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Affine Schur Duality

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Abstract. The Schur duality may be viewed as the study of the commuting actions of the symmetric group S_d and the general linear group $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{E}^{\otimes d}$ where $\mathbb{E}=\mathbb{C}^n$. Here we extend this duality to the context of the affine Weyl (or symmetric) group $\mathbb{Z}^d \rtimes S_d$ and the affine Lie (or Kac-Moody) algebra $\widetilde{\mathfrak{g}}=\mathcal{L}\mathfrak{g}\oplus\mathbb{C}c$, $\mathfrak{g}=\mathrm{sl}_n(\mathbb{C})$. Thus we construct a functor $\mathcal{F}: M\mapsto M\otimes_{S_d}\mathbb{E}^{\otimes d}$ from the category of finite dimensional $\mathbb{C}[\mathbb{Z}^d\rtimes S_d]$-modules M to that of finite dimensional $\widetilde{\mathfrak{g}}$-modules W of level 0 (the center $\mathbb{C}c$ of $\widetilde{\mathfrak{g}}$ acts as zero, thus these are representations of the loop group $\mathcal{L}\mathfrak{g}=\mathcal{L}\otimes_{\mathbb{C}}\mathfrak{g}$, where $\mathcal{L}=\mathbb{C}[t,t^{-1}]$, $\mathfrak{g}=\mathrm{sl}_n(\mathbb{C})$), the irreducible constituents of whose restriction to $\mathfrak{g}$ are subrepresentations of $\mathbb{E}^{\otimes d}$. When $d<n$ it is an equivalence of categories, but not for $d=n$, in contrast to the classical case. As an application we conclude that all irreducible finite dimensional representations of $\mathcal{L}\mathfrak{g}$, the irreducible constituents of whose restriction to $\mathfrak{g}$ are subquotients of $\mathbb{E}^{\otimes d}$, are tensor products of evaluation representations at distinct points of $\mathbb{C}^\times$.

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1. Introduction, and statement of result

An initial form of the Schur duality – promoted by Weyl in his book [30] – is the study of the commuting actions of the symmetric group S_d and the general linear group $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{E}^{\otimes d}$ where $\mathbb{E}=\mathbb{C}^n$ is the standard, or defining, representation of $\mathrm{GL}(n,\mathbb{C})$. Here we prove a natural extension of this duality from the context of the finite S_d and finite dimensional $\mathrm{GL}(n,\mathbb{C})$ to the context of the infinite, affine Weyl (or symmetric) group $\mathbb{Z}^d\rtimes S_d$ and the infinite dimensional, affine Lie (or Kac-Moody) algebra $\widetilde{\mathfrak{g}}=\mathcal{L}\mathfrak{g}\oplus\mathbb{C}c$, $\mathfrak{g}=\mathrm{sl}_n(\mathbb{C})$. Thus we construct a functor $\mathcal{F}: M\mapsto M\otimes_{S_d}\mathbb{E}^{\otimes d}$ from the category of finite dimensional $\mathbb{C}[\mathbb{Z}^d\rtimes S_d]$-modules M to that of finite dimensional $\mathbb{E}^{\otimes d}$-compatible $\widetilde{\mathfrak{g}}$-modules of level 0 (the center $\mathbb{C}c$ of $\widetilde{\mathfrak{g}}$ acts as zero, thus these are representations of the loop group $\mathcal{L}\mathfrak{g}=\mathcal{L}\otimes_{\mathbb{C}}\mathfrak{g}$, where $\mathcal{L}=\mathbb{C}[t,t^{-1}]$, $\mathfrak{g}=\mathrm{sl}_n(\mathbb{C})$). A finite dimensional $\mathcal{L}\mathfrak{g}$- or $\widetilde{\mathfrak{g}}$-module is called $\mathbb{E}^{\otimes d}$-*compatible* if the irreducible subquotients of its restriction to $\mathfrak{g}$ are isomorphic to subrepresentations of $\mathbb{E}^{\otimes d}$. When $d<n$ it is an equivalence of categories, but not when $d=n$. As an application we conclude that all irreducible finite dimensional $\mathbb{E}^{\otimes d}$-compatible representations of $\mathcal{L}\mathfrak{g}$ are tensor products of evaluation representations at distinct points of $\mathbb{C}^\times$.

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A *quantum* extension of the Schur duality had been given by Drinfeld [7] and Jimbo [16] in 1985, to the context of the finite Iwahori-Hecke algebra $H_d(q^2)$ and quantum algebras $\mathfrak{U}_q(\mathrm{gl}(n))$, on using universal R-matrices, which solve the Yang-Baxter equation. This was extended by Chari and Pressley to the *affine* quantum case [6]. The Schur duality was extended in the *super* direction by Sergeev [26] and later by Berele and Regev [2], and by Moon [22] and later by Mitsuhashi [23] – apparently unaware of [26] and of each other – to the *quantum super* case. This chain of works culminated in the recent [11] where we complete the cube, dealing with the general *affine super quantum* case. Here we relate the commuting actions of the affine Iwahori-Hecke algebra $H_d^a(q^2)$, and of the affine quantum Lie superalgebra $\mathfrak{U}_{q,a}^\sigma(\mathrm{sl}(m|n))$ using its presentation by Yamane [31] in terms of generators and relations, both acting on the dth tensor power of the superspace $\widetilde{\mathbb{E}} = \mathbb{C}^{m|n}$. Thus we construct there a functor and show it is an equivalence of categories of $H_d^a(q^2)$ and $\mathfrak{U}_{q,a}^\sigma(\mathrm{sl}(m|n))$-modules when $d < n' = m + n$. However, the non-, or pre-quantum affine case has not yet been proven, nor correctly stated. The aim of this work is to fill in this lacuna, as perhaps this initial affine case is the most important, and needed for extensions and applications. The quantum works mentioned above are deformations: the works of Drinfeld and Jimbo are deformations of the original Schur duality, the work of Chari-Pressley is a quantum deformation of the work here.

In more detail, the diagonal action of $\mathrm{GL}(n, \mathbb{C})$ on the $\mathbb{C}$-vector space $\mathbb{E}^{\otimes d}$, where $\mathbb{E} = \mathbb{C}^n$ is the standard $\mathrm{GL}(n, \mathbb{C})$-module and n, d are positive integers, commutes with the action of the symmetric group S_d on d letters by permutation of the factors. The *Schur duality* (Schur [24], [25], Weyl [30]; Fulton-Harris [13, §6.1, §15.3, §15.5], Green [14, (2.6c)], Etingof [8, 5.19]) decomposes the $S_d \times \mathrm{GL}(n, \mathbb{C})$-module $\mathbb{E}^{\otimes d}$ as a direct sum over all partitions $\lambda \vdash d$ of d of length $\ell(\lambda)$ at most n, of the tensor products $S^\lambda \otimes V^\lambda$. Here S^λ is the irreducible "Specht" representation of S_d associated with the partition λ (James [15], [13]); to be precise, it is $\mathbb{C}[S_d]c_\lambda$ of [13, Theorem 4.3], where it is denoted by V_λ, but perhaps S^λ is preferable, alluding to S_d and Specht. Also $\mathbb{E}$ is used here for $\mathbb{C}^n$ as V has many other uses. Further, V^λ is the irreducible $\mathrm{GL}(n, \mathbb{C})$-module of highest weight λ, the image of the Young symmetrizer $c_\lambda \in \mathbb{C}[S_d]$ on $\mathbb{E}^{\otimes d}$, thus the image of $\mathbb{E}$ under the Schur functor $\mathbb{S}_\lambda \mathbb{E} = \mathrm{Im}(c_\lambda | \mathbb{E}^{\otimes d})$ ([13, p. 76, section 6.1]). A reference to the decomposition formula $\mathbb{E}^{\otimes d} = \oplus_{\{\lambda \vdash d;\, \ell(\lambda) \leq n\}} S^\lambda \otimes V^\lambda$ is [13, Ex. 6.30, p. 87].

A *partition* λ of d (notation: $\lambda \vdash d$) is a weakly decreasing sequence $(\lambda_1, \dots, \lambda_n)$ of non-negative integers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, called *parts*, whose sum $\sum_{i \geq 1} \lambda_i$ is d; its *length* $\ell(\lambda)$ is the number of nonzero parts. The center $Z(\mathbb{C}) \simeq \mathbb{C}^\times$ of $\mathrm{GL}(n, \mathbb{C})$ acts on $\mathbb{E}$ by multiplication by $z \in Z(\mathbb{C})$, and $z \in Z(\mathbb{C})$ acts on $\mathbb{E}^{\otimes d}$ by multiplication by z^d. Also on V^λ, $\lambda \vdash d$, z acts by multiplication by $z^{\sum_i \lambda_i}$ $(1 \leq i \leq n)$, $= z^d$.

The Schur decomposition of $\mathbb{E}^{\otimes d}$ establishes a bijection between two sets of representations. The V^λ are precisely all the irreducible representations of $\mathrm{Aut}(\mathbb{E}) = \mathrm{GL}(n, \mathbb{C})$ with highest weight λ with $\lambda \vdash d$ and $\ell(\lambda) \leq n = n(\mathbb{E})$. These are the irreducible polynomial representations ρ of $\mathrm{GL}(n, \mathbb{C})$ that are homogeneous of degree d, meaning that the coefficients of $\rho(g)$ are homogeneous polynomials of degree d in the matrix coefficients of g.

More precisely, let F be an infinite field. Denote by $X_{i,j} : \mathrm{GL}(n,F) \to F$ the coordinate function that sends a matrix $g = (g_{i',j'})$ to its coefficient $g_{i,j}$ $(1 \le i,\ j \le n)$. The algebra $F[X_{i,j}] = F[X_{i,j};\ 1 \le i,\ j \le n]$ is canonically isomorphic to the polynomial ring in n^2 variables, as F is infinite.

A finite dimensional F-representation (ρ, W) of $\mathrm{GL}(n,F)$ is called *polynomial* if there is a basis $w_1, \dots, w_m$ of W such that the functions $f_{r,s} : \mathrm{GL}(n,F) \to F$ defined by $\rho(g)w_s = \sum_r f_{r,s}(g)w_r$ $(g \in \mathrm{GL}(n,F),\ 1 \le r,\ s \le m)$ lie in the polynomial algebra $F[X_{i,j}]$. If W is polynomial in one basis, it is polynomial in all bases. Alternatively, a finite dimensional F-representation (ρ, W) of $\mathrm{GL}(n,F)$ is called polynomial if it extends to a representation $(M_n(F), \cdot) \to (\mathrm{End}\, W, \cdot)$ of the algebraic monoid M_n of $n \times n$ matrices. It is called *homogeneous of degree* d if the $f_{r,s}$ are such. Thus a representation W of $\mathrm{GL}(n,F)$ is polynomial means that the action of each $g \in \mathrm{GL}(n,F)$ on W is given by a fixed family of polynomials in the coefficients of g.

Denote by $\mathrm{cf}(W) \subset F[X_{i,j}]$ the linear span of the coefficient functions $f_{r,s}$ of W. Note that if W' is a subquotient of W then $\mathrm{cf}(W') \subset \mathrm{cf}(W)$, and that $\mathrm{cf}(\mathbb{E}^{\otimes d})$ is the space of homogeneous $f \in F[X_{i,j}]$ of degree d, where $\mathbb{E} = F^n$ is the standard $\mathrm{GL}(n,F)$-module. Thus the representation $\mathbb{E}^{\otimes d}$ of $\mathrm{GL}(n,F)$, and each of its constituents, is homogeneous of degree d, and each irreducible polynomial representation of $\mathrm{GL}(n,F)$ homogeneous of degree d is a constituent of $\mathbb{E}^{\otimes d}$. For completeness, in section 14 we recall a geometric definition of polynomial representations.

Note that the highest weight of $V^\lambda \otimes \det^m$ is $\lambda + mI_n = (\lambda_1 + m, \dots, \lambda_n + m)$. When $\lambda_n + m < 0$ then $V^\lambda \otimes \det^m$ is not polynomial (in the matrix elements). Thus every irreducible polynomial representation of $\mathrm{GL}(n,\mathbb{C})$ that has highest weight λ that is a partition, say of d, with $\ell(\lambda) \le n$, occurs in $\mathbb{E}^{\otimes d}$, and corresponds to the irreducible representation S^λ of S_d associated with the partition $\lambda \vdash d$ of d; but not all S^λ are obtained: only those with length $\ell(\lambda) \le n$ occur.

