New Function Spaces Associated to Representations of Nilpotent Lie Groups and Generalized Time-Frequency Analysis

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Abstract.   We study function spaces that are related to square-integrable, irreducible, unitary representations of several low-dimensional nilpotent Lie groups. These are new examples of coorbit theory and yield new families of function spaces on $\mathbb{R}^d$. The concrete realization of the representation suggests that these function spaces are useful for generalized time-frequency analysis or phase-space analysis.

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1. Introduction

The theory of coorbit spaces offers a systematic construction of Banach spaces attached to a square-integrable representation of a locally compact group. Roughly speaking, to any square-integrable, irreducible, unitary representation π of a locally compact group on a Hilbert space $\mathcal{H}$ one can associate a family of Banach spaces by imposing a norm on the representation coefficients $x \in G \to \langle f, \pi(x)g \rangle$ for fixed non-zero $g \in \mathcal{H}$. For instance, the coorbit space $Co_\pi L^p(G)$ for $1 \leq p \leq 2$ is defined by the norm

$$\|f\|_{Co_\pi L^p} = \left(\int_G |\langle f, \pi(x)g \rangle|^p \, dx \right)^{1/p}.$$

Many important families of function spaces in analysis can be represented as coorbit spaces, among them are the Besov-Triebel-Lizorkin spaces and the modulation spaces on $\mathbb{R}^d$ or the Bergman spaces on the unit ball in $\mathbb{C}^d$. The main theorem of coorbit space theory provides atomic decompositions, sampling theorems, and frames for all these spaces[22, 23, 36].

In this paper we study coorbit spaces for representations of low-dimensional nilpotent Lie groups. Our emphasis is on the concrete realizations of these spaces (rather than abstraction) and on the proof that these families of spaces are different from each other.

Our motivation is twofold: (i) We explore generalizations of modulation spaces and time-frequency analysis. For this purpose we propose coorbit spaces with respect

to a square-integrable representation of a nilpotent Lie group modulo the center, as these function spaces come automatically with an underlying configuration space and a phase space. Phase-space analysis in quantum mechanics and time-frequency analysis in engineering are both based on the Heisenberg group, and the associated function spaces are the modulation spaces. These arise naturally as the coorbit spaces of the Schrödinger representation and are now standard for the treatment of pseudodifferential operators [39, 53], the Schrödinger equation [7, 55], evolution equations [15], uncertainty principles [37] etc.

Replacing the Heisenberg group by a nilpotent Lie group, many aspects of time-frequency analysis carry over. Every square-integrable representation (modulo the center) of a nilpotent Lie group comes automatically with a phase space and two sets of variables that play the role of position and momentum. Therefore function spaces associated to such a representation offer themselves as tools for a generalized phase-space analysis.

Yet, contributions to coorbit theory with respect to nilpotent groups are rare. Most notable is the sequence of papers by I. Beltita and D. Beltita [4, 5, 6]. They define generalized modulation spaces over the coadjoint orbit of a given irreducible unitary representation and substantially extend the Pedersen-Weyl calculus [50]. If the underlying representation is square-integrable modulo the center, then the coadjoint orbit is flat and serves as a phase space. The associated generalized modulation spaces then coincide with the coorbit spaces of the representation. In a different direction Fischer, Rottensteiner, and Ruzhansky [26, 52] formulate a time-frequency analysis on the Heisenberg group as configuration space. In this case the phase space is the Dynin-Folland group and the associated function spaces are coorbit spaces. In addition, there are several abstract attempts to use a nilpotent Lie group G or its Lie algebra $\mathfrak{g}$ as a configuration spaces and then construct a suitable phase space, for instance $G \times \widehat{G}$ or $\mathfrak{g} \times \mathfrak{g}^*$. The goal is then to associate generalized modulation spaces to these objects and to develop a pseudodifferential calculus for operators acting on G. For a sample of this abstract approach see [2, 10, 27, 44, 45].

By contrast, coorbit space theory with respect to other types of groups have received much more attention and are well investigated. The coorbit theory of certain semidirect products of $\mathbb{R}^d$ with dilation groups has been intensely studied [1, 9, 18, 19, 31, 34]. These theories should be considered as contemporary group-theoretic investigations of wavelet theory and carry a rather different flavor and applications. Coorbit spaces with respect to several semisimple Lie groups can be identified with Bergman spaces on certain domains [13, 12] and coorbit space theory yields the most general atomic decompositions known so far.

(ii) Our second motivation comes from a question in [26]: "precisely how can we prove the distinctness" [of different families of function spaces]? Likewise, [45] asks "to invest effort …in concreteness" and "to compare modulation spaces with other function spaces". In this regard we offer a modest contribution and show that the coorbit spaces with respect to three nilpotent Lie groups of dimensions 5 and 6 indeed lead to different families of function spaces on $\mathbb{R}^2$. In addition, we will give a short, different, self-contained proof for the main result of [26].

Our idea is to use the different invariance properties of coorbit spaces with respect to non-isomorphic groups. This idea leads to a simple, alternative proof of [26,

Thm. 7.6]. The main technical tool is the precise understanding of the multiplication operator $f(t) \rightarrow e^{-i\pi t^2} f(t)$ on modulation spaces, or equivalently, the Schrödinger evolution on modulation spaces.

The investigation of invariance properties of function spaces suggests that the coorbit spaces with respect to two different representations π_1 and π_2 on the same Hilbert space $\mathcal{H}$ are equal only when the groups of unitary operators $\pi_1(G_1)$ and $\pi_2(G_2)$ are equal. In this paper, we only treat explicit representations of several low-dimensional nilpotent Lie groups and defer the abstract question to a subsequent investigation.

The paper is organized as follows. In Section 2 we set up the definition of abstract coorbit spaces and list their main properties. As a model, we recall how the standard modulation spaces fit into the scheme of coorbit theory. We then explain where to find the phase space in every square-integrable irreducible unitary representation of a nilpotent Lie group. In Section 3 we walk through a list of concrete examples of coorbit spaces based on representations of nilpotent Lie groups. Our main insight is the discovery of several low-dimensional nilpotent Lie groups that yield a class of coorbit spaces that are different from the modulation spaces. In Section 4 we collect several facts about atomic decompositions of coorbit spaces and coherent frames and emphasize the case of nilpotent groups.

Throughout we will use the notation $f \asymp g$ to express an equivalence $C^{-1}f(x) \leq g(x) \leq Cf(x)$, $x \in X$, with some constant $C > 0$ independent of the relevant parameters.

2. Coorbit spaces

Let G be a simply connected nilpotent Lie group with Haar measure dx and with center $Z \subseteq G$. Let $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be an irreducible, unitary representation of G on a Hilbert space $\mathcal{H}$ that is square-integrable modulo its center, in brief $\pi \in SI(G/Z)$. This means that there exists a constant d_π, the formal dimension of π, such that

$$\int_{G/Z} \langle f_1, \pi(\dot{x})g_1 \rangle \overline{\langle f_2, \pi(\dot{x})g_2 \rangle} \, d\dot{x} = d_\pi^{-1} \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle} \tag{1}$$

for all $f_1, f_2, g_1, g_2 \in \mathcal{H}$. Since $\pi|_Z = \chi(z)\mathrm{I}_\mathcal{H}$ is a multiple of the identity with $\chi \in \widehat{Z}$, the map $x \in G \mapsto \langle f_1, \pi(zx)g_1 \rangle \overline{\langle f_2, \pi(zx)g_2 \rangle}$ is independent of $z \in Z$ and the integrand in (1) is indeed a function on G/Z.

For fixed non-zero $g \in \mathcal{H}$ the representation coefficient

$$V_g^\pi f(x) = \langle f, \pi(x)g \rangle \qquad x \in G \tag{2}$$

is a transform that maps elements $f \in \mathcal{H}$ to functions on G. Depending on the group and the application, this mapping occurs under various names, such as coherent state transform, generalized wavelet transform, or short-time Fourier transform.

To extend the domain of the map V_g^π, we restrict g to a space of test functions. For simplicity, we choose the space $\mathcal{H}_\pi^\infty$ of C^∞-vectors of π, where $g \in \mathcal{H}_\pi^\infty$ means that $x \in G \mapsto \pi(x)g$ is C^∞. For $g \in \mathcal{H}_\pi^\infty$ the representation coefficient $V_g^\pi f$ is well-defined for "distributions" in $(\mathcal{H}_\pi^\infty)^*$. Other choices of test function spaces are discussed in [22, 23].

2.1. Coorbit spaces

We first discuss the class of function and distribution spaces associated to a representation in $SI(G/Z)$. Fix a non-zero vector $g \in \mathcal{H}_\pi^\infty$. Let $m : G/Z \to (0, \infty)$ be a weight functions of polynomial growth on G/Z, and let $1 \leq p \leq \infty$. In the following we always assume that m satisfies the condition

$$v(x) := \sup_{y \in G} \left\{ \frac{m(xy)}{m(y)}, \frac{m(yx)}{m(y)} \right\} < \infty \qquad \text{for all } x \in G/Z, \qquad (3)$$

and that v grows polynomially on G/Z. We say that m is moderate with respect to v or v-moderate on G/Z. Note that (3) implies that $m(xy) \leq v(x)m(y)$ and $v(xy) \leq v(x)v(y)$.

For $p < \infty$ the coorbit space $Co_\pi L_m^p(G/Z)$ is the completion of the subspace of the space $\mathcal{H}_\pi^\infty$ of C^∞-vectors with respect to the norm

$$\|f\|_{Co_\pi L_m^p} = \left(\int_{G/Z} |\langle f, \pi(\dot{x})g \rangle|^p \, m(\dot{x})^p \, d\dot{x} \right)^{1/p} = \|V_g^\pi f\|_{L_m^p(G/Z)} . \qquad (4)$$

For $p = \infty$ one takes a weak closure as suggested in [29, 3]. A more general definition starts with a solid function space Y on G/Z satisfying certain natural conditions, and then one defines a norm on functions via pull-back, i.e., $Co_\pi Y$ is the completion of C^∞-vectors in $\mathcal{H}$ with respect to the norm

$$\|f\|_{Co_\pi Y} = \|V_g^\pi f\|_Y . \qquad (5)$$

We refer to [23] for the precise conditions and details. In this paper we will use only weighted L^p-spaces.

