-1}g(x)^{-1}g(y)^{-1})h^t(x,y)t(x*y). \tag{50}$$

Thus by comparing Equations (49) and (50), we have

$$h^s(x,y) = g(x)g(x*y)^{-1}g(y)\sigma_{x*y}(g(x)^{-1}g(y)^{-1})h^t(x,y). \tag{51}$$

These equations prompt us to the following definition:

**Definition 4.11.** Two multiplicative Lie center 2-cocycles $(f^s, h^s, \sigma)$ and $(f^t, h^t, \sigma)$ are said to be *equivalent* if there is an identity preserving map $g : K \longrightarrow H$ satisfying:

(1) $f^s(x,y) = g(x)g(xy)^{-1}\sigma_x(g(y))f^t(x,y)$,

(2) $h^s(x,y) = g(x*y)^{-1}g(x)g(y)\sigma_{x*y}(g(x)^{-1}g(y)^{-1})h^t(x,y)$.

The set $Z^2_{ML(\sigma)}(K,H)$ of all multiplicative Lie center 2-cocycles of $K$ with coefficients in $H$ is easily seen to be an abelian group with respect to coordinatewise operation given by $(f,h,\sigma)\cdot(f^{'},h^{'},\sigma) = (ff^{'},gg^{'},\sigma)$. Given any identity preserving map $g : K \longrightarrow H$, the triplet $(\delta g, g^*, \sigma)$ is a member of $Z^2_{ML(\sigma)}(K,H)$, where $\delta g, g^*$ are maps from $K \times K$ to $H$ given by $\delta g(x,y) = g(x)g(xy)^{-1}\sigma_x(g(y))$ and $g^*(x,y) = g(x*y)^{-1}g(x)g(y)\sigma_{x*y}(g(x)^{-1}g(y)^{-1})$.

Let $\mathrm{MAP}(K,H)$ denote the group of identity preserving maps from $K$ to $H$. Then we have a homomorphism $\chi : \mathrm{MAP}(K,H) \longrightarrow Z^2_{ML(\sigma)}(K,H)$ given by $\chi(g) = (\delta g, g^*, \sigma)$. The image of $\chi$ is called the group of multiplicative Lie center 2-coboundaries of $K$ with coefficient in $H$ and it is denoted by $B^2_{ML(\sigma)}(K,H)$. The quotient group $Z^2_{ML(\sigma)}(K,H)/B^2_{ML(\sigma)}(K,H)$ is called the second Lie center cohomology of $K$ with coefficient in $H$ and it is denoted by $H^2_{ML(\sigma)}(K,H)$. In turn, we get the following exact sequence of abelian groups:

$$1 \longrightarrow \mathrm{Hom}(K,H) \xrightarrow{i} \mathrm{MAP}(K,H) \xrightarrow{\chi} Z^2_{ML(\sigma)}(K,H) \xrightarrow{\nu} H^2_{ML(\sigma)}(K,H) \longrightarrow 1,$$

where $\nu$ is a quotient map. Now we have the following theorem, which proof is similar to the proof of the Theorem 3.13.

**Theorem 4.12.** *Let $H$ be an abelian group with trivial multiplicative Lie product, $K$ be a multiplicative Lie algebra and $\sigma$ be a group homomorphism from $K$ to the group of automorphisms $\mathrm{Aut}(H)$ of $H$. Then there is a bijective correspondence between the set $\mathrm{LExt}_\sigma(H,K)$ of equivalence classes of Lie center extensions of $H$ by $K$ with the given $\sigma$ and the second Lie center cohomology $H^2_{ML(\sigma)}(K,H)$.*

**Remark 4.13.** As in Section 3, we can define Baer sum on the class of Lie center extensions and with respect to that Baer sum $\mathrm{LExt}_{\sigma}(H,K)$ forms an abelian group which is isomorphic to $H^2_{ML(\sigma)}(K,H)$.

**Remark 4.14.** By Section 3 and Section 4, it is easy to see that for any extension $E(H,K) \equiv 1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$ of an abelian group $H$ with trivial multiplicative Lie product by an arbitrary multiplicative Lie algebra $K$, the group operation "$\cdot$" and the multiplicative Lie product "$*$" in $G$ are given by

$$ht(x) \cdot kt(y) = h\sigma^t_x(k)f^t(x,y)t(xy),$$
$$ht(x) * kt(y) = hk\Gamma^t_x(k)\sigma_{(x*y)}(h^{-1}k^{-1}\Gamma^t_y(h^{-1}))h^t(x,y)t(x*y).$$

where $\sigma^t_x(k) = t(x)kt(x)^{-1}$, $\Gamma^t_x(k) = t(x) * k$ are group homomorphisms on $H$ and $f^t, h^t : K \times K \longrightarrow H$ are maps satisfying Equations (26) and (27). One can discuss other properties of the extension $E(H,K)$ accordingly.

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