If $d \le n$ then $\lambda \vdash d$ implies $\ell(\lambda) \le n$, and all irreducible representations of S_d occur in $\mathbb{E}^{\otimes d}$. In this case the Schur ($\mathcal{S}$-)duality asserts then: the functor $\mathcal{S}$ *from* the category $\mathrm{Rep}\,\mathbb{C}[S_d]$ of finite dimensional $\mathbb{C}[S_d]$-modules, *to* the category $\mathrm{Rep}(\mathrm{GL}(n,\mathbb{C}), d)$ of finite dimensional polynomial homogeneous of degree d representations of $\mathrm{GL}(n,\mathbb{C})$, *defined by* $\mathcal{S}(M) = M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$ on objects and $\mathcal{S}(f) = f \otimes \mathrm{id}_{\mathbb{E}^{\otimes d}}$ on morphisms, *is an equivalence of categories.* In particular $\mathcal{S}$ takes S^λ to $S^\lambda \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d} = V^\lambda$ since for any G-modules V, W we have $V' \otimes_G W = \mathrm{Hom}_G(V, W)$, and S^λ is self-dual in characteristic 0 (only!; [15, Theorems 4.12, 6.7, 8.15, pages 16, 25, 33]), thus $S^\lambda \otimes_{\mathbb{C}[S_d]} S^\lambda \simeq \mathbb{C}$.

Dualities have the attraction of connecting seemingly unrelated areas, to the benefit of both. Since Young showed that the category of finite dimensional $\mathbb{C}[S_d]$-modules was semisimple, Schur deduced the same for the category of finite dimensional polynomial homogeneous of degree d representations of $\mathrm{GL}(n,\mathbb{C})$.

Now the restriction of V^λ to $\mathrm{SL}(n,\mathbb{C})$ remains irreducible, and V^λ is equivalent to V^μ as an $\mathrm{SL}(n,\mathbb{C})$-module if $\lambda - \mu$ is a $\mathbb{Z}$-multiple of I_n. A highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is called here ≥ 0 if it is a partition, thus $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, and minimal if it is ≥ 0 and $\sum_{1 \le i \le n} \lambda_i$ is minimal ≥ 0. Given an irreducible $\mathrm{SL}(n,\mathbb{C})$-module that occurs in, thus is a subquotient of, $\mathbb{E}^{\otimes d}$, it is of the form V^λ,

$\ell(\lambda) \leq n$, and when we choose λ minimal ≥ 0, then $\lambda \vdash d$. We extend this V^λ to a $\mathrm{GL}(n,\mathbb{C})$-module by letting $z \in Z(\mathbb{C})$ act by multiplication by z^d. Thus the equivalence can be stated to relate the category of finite dimensional $\mathbb{C}[S_d]$-modules, to the category of finite dimensional $\mathbb{E}^{\otimes d}$-compatible representations of $\mathrm{SL}(n,\mathbb{C})$.

An equivalent form of the $\mathcal{S}$-duality is in terms of the Lie algebra $\mathfrak{g} = \mathrm{gl}_n = \mathrm{gl}_n(\mathbb{C})$ of $\mathrm{GL}(n,\mathbb{C})$. If we write $g = I + \varepsilon X$ where we take $X \in \mathfrak{g}$, then the action $g : v = u_1 \otimes \cdots \otimes u_d \mapsto gu_1 \otimes \cdots \otimes gu_d$ becomes

$$\frac{1}{\varepsilon}((I+\varepsilon X)u_1 \otimes \cdots \otimes (I+\varepsilon X)u_d - u_1 \otimes \cdots \otimes u_d) = \Delta(X)v + O(\varepsilon),$$

where
$$\Delta(X) := \Delta_d(X) = \sum_{1\leq j\leq d} 1^{\otimes(j-1)} \otimes X \otimes 1^{\otimes(d-j)}.$$

As $\varepsilon \to 0$ we get that $\mathbb{E}^{\otimes d}$ is a $\mathbb{C}[S_d] \times \mathfrak{g}$-module. Note that $\Delta(zX) = z\Delta(X)$, $z \in \mathbb{C}$.

In fact the $\mathcal{S}$-duality is sometimes proven in this Lie algebra form, and the group form is deduced from that; see section 10. In this case the Schur duality asserts then: *when* $d \leq n$ the functor $\mathcal{S}$ *from* the category $\mathrm{Rep}\,\mathbb{C}[S_d]$ of finite dimensional $\mathbb{C}[S_d]$-modules, *to* the category $\mathrm{Rep}(\mathrm{gl}_n, d)$ of finite dimensional representations of the Lie algebra gl_n whose irreducible subquotients have highest weight a partition λ of d with at most n parts, *defined by* $\mathcal{S}(M) = M \otimes_{S_d} \mathbb{E}^{\otimes d}$, where $X \in \mathrm{gl}_n$ acts via Δ, *is an equivalence of categories.*

Note also that $\mathrm{gl}_n(\mathbb{C})$ is a reductive Lie algebra. Thus its radical (largest solvable ideal) equals its center. So it is isomorphic to the direct sum $\mathrm{sl}_n(\mathbb{C}) \oplus \mathbb{C}$ of its semisimple ideal $\mathrm{sl}_n(\mathbb{C})$ (its derived Lie algebra $[\mathrm{gl}_n(\mathbb{C}), \mathrm{gl}_n(\mathbb{C})]$, which is even simple), and its center $\mathbb{C}$. An element z in the center $\mathbb{C}$ of $\mathrm{gl}_n(\mathbb{C})$ acts on $\mathbb{E}^{\otimes d}$ by multiplication by z. As the categories of finite dimensional representations over $\mathbb{C}$ of $\mathrm{sl}_n(\mathbb{C})$ and of $\mathrm{SL}(n,\mathbb{C})$ are equivalent, the duality can be stated to assert: *when* $d \leq n$ the functor $\mathcal{S}$ *from* the category $\mathrm{Rep}\,\mathbb{C}[S_d]$ of finite dimensional $\mathbb{C}[S_d]$-modules, *to* the category $\mathrm{Rep}(\mathrm{sl}_n, d)$ of finite dimensional representations of the Lie algebra sl_n whose constituents have highest weight a partition λ (mod I_n) of d with at most n parts, *is an equivalence of categories.* Namely in the class mod I_n of partitions with at most n parts there is a representative $\lambda \vdash d$.

Our aim here is to extend this duality to the affine case. The symmetric group S_d is the Weyl group of the group $\mathrm{GL}(d)$; it will be extended to the affine Weyl group S_d^{aff}, which is the semidirect product $\mathbb{Z}^d \rtimes S_d$, where S_d acts on the lattice $\mathbb{Z}^d$ by permutation. Denote the d generators $(0,\dots,0,1,0,\dots,0)$ of $\mathbb{Z}^d$ by $y_1, \dots, y_d$, so that $\mathbb{C}[\mathbb{Z}^d] \simeq \mathbb{C}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]$. The Lie algebra $\mathfrak{g}$ will be taken to be the semisimple $\mathrm{sl}_n(\mathbb{C})$; it will be extended to the affine Lie (or Kac-Moody) algebra $\widetilde{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$, a central extension by $\mathbb{C}c$ of the loop algebra $\mathcal{L}\mathfrak{g} = \mathcal{L} \otimes_{\mathbb{C}} \mathfrak{g}$, where $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t. This $\widetilde{\mathfrak{g}}$ has a presentation in terms of generators E_i, F_i, H_i $(0 \leq i < n)$ and relations, where deleting E_0, F_0, H_0 would give a presentation of the Lie algebra $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$ by e_i, f_i $(1 \leq i < n)$, $E_i = 1 \otimes e_i$, $F_i = 1 \otimes f_i$. This is the reason why we work with the semisimple sl, rather than with gl. We shall first state our (affine) extension of the $\mathcal{S}$ duality to the affine case, then explain in more detail what is the affine Lie (or Kac-Moody) algebra $\widetilde{\mathfrak{g}}$ (we shall not need $\hat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathbb{C}d$). By Schur's Lemma, the central element c acts on an irreducible representation as multiplication by a scalar $k \in \mathbb{C}$, that is as kI. In this case the representation is said to be of *level* k. We denote the action of $\mathfrak{g}$ on $\mathbb{E} = \mathbb{C}^n$ by ρ, and on $\mathbb{E}^{\otimes d}$ by $\rho_d = \rho \otimes \cdots \otimes \rho$.

Denote by $\operatorname{Rep}\mathbb{C}[\mathbb{Z}^d \rtimes S_d]$ the category of finite dimensional $\mathbb{C}[\mathbb{Z}^d \rtimes S_d]$-modules. Denote by $\operatorname{Rep}(\mathcal{L}\mathfrak{g}; d)$ the category of finite dimensional $\mathcal{L}\mathfrak{g}$-modules, namely $\widetilde{\mathfrak{g}}$-modules of level zero (the central element c acts as 0) whose constituents as $\mathfrak{g}$-modules have highest weight $\lambda \vdash d$ with $\ell(\lambda) \leq n \bmod I_n$ (that is, each of its irreducible subquotients as a $\mathfrak{g}$-module is a constituent of $\mathbb{E}^{\otimes d}$).

Theorem 1.1. *There exists a functor $\mathcal{F}$ from the category $\operatorname{Rep}\mathbb{C}[\mathbb{Z}^d \rtimes S_d]$ to the category $\operatorname{Rep}(\mathcal{L}\mathfrak{g}; d)$, $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$, defined as follows. If M is a $\mathbb{C}[\mathbb{Z}^d \rtimes S_d]$-module, then $\mathcal{F}(M)$ as a $\mathfrak{g}$-module is $\mathcal{S}(M) = M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$ by the $\mathcal{S}$-correspondence, with $x \in \mathfrak{g}$ acting via*

$$\rho_d(x) \cdot m \otimes v = m \otimes \rho_d(\Delta(x))v = m \otimes \sum_{1 \leq j \leq d} (1^{\otimes(j-1)} \otimes \rho(x) \otimes 1^{\otimes(d-j)})v.$$

The action of the remaining generators of $\widetilde{\mathfrak{g}}$,

$$E_0 = t \otimes e_0, \qquad F_0 = t^{-1} \otimes f_0, \qquad H_0 = [E_0, F_0] = [e_0, f_0] - (E_0, F_0)c$$

is given by

$$\rho_d(E_0) \cdot m \otimes v = \sum_{1 \leq j \leq d} m y_j \otimes (1^{\otimes(j-1)} \otimes \rho(e_0) \otimes 1^{\otimes(d-j)})v,$$

$$\rho_d(F_0) \cdot m \otimes v = \sum_{1 \leq j \leq d} m y_j^{-1} \otimes (1^{\otimes(j-1)} \otimes \rho(f_0) \otimes 1^{\otimes(d-j)})v.$$

When $0 \leq d < n$ the functor $\mathcal{F}$ is an equivalence of categories, but not when $d = n$.

The condition $d < n$ in the affine case, rather than $d \leq n$ of the finite dimensional case, is used in the proof, and it is optimal: $\mathcal{F}$ is not an equivalence when $n = d$. In contrast to the cases of the finite group S_d and the finite dimensional Lie algebra $\mathfrak{g}$, where the categories of finite dimensional representations are semisimple, in the infinite dimensional affine cases the categories of finite dimensional representation that appear in the theorem have nontrivial extensions. This is used in section 15 to show that when $d = n$, and λ is the partition $\omega = (1, \ldots, 1)$ of d, the $\mathfrak{U}(\mathcal{L}\mathfrak{g})$-module $V^\omega = \mathbb{C}$ (on which t acts as $a \in \mathbb{C}^\times$) has no nontrivial self extensions. Yet the corresponding $\mathbb{C}[\mathbb{Z}^d \rtimes S_d]$-module lifted from $S^\omega = \mathrm{sgn}$ has. So the theorem as stated for $d < n$ does not extend to $d = n$. But it might extend to $\overline{\mathfrak{g}} = \mathrm{gl}_n(\mathbb{C})$.

When $d = 0$ the category on the S_d-side is that of finite dimensional complex vector spaces, and the theorem asserts that there are no nontrivial extensions of $\mathcal{L}\mathfrak{g}$-modules lifted from the trivial $\mathfrak{g}$-module $\mathbb{C}$. By a $\mathfrak{g}$ or $\mathcal{L}\mathfrak{g}$-module we mean a module for their universal enveloping algebras $\mathfrak{U}(\mathfrak{g})$ and $\mathfrak{U}(\mathcal{L}\mathfrak{g})$.

When $d = 1$ an irreducible representation of $\mathbb{C}[S_d \ltimes \mathbb{Z}^d] = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[t, t^{-1}]$ is a $\mathbb{C}$-linear homomorphism $\chi : \mathbb{C}[t^{\pm 1}] \to \mathbb{C}$ determined by the value $\chi(t) \in \mathbb{C}^\times$ of χ at t, or at $1 \in \mathbb{Z}$. An $\mathbb{E}$-compatible irreducible representation of $\mathcal{L}\mathfrak{g} = \mathcal{L} \otimes \mathrm{sl}_n(\mathbb{C})$ (i.e., whose restriction to $\mathrm{sl}_n(\mathbb{C})$ is the standard representation ρ on $\mathbb{E} = \mathbb{C}^n$) is then of the form $\chi \otimes \rho$, where $\chi : \mathcal{L} \to \mathbb{C}$ is a $\mathbb{C}$-linear algebra homomorphism determined by the value $\chi(t) \in \mathbb{C}^\times$ (see Corollary 9.3). On irreducibles the correspondence defined by $\mathcal{F}$ is then $\chi \mapsto \chi \otimes \rho$. Both categories, of finite dimensional $\mathcal{L}$-modules, and of finite dimensional $\mathbb{E}$-compatible $\mathcal{L}\mathfrak{g}$-modules, are not semisimple, see Corollary 15.3.