For nilpotent groups the Hilbert space of π can always be realized as $L^2(\mathbb{R}^d)$, where $\mathbb{R}^d$ is to be interpreted as a homogeneous space G/M for a polarization M. Then g can be taken from the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ by [17, 4.1.2]. In this realization, $Co_\pi L_m^p(G/Z)$ is the subspace of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the representation coefficient $\dot{x} \to \langle f, \pi(\dot{x})g \rangle$ belongs to $L_m^p(G/Z)$.

We summarize the main properties of a coorbit space from [23].

Proposition 2.1. *Assume that $1 \leq p \leq \infty$ and that m is v-moderate.*

(i) Invariance properties: *The coorbit space $Co_\pi L_m^p(G/Z)$, $1 \leq p \leq \infty$ is a Banach space with the norm (4). It is invariant with respect to the representation π, precisely, if $f \in Co_\pi L_m^p(G/Z)$ and $y \in G$, then $\pi(y)f \in Co_\pi L_m^p(G/Z)$ and $\|\pi(y)f\|_{Co_\pi L_m^p} \leq v(y)\|f\|_{Co_\pi L_m^p}$. In particular, $\pi(y)$ is an isometry on the unweighted coorbit space $Co_\pi L^p(G/Z)$.*

(ii) *If $h \in Co_\pi L_v^1$, then $\|V_h^\pi f\|_{L_m^p}$ is an equivalent norm on $Co_\pi L_m^p(G/Z)$.*

(iii) Duality: *The dual space of $Co_\pi L_m^p(G/Z)$ for $1 \leq p < \infty$ is $Co_\pi L_{1/m}^{p'}(G/Z)$ with respect to the duality $\langle f, h \rangle = \int_{G/Z} V_g^\pi f(\dot{x}) \overline{V_g^\pi h(\dot{x})} \, d\dot{x}$.*

(iv) *As a Banach space, $Co_\pi L_m^p(G/Z)$ is isomorphic to ℓ^p.*

(v) *For $p = 2$ we have $Co_\pi L^2(G/Z) = \mathcal{H}$. If $m \geq 1$ on G/Z and $1 \leq p \leq 2$, then $Co_\pi L_m^p(G/Z) \subseteq \mathcal{H}$.*

The results in [23] are formulated for a square-integrable representation of a locally compact group G. For a simply connected nilpotent Lie group we have to consider square-integrability modulo the center and thus subsequently we work with G/Z instead of G. Alternatively, one could work with projective representations of G/Z as mentioned in [11, 22]. This is a matter of convenience and taste, and we prefer to work with representation of a given group G.

2.2. Modulation spaces

Most classical function spaces, such as Sobolev spaces and Besov spaces on $\mathbb{R}^d$ can be interpreted as coorbit spaces with respect to a solvable group of affine transformations. The most important coorbit spaces attached to a nilpotent group are the modulation spaces originally introduced by Feichtinger [20]. We briefly recall their definition. In the following we write

$$T_x f(t) = f(t - x) \qquad \text{and} \qquad M_\xi f(t) = e^{2\pi i \xi \cdot t} \qquad x, \xi, t \in \mathbb{R}^d, \qquad (6)$$

for the operators of translation and modulation. The short-time Fourier transform of a function f with respect to a fixed non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$S_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \xi} \, dt. \qquad (7)$$

In signal processing the variable x is interpreted as "time" and ξ as "frequency", and the pair (x, ξ) is a point in time-frequency space $\mathbb{R}^{2d}$. In the language of physics, in particular in quantum mechanics, x corresponds to the position and ξ to the momentum, and (x, ξ) is a point in phase space $\mathbb{R}^{2d}$. The short-time Fourier transform is an important tool to study the simultaneous time-frequency distribution (or phase-space distribution) of a signal f.

The modulation spaces are defined by imposing a norm on the short-time Fourier transform $S_g f$. Every solid norm $\| \cdot \|_Y$ for functions on phase space $\mathbb{R}^{2d}$ induces a norm on functions or distributions on $\mathbb{R}^d$ by the rule

$$\|f\|_{M(Y)} = \|S_g f\|_Y.$$

The resulting space $M(Y)$ is the modulation space corresponding to Y. The standard choice for Y is a weighted L^p-space. Let $m \geq 0$ be a weight function on $\mathbb{R}^{2d}$, then the modulation space $M_m^p(\mathbb{R}^d)$ is defined by the norm

$$\|f\|_{M_m^p} = \left(\int_{\mathbb{R}^{2d}} |S_g f(x, \xi)|^p m(x, \xi)^p \, dx d\xi \right)^{1/p}. \qquad (8)$$

The needs of time-frequency analysis often require mixed L^p-spaces, so one often looks at the modulation space $M_m^{p,q}(\mathbb{R}^d)$ defined by the norm

$$\|f\|_{M_m^{p,q}} = \left(\left(\int_{\mathbb{R}^d} |S_g f(x, \xi)|^p m(x, \xi)^p \, dx \right)^{q/p} d\xi \right)^{1/p}.$$

Thus the M_m^p and $M_m^{p,q}$-norms quantify the phase-space content of f.

The theory of modulation spaces forms a branch of time-frequency analysis and hardly needs any further advertisement in this context. Modulation spaces have

become indispensible for time-frequency analysis and phase-space analysis and for the study of pseudodifferential operators. The original paper is Feichtinger's [20], for the history see [21], a detailed exposition is contained in [38] and the forthcoming books [8, 16].

In our context modulation spaces serve as the prototype of function spaces associated to a representation of a nilpotent Lie group.

2.3. Chirps

Our main tool for distinguishing various coorbit spaces is the behavior of a certain multiplication operator on these spaces. Let $C = C^T$ be a real-valued symmetric $d \times d$-matrix and

$$\mathcal{N}_C f(t) = e^{-i\pi Ct \cdot t} f(t) \tag{9}$$

be the corresponding multiplication operator. In engineering terminology the multiplier $e^{-i\pi Ct \cdot t}$ is called a "chirp", in quantum mechanics and PDE $\mathcal{N}_C$ is an ingredient in the solution formula for the free Schrödinger equation. Given C, we define $D = (4\mathrm{I} + C^2)^{-1}$ and the $2d \times 2d$-matrix $\Delta = \begin{pmatrix} 2D & -DC \\ -DC & \mathrm{I}-2D \end{pmatrix}$ acting on $z \in \mathbb{R}^{2d}$. We will need and use a precise estimate for action of $\mathcal{N}_C$ on modulation spaces.

Proposition 2.2. *Let $\phi(t) = e^{-\pi t \cdot t}$ be the standard Gaussian on $\mathbb{R}^d$ and $1 \leq p \leq 2$. Then the modulus of the short-time Fourier transform is*

$$|S_\phi(\mathcal{N}_C \phi)(x, \xi)| = \det(4\mathrm{I} + C^2)^{-1/4} e^{-\pi \Delta(\xi, x)^T \cdot (\xi, x)^T}, \tag{10}$$

and its p-norm is therefore $\qquad \|\mathcal{N}_C \phi\|_{M^p(\mathbb{R}^d)} = \det(4\mathrm{I} + C^2)^{\frac{1}{2p} - \frac{1}{4}}. \tag{11}$

For $d = 1$, $C = u$, and $\mathcal{N}_u = e^{-\pi i u t^2}$ we obtain

$$|S_\phi(\mathcal{N}_u \phi)(x, \xi)| = (4 + u^2)^{-1/4} e^{-\pi|x|^2/2} e^{-2\pi(\xi + ux/2)^2/(4+u^2)} \tag{12}$$

$$\|S_\phi(\mathcal{N}_u \phi)\|_{L^p(\mathbb{R}^2)} = c_p (4 + u^2)^{\frac{1}{2p} - \frac{1}{4}} \asymp |u|^{\frac{1}{p} - \frac{1}{2}}. \tag{13}$$

For $d = 2$, $uC = \begin{pmatrix} 0 & u/2 \\ u/2 & 0 \end{pmatrix}$, and $\mathcal{N}_{uC} = e^{-\pi i u Ct \cdot t}$ we have $(uC)^2 = \frac{u^2}{4}\mathrm{I}$ and therefore $\det(4\mathrm{I} + u^2/4\mathrm{I}) = (4 + u^2/4)^2$ and thus

$$\|S_\phi(\mathcal{N}_{uC} \phi)\|_{L^p(\mathbb{R}^4)} = (4 + u^2/4)^{\frac{1}{p} - \frac{1}{2}} \asymp |u|^{\frac{2}{p} - 1}. \tag{14}$$

The proof of Proposition 2.2 consists of the manipulation of Gaussian integrals as in [28]. In the context of modulation spaces the operator norm of $\mathcal{N}_C$ has been calculated in [14, Lemma 5.3], among others. For convenience we offer a short proof in our notation in the appendix.

2.4. Where is phase space?

The construction of coorbit spaces works for arbitrary integrable, irreducible, unitary representations of a locally compact group. To understand why the representations of a nilpotent Lie group yield a form of time-frequency analysis or phase-space analysis, we need to look at the general form of the irreducible unitary representations of nilpotent Lie groups, e.g., in [17].

By Kirillov's theory every irreducible representation of a (simply connected, connected) nilpotent Lie group is induced from a character of a particular subgroup M of G. Fix a functional $\ell \in \mathfrak{g}^*$ and a maximal Lie subalgebra $\mathfrak{m} \subseteq \mathfrak{g}$, a so-called polarization, such that $\ell([X, Y]) = 0$ for all $X, Y \in \mathfrak{m}$. Then ℓ defines a character

χ on the subgroup $M = \exp(\mathfrak{m})$ by $\chi(m) = e^{2\pi i \langle \ell, \log m \rangle}$ for $m \in M$. The representation $\pi = \pi_\ell$ is the representation induced from (χ, M) to G. Every irreducible unitary representation of G can be obtained in this way from some $\ell \in \mathfrak{g}^*$.