We write y_j^{-1} for the inverse of $y_j \in \mathbb{Z}^d \subset S_d^{\mathrm{aff}}$ to emphasize that my_j^{-1} is not $-my_j$, but it is the m' in M such that $m'y_j = m$. A statement of an extension of the $\mathcal{S}$-duality to the affine Lie (or Kac-Moody) case is attempted at [6, §4.9]. Surprisingly, our proof that our functor $\mathcal{F}$ is an equivalence of the suitable categories, is not really simpler than that of [6] in the affine-quantum case, that we adapt to our situation. Also note that there is no "homomorphism $\widetilde{ev}_a^0 : S_\ell \ltimes \mathbb{Z}^\ell \to S_\ell$ that is the identity on S_ℓ and for which $\widetilde{ev}_a^0(z_j) = a$ for all j", contrary to [6, p. 314, l. 10-12], simply since $a \in \mathbb{C}^\times$ does not lie in S_ℓ. The group algebra should be used.

Put $Y = \mathbb{C}[y_1^{\pm 1}, \dots, y_d^{\pm 1}] = \Gamma(\mathbb{G}_m^d, \mathcal{O})$. Note that

$$\mathbb{C}[S_d \ltimes \mathbb{Z}^d] \simeq \mathbb{C}[S_d] \ltimes Y, \quad \text{and} \quad (\mathbb{E} \otimes \mathbb{C}[t, t^{-1}])^{\otimes d} \simeq \mathbb{E}^{\otimes d} \otimes Y.$$

An alternative phrasing of the functor $\mathcal{F}$, and of Theorem 1.1, extended geometrically in section 12, is given by the following clearer statement. It also suggests an extension to $\mathcal{L} \otimes \mathrm{gl}(n, \mathbb{C})$, although our proof is carried out here only for $\mathcal{L} \otimes \mathrm{sl}_n(\mathbb{C})$, and it is via the statement of Theorem 1.1.

Theorem 1.2. *There exists a functor $\mathcal{F}$ from the category* $\mathrm{Rep}\, \mathbb{C}[\mathbb{Z}^d \rtimes S_d]$ *to the category* $\mathrm{Rep}(\mathcal{L}\mathfrak{g}; d)$*,* $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$*, defined by*

$$M \mapsto M \otimes_{\mathbb{C}[S_d \ltimes \mathbb{Z}^d]} (\mathbb{E} \otimes \mathbb{C}[t, t^{-1}])^{\otimes d} = M \otimes_{\mathbb{C}[S_d] \ltimes Y} (\mathbb{E}^{\otimes d} \otimes Y),$$

with $x \otimes t^k$*,* $x \in \mathfrak{g}$*,* $k \in \mathbb{Z}$*, acting on* $M \otimes_{\mathbb{C}[S_d \ltimes \mathbb{Z}^d]} (\mathbb{E} \otimes \mathbb{C}[t, t^{-1}])^{\otimes d}$ *by*

$$m \otimes (v_1 \otimes \cdots \otimes v_d) \mapsto m \otimes \sum_{1 \le j \le d} v_1 \otimes \cdots \otimes v_{j-1} \otimes \rho(x) v_j y_j^k \otimes v_{j+1} \otimes \cdots \otimes v_d.$$

Here $m \in M$*,* $v = v_1 \otimes \cdots \otimes v_d \in (\mathbb{E} \otimes \mathbb{C}[t, t^{-1}])^{\otimes d}$.
When $0 \le d < n$ *the functor* $\mathcal{F}$ *is an equivalence of categories, but not when* $d = n$.

The statement of Theorem 1.2 extends to $\mathcal{L} \otimes \mathrm{gl}(n, \mathbb{C})$, but this is not discussed in this paper. If proven, using $\mathrm{GL}(n, \mathcal{L}) \subset \mathcal{L}\overline{\mathfrak{g}}$, we obtain an action of $\mathrm{GL}(n, \mathcal{L})$ on each finite dimensional $\mathcal{L}\overline{\mathfrak{g}}$-module. By restriction, an $\mathrm{SL}(n, \mathcal{L})$-action is obtained. As $\mathrm{GL}(n, \mathbb{C}) = \mathrm{SL}(n, \mathbb{C}) Z(\mathbb{C})$, where $Z(\mathbb{C})$ is the center $\{zI_n;\, z \in \mathbb{C}^\times\}$, an irreducible $\mathrm{SL}(n, \mathbb{C})$-module extends to a $\mathrm{GL}(n, \mathbb{C})$-module on choosing the action of the central $Z(\mathbb{C}) \simeq \mathbb{C}^\times$. Over the ring $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ we have $\mathrm{GL}(n, \mathcal{L}) = \mathrm{SL}(n, \mathcal{L}) \rtimes T$, $T = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & I' \end{smallmatrix} \right);\, a \in \mathbb{C}^\times t^{\mathbb{Z}} = \mathcal{L}^\times \right\}$, $I' = I_{n-1}$. An irreducible finite dimensional $\mathrm{SL}(n, \mathcal{L})$-module can be extended to $\mathrm{SL}(n, \mathcal{L}) Z(\mathcal{L})$ (where $Z(\mathcal{L}) \simeq \mathcal{L}^\times$), a subgroup of index n in $\mathrm{GL}(n, \mathcal{L})$, then induced to $\mathrm{GL}(n, \mathcal{L})$. A finite dimensional $\mathcal{L} \otimes \mathrm{sl}_n(\mathbb{C})$-module integrates to an $\mathrm{SL}(n, \mathcal{L})$-module, see section 13. In any case, Theorem 1.1 and its proof concern only $\mathcal{L} \otimes \mathrm{sl}_n(\mathbb{C})$, and it is stated in the form in which it is proven, that is, giving an equivalence of categories.

More generally, for a ring Y over $\mathbb{C}$ one may study $M \mapsto M \otimes_{\mathbb{C}[S_d] \ltimes Y} (\mathbb{E}^{\otimes d} \otimes Y)$ as a functor from the category of finite dimensional $\mathbb{C}[S_d] \ltimes Y$-modules to the category of finite dimensional $\mathfrak{g} \otimes Y$-modules compatible with $\mathbb{E}^{\otimes d}$, and ask whether it defines an equivalence of categories, and even state it geometrically.

An analogue of the Schur duality for current algebras, thus the Y above is replaced by $\mathbb{C}[y_1,\dots,y_d] = \Gamma(\mathbb{G}_a^d,\mathcal{O})$ (the y_j occur only with non-negative exponents), is deduced in [9] from a current analogue of the Howe duality. The latter is viewed as a version of a current analogue of the Peter-Weyl theorem for the space $M(n\times d,\mathbb{C})$ of matrices. The Peter-Weyl theorem for current groups $G(\mathbb{C}[[t]])$ (see [20, section 13.2]) is proven there on using global and local Weyl modules, and highest weight categories. This technique may extend to our case, but our proof is perhaps more elementary.

From numerous interesting works related to the subject matter we mention here [1], [4], [9], [19], [21].

We hope to report on applications of the theorem in a subsequent work.

On the affine case, [18] Kac opines: "It is a well kept secret that the theory of Kac-Moody algebras has been a disaster. ... However, there are two notable exceptions. The best known one is, of course, the theory of affine Kac-Moody algebras. This part of the Kac-Moody theory has deeply penetrated many branches of mathematics and physics. The most important single reason for this success is undoubtedly the isomorphism of affine algebras and central extensions of loop algebras, ... "

An analogue of the present work for affine super Lie algebras, of which our [11] is a quantum deformation, is given at [12]. The statement, settings and proof in the super case are considerably more involved. Uniting the present work in the affine case – that is the simplest infinite dimensional case – with its "super" extension, will lead to a work too complicated for a potential user of the present, affine case. This affine case anyway most likely to have most current applications, and is sufficiently rich to exhibit most key features.

We briefly review the theory of finite dimensional Lie algebras in section 2, and that of affine Lie algebras in section 3. The example of $\mathfrak{g} = \mathrm{sl}_n$ is detailed in section 4, except that in the proof we use n for the type $(A_n^{(1)})$ so n becomes $n+1$ in sections 5-7 where the theorem is proven. One has to show the functor is well defined, defines a representation of the affine Lie algebra, and defines an equivalence of categories. Section 8 shows the functor commutes with induction. Section 9 introduces evaluation representations, and concludes from the description (by Mackey theory) of the finite dimensional irreducible $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-modules:

Each finite dimensional irreducible $\mathbb{E}^{\otimes d}$-compatible representations of the loop algebra $\mathcal{L}\mathfrak{g}$ is a tensor product of evaluation representations $W_{p_i} = \mathrm{ev}_{p_i}^*(\mathcal{S}(M_i))$ at distinct points $p_1,\dots,p_k \in \mathbb{C}^\times$, where M_i are irreducible $\mathbb{C}[S_{d_i}]$-modules, and $d = d_1+\cdots+d_k$.

It is a natural question, that was in fact part of our motivation for the present work, and it might be needed for the applications we have in mind, to state the equivalence of categories in group theoretic terms, rather than Lie algebra language. In section 10 we recall how to translate the result from the context of Lie algebras to that of Lie groups, in the classical case of finite dimensional Lie algebras and groups.

In section 11 we find a set of Chevalley-type generators of the loop group $\mathrm{SL}(n,\mathcal{L})$, analogous to that of the loop algebra $\mathcal{L}\mathfrak{g}$. These generators are subject to Steinberg-like relations. Then we explain that the theorem cannot simply be translated from the context of affine Lie algebras $\mathcal{L}\mathfrak{g}$ to that of affine Lie groups $\mathrm{SL}(n,\mathcal{L})$,

$\mathcal{L} = \mathbb{C}[t, t^{-1}]$, on defining an exponential type isomorphism by means of these generators, the reason being that the image of such a map would not be polynomial, and not land in $\mathrm{SL}(n, \mathcal{L})$. I do not know to define an $\mathrm{SL}(n, \mathcal{L})$-action on $M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$ extending the diagonal $\mathrm{SL}(n, \mathbb{C})$-action on $\mathbb{E}^{\otimes d}$ by means of the extra generators.

In section 12, an algebro-geometric definition of the functor suggested to me by P. Deligne, that is group-theoretic, and purely in terms of finite dimensional schemes, is described. Taking the hint that the group algebra $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ is $\mathbb{C}[S_d] \ltimes \mathbb{C}[\mathbb{Z}^d]$ and $\mathbb{C}[\mathbb{Z}^d]$ is the ring $\Gamma(\mathbb{G}_m^d, \mathcal{O})$ of global sections of the torus $\mathbb{G}_m^d = \mathrm{Spec}\, \mathbb{C}[\mathbb{Z}^d]$, one notes that a finite dimensional $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-module M can be viewed as the module of global sections $\Gamma(\mathcal{M}, \mathcal{O})$ of an S_d-equivariant quasi-coherent sheaf of modules $\mathcal{M} = \pi^* M$ over $\mathbb{G}_m^d = \mathrm{Spec}\, \mathbb{C}[\mathbb{Z}^d]$, pulled-back from a point: $\pi : \mathbb{G}_m^d \to \{*\}$. The role of the affine algebra $\mathcal{L}\mathfrak{g}$ is replaced by the affine group $\mathrm{SL}(n, \mathcal{L})$. Suitably interpreted, the functor $\mathcal{F}$ takes the (modified to be a limit of subsheaves with finite support) form $M = \Gamma(\mathcal{M}, \mathcal{O}) \mapsto \Gamma((\mathcal{M} \otimes_{\mathbb{G}_m^d} (\mathbb{E}^{\otimes d} \otimes \mathbb{G}_m^d)), \mathcal{O})_{\mathbb{C}[S_d]}$.

In section 13, following correspondence with Shrawan Kumar, using [4, Lemma 1, §2.2], that asserts that an ideal I in $\mathcal{L}\mathfrak{g}$, in particular the kernel $I = I_W$ of a finite dimensional representation $\rho_W : \mathcal{L}\mathfrak{g} \to W$, has the form $I = \mathfrak{g} \otimes I'$, where I' is an ideal of the PID $\mathbb{C}[t, t^{-1}]$, we show: a finite dimensional representation of the loop algebra $\mathcal{L}(\mathrm{sl}_n)$ integrates to a representation of the loop group $\mathrm{SL}(n, \mathcal{L})$, $\mathcal{L} = \mathbb{C}[t, t^{-1}]$.

This permits restating Theorem 1.1 as asserting: the functor $\mathcal{F}$ is, for $d < n$, an equivalence of categories *between* the category of finite dimensional $\mathbb{C}[S_d^{\mathrm{aff}}]$-modules, *and* the category of finite dimensional representations of $G(\mathcal{L})$ all of whose subquotients as representations of $\mathrm{SL}(n, \mathbb{C})$ ($\subset \mathrm{SL}(\mathbb{C}[t, t^{-1}])$ via $\mathbb{C} \hookrightarrow \mathbb{C}[t, t^{-1}]$) occur in $\mathbb{E}^{\otimes d}$. As $\mathbb{E}^{\otimes d}$ is a semisimple $\mathrm{SL}(n, \mathbb{C})$-module, "occur in" means "are subrepresentations of", or "are subquotients of".

Note that for loop groups $G(\mathcal{L})$, the affine Weyl group S_d^{aff} of $\mathrm{GL}(d, \mathcal{L})$ is

$$N(T)/T(\mathbb{C}) \simeq N(T)/T(\mathcal{L}) \ltimes T(\mathcal{L})/T(\mathbb{C}) \simeq S_d \ltimes \mathbb{Z}^d,$$

where $N(T)$ is the normalizer of $T(\mathcal{L})$ in $\mathrm{GL}(d, \mathcal{L})$. The centralizer of $T(\mathcal{L})$ in $\mathrm{GL}(d, \mathcal{L})$ is itself, the group of diagonal matrices with diagonal entries in $\mathbb{C}^\times t^{\mathbb{Z}}$.

The functor in the opposite direction, from $\mathrm{GL}(n, \mathcal{L})$-modules to S_d^{aff}-modules, $d \le n$, is simpler. Denote by $e_1, \dots, e_n$ the canonical basis of the standard representation $\mathbb{E} = \mathbb{C}^n$ of $\mathrm{GL}(n, \mathbb{C})$. Embed $\mathrm{GL}(d, \mathbb{C})$ in $\mathrm{GL}(n, \mathbb{C})$ as the subgroup of automorphisms of the subspace $\mathbb{E}_d$ of $\mathbb{E}$ spanned by $\{e_1, \dots, e_d\}$, that fix the basis elements $e_{d+1}, \dots, e_n$. This way we view the affine Weyl group S_d^{aff} of $\mathrm{GL}(d, \mathcal{L})$ as a subgroup of $\mathrm{GL}(n, \mathcal{L})$. Let (ρ, W) be a finite dimensional $\mathrm{GL}(n, \mathbb{C})$-module. By a *weight* we mean $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}_{\ge 0}$, and the α-*weight space* of W is $W^\alpha = \{w \in W;\, \rho(d(a))w = a_1^{\alpha_1} \dots a_n^{\alpha_n} w$ for all $d(a) = \mathrm{diag}(a_1, \dots, a_n) \in T(\mathbb{C})\}$, $T(\mathbb{C})$ denotes the diagonal subgroup of $\mathrm{GL}(n, \mathbb{C})$. Then the weight space W^ω for $\omega = (1, \dots, 1, 0, \dots, 0) \in \mathbb{Z}^n$ (1 occurs d times) is an S_d-module, by restriction of ρ, since S_d fixes ω. Schur proved in his thesis ([24], see [14, section 6]) that: *when $d \le n$ the functor $W \mapsto W^\omega$ defines an equivalence from the category of finite dimensional $\mathrm{GL}(n, \mathbb{C})$-modules whose irreducible subquotients are subrepresentations of $\mathbb{E}^{\otimes d}$, to the category of finite dimensional $\mathbb{C}[S_d]$-modules. Moreover, this functor is an inverse to the functor $\mathcal{S} : M \mapsto M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$.* Now if we start with a finite dimensional $\mathrm{GL}(n, \mathcal{L})$-module (ρ, W), the weight space W^ω is invariant under the action of the affine Weyl group S_d^{aff}, since the diagonal subgroup $T_d(\mathcal{L})$ of $\mathrm{GL}(d,\mathcal{L})$ stabilizes W^ω. When $d < n$ we embed $\mathrm{GL}(d) \subset \mathrm{SL}(d+1) \subset \mathrm{SL}(n)$ via $g \mapsto (g, 1/\det(g)) \mapsto (g, 1/\det(g), 1, \dots, 1)$, in particular we get an action of S_d^{aff} on W^ω from the action of $\mathrm{SL}(n,\mathcal{L})$ on W. Then we conclude: *the functor* $\mathcal{G}: W \to W^\omega$ *is an equivalence from* the category of finite dimensional $\mathrm{SL}(n,\mathcal{L})$-modules whose restriction to $\mathrm{SL}(n,\mathbb{C})$ has the property that its irreducible subquotients are subrepresentations of $\mathbb{E}^{\otimes d}$, *to* the category of finite dimensional $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-modules, inverse to the functor $\mathcal{F}$.

From the perspective of p-adic groups (for an introduction to the Bruhat-Tits theory see [28]), the affine Weyl group W^{aff} of a simply connected and split, or more generally quasisplit unramified, linear algebraic group G over a local non-Archimedean field F, can be defined to be: the quotient of the normalizer $N_T(F)$ in $G(F)$ of a maximal split torus T over F in G, by the unique maximal compact subgroup M of the centralizer $T'(F)$ of T in $G(F)$. Here T' is the unique maximal torus over F in G containing T. "Unramified" means that T splits over a finite unramified field extension F' of F, in which case the quotient of $T(F)$ by its maximal compact subgroup is isomorphic to $T'(F)/M$. Then $W^{\mathrm{aff}} = N_T(F)/M$ is an extension of the Weyl group $W = N_T(F)/T'(F)$ by the lattice $T'(F)/M$; see also [10]. For example, let $G = \mathrm{SU}(n, F'/F)$ be the quasisplit special unitary group over F that splits over the quadratic field extension F' of F and is an outer form of $\mathrm{SL}(n)$, thus $G(F') = \mathrm{SL}(n, F')$. Denote by $n' = [n/2]$ the largest integer $\le n/2$ and by $\mathcal{O}_F^\times$ the group of units in the ring of integers of F. Than the quotient of $T(F)$ by its maximal compact subgroup is $(F^\times/\mathcal{O}_F^\times)^{n'}$, a subgroup of index ι in $T'(F)/M \simeq (F'^\times/\mathcal{O}_{F'}^\times)^{n'}$; here ι is $2^{n'}$ when F'/F is ramified (and 1 if F'/F is unramified). The analogue for loop groups is three paragraphs above.

For example when $G = \mathrm{GL}(d)$, T the diagonal subgroup, M the maximal compact subgroup $T(R)$ of $T(F)$ of diagonal elements whose diagonal coefficients are all in the group of units of the ring of integers R of F, we have: $W = S_d$ is the symmetric group on d letters, and the lattice $T(F)/M$ is $\mathbb{Z}^d$. So W^{aff} is the semidirect product $\mathbb{Z}^d \rtimes S_d$, where S_d acts on the lattice $\mathbb{Z}^d$ by permutations.

More interesting is the quantum case (see [6], and further the super case [11]), corresponding to that of the Iwahori-Hecke convolution algebra $C_c[I\backslash G(F)/I]$ of compactly supported I-biinvariant $\mathbb{C}$-valued measures on $G(F)$. Here I is an Iwahori subgroup (the pullback $\mathrm{pr}^{-1}(B(\mathbb{F}_q))$ of a Borel subgroup by the reduction modulo the maximal ideal M map $\mathrm{pr}: G(R) \to G(\mathbb{F}_q)$, where $\mathbb{F}_q = R/M$ is the residue field of the local ring R, a finite field of q elements).

2. Finite dimensional semisimple Lie algebras

We begin with a brief introduction to the terminology of affine Lie (or Kac-Moody, KM) algebras $\widetilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}$, to fix notation and explain what is the theorem about; we use only the case of sl, but a more general introduction would put this case in perspective. We follow [17], and work over the base field $\mathbb{C}$. The affine case is an extension of the finite dimensional case of semisimple Lie algebras, that we review first.

A *Lie algebra* (LA) $\mathfrak{g}$ is a vector space with a bilinear map $[.,.]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called *bracket*, satisfying for x, y, $z \in \mathfrak{g}$, *antisymmetry*: $[y,x] = -[x,y]$, and the

Jacobi identity: $$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

For example, an algebra A with the bracket $[a,b]=ab-ba$ is a Lie algebra. Write LA for "Lie algebra".

A LA $\mathfrak{g}$ is called *semisimple* if $\{0\}$ is the unique solvable ideal of $\mathfrak{g}$. The example of interest to us is the semisimple LA $\mathrm{sl}_n=\{X\in \mathrm{gl}_n;\ \mathrm{tr}(X)=0\}$. It is a Lie subalgebra of $\mathrm{gl}_n=M_n(\mathbb{C})$, the LA of $n\times n$-matrices over $\mathbb{C}$ with $[A,B]=AB-BA$, but sl_n is not a subalgebra of the matrix algebra $M_n(\mathbb{C})$, and gl_n is not semisimple.

A $\mathfrak{g}$-*submodule* of a $\mathfrak{g}$-module V is a $\mathfrak{g}$-invariant subspace V' of V (thus $x\cdot v'\in V'$ for all $x\in\mathfrak{g}$, $v'\in V'$). The submodules $\{0\}$, V are called trivial. Then V is called *simple* if V has no nontrivial submodules; V is *semisimple* if it is a direct sum of simple modules. The complete reducibility result asserts: *A finite dimensional representation of a finite dimensional semisimple LA is semisimple.*

A semisimple Lie algebra has a presentation in terms of Chevalley generators $\{e_i,\ f_i,\ h_i;\ 1\le i\le n\}$. To state the relations they satisfy, define a *Cartan matrix* to be an $n\times n$ matrix $C=(C_{i,j};\ 1\le i,\ j\le n)$ whose coefficients $C_{i,j}$ are in $\mathbb{Z}$, its diagonal entries $C_{i,i}$ are 2, the off diagonal entries $C_{i,j}$, $i\ne j$, are nonpositive, and satisfy $C_{i,j}=0$ if and only if $C_{j,i}=0$, and all its principal minors are strictly positive: $\det(C_{i,j};\ 1\le i,\ j\le r)>0$ for all r $(1\le r\le n)$. The relations are:

$$[h_i,h_j]=0;\quad [e_i,f_j]=\delta_{i,j}h_i;\quad [h_i,e_j]=C_{i,j}e_j,\quad [h_i,f_j]=-C_{i,j}f_j;$$
$$(1-\delta_{i,j})(\mathrm{Ad}_{e_i})^{1-C_{i,j}}(e_j)=0=(1-\delta_{i,j})(\mathrm{Ad}_{f_i})^{1-C_{i,j}}(f_j).$$

The relations in the last line are called the *Serre relations*. For example $e=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $f=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $h=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ span $\mathrm{sl}_2=\mathbb{C}e\oplus\mathbb{C}f\oplus\mathbb{C}h$, and satisfy $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$. The Cartan matrix is $C=(2)$. As usual, $\delta_{i,j}$ is 1 if $i=j$ and 0 if $i\ne j$.

To consider the finite dimensional representations of the LA $\mathfrak{g}$, note that the direct sum $\mathfrak{h}=\oplus_{1\le i\le n}\mathbb{C}h_i$ is a Lie subalgebra of $\mathfrak{g}$. It is *commutative* ($[h,h']=0$ for all $h,\ h'\in\mathfrak{h}$), and called a *Cartan subalgebra* of $\mathfrak{g}$. The *lattice of weights* is

$$P=\{\varpi\in\mathfrak{h}^*;\ \varpi(h_i)\in\mathbb{Z},\ \ 1\le\forall i\le n\},$$

and the *set of non-negative weights* is

$$P^+=\{\varpi\in\mathfrak{h}^*;\ \varpi(h_i)\ge 0,\ \ 1\le\forall i\le n\}.$$

For each $\lambda\in\mathfrak{h}^*$ there exists a unique simple representation $L(\lambda)$ of the LA $\mathfrak{g}$ such that there exists $v\ne 0$ in $L(\lambda)$ satisfying $h_i\cdot v=\lambda(h_i)v$ and $e_i\cdot v=0$ for all i, $1\le i\le n$. Call λ the *highest weight* of $L(\lambda)$. This $L(\lambda)$ is finite dimensional if and only if $\lambda\in P^+$. Each simple finite dimensional representation of $\mathfrak{g}$ is of the form $L(\lambda)$ for some $\lambda\in P^+$.

The LA $\mathfrak{g}$ has a *root space decomposition*

$$\mathfrak{g}=\mathfrak{h}\oplus(\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha),$$

where for each $\alpha\in\mathfrak{h}^*$ the eigenspace of the action of $\mathfrak{h}$:

$$\mathfrak{g}_\alpha=\{x\in\mathfrak{g};\ [h,x]=\alpha(h)x,\ \forall h\in\mathfrak{h}\},$$

is called the *root space*, and the *set of roots* is $\Delta=\{\alpha\in\mathfrak{h}^*-\{0\};\ \mathfrak{g}_\alpha\ne\{0\}\}$. It is also called the *root system*. We have $e_i\in\mathfrak{g}_{\alpha_i}$, $f_i\in\mathfrak{g}_{-\alpha_i}$, where $\alpha_i(h_j)=C_{i,j}$.