For simplicity we describe the induced representation only under the additional assumption that the polarization $\mathfrak{m}$ is an ideal and M is a normal subgroup of G. This assumption is satisfied in all our examples. The induced representation can be realized explicitly on the representation space $L^2(M \backslash G)$ with a G-invariant measure on $M \backslash G$. Let $q : G \to M \backslash G$ denote the projection and let $\sigma : M \backslash G \to G$ be a (continuous) section with image $H = \sigma(M \backslash G)$. Then every element in G can be written uniquely as $g = mh$ with $m = g\sigma(q(g))^{-1} \in M$ and $h = \sigma(q(g)) \in H$. The induced representation is then

$$(\pi(g)f)(q(h')) = e^{2\pi i \langle l, \log \left(h'mh'^{-1}(h'h\sigma(h'h)^{-1}) \right) \rangle} f\left(q(h'\,h) \right). \tag{15}$$

If π is square-integrable modulo the center, then the radical of ℓ coincides with the center [17] and $\dim \mathfrak{g}/\mathfrak{z} = 2d$ is even. Let $r = \dim \mathfrak{z}$ and $\dim \mathfrak{m} = r + d$, so that $\dim \mathfrak{g}/\mathfrak{m} = d$. Now choose a (strong) Malcev basis

$$\{Z_1, \ldots, Z_r, Y_1, \ldots, Y_d, X_1, \ldots, X_d\}$$

of $\mathfrak{g}$ passing through $\mathfrak{z}$ and $\mathfrak{m}$, such that $\mathfrak{m} = \operatorname{span}\{Z_1, \ldots, Z_r, Y_1, \ldots, Y_d\}$. Then every $m \in M$ can be written as $m = e^{z_1 Z_1} \ldots e^{z_r Z_r} e^{y_1 Y_1} \ldots e^{y_d Y_d}$, and a distinguished section $\sigma : M \backslash G \to G$ is $\sigma(Mh) = e^{t_1 X_1} \ldots e^{t_d X_d}$. With this choice of coordinates, the multiplicative term in (15) is $e^{2\pi i \langle l, \log \left(h'mh'^{-1} h'h\sigma(h'h)^{-1} \right) \rangle} = e^{2\pi i (P(y,t) + Q(x,t))}$ where P a polynomial in the coordinates y of m and t of $h' \in H = \sigma(M \backslash G)$ and Q is a polynomial in the coordinates x of h and t of $h' \in H$. Thus $\pi(m), m \in M$, is a multiplication operator and can be interpreted as a generalized modulation, with the polynomial expression $e^{2\pi i P(y,t)}$ replacing the linear expression $M_\xi = e^{2\pi i \xi \cdot t}$. Similarly, the operator $\pi(h)$ for $h \in H$ consists of the group action $Mh' \mapsto Mh'h$ and the multiplication by $e^{2\pi i Q(x,t)}$ and can be considered as a generalized translation. The representation is therefore of the form

$$g(t) \to \pi(x, y)g(t) = e^{2\pi i P(t,y)} e^{2\pi i Q(x,t)} g(tx), \tag{16}$$

where P and Q are polynomials in t, x, y and $t \to tx$ is a group action. Thus π splits into a generalized modulation and a generalized translation. In this sense the induced representation can be viewed as a generalized time-frequency shift on the configuration space $M \backslash G$.

Usually one cannot neglect the factor $e^{2\pi i \langle l, \log \left(h'h\sigma(h'h)^{-1} \right) \rangle}$ depending on $M \backslash G$. More precisely, if G splits as a semidirect product $G = M \ltimes H$ for a subgroup H, then $h'h\sigma(h'h) = e \in M$ and this factor is absent. We will see that in most of our concrete examples it disappears.

Omitting the central coordinates, we interpret the variables $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ parametrizing $m \in M$ as "frequency/momentum" and the variables $(x_1, \ldots, x_d) \in \mathbb{R}^d$ parametrizing $h = e^{x_1 X_1} \ldots e^{x_d X_d} \in \sigma(M \backslash G)$ as "time/position".
In this analogy the associated representation coefficient

$$V_g^\pi f(\dot{x}) = \langle f, \pi(\dot{x})g \rangle$$

is a version of the short-time Fourier transform measuring a new kind of phase space concentration on $M \backslash G$.

3. Concrete examples

We give some concrete examples of coorbit spaces with respect to nilpotent groups, some already known, some new.

For the low dimensional examples we use the classification of Nielsen [46]. This source lists all nilpotent Lie groups of dimension ≤ 6 with their Lie algebras, the explicit group multiplications, descriptions of the coadjoint orbits, and the associated irreducible representations. For our purpose it is not necessary to know Kirillov's theory of nilpotent groups or to rederive all formulas. We will use Nielsen's list as a black box and simply write down the groups and their representations[1].

3.1. The Heisenberg group and modulation spaces

Let $\mathbb{H}_d = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ with multiplication $(z, y, x) \cdot (w, v, u) = (z+w+x \cdot v, y+v, x+u)$ defined by the Lie brackets $[X_j, Y_k] = \delta_{j,k} Z$. It possesses the irreducible unitary representations $\pi_\lambda, \lambda \neq 0$, acting on $L^2(\mathbb{R}^d)$ by the operators

$$\pi_\lambda(z, y, x) f(t) = e^{2\pi i \lambda z} e^{-2\pi i \lambda y \cdot t} f(t - x) \tag{17}$$

for $f \in L^2(\mathbb{R}^d)$. This representation, the Schrödinger representation, is square-integrable modulo the center $\mathbb{R} \times \{0\} \times \{0\}$.

Omitting the center (or setting $z = 0$) and comparing to (7), we see that the representation coefficients of π_λ are simply a scaled version of the short-time Fourier transform:

$$V_g^{\pi_\lambda} f(0, y, x) = \langle f, M_{-\lambda y} T_x g \rangle = S_g f(x, -\lambda y) \,.$$

Since by definition the coorbit space norm on $Co_\pi L_m^p$ is

$$\|f\|_{Co_\pi L_m^p}^p = \int_{\mathbb{R}^{2d}} |V_g^{\pi_\lambda} f(y, x)|^p m(x, y)^p \, dx dy$$

$$= \int_{\mathbb{R}^{2d}} |S_g f(x, -\lambda y)|^p m(x, y)^p \, dx dy = \lambda^{-p} \|f\|_{M_{m_\lambda}^p}^p$$

with the weight $m_\lambda(x, y) = m(x, -\lambda^{-1} y)$. The class of coorbit spaces of the Heisenberg group is therefore identical to the class of the modulation spaces. This group-theoretic point of view on modulation spaces is explained in detail in [25].

3.2. The group $G_{6,16}$

This group is the six-dimensional group $G_{6,16} \simeq \mathbb{R}^6$ defined by the Lie brackets

$$[X_6, X_5] = X_2, \quad [X_6, X_4] = X_1, \quad [X_5, X_3] = X_1$$

with group multiplication

$$x \cdot y = (x_1, \ldots, x_6) \cdot (y_1, \ldots, y_6)$$
$$= (x_1 + y_1 + x_5 y_3 + x_6 y_4, x_2 + y_2 + x_6 y_5, x_3 + y_3, x_4 + y_4, x_5 + y_5, x_6 + y_6) \,.$$

The group possesses the two-parameter family of representations modulo the center $\mathbb{R}^2 \times \{(0,0,0,0)\}$ acting on $L^2(\mathbb{R}^2)$ by the operators

$$\pi_{\lambda,\mu}(x_1, \ldots, x_6) g(s, t) = \exp 2\pi i \Big(\lambda(x_1 - x_3 s - x_4 t) + \mu(x_2 - x_5 x_6 + x_6 s) \Big) g(s - x_5, t - x_6) \,.$$

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[1] Nielsen uses coordinates of the second kind and induced representations with left shifts rather than right shifts.

For $\lambda \neq 0, \mu \in \mathbb{R}$ the representation $\pi_{\lambda,\mu}$ is square-integrable modulo the center. When using the absolute values $|V_g^{\pi_{\lambda,\mu}} f|$, the phase factor $e^{2\pi i(\lambda x_1 + \mu(x_2 - x_5 x_6))}$ disappears, and we will therefore omit it in the definition of the transform V_g^π. In addition, we identify the quotient $G_{6,16}/Z$ with a convenient section $G_{6,16}/Z \to G_{6,16}$ and will write $\dot{x} = (0, 0, x_3, \ldots, x_6) \in G_{6,16}/Z$. Then we can write the representation with the help of time-frequency shifts as follows:

$$\pi_{\lambda,\mu}(\dot{x}) = M_{(-\lambda x_3 + \mu x_6, -\lambda x_4)} T_{(x_5, x_6)} \,. \tag{18}$$

So the cleaned-up representation coefficient is just a scaled version of the short-time Fourier transform

$$V_g^{\pi_{\lambda,\mu}} f(\dot{x}) = S_g f\big((x_5, x_6), (-\lambda x_3 + \mu x_6, -\lambda x_4)\big) \,.$$

The resulting coorbit space norm $Co_\pi L_m^p$ is then

$$\|f\|_{Co_\pi L_m^p}^p = \int_{\mathbb{R}^4} |V_g^{\pi_{\lambda,\mu}} f(\dot{x})|^p \, m(\dot{x})^p \, d\dot{x} \,.$$

Using the linear coordinate transform $T\dot{x} = y$, $y_3 = -\lambda x_3 + \mu x_6$, $y_4 = -\lambda x_4$, $y_5 = x_5$, $y_6 = x_6$, we find that

$$\|f\|_{Co_\pi L_m^p(G/Z)}^p = \int_{G/Z} |S_g f(T\dot{x})|^p m(\dot{x})^p \, d\dot{x} = c \int_{\mathbb{R}^4} |S_g f(y)|^p m(T^{-1}y) \, dy \,.$$

Therefore $Co_{\pi_{\lambda,\mu}} L^p(G_{6,16}/Z) = M_{m \circ T^{-1}}^p(\mathbb{R}^2)$ is just a modulation space with a modified weight that depends on the parameters λ, μ of the representation $\pi_{\lambda,\mu}$. Thus the coorbit spaces associated to a representation $\pi_{\lambda,\mu}$ of the group $G_{6,16}$ all belong to the class of modulation spaces, and no new spaces arise from this group. For the special case of a polynomial weight $m(\dot{x}) = (1 + |\dot{x}|)^s, s \in \mathbb{R}$, we have $m \circ T^{-1} \asymp m$, and we see that (18) is just an equivalent norm for the modulation space $M_m^p(\mathbb{R}^2)$ independent of the parameters of the representation $\pi_{\lambda,\mu}$.