Define $\alpha_i^\vee \in \mathfrak{h}$ by $\langle \alpha_i^\vee, \alpha_j \rangle = C_{i,j}$ $(1 \le i, j \le n)$. The set $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ of *simple roots*, and the set $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ of *simple coroots* are linearly independent, and $h_i = \alpha_i^\vee$. Also write $Q = \sum \mathbb{Z}\alpha$, $\alpha \in \Pi$, for the *root lattice*, and $Q^\vee = \sum \mathbb{Z}\alpha^\vee$, $\alpha^\vee \in \Pi^\vee$, for the *coroot lattice.*

It is known that $\dim(\mathfrak{g}_\alpha) = 1$ for $\alpha \in \Delta$, and there is a unique $\theta \in \Delta$ such that $\theta + \alpha_i \notin \Delta \cup \{0\}$ for all i $(1 \le i \le n)$. This θ is called the *highest root* of $\mathfrak{g}$.

3. Affine Lie (Kac-Moody) algebras

We are interested in infinite dimensional Lie (or Kac-Moody, KM) algebras associated to Cartan matrices whose principal minors are not all strictly positive. The simplest case is the affine case, where $\det(C_{i,j};\, 1 \le i, j \le r) > 0$ for all r $(1 \le r < n)$, but $\det(C) = 0$. They have a simple geometric construction as central extensions of loop algebras of semisimple LAs, that we review next, following [17, §7]. Theorem 1.1 deals only with the loop algebra $\mathcal{L}\mathfrak{g}$, but the natural object is the affine Lie algebra $\widetilde{\mathfrak{g}}$, of which $\mathcal{L}\mathfrak{g}$ is a quotient; it has a presentation that we use in the proof.

Let $\mathcal{L}$ denote the algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials in t. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. The *loop algebra* of $\mathfrak{g}$ is the algebra $\mathcal{L}\mathfrak{g} = \mathcal{L} \otimes_{\mathbb{C}} \mathfrak{g}$. It is an infinite dimensional complex Lie algebra with the bracket $[.,.]_0$ defined by

$$[P \otimes x, Q \otimes y]_0 = PQ \otimes [x, y], \qquad P, Q \in \mathcal{L}; \quad x, y \in \mathfrak{g}.$$

It can be viewed as the Lie algebra of regular rational maps $\mathbb{C}^\times \to \mathfrak{g}$, the element $\sum_i (t^i \otimes x_i)$ defines the map $z \mapsto \sum_i z^i x_i$.

Fix a *nondegenerate* (on $\mathfrak{h}$, thus for $h \in \mathfrak{h}$, $(\mathfrak{h}, h) = 0$ if and only if $h = 0$) *invariant* ($([x, y], z) = (x, [y, z])$) *symmetric* ($(x, y) = (y, x)$) bilinear $\mathbb{C}$-valued form $(.,.) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on $\mathfrak{g}$. Such a form exists ([17, Theorem 2.2]). It is unique up to a scalar multiple when $\mathfrak{g}$ is simple, as we now assume. When C is symmetric we may and we do choose the scalar so that the form $(.,.)$ satisfies $(\alpha_i^\vee, \alpha_j^\vee) = C_{i,j}$, and $(e_i, f_j) = \delta_{i,j}$ $(1 \le i, j \le n)$. Extend it by linearity to an $\mathcal{L}$-valued bilinear form $(.,.)_t$ on $\mathcal{L}\mathfrak{g}$ by

$$(.,.)_t : \mathcal{L}\mathfrak{g} \times \mathcal{L}\mathfrak{g} \to \mathcal{L}, \qquad (P \otimes x, Q \otimes y)_t = PQ(x, y).$$

Define linear maps $\frac{d}{dt} : \mathcal{L}\mathfrak{g} \to \mathcal{L}\mathfrak{g}$ and $\mathrm{Res} : \mathcal{L} \to \mathbb{C}$ by $\frac{d}{dt}(P \otimes x) = \frac{dP}{dt} \otimes x$ and $\mathrm{Res}(t^r) = \delta_{r,-1}$, for $P \in \mathcal{L}$, $x \in \mathfrak{g}$, $r \in \mathbb{Z}$. Res is the unique functional on $\mathcal{L}$ satisfying $\mathrm{Res}\, t^{-1} = 1$ and $\mathrm{Res} \frac{dP}{dt} = 0$ for all $P \in \mathcal{L}$. A more natural presentation of the residue would be to view $f \in \mathcal{L}\mathfrak{g}$ as a morphism $f : \mathbb{C}^\times \to \mathfrak{g}$, where $\mathbb{C}^\times$ is the multiplicative group $\mathbb{C} - \{0\}$ of $\mathbb{C}$. The differential df of f is a 1-form with values in $\mathfrak{g}$. Then $(df, g)_t$ is a 1-form on $\mathbb{C}^\times$, whose residue at 0 is denoted by $\mathrm{Res}_0((df, g)_t)$ $(= \mathrm{Res}((\frac{df}{dt}, g)_t))$. In particular $\mathrm{Res}_0(dP) = 0$ for all $P \in \mathcal{L}$.

Lemma 3.1. *Define a bilinear map* $\psi : \mathcal{L}\mathfrak{g} \times \mathcal{L}\mathfrak{g} \to \mathbb{C}$ *by* $\psi(f, g) = \mathrm{Res}_0((df, g)_t)$. *Then* ψ *is a* 2*-cocycle on* $\mathcal{L}\mathfrak{g}$*, namely it satisfies for all* f, g, $h \in \mathcal{L}\mathfrak{g}$*:*

$$\psi(f, g) = -\psi(g, f), \qquad \psi([f, g], h) + \psi([g, h], f) + \psi([h, f], g) = 0.$$

Proof. It suffices to check these two claims for $f = P\otimes x$, $g = Q\otimes y$, $h = R\otimes z$. Then we have

$$\psi(P\otimes x,Q\otimes y)+\psi(Q\otimes y,P\otimes x) = \mathrm{Res}_0(dP\cdot Q+PdQ)(x,y) = \mathrm{Res}_0(d(PQ))(x,y) = 0,$$

and in view of the invariance of $(.,.)$: $([x,y],z) = (x,[y,z])$, and its being symmetric,

$$\begin{aligned}&\psi([P\otimes x,Q\otimes y],R\otimes z)+\psi([Q\otimes y,R\otimes z],P\otimes x)+\psi([R\otimes z,P\otimes x],Q\otimes y)\\&=\mathrm{Res}_0(d(PQ)R)([x,y],z)+\mathrm{Res}_0(d(QR)P)([y,z],x)+\mathrm{Res}_0(d(RP)Q)([z,x],y)\\&=(\mathrm{Res}_0(d(PQ)R)+\mathrm{Res}_0(d(QR)P)+\mathrm{Res}_0(d(RP)Q))([x,y],z)\\&=\mathrm{Res}_0(d(PQR))([x,y],z)=0.\end{aligned}$$ ■

Denote by $\widetilde{\mathfrak{g}} = \widetilde{\mathcal{L}\mathfrak{g}}$ the extension of the loop Lie algebra $\mathcal{L}\mathfrak{g}$ by a 1-dimensional center, associated to the cocycle ψ. Thus it is $\widetilde{\mathfrak{g}} = \mathcal{L}\mathfrak{g}\oplus\mathbb{C}c$, where c is a formal central element ($[c,f]=0$ for all $f\in\mathcal{L}\mathfrak{g}$), with Lie algebra structure defined by the bracket $[f,g]=[f,g]_0+\psi(f,g)c$, f, $g\in\mathcal{L}\mathfrak{g}$. The skew symmetry for $[.,.]$ on $\widetilde{\mathfrak{g}}$ follows from the first property in the Lemma, the Jacobi identity follows from the Jacobi identity for $\mathcal{L}\mathfrak{g}$ and the last claim in the Lemma.

The *affine Lie* (*or Kac-Moody, KM*) *algebra* $\hat{\mathfrak{g}} = \hat{\mathcal{L}\mathfrak{g}}$ is obtained on adding to $\widetilde{\mathfrak{g}}$ the derivation d that acts on $\mathcal{L}\mathfrak{g}$ as $t\frac{d}{dt} : P\otimes x\mapsto t\frac{dP}{dt}\otimes x$, and maps c to 0. Thus $\hat{\mathfrak{g}} = \mathcal{L}\mathfrak{g}\oplus\mathbb{C}c\oplus\mathbb{C}d$, with bracket extended by $[d,P(t)\otimes x]=t\frac{dP(t)}{dt}\otimes x$, $[d,c]=0$.

The affine Lie algebra $\widetilde{\mathfrak{g}}$ has a presentation in terms of generators and relations, associated to a *generalized Cartan matrix*. The latter is an $(n+1)\times(n+1)$ matrix $C=(C_{i,j};\,0\le i,\,j\le n)$ whose coefficients $C_{i,j}$ are in $\mathbb{Z}$, its diagonal entries $C_{i,i}$ are 2, the off diagonal entries $C_{i,j}$, $i\neq j$, are nonpositive, and satisfy $C_{i,j}=0$ if and only if $C_{j,i}=0$, all its proper principal minors are strictly positive: $\det(C_{i,j};\,1\le i,\,j\le r)>0$ for all r $(1\le r<n)$, but $\det(C)=0$. Then $\widetilde{\mathfrak{g}}$ is generated by $(E_i,\,F_i,\,H_i;\,0\le i\le n)$, subject to the relations

$$[H_i,H_j]=0;\quad [E_i,F_j]=\delta_{i,j}H_i;\quad [H_i,E_j]=C_{i,j}E_j,\quad [H_i,F_j]=-C_{i,j}F_j;$$

$$(1-\delta_{i,j})(\mathrm{Ad}_{E_i})^{1-C_{i,j}}(E_j)=0=(1-\delta_{i,j})(\mathrm{Ad}_{F_i})^{1-C_{i,j}}(F_j).$$

The relations in the last line are called the *Serre relations*. Moreover, the labeling $\{0,\dots,n\}$ can and will be chosen so that the subalgebra generated by the E_i, F_i, H_i $(1\le i\le n)$ is isomorphic to $\mathfrak{g}$, namely $(C_{i,j};\,1\le i,j\le n)$ is the Cartan matric of $\mathfrak{g}$. The affine Lie algebra $\hat{\mathfrak{g}}$ is obtained on adding a generator d to the Cartan algebra $\mathfrak{h}$, but it does not appear in the theorem.

The *Chevalley involution* ω is the involution of $\mathfrak{g}$ defined by $\omega(e_i)=-f_i$, $\omega(f_i)=-e_i$, $\omega(h_i)=-h_i$. Recall that θ is defined at the end of last section. Choose $f_0\in\mathfrak{g}_\theta$ with $(f_0,\omega(f_0))=-2/(\theta,\theta)$. Define $e_0=-\omega(f_0)\in\mathfrak{g}_{-\theta}$. Then we have $[e_0,f_0]=-\theta^\vee$. The e_i $(0\le i\le n)$ generate the Lie algebra $\mathfrak{g}$, as in the adjoint representation $\mathfrak{g}=(\mathfrak{U}(\mathfrak{g}))(e_0)=(\mathfrak{U}(\mathfrak{n}_+))(e_0)$ since $\mathfrak{g}$ is simple ($\mathfrak{U}$ for the universal enveloping algebra, $\mathfrak{n}_+$ the sub LA generated by the e_i $(1\le i\le n)$).

The (infinite dimensional) Lie algebra $\hat{\mathfrak{g}}=\hat{\mathcal{L}\mathfrak{g}}$ can now be described. Its center is $\mathbb{C}c$. The centralizer of d in $\hat{\mathfrak{g}}$ is $\mathbb{C}c\oplus\mathbb{C}d\oplus(1\otimes\mathfrak{g})$. In particular $1\otimes\mathfrak{g}$ ia a subalgebra of $\hat{\mathfrak{g}}$, that we identify with $\mathfrak{g}$ by $x\mapsto 1\otimes x$. Then $\hat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d$ is an

$(n+2)$-dimensional commutative subalgebra in $\hat{\mathfrak{g}}$. Extend any $\lambda \in \mathfrak{h}^*$ to a linear function on $\hat{\mathfrak{h}}$ by $\langle \lambda, c\rangle = 0 = \langle \lambda, d\rangle$, so that $\mathfrak{h}^*$ is identified with a subspace of $\hat{\mathfrak{h}}^*$. Denote by $\delta \in \hat{\mathfrak{h}}^*$ the linear form on $\hat{\mathfrak{h}}$ that is zero on $\mathfrak{h} \oplus \mathbb{C}c$ and with $\langle \delta, d\rangle = 1$. Now introduce

$$E_0 = t \otimes e_0, \quad F_0 = t^{-1} \otimes f_0, \quad E_i = 1 \otimes e_i, \quad F_i = 1 \otimes f_i \quad (1 \le i \le n).$$

From $[e_0, f_0] = -\theta^\vee$ we deduce that $[E_0, F_0] = \frac{2}{(\theta,\theta)}c - \theta^\vee$.

The root system and the root space decomposition of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$ are as follows.

$$\hat{\Delta} = \{j\delta + \gamma;\ j \in \mathbb{Z},\ \gamma \in \Delta\} \cup \{j\delta;\ j \in \mathbb{Z} - \{0\}\}.$$

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \oplus_{\alpha\in\hat{\Delta}} \mathcal{L}\mathfrak{g}_\alpha, \quad \mathcal{L}\mathfrak{g}_{j\delta+\gamma} = t^j \otimes \mathfrak{g}_\gamma, \quad \mathcal{L}\mathfrak{g}_{j\delta} = t^j \otimes \mathfrak{h}.$$

Put $\Pi_0 = \{\alpha_0 := \delta - \theta,\ \alpha_1, \dots, \alpha_n\}$ and

$$\Pi_0^\vee = \{\alpha_0^\vee := \frac{2}{(\theta,\theta)}c - 1 \otimes \theta^\vee,\ \alpha_1^\vee := 1 \otimes h_1, \dots, \alpha_n^\vee := 1 \otimes h_n\}.$$

Then $C = (\langle \alpha_j, \alpha_i^\vee\rangle;\ 0 \le i, j \le n)$, and $(\hat{\mathfrak{h}}, \Pi_0, \Pi_0^\vee)$ is a realization of the affine matrix C in the terminology of [17, §1.1].

4. Case of sl_{n+1}

The classical Schur duality concerns the case of (diagram of) a Cartan matrix C of type (A_n). We extend it to that of a generalized Cartan matrix – denoted here by C_0 – of type $(A_n^{(1)})$. The matrix C is obtained from C_0 on erasing its first row and column. The generalized Cartan matrix C_0 is symmetric, of size $(n+1) \times (n+1)$. We label its rows and columns from 0 to n (this explains the subscript 0), and those of C from 1 to n. When $n = 1$, C_0 is $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. When $n \ge 2$, C_0 is the matrix with diagonal entries $C_{i,i} = 2$ $(0 \le i \le n)$, the entries above and below diagonal are $C_{i,i+1} = -1 = C_{i+1,i}$ $(0 \le i < n)$, and so are the entries at the top right and bottom left positions $(0, n)$ and $(n, 0)$, all other entries are 0:

$$C_0 = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ is sl_{n+1}, the space of $(n+1) \times (n+1)$ matrices with entries in $\mathbb{C}$ and trace zero, with the bracket $[x, y] = xy - yx$. Denote by $E_{i,j}$ $(1 \le i, j \le n+1)$ the matrix whose only nonzero entry is 1 on row i and column j. These give the standard basis for the space of $(n+1) \times (n+1)$-matrices. Let $\mathfrak{h}$ be the subspace of diagonal matrices in $\mathfrak{g}$. Then $h_i = \alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$ $(1 \le i \le n)$ makes a basis of $\mathfrak{h}$. Define $\varepsilon_i \in \mathfrak{h}^*$ $(1 \le i \le n+1)$ by

$$\varepsilon_i(\mathrm{diag}(a_1, \dots, a_{n+1})) = a_i.$$

(This is perhaps what is meant by [6, p. 299, l. -11 to -8]). Then $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le n)$ makes a basis of $\mathfrak{h}^*$, and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is the set of roots (in $\mathfrak{h}^*$), $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ the set of coroots (in $\mathfrak{h}$).

The root space decomposition with respect to $\mathfrak{h}$ is $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{i \neq j} \mathbb{C} E_{i,j})$. The Chevalley generators of $\mathfrak{g}$ are $e_i = E_{i,i+1}$, $f_i = E_{i+1,1}$ $(1 \le i \le n)$. The set of roots is $\{\varepsilon_i - \varepsilon_j;\, i \neq j\}$, the set of positive roots (nonzero linear combinations of the α_i with integral non-negative coefficients) is $\{\varepsilon_i - \varepsilon_j;\, i < j\}$, and the Chevalley involution is $x \mapsto -{}^t x$. Further we have that $f_0 = E_{1,n+1}$ and $e_0 = E_{n+1,1}$, and $\theta^\vee = E_{1,1} - E_{n+1,n+1} = \alpha_1 + \cdots + \alpha_n$.

The weights $\lambda_i = \sum_{1 \le j \le i} \varepsilon_j$, $1 \le i \le n$, are called *fundamental*, and the corresponding representations $V(\lambda_i)$ of $\mathfrak{U}(\mathfrak{g})$ are the *fundamental representations.* The natural representation of $\mathfrak{U}(\mathfrak{g})$ is $\mathbb{E} = V(\lambda_1)$. In its standard basis $\{u_1, \dots, u_{n+1}\}$, the action ρ of $\mathfrak{g}$ on $\mathbb{E} = \mathbb{C}^n$ is given by: $\rho(e_i)u_r = \delta_{r,i+1} u_{r-1}$, $\rho(f_i)u_r = \delta_{r,i} u_{r+1}$, $\rho(h_i)u_r$ is u_i if $r = i$, $-u_{i+1}$ if $r = i+1$, and 0 otherwise. Further, $\rho(e_0)u_r = \delta_{r,1} u_{n+1}$, $\rho(f_0)u_r = \delta_{r,n+1} u_1$. However, we can make $\mathbb{E}$ into a $\mathcal{L}\mathfrak{g}$-module, that is a $\widetilde{\mathfrak{g}}$-module of level 0 (c act as 0) $\mathbb{E}_a$ for each $a \in \mathbb{C}^\times$, by letting $E_i = 1 \otimes e_i$ act as e_i and $F_i = 1 \otimes f_i$ as f_i for $1 \le i \le n$, and $E_0 = t \otimes e_0$ as ae_0 and $F_0 = t^{-1} \otimes f_0$ as $a^{-1} f_0$. In other words, $\mathbb{E}_a$ is the pullback via the evaluation at $a \in \mathbb{C}^\times$ map $\mathrm{ev}_a : \mathcal{L}\mathfrak{g} \to \mathfrak{g}$, $t \mapsto a$, of the $\mathfrak{g}$-module $\mathbb{E}$. (This is perhaps what is meant by [6, last sentence in subsection 2.4]).

This completes our review of the affine Lie algebra $\widetilde{\mathrm{sl}}_{n+1}$. It is of (untwisted) type $(A_n^{(1)})$. We shall use its presentation in terms of generators E_i, F_i, H_i $(0 \le i \le n)$ and relations. The first step will be to show that the representation defined in Theorem 1.1 satisfies the relations. Then we shall need to verify it gives an equivalence of categories. *Note that the theorem concerns* $\widetilde{\mathrm{sl}}_n$*, that is of type* $(A_{n-1}^{(1)})$*. It is more convenient to discuss* $(A_n^{(1)})$*, so we are led to* $\widetilde{\mathrm{sl}}_{n+1}$*. To get the theorem we simply change* n *to* $n-1$ *at the end.*

5. The functor of the theorem is well defined

We can now return to the theorem in an attempt to verify it. Recall that the affine Lie algebra $\widetilde{\mathfrak{g}}$ is generated by two generators: $E_0 = t \otimes e_0$ and $F_0 = t^{-1} \otimes f_0$, in addition to the generators $E_i = 1 \otimes e_i$ and $F_i = 1 \otimes f_i$ $(1 \le i \le n)$ that come from the generators e_i and f_i $(1 \le i \le n)$ of $\mathfrak{g} = \mathrm{sl}_{n+1}$, and our proposed functor $\mathcal{F}$ maps the $S_d^{\mathrm{aff}} = \mathbb{Z}^d \rtimes S_d$-module M to the $\mathfrak{g}$-module $\mathcal{S}(M) = M \otimes_{S_d} \mathbb{E}^{\otimes d}$, on which $x \in \mathfrak{g}$ acts as

$$\rho_d(x) \cdot m \otimes v = m \otimes \rho_d(\Delta(x))v = m \otimes \sum_{1 \le j \le d} Y_j(x)v,$$

$$Y_j(x) = Y_j(\rho(x)) = 1^{\otimes(j-1)} \otimes \rho(x) \otimes 1^{\otimes(d-j)}.$$

The new generators act using the additional structure of M as an S_d^{aff}-module, not only an S_d-module, thus the right action of the generators $y_j = (0, \dots, 0, 1, 0, \dots, 0)$ $(1 \le i \le d)$ of the translation group $\mathbb{Z}^d$ of $S_d^{\mathrm{aff}} = \mathbb{Z}^d \rtimes S_d$. We write this subgroup multiplicatively: $y_j y_k$ and y_j^{-1}, rather than additively, to prevent possible confusion. The theorem defines the action of the new generators by

$$\rho_d(E_0) \cdot m \otimes v = \sum_{1 \le j \le d} m y_j \otimes Y_j(e_0)v, \qquad \rho_d(F_0) \cdot m \otimes v = \sum_{1 \le j \le d} m y_j^{-1} \otimes Y_j(f_0)v.$$

Our first task is to show that these last formulae that define the action of E_0 and F_0 on $M \otimes_{S_d} \mathbb{E}^{\otimes d}$ are well defined. Consider the case of $\rho_d(E_0)$, that of $\rho_d(F_0)$ is similar. As $m \otimes v$ lies in $M \otimes_{S_d} \mathbb{E}^{\otimes d}$, we need to verify that $(\rho_d(E_0))(m\sigma \otimes v) = (\rho_d(E_0))(m \otimes \sigma v)$ for all generators $\sigma = \sigma_i = (i, i+1)$, $1 \leq i < d$, of the symmetric group S_d. Namely we have to show that as operators on $M \otimes_{S_d} \mathbb{E}^{\otimes d}$ we have

$$\sum_{1 \leq j \leq d} \sigma_i y_j \otimes Y_j(e_0) = \sum_{1 \leq j \leq d} y_j \otimes Y_j(e_0)\sigma_i.$$

If $j \neq i$, $i+1$, then the jth terms on the left and the right sides are equal, since in this case $\sigma_i y_j = y_j \sigma_i$, and $Y_j(e_0)\sigma_i = \sigma_i Y_j(e_0)$. So it remains to show that the sum of the remaining ith and $(i+1)$th terms on the left is equal to that sum on the right:

$$\sigma_i y_i \otimes Y_i(e_0) + \sigma_i y_{i+1} \otimes Y_{i+1}(e_0) = y_i \otimes Y_i(e_0)\sigma_i + y_{i+1} \otimes Y_{i+1}(e_0)\sigma_i.$$

But $\sigma_i y_i = y_{i+1}\sigma_i$, $\sigma_i y_{i+1} = y_i \sigma_i$, $\sigma_i Y_i(e_0) = Y_{i+1}(e_0)\sigma_i$, $\sigma_i Y_{i+1}(e_0) = Y_i(e_0)\sigma_i$, so in fact $\sigma_i y_i \otimes Y_i(e_0) = y_{i+1} \otimes Y_{i+1}(e_0)\sigma_i$ and $\sigma_i y_{i+1} \otimes Y_{i+1}(e_0) = y_i \otimes Y_i(e_0)\sigma_i$, as required.

6. The representation extends to the affine algebra

In order to show that *the formulae defining the action of $\rho_d(E_0)$ and $\rho_d(F_0)$ define a representation of the affine Lie algebra $\widetilde{\mathfrak{g}}$ of level* 0, thus the center c acts as zero, we need to check they satisfy the relations that the generators $(E_i, F_i, H_i; 0 \leq i \leq n)$ of $\widetilde{\mathfrak{g}}$ satisfy. Only the relations involving E_0, F_0 are new and need to be checked. These are all simple, except for the Serre relations. Suppose $n \geq 2$. The Serre relations involving E_0 are

$$[E_0, [E_0, E_1]] = 0 = [E_0, [E_0, E_n]] \text{ and } [E_1, [E_0, E_0]] = 0 = [E_n, [E_0, E_0]],$$

those involving F_0 are analogous, obtained on replacing E by F in the Serre relations for E just spelt out. This is because the coefficients in the Cartan matrix of relevance are $C_{0,1} = -1 = C_{1,0}$, $C_{0,n} = -1 = C_{n,0}$. When $n = 1$ the relations are $[E_0, [E_0, [E_0, E_1]]] = 0 = [E_1, [E_1, [E_1, E_0]]]$ as $C_{0,1} = -2 = C_{1,0}$, and they are dealt with in the same way.

As a representative case, consider $n \geq 2$, $[E_0, [E_0, E_1]]$. First of all, $[\rho_d(E_0), \rho_d(E_1)]$ maps $m \otimes v$ to the sum of $my_j \otimes Y_j([\rho(e_0), \rho(e_1)])v$, as all other terms vanish. Applying then $\operatorname{Ad}\rho(E_0)$ would give a sum over $j < k$ of terms

$$my_j y_k \otimes (1^{\otimes(j-1)} \otimes [\rho(e_0), \rho(e_1)] \otimes 1^{\otimes(k-j-1)} \otimes \rho(e_0) \otimes 1^{\otimes(d-k)})v$$

$$-my_k y_j \otimes (1^{\otimes(j-1)} \otimes [\rho(e_0), \rho(e_1)] \otimes 1^{\otimes(k-j-1)} \otimes \rho(e_0) \otimes 1^{\otimes(d-k)})v,$$

that is 0 since y_j, $y_k \in \mathbb{Z}^d \subset \mathbb{Z}^d \rtimes S_d$ commute; plus the analogous sum for $k < j$, that is equally 0; plus

$$my_j y_j \otimes (1^{\otimes(j-1)} \otimes [\rho(e_0), [\rho(e_0), \rho(e_1)]] \otimes 1^{\otimes(d-j)})v,$$

that equals 0 since $[\rho(e_0), [\rho(e_0), \rho(e_1)]] = 0$. This, and the analogous formulae for the remaining Serre relations that are proven analogously, shows that the formulae for $\rho_d(E_0)$ and $\rho_d(F_0)$ indeed define a $\widetilde{\mathfrak{g}}$-module structure on $\mathcal{S}(M) = M \otimes_{S_d} \mathbb{E}^{\otimes d}$.