In conclusion, the group $G_{6,16}$ does not yield any new coorbit spaces, nor a new version of time-frequency analysis. This seems intuitive, because the quotient $G_{6,16}/Z$ is the abelian group $\mathbb{R}^4$, as is the quotient $\mathbb{H}_2/Z$. Likewise the groups studied in [2, 10] for the construction of general coherent states lead to the standard modulation spaces (because $G/Z \simeq \mathbb{R}^{2d}$ and G is 2-step nilpotent).

In the following we consider some k-step nilpotent groups with $k \geq 3$ and will see that they lead to function spaces different from the modulation spaces.

3.3. The group $G_{5,3}$

This is the (simply connected) nilpotent Lie group generated by the Lie algebra with the Lie brackets

$$[X_5, X_4] = X_2, \quad [X_5, X_2] = X_1, \quad [X_4, X_3] = X_1 \,. \tag{19}$$

For $x = (x_1, x_2, \ldots, x_5), y = (y_1, \ldots, y_5) \in \mathbb{R}^5$ the group multiplication of $G_{5,3}$ is given by

$$x \cdot y = (x_1 + y_1 + x_4 y_3 + x_5 y_2 + \tfrac{1}{2} x_5^2 y_4, x_2 + y_2 + x_5 y_4, x_3 + y_3, x_4 + y_4, x_5 + y_5) \,. \tag{20}$$

The center of $G_{5,3}$ is $\mathbb{R} \times \{(0,0,0,0)\}$. The analysis of all irreducible unitary representations yields a one-parameter family of representations square-integrable modulo the center $\pi_\lambda \in SI(G/Z), \lambda \neq 0$. For $x \in G_{5,3}$ and $(s,t) \in \mathbb{R}^2$, π_λ acts on $g \in L^2(\mathbb{R}^2)$ as follows:

$$\pi_\lambda(x)g(s,t) = \exp\left(2\pi i\lambda(x_1 - x_3 x_4 + x_4 s - x_2 t + \tfrac{1}{2}x_4 t^2)\right) g(s - x_3, t - x_5). \quad (21)$$

Let us briefly analyze this representation from the perspective of time-frequency analysis. The operators $\pi_\lambda(x)$ act on g by the translations $T_{(x_3,x_5)}$ and the modulations $M_{(\lambda x_4, -\lambda x_2)}$. The additional factor that makes this representation interesting and different from the Schrödinger representation (17) is the multiplication by the chirp $e^{\pi i\lambda x_4 t^2}$. We see that π_λ is indeed of the form $e^{2\pi i P(x;s,t)}T_u$ as motivated in (16) for some polynomial P and some translation by $u = (x_3, x_5)$.

The polarization used for π_λ is $\mathfrak{m} = \mathbb{R} - \mathrm{span}\{X_1, X_2, X_4\}$. As explained in Section 2.4, the underlying phase space consists of the "frequency" variables x_2, x_4 and the "time" variables x_3, x_5, and the pairs (x_2, x_5) and (x_3, x_4) are conjugate variables, since $[X_5, X_2], [X_3, X_4] \in \mathfrak{z} \setminus \{0\}$.

Omitting the phase factor $e^{2\pi i\lambda(x_1 - x_3 x_4)}$, which disappears in absolute values of $V_g^{\pi_\lambda} f$, and choosing the section $\dot{x} = (0, x_2, x_3, x_4, x_5)$, the associated representation coefficient with respect to a fixed $g \in L^2(\mathbb{R}^2)$ is given by

$$V_g^{\pi_\lambda} f(\dot{x}) = \int_{\mathbb{R}^2} f(s,t)\,\bar{g}(s - x_3, t - x_5)\exp\left(-2\pi i\lambda(x_4 s - x_2 t + \tfrac{1}{2}x_4 t^2)\right) ds\,dt. \quad (22)$$

This formula certainly justifies the interpretation of $V_g^{\pi_\lambda} g(x_2, x_3, x_4, x_5)$ as a generalized short-time Fourier transform.

Let m be a moderate, polynomially growing weight on $\mathbb{R}^4$, $1 \leq p \leq \infty$, and fix $g \in \mathcal{S}(\mathbb{R}^2), g \neq 0$. The family of coorbit spaces with respect to π_λ is given by

$$Co_{\pi_\lambda} L_m^p(G_{5,3}/Z) = \{f \in \mathcal{S}'(\mathbb{R}^2) : V_g^{\pi_\lambda} f \in L_m^p(G_{5,3}/Z)\} \quad (23)$$

with norm

$$\|f\|_{Co_{\pi_\lambda} L_m^p}^p = \int_{\mathbb{R}^4} |V_g^{\pi_\lambda} f(x_2, x_3, x_4, x_5)|^p\, m(x_2, x_3, x_4, x_5)^p\, dx_2 dx_3 dx_4 dx_5. \quad (24)$$

In analogy to the mixed modulation spaces $M_m^{p,q}(\mathbb{R}^d)$ one might also consider the spaces with norm

$$\|f\|_{Co_{\pi_\lambda} L_m^{p,q}}^p = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |V_g^{\pi_\lambda} f(x_2, x_3, x_4, x_5)|^p\, m(x_2, x_3, x_4, x_5)^p\, dx_3 dx_5\right)^{q/p} dx_2 dx_4.$$

Thus the associated coorbit spaces bear some resemblance to the modulation spaces and can be rightly considered generalized modulation spaces. This class of function spaces should be well suited for all questions concerning the operators π_λ.

Note that $\qquad |V_g^{\pi_\lambda} f(x_2, x_3, x_4, x_5)| = |V_g^{\pi_1} f(\lambda x_2, x_3, \lambda x_4, x_5)|,$

so that $V_g^{\pi_\lambda} f \in L_m^p(G_{5,3}/Z)$ if and only if $V_g^{\pi_1} f \in L_{m_\lambda}^p(G_{5,3}/Z)$ with the scaled weight $m_\lambda(x_2, x_3, x_4, x_5) = m(\lambda^{-1}x_2, x_3, \lambda^{-1}x_4, x_5)$. Thus different parameters λ yield the same family of spaces, possibly with a change of the weight. We may therefore assume without loss of generality that $\lambda = 1$ and write $\pi = \pi_1$ in the following.

Our main insight is that the coorbit spaces $Co_\pi L_m^p(G_{5,3}/Z)$ form a new class of function spaces on $\mathbb{R}^2$ and differ from the standard modulation spaces.

Proposition 3.1. *Let* $p, q \in [1, \infty]$, $(p, q) \neq (2, 2)$. *Then* $Co_\pi L^p(G_{5,3}) \neq M^q(\mathbb{R}^2)$.

First proof. Because of the structure of the representation we use tensor products $f(s, t) = f_1(s) f_2(t) = (f_1 \otimes f_2)(s, t)$. To measure the norm, we choose a Gaussian window $g(s, t) = e^{-\pi s^2} e^{-\pi t^2} = \phi \otimes \phi(s, t)$ with the Gaussian $\phi(s) = e^{-\pi s^2}$ for $s \in \mathbb{R}$.

Let
$$\mathcal{N}_u f(t) = e^{-i\pi u t^2} f(t)$$

be the multiplication operator with the "chirp" $e^{-i\pi u t^2}$ acting on $L^2(\mathbb{R})$, and T_x, M_ξ be the time-frequency shifts on $L^2(\mathbb{R})$. Then the representation coefficients of a tensor product can be written as

$$
\begin{aligned}
&V_g^\pi(f_1 \otimes f_2)(x_2, x_3, x_4, x_5) \\
&= \int_\mathbb{R} f_1(s)\phi(s - x_3)e^{-2\pi i x_4 s}\, ds \int_\mathbb{R} e^{-i\pi x_4 t^2} f_2(t)\phi(t - x_5)e^{2\pi i x_2 t}\, dt \\
&= S_\phi f_1(x_3, x_4)\, S_\phi(\mathcal{N}_{x_4} f_2)(x_5, -x_2)\, dt\,.
\end{aligned}
$$

Taking the p-norm first with respect to x_2, x_5, we obtain

$$
\begin{aligned}
\|V_g^\pi(f_1 \otimes f_2)\|_{L^p(G/Z)}^p &= \int_{\mathbb{R}^2} \|S_\phi(\mathcal{N}_{x_4} f_2)\|_{L^p(\mathbb{R}^2)}^p |S_\phi f_1(x_3, x_4)|^p\, dx_3 dx_4 \\
&= \int_{\mathbb{R}^2} \|\mathcal{N}_{x_4} f_2\|_{M^p(\mathbb{R})}^p |S_\phi f_1(x_3, x_4)|^p\, dx_3 dx_4\,. \qquad (25)
\end{aligned}
$$

We now set $v(x_3, x_4) = (1 + |x_4|)^{\frac{1}{2p} - \frac{1}{4}}$ and choose $f_2 = \phi$ and $f_1 \in M^p(\mathbb{R})$ arbitrary.