To show that $\mathcal{F}$ *is a functor* we need to deal also with morphisms. Thus if $f: M \to M'$ is a homomorphism of $S_d^{\mathrm{aff}} = \mathbb{Z}^d \rtimes S_d$-modules, define the map $\mathcal{F}f : \mathcal{F}M \to \mathcal{F}M'$ by $(\mathcal{F}f)(m \otimes v) = f(m) \otimes v$. To check that $\mathcal{F}f$ is a well-defined homomorphism of S_d^{aff}-modules we note for example that

$$\rho_d(E_0)((\mathcal{F}f)(m \otimes v)) = \sum_{1 \le j \le d} f(m)y_j \otimes Y_j(e_0)v$$

$$= \sum_{1 \le j \le d} f(my_j) \otimes Y_j(e_0)v = (\mathcal{F}f)((\rho_d(E_0))(m \otimes v)).$$

Hence $\mathcal{F}$ is a functor from the category of $\mathbb{C}[S_d^{\mathrm{aff}}]$-modules to the category of finite dimensional $\mathfrak{U}(\widetilde{\mathfrak{g}})$-modules of level 0 as specified in the theorem.

7. The functor is an equivalence of categories

Assume in this section that $d < n+1$. This condition is used in the proof, and it is optimal: the theorem does not hold for $d = n+1$, see section 15. To show that the functor $\mathcal{F}$ – that we have seen is a well-defined functor between the categories specified in the theorem – is an equivalence, one has to show:

(a) Every finite dimensional $\mathcal{L}\mathfrak{g}$-module W, thus $\widetilde{\mathfrak{g}}$-module W of level 0 (the one dimensional center spanned by c acts as 0), each of whose irreducible subquotients when restricted to $\mathfrak{g}$ is a constituent of $\mathbb{E}^{\otimes d}$, is isomorphic to $\mathcal{F}(M) = M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$ for some $\mathbb{C}[\mathbb{Z}^d \rtimes S_d]$-module M.

(b) $\mathcal{F}$ is bijective on sets of morphisms.

To prove (a), by the Schur duality we assume that the restriction $\mathrm{Res}_{\mathfrak{g}}^{\mathcal{L}\mathfrak{g}} W$ is of the form $M \otimes_{S_d} \mathbb{E}^{\otimes d}$ for some $\mathbb{C}[S_d]$-module M. We shall construct the action of the generators $y_j^{\pm 1}$ of $\mathbb{Z}^d \subset S_d^{\mathrm{aff}} = \mathbb{Z}^d \rtimes S_d$ on M from the given action of $\rho_d(E_0)$, $\rho_d(F_0)$, $\rho_d(\mathfrak{g})$ on W, so that M becomes a $\mathbb{C}[S_d^{\mathrm{aff}}]$-module.

Proposition 7.1. (a) *Let M be a finite dimensional $\mathbb{C}[S_d]$-module. Fix $v \in \mathbb{E}^{\otimes d}$ such that $\mathbb{E}^{\otimes d} = \rho_d(\mathfrak{U}(\mathfrak{g})) \cdot v$. Then the map $M \to \mathcal{S}(M)$, $m \mapsto m \otimes_{\mathbb{C}[S_d]} v$, is injective.*

(b) *Recall that $\{u_1, \dots, u_{n+1}\}$ denotes the standard basis of $\mathbb{E}$. Suppose now $v = u_{i_1} \otimes \cdots \otimes u_{i_d} \in \mathbb{E}^{\otimes d}$, where $i_1, \dots, i_d \in \{1, \dots, n+1\}$ are distinct. Then $\mathbb{E}^{\otimes d} = \rho_d(\mathfrak{U}(\mathfrak{g})) \cdot v$. In particular v satisfies the condition stated in* (a).

Proof. Choose an isomorphism $\mathbb{E}^{\otimes d} = \oplus_\lambda S^\lambda \otimes V^\lambda$, where $\lambda \vdash d$ ranges over the set of partitions of d, and the length $\ell(\lambda)$ of λ is $\le n$ since $d \le n$. Here S^λ is the λ-Specht representation of S_d and $V^\lambda = L(\lambda)$ the $\mathrm{sl}_{n+1}(\mathbb{C})$-module parametrized by λ. The vector $v = \sum_\lambda x_\lambda$ spans $\mathbb{E}^{\otimes d}$ under the action of $\rho_d(\mathfrak{U}(\mathfrak{g}))$, in particular $\rho_d(\mathfrak{U}(\mathfrak{g})) \cdot x_\lambda = S^\lambda \otimes L(\lambda) = \mathrm{Hom}_{\mathbb{C}}(L(\lambda)^\vee, S^\lambda)$, so we may assume $x_\lambda : L(\lambda)^\vee \to S^\lambda$ is onto, for each λ. Note that $\dim S^\lambda \le \dim L(\lambda)$ by [13, Ex. 6.4*, p. 78].

Since $\mathbb{C}[S_d]$ is semisimple, by Maschke theorem the finite dimensional $\mathbb{C}[S_d]$-module M is completely reducible. Thus $M = \oplus_{\mu \vdash d} M_\mu$, where M_μ are the μ-isotypical components of M. Hence $M_\mu \simeq S^\mu \otimes A_\mu$, where $A_\mu = \mathrm{Hom}_{\mathbb{C}[S_d]}(S^\mu, M)$ is a vector space. Since $S^\mu \simeq (S^\mu)'$ is self dual, $M_\mu \simeq \mathrm{Hom}_{\mathbb{C}}(S^\mu, A_\mu)$.

Next we use the fact that S^λ is self dual, and Schur's lemma: $V' \otimes_G W \simeq \mathrm{Hom}_G(V, W)$ is $\mathbb{C}$ if the irreducible G-modules V, W are isomorphic, and 0 if not; $G = S_d$. Consider the map

$$M \times \mathbb{E}^{\otimes d} \to M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d} = (\oplus_\mu \operatorname{Hom}_{\mathbb{C}}(S^\mu, A_\mu)) \otimes_{\mathbb{C}[S_d]} (\oplus_\lambda S^\lambda \otimes L(\lambda))$$
$$= \oplus_\lambda A_\lambda \otimes L(\lambda) = \oplus_\lambda \operatorname{Hom}_{\mathbb{C}}(L(\lambda)^\vee, A_\lambda),$$
$$((f_\lambda : S^\lambda \to A_\lambda), (x_\lambda \in S^\lambda \otimes L(\lambda))) \mapsto (f_\lambda(x_\lambda) \in A_\lambda \otimes L(\lambda) = \operatorname{Hom}_{\mathbb{C}}(L(\lambda)^\vee, A_\lambda)),$$

where $f_\lambda(x_\lambda)$ is $f_\lambda \circ x_\lambda : L(\lambda)^\vee \to S^\lambda \to A_\lambda$.

Injectivity of the map $M \mapsto M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$, $m \mapsto m \otimes v$, means that $f_\lambda \circ x_\lambda = 0$ implies $f_\lambda = 0$, for all λ. This holds when $x_\lambda : L(\lambda)^\vee \to S^\lambda$ is surjective, as assumed; (a) follows.

It does not suffice to assume that $x_\lambda \neq 0$ for all λ as assumed in [6, Lemma 4.3(a)]: if for example $x_\lambda = a \otimes b$, $a \in S^\lambda$, $b \in L(\lambda)$, $\dim S^\lambda \geq 2$ and $A_\lambda \neq 0$, there are nonzero $f_\lambda : S^\lambda \to A_\lambda$ that send a to 0. For this reason we write up a proof of the proposition.

Claim (b) is elementary. We need to show that under the action of the universal enveloping algebra $\rho_d(\mathfrak{U}(\mathfrak{g}))$, each basis vector $u_{j_1} \otimes \cdots \otimes u_{j_d}$ of $\mathbb{E}^{\otimes d}$ can be obtained from $u_{i_1} \otimes \cdots \otimes u_{i_d}$ with distinct $i_1, \ldots, i_d \in \{1, \ldots, n+1\}$. To simplify the notation it suffices to show this for $d = 2$. Then $n+1 > 2$ by our standing assumption $d \leq n$. Thus it suffices to show that $\rho_d(\mathfrak{U}(\mathfrak{g}))$ takes $u_1 \otimes u_2$ to $u_k \otimes u_m$, for any k, m between 1 and $n+1$. Recall that $\rho_d(g)$ acts as $\Delta_d(g) = \sum_{1 \leq j \leq d} I^{\otimes(j-1)} \otimes g \otimes I^{\otimes(d-j)}$. Write $g = (a, b, c)$ for the $(n+1) \times (n+1)$-matrix with 1st column a, 2nd column b, kth column c, $k > 2$; the other columns are not written, to save on notation; it suffices to take $n+1 = 3$ and $k = 3$. Then $\Delta_d(u_k, 0, *)$ takes $u_1 \otimes u_2$ to $u_k \otimes u_2$, where $*$ means any column; $\Delta_d(*, u_m, 0)$ takes $u_k \otimes u_2$ to $u_k \otimes u_m$ $(k \neq 2)$; $\Delta_d(0, *, u_2)$ takes $u_k \otimes u_1$ to $u_2 \otimes u_1$; $\Delta_d(u_m, 0, *)$ takes $u_2 \otimes u_1$ to $u_2 \otimes u_m$; $\Delta_d(0, u_k, *)$ takes $u_1 \otimes u_2$ to $u_1 \otimes u_k$. ∎

Proposition 7.2. *For j $(1 \leq j < n)$ put $a(j) = u_2 \otimes \cdots \otimes u_j$,*

$$b(j) = u_{j+1} \otimes \cdots \otimes u_d, \quad v^{(j)} = a(j) \otimes u_{n+1} \otimes b(j), \quad w^{(j)} = a(j) \otimes u_1 \otimes b(j).$$

Then there exists $\alpha_{jF} \in \operatorname{End}_{\mathbb{C}}(M)$ with

$$(\rho_d(F_0))(m \otimes v^{(j)}) = \alpha_{jF}(m) \otimes \rho^{\otimes d}(Y_j(f_0))v^{(j)},$$

and $\alpha_{jE} \in \operatorname{End}_{\mathbb{C}}(M)$ with $(\rho_d(E_0))(m \otimes w^{(j)}) = \alpha_{jE}(m) \otimes \rho^{\otimes d}(Y_j(e_0))w^{(j)}$.
We have $\rho^{\otimes d}(Y_j(f_0))v^{(j)} = w^{(j)}$, and $\rho^{\otimes d}(Y_j(e_0))w^{(j)} = v^{(j)}$.