Since
$$S_{\phi \otimes \phi}(f_1 \otimes f_2)(x_1, x_2, \xi_1, \xi_2) = S_\phi f_1(x_1, \xi_1) S_\phi f_2(x_2, \xi_2)\,, \qquad (26)$$

we see that $f_1 \otimes \phi \in M^p(\mathbb{R}^2)$. On the other hand, using Proposition 2.2 the modulation space norm of $\mathcal{N}_{x_4}\phi$ is

$$\|\mathcal{N}_{x_4}\phi\|_{M^p(\mathbb{R})} \asymp \|S_\phi(\mathcal{N}_{x_4}\phi)\|_{L^p(\mathbb{R}^2)} \asymp (4 + x_4^2)^{\frac{1}{2p} - \frac{1}{4}} \asymp |x_4|^{\frac{1}{p} - \frac{1}{2}}\,. \qquad (27)$$

Continuing (25), we find that

$$\|f_1 \otimes \phi\|_{Co_\pi L^p}^p \asymp \int_{\mathbb{R}^2} (1 + x_4^2)^{1/2 - p/4} |S_\phi f_1(x_3, x_4)|^p\, dx_3 dx_4 \asymp \|f_1\|_{M_v^p}^p \|\phi\|_{M^p}^p\,.$$

Thus $f_1 \otimes \phi \in Co_\pi L^p(G_{5,3}/Z)$, if and only if $f_1 \in M_v^p(\mathbb{R})$. If $1 \leq p < 2$, then M_v^p is a proper subspace of M^p, e.g., by [38, Cor. 12.3.5]. Therefore there exists $f_1 \in M^p \setminus M_v^p$, and consequently, we have constructed an element $f_1 \otimes \phi \in M^p(\mathbb{R}^2)$, but $f_1 \otimes \phi \notin Co_\pi L^p$.

So far, we have proved that $Co_\pi L^p \neq M^p$ for $1 \leq p < 2$. For $p > 2$ and thus $p' = p/(p-1) < 2$, we use the duality. By Proposition 2.1(iii), $Co_\pi L^p \simeq (Co_\pi L^{p'})^* \neq (M^{p'})^* \simeq M^p$.

Finally, we argue that $Co_\pi L^p \neq M^q$ for $q \neq p$. By [24, Prop. 9.3] $Co_\pi L^p$ is isomorphic to ℓ^p, whereas M^q is isomorphic to ℓ^q, whence these spaces must be different. ∎

The above argument does not work for the case $p = 2$. This is clear, because we always have $Co_\pi L^2(G/Z) = \mathcal{H} = L^2(\mathbb{R}^2)$ by Proposition 2.1(v).

Next we recast the proof in a different form that is more suitable for generalization.

Second proof. By Proposition 2.1 the representation π is an isometry on the space $Co_\pi L^p(G_{5,3}/Z)$, Therefore we have

$$\|\pi(0, 0, x_4, 0)f\|_{Co_\pi L^p} = \|f\|_{Co_\pi L^p}. \tag{28}$$

By contrast, since by (21) $\pi(0, 0, x_4, 0)(f_1 \otimes f_2) = M_{x_4} f_1 \otimes \mathcal{N}_{x_4} f_2$ and the modulation is an isometry on M^p, we have, for x_4 large enough,

$$\|\pi(0, 0, x_4, 0)(f_1 \otimes \phi)\|_{M^p(\mathbb{R}^2)} = \|f_1\|_{M^p(\mathbb{R})} \|\mathcal{N}_{x_4}\phi\|_{M^p(\mathbb{R})}$$
$$= c_p(4 + x_4^2)^{\frac{1}{2p} - \frac{1}{4}} \|f_1\|_{M^p} \|\phi\|_{M^p} \leq C|x_4|^{\frac{1}{p} - \frac{1}{2}} \|f_1 \otimes \phi\|_{M^p}.$$

Thus the one-parameter group of (unitary) operators $\pi(0, 0, 0, x_4, 0)$ does not act by isometries on $M^p(\mathbb{R}^2)$ for $1 \leq p < 2$, but is unbounded on $M^p(\mathbb{R}^2)$. We conclude that $Co_\pi L^p(G_{5,3}/Z) \neq M^p(\mathbb{R}^2)$. If $p > 2$, we use duality to achieve the same conclusion.

In fact, using a standard argument we construct an element $h \in Co_\pi L^p(G_{5,3}/Z)$ that is not in $M^p(\mathbb{R}^2)$. Simply choose an increasing sequence $u_n > 0$, such that

$$\frac{1}{n^2} u_n^{\frac{1}{p} - \frac{1}{2}} \to \infty \qquad \text{and} \qquad c_p u_n^{\frac{1}{p} - \frac{1}{2}} > 2Cn^2 \sum_{j=1}^{n-1} \frac{1}{j^2} u_j^{\frac{1}{p} - \frac{1}{2}} \qquad \forall n.$$

Fix $f = f_1 \otimes \phi$. Then by absolute convergence and (28)

$$h = \sum_{j=1}^{\infty} \frac{1}{j^2} \pi(0, 0, u_j, 0)f \in Co_\pi L^p(G_{5,3}/Z) \subseteq L^2(\mathbb{R}^2),$$

but the sequence of partial sums of h is unbounded in $M^p(\mathbb{R}^2)$:

$$\|\sum_{j=1}^{n} \frac{1}{j^2} \pi(0, 0, u_j, 0)f\|_{M^p} \geq \frac{1}{n^2} \|\pi(0, 0, u_j, 0)f\|_{M^p} - \sum_{j=1}^{n-1} \frac{1}{j^2} \|\pi(0, 0, u_j, 0)f\|_{M^p}$$

$$\geq c_p \frac{1}{n^2} u_n^{\frac{1}{p} - \frac{1}{2}} - C \sum_{j=1}^{n-1} \frac{1}{j^2} u_j^{\frac{1}{p} - \frac{1}{2}} > \frac{1}{2n^2} u_n^{\frac{1}{p} - \frac{1}{2}} \to \infty.$$

Thus $h \notin M^p(\mathbb{R})$.   ∎

The above argument does not exclude the possibility that $Co_\pi L^p(G_{5,3}/Z) = M_m^p(\mathbb{R}^2)$ for some moderate weight m. An extension of Proposition 3.1 yields the general result.

Theorem 3.2.   *Let $1 \leq p \leq \infty, p \neq 2$, and m be an arbitrary moderate weight function on $\mathbb{R}^4$. Then $Co_\pi L^p(G_{5,3}/Z) \neq M_m^p(\mathbb{R}^2)$.*

Proof.   Again, we only treat $1 \leq p < 2$ and use duality for $p > 2$. Since $Co_\pi L^p(G/Z) \subseteq L^2(\mathbb{R}^2)$ by Proposition 2.1(v), we only need to compare to those modulation spaces $M_m^p(\mathbb{R}^2)$ that are embedded in $L^2(\mathbb{R}^2)$. Indeed, if $M_m^p(\mathbb{R}^2) \not\subseteq L^2(\mathbb{R}^2)$, then there exists $h \in M_m^p(\mathbb{R}^2) \setminus L^2(\mathbb{R}^2)$ and this function h cannot be in $Co_\pi L^p(G_{5,3}/Z) \subseteq L^2(\mathbb{R}^2)$.

For $p < 2$ the embedding $M_m^p(\mathbb{R}^2) \subseteq L^2(\mathbb{R}^2) = M^2(\mathbb{R}^2)$ implies that m is bounded below, $m(\dot{x}) \geq C_0 > 0$ for all $\dot{x} \in \mathbb{R}^4$ (check for instance [38, Thm. 12.2.2] and [24, Thm. 4]). In particular, we also have the embedding $M_m^p(\mathbb{R}^2) \subseteq M^p(\mathbb{R}^2)$. We now use the second proof and compute the norm of $M_u f_1 \otimes \mathcal{N}_u \phi$ in $M_m^p(\mathbb{R}^2)$. Since the short-time Fourier transform factors as

$$S_{\phi \otimes \phi}(M_u f_1 \otimes \mathcal{N}_u \phi)(x_1, x_2, \xi_1, \xi_2) = S_\phi(M_u f_1)(x_1, \xi_1)\, S_\phi(\mathcal{N}_u \phi)(x_2, \xi_2)$$

$$= S_\phi f_1(x_1, \xi_1 - u)\, S_\phi(\mathcal{N}_u \phi)(x_2, \xi_2)\,,$$

the norm of $M_u f_1 \otimes \mathcal{N}_u \phi$ is

$$\|M_u f_1 \otimes \mathcal{N}_u \phi\|_{M_m^p}^p = \int_{\mathbb{R}^4} |S_\phi f_1(x_1, \xi_1 - u)|^p\, |S_\phi(\mathcal{N}_u \phi)(x_2, \xi_2)|^p m(x_1, x_2, \xi_1, \xi_2)^p\, dx d\xi$$

$$\geq C_0^p \int_{\mathbb{R}^4} |S_\phi f_1(x_1, \xi_1 - u)|^p\, |S_\phi(\mathcal{N}_u \phi)(x_2, \xi_2)|^p\, dx_1 dx_2 d\xi_1 d\xi_2$$

$$\asymp \int_{\mathbb{R}^2} |S_\phi f_1(x_1, \xi_1)|^p\, \|\mathcal{N}_u \phi\|_{M^p}^p\, dx_1 d\xi_1$$

$$\asymp \int_{\mathbb{R}^2} |S_\phi f_1(x_1, \xi_1)|^p\, (1 + u^2)^{\frac{1}{2} - \frac{p}{4}}\, dx_1 d\xi_1$$

$$= (1 + u^2)^{\frac{1}{2} - \frac{p}{4}} \|f_1\|_{M^p}^p \asymp (1 + u^2)^{\frac{1}{2} - \frac{p}{4}} \|f_1 \otimes \phi\|_{M^p}^p\,.$$

Again, we have used Proposition 2.2 and (13). We conclude that the orbit $u \to M_u f_1 \otimes \mathcal{N}_u \phi$ is bounded in $Co_\pi L^p(G_{5,3}/Z)$, but unbounded in $M_m^p(\mathbb{R}^2)$. Therefore the two spaces cannot be equal. ∎

The above arguments are typical and can be applied to several other groups in Nielsen's list. In the following we only deal with unweighted versions of $Co_\pi L^p$.

3.4. The group $G_{6,19}$

This is the six-dimensional group $G_{6,19} \simeq \mathbb{R}^6$ defined by the Lie brackets

$$[X_6, X_5] = X_4, \quad [X_6, X_3] = X_1, \quad [X_5, X_4] = X_2$$

with group multiplication

$$x \cdot y = (x_1, \ldots, x_6) \cdot (y_1, \ldots, y_6) =$$

$$(x_1 + y_1 + x_6 y_3, x_2 + y_2 + x_5 y_4 + x_5 x_6 y_5 + \tfrac{1}{2} x_6 y_5^2, x_3 + y_3, x_4 + y_4 + x_6 y_5, x_5 + y_5, x_6 + y_6)\,.$$

This group possesses a two-parameter family of square-integrable representations $\pi_{\lambda, \mu}, \lambda \mu \neq 0$, modulo the center $\mathbb{R}^2 \times \{(0, 0, 0, 0)\}$ acting on $L^2(\mathbb{R}^2)$ by the operators

$$\pi_{\lambda, \mu}(x_1, \ldots, x_6) g(s, t) =$$

$$\exp 2\pi i \Big(\lambda(x_1 - x_3 t) + \mu(x_2 - \tfrac{1}{2} x_5^2 x_6 - x_4 s + x_5 x_6 s - \tfrac{1}{2} x_6 s^2) \Big) g(s - x_5, t - x_6)\,.$$

When using the absolute values $|V_g^{\pi_{\lambda, \mu}} f|$, the phase factor $e^{2\pi i(\lambda x_1 + \mu(x_2 - x_5^2 x_6/2))}$ disappears, and we may omit it in the definition of the transform V_g^π. In addition, we identify the quotient $G_{6,19}/Z$ with a convenient section $G_{6,19}/Z \to G_{6,19}$ and will write $\dot{x} = (0, 0, x_3, \ldots, x_6) \in G_{6,19}/Z$.

Then the representation can be written with the help of time-frequency shifts as

$$\pi_{\lambda,\mu}(\dot{x}) = M_{(\mu(-x_4+x_5x_6),-\lambda x_3)}\mathcal{N}^{(1)}_{\mu x_6}T_{(x_5,x_6)} = e^{i\pi\mu x_5^2}M_{(-\mu x_4,-\lambda x_3)}T_{(x_5,x_6)}\mathcal{N}^{(1)}_{\mu x_6}, \quad (29)$$

where $\mathcal{N}^{(1)}_u g(s,t) = e^{-i\pi u s^2}g(s,t)$ is a multiplication by a chirp in the variable s. Using the short-time Fourier transform, the cleaned-up representation coefficient is

$$V_g^{\pi_{\lambda,\mu}}f(\dot{x}) = S_{\mathcal{N}^{(1)}_{\mu x_6}g}f\big((x_5,x_6),(-\mu x_4,-\lambda x_3)\big).$$

The underlying phase space consists of the "frequency/momentum" variables x_3, x_4 and the "time/position" variables x_5, x_6. The pairs (x_4, x_5) and (x_3, x_6) are conjugate, because $[X_4, X_5]$ and $[X_3, X_6] \in \mathfrak{z} \setminus \{0\}$. Given a weight function m on G/Z, the coorbit space with respect to $\pi_{\lambda,\mu}$ is defined by the norm

$$\|f\|^p_{Co_\pi L^p_m} = \int_{\mathbb{R}^4}|V_g^{\pi_{\lambda,\mu}}f(\dot{x})|^p\,m(\dot{x})^p\,d\dot{x}$$

$$= \int_{\mathbb{R}^4}|S_{\mathcal{N}^{(1)}_{\mu x_6}g}f\big((x_5,x_6),(-\mu x_4,-\lambda x_3)\big)|^p\,m(\dot{x})^p dx_3\ldots dx_6.$$

Again the question arises whether we have defined a new class of function spaces or not.

Proposition 3.3. *Let $\lambda\mu \neq 0$ and $p, q \in [1,\infty], (p,q) \neq (2,2)$. Then*
(i) $Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z) \neq M^q(\mathbb{R}^2)$, *and*
(ii) $Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z) \neq Co_\pi L^q(G_{5,3}/Z)$.

Proof.   (i) As in the case of $G_{5,3}$ we use tensor products, namely $g(s,t) = e^{-\pi(s^2+t^2)} = (\phi\otimes\phi)(s,t)$ and $f = \phi \otimes f_2$ for suitable f_2. Then

$$V_g^{\pi_{\lambda,\mu}}f(\dot{x}) = \langle\phi, M_{-\mu x_4}T_{x_5}\mathcal{N}_{\mu x_6}\phi\rangle\,\langle f_2, M_{-\lambda x_3}T_{x_6}\phi\rangle$$

$$= e^{2\pi i\mu x_4 x_5}\overline{S_\phi(\mathcal{N}_{\mu x_6}\phi)(-x_5,\mu x_4)}\,S_\phi f_2(x_6,-\lambda x_3). \quad (30)$$

Hence, $$\|f\|^p_{Co_{\pi_{\lambda,\mu}}L^p} = \int_{\mathbb{R}^4}|V_g^{\pi_{\lambda,\mu}}f(\dot{x})|^p\,dx_3 dx_4 dx_5 dx_6$$

$$= \int_{\mathbb{R}^2}\|\mathcal{N}_{\mu x_6}\phi\|^p_{M^p(\mathbb{R})}\,|S_\phi f_2(x_6,-\lambda x_3)|^p\,dx_3 dx_6. \quad (31)$$

Since by (27) we have $\|\mathcal{N}_{\mu x_6}\phi\|_{M^p(\mathbb{R})} \asymp (4+\mu^2 x_6^2)^{\frac{1}{2p}-\frac{1}{4}} \asymp (1+x_6^2)^{\frac{1}{2p}-\frac{1}{4}} := v(x_6,x_3)$, the $Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z)$-norm of $\phi\otimes f_2$ is

$$\|\phi\otimes f_2\|^p_{Co_{\pi_{\lambda,\mu}}L^p} \asymp \int_{\mathbb{R}^2}|S_\phi f_2(x_6,-\lambda x_3)|^p v(x_6,-\lambda x_3)^p\,dx_6 dx_3.$$

We see that $\phi\otimes f_2$ is in $Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z)$, if and only if $f_2 \in M^p_v(\mathbb{R})$. By choosing $f_2 \in M^p(\mathbb{R}) \setminus M^p_v(\mathbb{R})$, we have constructed a function $\phi\otimes f_2 \in M^p(\mathbb{R}^2)$, but $\phi\otimes f_2 \notin Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z)$.

(ii) We take the one-parameter subgroup $\pi(0,0,0,u,0)$ of $G_{5,3}$ acting on $f = f_1\otimes f_2$ and show that it acts unboundedly on $Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z)$. Using (21), we write $\pi(0,0,0,u,0)(f_1\otimes f_2) = M_u f_1 \otimes \mathcal{N}_u f_2\phi$ and

$$\pi_{-\lambda,\mu}(0,0,x_3,\ldots,x_6)(\phi\otimes\phi) = M_{-x_4}T_{x_5}\mathcal{N}_{x_6}\phi \otimes M_{-x_3}T_{x_6}\phi.$$

The corresponding representation coefficient is then

$$V_{\phi\otimes\phi}^{\pi_{\lambda,\mu}}(\pi(0,0,0,u,0)(f_1\otimes f_2))(x_3,\ldots,x_6)$$
$$= \langle M_u f_1 \otimes \mathcal{N}_u f_2, M_{-\mu x_4} T_{x_5} \mathcal{N}_{\mu x_6}\phi \otimes M_{-\lambda x_3} T_{x_6}\phi\rangle$$
$$= \langle T_{-x_5} M_{u+\mu x_4} f_1, \mathcal{N}_{\mu x_6}\phi\rangle \langle \mathcal{N}_u f_2, M_{-\lambda x_3} T_{x_6}\phi\rangle$$
$$= e^{2\pi i x_5(u+\mu x_4)}\overline{S_{f_1}(\mathcal{N}_{\mu x_6}\phi)}(-x_5,\mu x_4+u)\, S_\phi(\mathcal{N}_u f_2)(x_6,-\lambda x_3)\,.$$

We now choose $f_1 = f_2 = \phi$ and obtain

$$\|\pi(0,0,0,u,0)(\phi\otimes\phi)\|_{Co_{\pi_{\lambda,\mu}}L^p}^p =$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |S_\phi(\mathcal{N}_{\mu x_6}\phi)(-x_5,\mu x_4+u)|^p\, |S_\phi(\mathcal{N}_u\phi)(x_6,-\lambda x_3)|^p\, dx_4 dx_5\, dx_3 dx_6$$
$$= \mu^{-1}\int_{\mathbb{R}^2} \|\mathcal{N}_{x_6}\phi\|_{M^p}^p |S_\phi(\mathcal{N}_u\phi)(x_6,-\lambda x_3)|^p\, dx_3 dx_6$$
$$\asymp \int_{\mathbb{R}^2} (1+|x_6^2|)^{\frac{1}{2}-\frac{p}{4}} |S_\phi(\mathcal{N}_u\phi)(x_6,-\lambda x_3)|^p\, dx_3 dx_6\,.$$

In the last expression we substitute (12) for $|S_\phi(\mathcal{N}_u\phi)|$ and continue with

$$\|\pi(0,0,0,u,0)(\phi\otimes\phi)\|_{Co_{\pi_{\lambda,\mu}}L^p}^p \asymp$$
$$\asymp \int_{\mathbb{R}^2} (1+|x_6^2|)^{\frac{1}{2}-\frac{p}{4}}(4+u^2)^{-p/4}e^{-\pi p x_6^2/2}e^{-2\pi p(-\lambda x_3+u x_6/2)^2/(4+u^2)}\, dx_3 dx_6$$
$$= \lambda^{-1/2}\int_{\mathbb{R}} (1+|x_6^2|)^{\frac{1}{2}-\frac{p}{4}}(4+u^2)^{-\frac{p}{4}+\frac{1}{2}}\, e^{-\pi p x_6^2/2}\, dx_6 \asymp (4+u^2)^{-\frac{p}{4}+\frac{1}{2}}\,.$$

We conclude that $\|\pi(0,0,0,u,0)(\phi\otimes\phi)\|_{Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z)} \asymp (1+u^2)^{\frac{1}{2p}-\frac{1}{4}}\,.$
Since $\pi(0,0,0,u,0)$ is an isometry on $Co_\pi L^p(G_{5,3}/Z)$, but not on $Co_{\pi_{\lambda,\mu}}L^p(G_{6,19}/Z)$, these two spaces cannot be equal. ∎

To summarize, we have constructed three families of function spaces on $\mathbb{R}^2$ that are obtained as coorbit spaces with respect to the nilpotent Lie groups $\mathbb{H}_2, G_{5,3}$ and $G_{6,19}$. After quotienting out their centers, these groups are non-isomorphic. As these three families of coorbit spaces have different invariance properties, we were able to show that they are distinct. From the point of view of the theory of function spaces, we have discovered two brand-new families of function spaces.

3.5. The Dynin-Folland group

Let $\mathfrak{g} := \mathbb{R} - \mathrm{span}\,\{Z, Y_1, Y_2, Y_3, X_1, X_2, X_3\}$ with non-trivial Lie brackets

$$[X_3, Y_1] = [X_2, Y_2] = [X_1, Y_3] = Z,\ [X_2, Y_3] = \tfrac{1}{2}Y_1,\ [X_3, Y_3] = -\tfrac{1}{2}Y_2,\ [X_3, X_2] = X_1.$$

We label the elements of G as $(z, y_1, y_2, y_3, x_1, x_2, x_3)$. This group possesses a one-parameter family of square-integrable representations modulo center $\pi_\lambda \in SI(G/Z)$ that act on $L^2(\mathbb{R}^3)$, or more precisely, on $L^2(\mathbb{H}_1)$ of the Heisenberg group $\mathbb{H}_1$. As they are obtained by a dilation from each other, we may restrict to $\lambda = 1$ and set

$$\big(\pi(z, y_1, y_2, y_3, x_1, x_2, x_3)f\big)(t_3, t_2, t_1) =$$
$$= e^{2\pi i(z+\sum_{k=1}^{3} t_j y_j - t_1 t_2 y_3/2)} f(t_3+x_1+t_1 x_2, t_2+x_2, t_1+x_3)\,. \qquad (32)$$

Again, this representation acts on $f \in L^2(\mathbb{R}^3)$ by means of generalized modulations and translations as in (16). In fact, the translations are just the multiplication in the Heisenberg group $\mathbb{H}_1$, the modulations are just the standard modulations. The most interesting item is the multiplication by the chirp $f(t_3, t_2, t_1) \to e^{-\pi i t_1 t_2 u} f(t_3, t_2, t_1)$. For background and detailed derivations of the multiplication and the representations of G we refer to [26, 52].

We now show that the resulting coorbit spaces are different from the modulation spaces. This was one of the main results of [26], where it was proved in much greater generality. The proof in [26] requires substantial parts of the theory of decomposition spaces [35, 54] and is based on the subtle identification of $Co_\pi L^p$ with a decomposition space. It seems therefore worthwhile to have a simple, short, and self-contained proof. Ours is based on the different invariance properties of the coorbit spaces.

Proposition 3.4. *For $p \in [1, \infty]$, $p \neq 2$ we have $Co_\pi L^p(G/Z) \neq M^p(\mathbb{R}^3)$.*

Proof. The argument is similar to the proof of Proposition 3.1. By inspection of (32) we see that the one-parameter subgroup $(0,0,0,u,0,0,0) = e^{uY_3}$ acts on $f \in L^2(\mathbb{R})$ as

$$\pi(e^{uY_3})f(t_3, t_2, t_1) = e^{2\pi i u t_3} e^{-\pi i u t_1 t_2} f(t_3, t_2, t_1). \tag{33}$$

We recognize the first exponential as a modulation with respect to the variable t_3 and the second exponential as a chirp with respect to the variables t_2, t_1. To simplify, we write $\bar{t} = (t_2, t_1)$ and the quadratic form $(t_1, t_2) \to t_1 t_2$ as $C\bar{t} \cdot \bar{t}$ with the matrix $C = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Again we take f to be a tensor product $f(t_3, t_2, t_1) = f_3(t_3)\phi_2(\bar{t})$ with the two-dimensional Gaussian $\phi_2(\bar{t}) = e^{-\pi \bar{t} \cdot \bar{t}}$. Then the action of the one-parameter group generated by Y_3 is

$$\pi(e^{uY_3})(f_3 \otimes \phi) = M_u f_3 \otimes \mathcal{N}_{uC}\phi_2, \tag{34}$$

and its short-time Fourier transform is

$$S_{\phi \otimes \phi_2}\big(\pi(e^{uY_3})(f_3 \otimes \phi_2)\big)(x_3, \bar{x}, \xi_3, \bar{\xi}) = S_\phi(M_u f_3)(x_3, \xi_3)\, S_{\phi_2}(\mathcal{N}_{uC}\phi_2)(\bar{x}, \bar{\xi}). \tag{35}$$

Using the general version of Proposition 2.2, specifically (14), the modulation space-norm on $\mathbb{R}^3$ is therefore

$$\|S_{\phi \otimes \phi_2}\big(\pi(e^{uY_3})(f_3 \otimes \phi_2)\big)\|_{M^p} = \|M_u f_3\|_{M^p(\mathbb{R})}\, \|\mathcal{N}_{uC}\phi_2\|_{M^p(\mathbb{R}^2)} \asymp \|f_3\|_{M^p(\mathbb{R})}\, (4 + \tfrac{u^2}{4})^{\frac{1}{p} - \frac{1}{2}}.$$

For $1 \leq p < 2$ the one-parameter group e^{uY_3} acts unboundedly on $M^p(\mathbb{R}^3)$, and we conclude that $Co_\pi L^p \neq M^p(\mathbb{R}^3)$ for $1 \leq p < 2$. For $p > 2$ we use duality. ∎

4. Atomic decompositions

The abstract version of atomic decompositions in coorbit spaces was developed in [23, 24, 36]. They yield series expansions of elements in a coorbit space with respect to elements in the orbit of the representation. Following a costum in coorbit theory, we briefly summarize these results and update them with recent results that are specific for nilpotent groups.

The first goal is to construct a "window" function $g \in \mathcal{H}$ and (relatively) discrete set $\Lambda \subseteq G$, such that there exist constants $A, B > 0$, such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|_{\mathcal{H}}^2 \qquad \forall f \in \mathcal{H}. \tag{36}$$

A set $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ satisfying (36) is called a (coherent) frame.

If G is nilpotent with center Z and π is irreducible, then, for arbitrary $z_\lambda \in Z$, the set $\{\pi(z_\lambda \lambda)g : \lambda \in \Lambda\}$ is also a frame with the same bounds A, B. We could have started with a projective representation of G/Z and formulated everything in G/Z. However, since the construction of irreducible unitary representations of a nilpotent Lie group is on the level of G, we prefer to work on the group level. To take into account this ambiguity, we will use discrete sets $\tilde{\Lambda} \subseteq G/Z$ and an arbitrary section $\tilde{\Lambda} \to \Lambda \subseteq G$.

We also need the notion of "nice" vectors. We say that $g \in \mathcal{H}$ is nice, if $g = \pi(k)g_0 = \int_G k(x)\pi(x)g_0 \, dx$ for some $g_0 \in \mathcal{H}$ and k continuous with compact support (or k in the amalgam space $W(C, \ell^1)(G)$).

Theorem 4.1 (Existence of coherent frames). *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G that is square-integrable modulo center and let $g \in \mathcal{H}$ be a nice vector. Then there exists a neighborhood $U \subseteq G/Z$ of e with the following property: Assume that $\tilde{\Lambda} \subseteq G/Z$ is U-dense and relatively separated, i.e., $\bigcup_{\tilde{\lambda} \in \tilde{\Lambda}} \tilde{\lambda}U = G/Z$ and $\max_{x \in G/Z} \#(\tilde{\Lambda} \cap xU) < \infty$, and let $\Lambda \subseteq G$ be some preimage of $\tilde{\Lambda}$.*
Then the set $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a frame for $\mathcal{H}$.

The existence of coherent frames was first proved in [32, Thm. 4.1] for arbitrary square-integrable representations. Note that the original coorbit theory [23] required somewhat stronger assumptions on g and π, namely the integrability of π and $V_g g \in W(C, \ell^1)(G)$. For nilpotent Lie groups the square-integrability of π modulo the center automatically implies automatically its integrability [17]. Theorem 4.1 also follows from the recent work [30] on the discretization of arbitrary reproducing kernel Hilbert spaces.

Theorem 4.1 amounts to a non-uniform sampling theorem for the transform $V_g^\pi f$. For nilpotent Lie groups one can choose the set Λ to be a lattice or a quasi-lattice. We use the following terminology: a set $\Lambda \subseteq G$ is called a quasi-lattice of G with relatively compact fundamental domain K, if $G = \bigcup_{\lambda \in \Lambda} \lambda K = G$ and $\lambda K \cap \lambda' K = \emptyset$ for $\lambda \neq \lambda'$. If, in addition, Λ is a subgroup, then Λ is called a lattice of G. (Note subtle differences of terminology in the literature!). Quasi-lattices exist in every simply connected nilpotent Lie group [32], whereas the existence of a lattice requires a rational structure of the Lie algebra of G [17].

For nilpotent groups we can add more structure in Theorem 4.1.

Proposition 4.2. *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G that is square-integrable modulo center and let $g \in \mathcal{H}$ be a nice vector. Then there exists a quasi-lattice $\tilde{\Lambda} \subseteq G/Z$, such that $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a frame for $\mathcal{H}$.*

Proof. In view of Theorem 4.1 we only need the existence of sufficiently fine quasi-lattices. This is essentially proved in [41, Lemma 3.9]. In brief, fix a strong Malcev basis $X_1, \ldots X_n$ of $\mathfrak{g}/\mathfrak{z}$. Given a neighborhood $U \subseteq G/Z$, choose $\epsilon > 0$,

such that $K = \{e^{t_1 X_1} \ldots e^{t_n X_n} : -\epsilon/2 \leq t_j < \epsilon/2\}$ is contained in U. Then the set $\Gamma = \{e^{k_n \epsilon X_n} \ldots e^{k_1 \epsilon X_1} : k_n \in \mathbb{Z}\}$ is a quasi-lattice with fundamental domain K. The proof by induction is identical to [41]. ∎

This qualitative result can be complemented by a necessary density condition in the style of Landau [43]. As an appropriate metric on G/Z we choose a word metric: fix a symmetric neighborhood $W = W^{-1}$ of e in G/Z and let

$$d(x,y) = \min\{n \in \mathbb{N} : y^{-1}x \in W^n\} \text{ for } x \neq y.$$

Denoting the balls with respect to this metric by

$$\bar{B}(x,r) = \{y \in G/Z : d(x,y) \leq r\} = x\bar{B}(e,r),$$

the lower Beurling density of a set $\Lambda \subseteq G/Z$ is given by

$$D^-(\tilde{\Lambda}) = \liminf_{r \to \infty} \inf_{x \in G/Z} \frac{\#\tilde{\Lambda} \cap \bar{B}(x,r)}{|\bar{B}(x,r)|}.$$

As in $\mathbb{R}^d$ the density $D^-(\tilde{\Lambda})$ is the average number of points per unit volume. For coherent frames with respect to a square-integrable irreducible representation the following density result was proved in [33].

Theorem 4.3 (Necessary density condition). *Let G be a nilpotent Lie group and $\pi \in SI(G/Z)$ be a square-integrable representation of G modulo the center with formal dimension d_π. Let $g \in \mathcal{H}$ be a nice vector and $\tilde{\Lambda} \subseteq G/Z$. If $\mathcal{G}(g, \Lambda)$ is a frame for $\mathcal{H}$, then $D^-(\tilde{\Lambda}) \geq d_\pi$*

For nilpotent Lie group one can even prove the existence of orthonormal bases in the orbit of every square-integrable representation modulo the center.

Theorem 4.4. *Let G be a nilpotent Lie group with one-dimensional center and $(\pi, \mathcal{H})$ be a square-integrable irreducible unitary representation modulo center. Then there exists a discrete set $\Lambda \subseteq G$ and a function $g \in \mathcal{H}$, such that $\mathcal{G}(g, \Lambda)$ is an orthonormal basis for $\mathcal{H}$.*

The first examples of Theorem 4.4 were derived in [42] based on Nielsen's list [46], with more examples found in [47, 48]. Under the additional assumption of a gradation on G, Theorem 4.4 was proved in [41]. The breakthrough to full generality was recently achieved by V. Oussa [49].

Theorem 4.1 possesses a version for coorbit spaces in [23] as follows:

Theorem 4.5 (Banach frames). *Let g be a nice vector in $Co_\pi L_v^1(G)$, e.g., $g \in \mathcal{S}(G/Z)$. Assume that $1 \leq p \leq \infty$ and that m is v-moderate. Then there exists a neighborhood U in G/Z with the following property: If $\tilde{\Lambda} \subseteq G/Z$ is U-dense and relatively separated and $\Lambda \subseteq G$ is a section of $\tilde{\Lambda}$, then there exists a dual frame $\{e_\lambda : \lambda \in \Lambda\}$ in $Co_\pi L_v^1(G/Z)$, such that for all $f \in Co_\pi L_m^p(G/Z)$*

$$f = \sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle \pi(\lambda) g = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle e_\lambda \tag{37}$$

$$\|(\langle f, \pi(\lambda) g \rangle)_{\lambda \in \Lambda}\|_{\ell_m^p} \asymp \|(\langle f, e_\lambda \rangle)_{\lambda \in \Lambda}\|_{\ell_m^p} \asymp \|f\|_{Co_\pi L_m^p}. \tag{38}$$

The convergence in (37) is in the norm of $Co_\pi L_m^p$ for $p < \infty$ and in $\sigma(Co_\pi L_{1/v}^\infty, Co_\pi L_v^1)$ for $p = \infty$.

Remark 4.6. (1) If Λ is a lattice in G/Z, then there exists a dual window $\gamma \in Co_\pi L_v^1$, such that the dual frame is given by $\{\pi(\lambda)\gamma : \lambda \in \Lambda\}$. The proof is the same as for the Schrödinger representation of the Heisenberg group.

(2) More information about the dual $\{e_\lambda\}$ is derived in [40, 51].

Appendix

For quick reference we provide the computation of the short-time Fourier transform of chirps, as it can be found, for instance in [14]. Let B, C two *real*-valued symmetric $d \times d$-matrices with B positive semi-definite and $A = B + iC$. Then $A^T = A$ as well. The associated Gaussian is $\phi_A(x) = e^{-\pi Ax \cdot x}$, where $x \cdot y = \sum_j x_j y_j$ is the inner product on $\mathbb{R}^d$. The Fourier transform of ϕ_A is given as

$$\widehat{\phi_A}(\xi) = (\det A)^{-1/2} e^{-\pi A^{-1}\xi \cdot \xi}, \qquad \xi \in \mathbb{R}^d. \tag{39}$$

This formula holds for $\xi \in \mathbb{R}^d$ and real-valued positive-definite A. By analytic continuation (39) extends to $\xi \in \mathbb{C}^d$ and complex-valued matrices with the understanding that the branch of the square-root is determined by the requirement that $(\det A)^{-1/2} > 0$ for real-valued positive-definite A. See [28].

Now let $\mathcal{N}_C f(t) = e^{-\pi i Ct \cdot t} f(t), t \in \mathbb{R}^d$, be the operator of multiplication by the chirp $e^{-\pi i Ct \cdot t}$ (with $C^T = C$). We compute the modulus of the short-time Fourier transform with respect to the Gaussian $\phi = \phi_I$.

$$S_\phi(\mathcal{N}_C\phi)(x, \xi) = \int_{\mathbb{R}^d} e^{-\pi i Ct \cdot t} e^{-\pi|t|^2} e^{-\pi|t-x|^2} e^{-2\pi i\xi \cdot t} dt$$

$$= e^{-\pi|x|^2} \int_{\mathbb{R}^d} e^{-\pi(2I+iC)t \cdot t} e^{-2\pi i t \cdot (\xi+ix)} dt$$

$$= \det(2I + iC)^{-1/2} e^{-\pi|x|^2} e^{-\pi(2I(\xi+ix) \cdot (\xi+ix)}. \tag{40}$$

The determinant is

$$|\det(2I + iC)| = |\det(2I + iC)\det(2I - iC)|^{1/2} = \det(4I + C^2)^{1/2}.$$

Writing $D = (4I + C^2)^{-1}$ and $(2I + iC)^{-1} = (4I + C^2)^{-1}(2I - iC) = 2D - iDC$, we find after some algebraic manipulations that the real part of the exponent in (40) is given by

$$\mathrm{Re}\, D(2I - iC)(\xi + ix) \cdot (\xi + ix) + x \cdot x = 2D\xi \cdot \xi + (I - 2D)x \cdot x + 2DC\xi \cdot x.$$

This is again a quadratic form, now on $\mathbb{R}^{4d}$, and for its description we use the following abbreviations: $z = (\xi, x) \in \mathbb{R}^{4d}$ and

$$\Delta = \begin{pmatrix} 2D & DC \\ DC & I - 2D \end{pmatrix}.$$

With this notation the modulus of the short-time Fourier transform is

$$|S_\phi(\mathcal{N}_C\phi)(x, \xi)| = \det(4I + C^2)^{-1/4} e^{-\pi\Delta z \cdot z}, \tag{41}$$

and its p-norm is therefore

$$\|\mathcal{N}_C\phi\|_{M^p} = \|S_\phi(\mathcal{N}_C\phi)\|_{L^p(\mathbb{R}^{4d})} = \det(4I + C^2)^{-1/4}(\det p\Delta)^{-\frac{1}{2p}}, \tag{42}$$

where the last identity is obtained from (39) applied to the Gaussian $e^{-\pi p\Delta z \cdot z}$ at $z = 0$.

To compute the determinant of Δ we use the (block) factorization

$$\Delta \begin{pmatrix} 0 & C \\ I & -2\mathrm{I} \end{pmatrix} = \begin{pmatrix} DC & 0 \\ \mathrm{I} - 2D & -\mathrm{I} \end{pmatrix}$$

As the determinant of a matrix with a 0-block is easy to compute, we obtain

$$\det \Delta \, \det C \, \det(-\mathrm{I}) = \det(DC) \, \det(-\mathrm{I}) \,,$$

and this implies that
$$\det \Delta = \det D = \det(4\mathrm{I} + C^2)^{-1} \,. \tag{43}$$

This derivation is rigorous for invertible C and extends to arbitrary C by continuity. See [28], Appendix. The final result is therefore

$$\|S_\phi(\mathcal{N}_C \phi)\|_{L^p(\mathbb{R}^{2d})} = (p^{-\frac{1}{2p}})^{4d} \det(4\mathrm{I} + C^2)^{\frac{1}{2p}-\frac{1}{4}} \,. \tag{44}$$

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