Proof. For τ in the symmetric group S_d on d letters, put

$$w_\tau^{(j)} = (u_{\tau(2)} \otimes \cdots \otimes u_{\tau(j)}) \otimes u_{\tau(1)} \otimes (u_{\tau(j+1)} \otimes \cdots \otimes u_{\tau(d)}).$$

The set $\{w_\tau^{(j)}; \tau \in S_d\}$ spans the subspace of $\mathbb{E}^{\otimes d}$ of weight $\lambda_d = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d$. Indeed, $(\rho_d(H))u_i = \langle \varepsilon_i, H \rangle u_i$, so $(\rho_d(H))w_\tau^{(j)} = \langle H, \varepsilon_1 + \cdots + \varepsilon_d \rangle w_\tau^{(j)}$. Note that $\rho_d(H)\rho_d(F_0) = \langle H, \alpha_0 \rangle \rho_d(F_0)\rho_d(H)$, hence $\rho_d(F_0)$ adds $\varepsilon_1 - \varepsilon_{n+1}$ to the weight, hence it takes ε_{n+1} to ε_1. Hence for every $m \in M$ we have

$$(\rho_d(F_0))(m \otimes v^{(j)}) = \sum_{\tau \in S_d} m_\tau \otimes w_\tau^{(j)}$$

for some $m_\tau \in M$. Now $w_\tau^{(j)}$ is $h \cdot w^{(j)}$ for some $h \in \mathbb{C}[S_d]$, $h = h(\tau)$. Hence $(\rho_d(F_0))(m \otimes v^{(j)})$ equals $m' \otimes w^{(j)}$ for some $m' \in M$. Proposition 7.1 implies that m' can be reconstructed from $m' \otimes w^{(j)}$. Then there exists $\alpha_{jF} \in \operatorname{End}_{\mathbb{C}}(M)$ with $m' = \alpha_{jF}(m)$ for all $m \in M$. The existence $\alpha_{jE} \in \operatorname{End}_{\mathbb{C}}(M)$ is proven analogously. ∎

Proposition 7.3. *For all $m \in M$ and $v \in \mathbb{E}^{\otimes d}$ we have*

$$(\rho_d(E_0))(m \otimes v) = \sum_{1 \le j \le d} \alpha_{jE}(m) \otimes \rho^{\otimes d}(Y_j(e_0))v,$$

$$(\rho_d(F_0))(m \otimes v) = \sum_{1 \le j \le d} \alpha_{jF}(m) \otimes \rho^{\otimes d}(Y_j(f_0))v.$$

Proof. Recall that we have $HF_0 = -\langle H, \alpha_0 \rangle F_0 H$, where $\alpha_0 = \varepsilon_{n+1} - \varepsilon_1$, and $\rho(H)u_i = \langle \varepsilon_i, H \rangle u_i$. In consequence $\rho_d(H)\rho_d(F_0)(m \otimes v)$, where $v = u_{i_1} \otimes \cdots \otimes u_{i_d}$, is $\langle H, -\alpha_0 + \varepsilon_{i_1} + \cdots + \varepsilon_{i_d} \rangle \rho_d(F_0)(m \otimes v)$, and this will be 0 if no i_j is $n+1$, as then $-\varepsilon_{n+1} + \varepsilon_1 + \varepsilon_{i_1} + \cdots + \varepsilon_{i_d}$ cannot be a weight of $\mathbb{E}^{\otimes d}$. So we may assume some component of v is u_{n+1}.

Let $r \ge 0$, $s \ge 1$, $r+s \le d$, $1 \le j_1 < j_2 < \cdots < j_r \le d$, $1 \le j'_1 < j'_2 < \cdots < j'_s \le d$. Assume $\{j_1, \ldots, j_r\} \cap \{j'_1, \ldots, j'_s\} = \emptyset$. Write $j = (j_1, \ldots, j_r)$, $j' = (j'_1, \ldots, j'_s)$. Let $V^{(j,j')}$ be the subspace of $\mathbb{E}^{\otimes d}$ spanned by the vectors that have u_1 in positions $j_1, \ldots, j_r$; u_n in positions $j'_1, \ldots, j'_s$; and vectors from $\{u_2, \ldots, u_n\}$ in the remaining positions. We prove the proposition when v is in $V^{(j,j')}$ for all j, j' in two steps.

(i) For $s = 1$, by induction on r.

(ii) For all r, by induction on s.

By Proposition 7.1, applied to the subalgebra of $\mathfrak{g}$ generated by the E_i, F_i, H_i for $i \in \{2, \ldots, n-1\}$, to prove our proposition for all $v \in V^{(j,j')}$ it suffices to prove it for one $0 \ne v \in V^{(j,j')}$ whose components have no vector from $\{u_2, \ldots, u_n\}$ twice. Such vectors exist since $1 \le d + 1 - r - s \le d \le n$.

Proof of step (i): Here $s = 1$. The case of $r = 0$ follows from Proposition 7.2: take

$$v = a(j'_1) \otimes u_{n+1} \otimes b(j'_1), \qquad w = a(j'_1) \otimes u_1 \otimes b(j'_1),$$

(recall: $a(j) = u_2 \otimes \cdots \otimes u_j$, $b(j) = u_{j+1} \otimes \cdots \otimes u_d$). As we have

$$Y_j(f_0) = 1^{\otimes(j-1)} \otimes \rho(f_0) \otimes 1^{\otimes(d-j)}, \quad \text{and} \quad \rho(f_0) = E_{1,n+1},$$

we have $\rho^{\otimes d}(Y_{j'_1}(f_0))v = w$, and $\rho^{\otimes d}(Y_j(f_0))v = 0$ for all $j \ne j'_1$, hence

$$(\rho_d(F_0))(m \otimes v) = \sum_{1 \le j \le d} \alpha_{jF}(m) \otimes \rho^{\otimes d}(Y_j(f_0))v, \quad \text{where} \quad \alpha_{jF}(m) = 1.$$

Assume step (i) holds for $r - 1$. Put $\widetilde{j} = (j_2, \ldots, j_r)$. Define $v' \in V^{(\widetilde{j},j')}$ to be a pure tensor with u_2 in the j_1 position, and distinct vectors from $\{u_3, \ldots, u_n\}$ in the remaining positions. Then $v = \rho_d(E_1)v'$. Indeed, recall that $\rho_d(E_1) = \sum_k 1^{\otimes(k-1)} \otimes \rho(e_1) \otimes 1^{\otimes(d-k)}$, that $\rho(e_1)u_j = \delta(2,j)u_1$, and that v' has u_2 only at position j_1 (and u_1 only at positions $j_2, \ldots, j_r$), so only $k = j_1$ survives in the sum over k that defines $\rho_d(E_1)$, and $(\rho_d(E_1))v' = v$.

Define v'' by replacing u_{n+1} in position $j' = j'_1$ in v' by u_1, and v''' by replacing u_2 in position j_1 in v'' by u_1. Now $r(v') = r - 1$, so we can apply the induction on r (in the third equality below, and a Serre relation in the second):

$$(\rho_d(F_0))(m \otimes v) = \rho_d(F_0)\rho_d(E_1)(m \otimes v') = \rho_d(E_1)\rho_d(F_0)(m \otimes v')$$
$$= \rho_d(E_1) \sum_{1 \le \ell \le d} \alpha_{\ell,F}(m) \otimes \rho^{\otimes d}(Y_\ell(f_0))v'.$$

Recall again that $Y_\ell(f_0)$ is $1^{\otimes(\ell-1)} \otimes \rho(f_0) \otimes 1^{\otimes(d-\ell)}$, and $\rho(f_0) = E_{1,n+1}$, and u_{n+1} occurs only at position j'_1 in v'. Then only $\ell = j'_1$ survives in the sum, that becomes v''. Since u_2 occurs in v'' only in position j_1, in the sum defining $\rho_d(E_1)$ only the summand indexed by $k = j_1$ survives when acting on v'', and it is $1^{\otimes(j_r-1)} \otimes \rho(e_1) \otimes 1^{\otimes(d-j_r)}$. So $\rho_d(E_1)$ maps v'' to v'''. We obtain $\alpha_{j'_1,F}(m)$ times $v''' = \rho^{\otimes d}(Y_{j'_1}(f_0))v$. For other j we have $0 = \rho^{\otimes d}(Y_j(f_0))v$. So we end up with $\sum_j \alpha_{j,F}(m) \otimes \rho^{\otimes d}(Y_j(f_0))v$, completing step (i).

Proof of step (ii): Assume the proposition holds for all $v \in V^{(j,j')}$ with less than s components u_{n+1}. As in step (i), it suffices to prove the claim for one element $v \neq 0$ in $V^{(j,j')}$ that has distinct entries from $\{u_2, \ldots, u_{n-1}\}$ in the remaining positions. Fix such a v. Let v' be the tensor obtained from v on replacing u_{n+1} in positions j'_{s-1} and j'_s by u_n. We claim that

$$\rho_d(F_n)^2 v' = 2v.$$

To see this, recall that $\rho(f_n) = E_{n+1,n}$,

$$\rho_d(F_n) = \sum_{1 \le k \le d} 1^{\otimes(k-1)} \otimes \rho(f_n) \otimes 1^{\otimes(d-k)}.$$

So in $\rho_d(F_n)^2 v'$ the sum over k in each $\rho_d(F_n)$ reduces to $k = j'_{s-1}$, j'_s, and all factors in positions $\neq j'_{s-1}$, j'_s in each summand, commute. At these two positions the components of v' are $u_n \otimes u_n$ and those of $\rho_d(F_n)^2$ are

$$(\rho(f_n) \otimes 1 + 1 \otimes \rho(f_n))(\rho(f_n) \otimes 1 + 1 \otimes \rho(f_n))$$
$$= \rho(f_n) \otimes \rho(f_n) + \rho(f_n) \otimes \rho(f_n)$$

as $\rho(f_n)^2 = 0$. So $\rho_d(F_n)^2 v'$ equals

$$(1^{\otimes(j'_{s-1}-1)} \otimes \rho(f_n) \otimes 1^{\otimes(j'_s-1-j'_{s-1})} \otimes (\rho(f_n) + \rho(f_n)) \otimes 1^{\otimes(d-j'_s)})v'.$$

Now $\rho(f_n)u_n = u_{n+1}$, so in conclusion $v = \frac{1}{2}\rho_d(F_n)^2 v'$, as claimed.
To continue we use the Serre relation $\rho_d([F_n, [F_n, F_0]]) = 0$, thus:

$$\rho_d(F_0)\rho_d(F_n)^2 = 2\rho_d(F_n)\rho_d(F_0)\rho_d(F_n) - \rho_d(F_n)^2\rho_d(F_0)$$

in the second equality below:

$$(\rho_d(F_0))(m \otimes v) = \frac{1}{2}\rho_d(F_0)\rho_d(F_n)^2(m \otimes v') = A + B,$$

$$A = \rho_d(F_n)\rho_d(F_0)\rho_d(F_n)(m \otimes v'), \quad B = -\frac{1}{2}\rho_d(F_n)^2\rho_d(F_0)(m \otimes v').$$

To find B, we write by induction

$$(\rho_d(F_0))(m \otimes v') = \sum_{1 \le k \le s-2} \alpha_{j'_k,F}(m) \otimes \rho^{\otimes d}(Y_{j'_k}(f_0))v', \;\; Y_j(f_0) = 1^{\otimes(j-1)} \otimes \rho(f_0) \otimes 1^{\otimes(d-j)},$$

as u_{n+1} occurs only at the $s - 2 < s$ positions $j'_1, \ldots, j'_{s-2}$ in v'. Recall that $\rho(f_0) = E_{1,n+1}$. Note that $\rho_d(F_n)$ changes the factors (u_n to u_{n+1}) of v' only at the positions j'_{s-1}, j'_s.

Applying $\rho_d(F_n)$ to $(\rho_d(F_0))(m\otimes v')$ would send the part $u_n\otimes u_n$ at the positions j'_{s-1} and j'_s to $u_{n+1}\otimes u_n$ (from the summand of $\rho_d(F_n)$ with (j'_{s-1},j'_s)-parts $\rho(f_n)\otimes 1$), plus $u_n\otimes u_{n+1}$ (from the summand of $\rho_d(F_n)$ with (j'_{s-1},j'_s)-parts $1\otimes\rho(f_n)$). Applying $\rho_d(F_n)$ again we obtain

$$u_{n+1}\otimes u_{n+1}+u_{n+1}\otimes u_{n+1}=2u_{n+1}\otimes u_{n+1}.$$

Now $\rho^{\otimes d}(Y_{j'_k}(f_0))$ acts on the two factors $u_{n+1}\otimes u_{n+1}$ of v at the positions (j'_{s-1},j'_s) trivially. So in summary,

$$B=-\sum_{1\le k\le s-2}\alpha_{j'_k}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v.$$

To compute A, let v'' (resp. v''') be obtained from v' on replacing the vector u_n at the j'_{s-1} (resp. j'_s) position by u_{n+1}. Observe that

$$(\rho_d(F_n))(m\otimes v')=m\otimes v''+m\otimes v'''.$$

(Applying $\rho_d(F_n)$ again we recover the result of the start of the proof, namely that $(\rho_d(F_n)^2)(m\otimes v')=2(m\otimes v)$.) As $s(v'')=s-1=s(v''')<s$, by induction we get

$$\rho_d(F_0)\rho_d(F_n)(m\otimes v')=\sum_{k\ne s}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v''+\sum_{k\ne s-1}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v'''.$$

Now we apply $\rho_d(F_n)$. As v'' has u_n only at the j'_s-position, we get

$$\rho_d(F_n)\sum_{k\ne s}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v''=\sum_{k\le s-1}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v.$$

Denote this by A_1. As v''' has u_n only at the j'_{s-1} position,

$$\rho_d(F_n)\sum_{k\ne s-1}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v'''=A_2+A_3,$$

$$A_2=\alpha_{j'_s,F}(m)\otimes\rho^{\otimes d}(Y_{j'_s}(f_0))v,\qquad A_3=\sum_{k\le s-2}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v.$$

Then $A=A_1+A_2+A_3=A_3+A_2+A_1$. So $B+A$ is

$$\begin{aligned}(\rho_d(F_0))(m\otimes v)=&-\sum_{1\le k\le s-2}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v\\&+\sum_{k\le s-2}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v+\alpha_{j'_s,F}(m)\otimes\rho^{\otimes d}(Y_{j'_s}(f_0))v\\&+\sum_{1\le k\le s-1}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v=\sum_{1\le k\le s}\alpha_{j'_k,F}(m)\otimes\rho^{\otimes d}(Y_{j'_k}(f_0))v.\end{aligned}$$ ■

Recall that $S_d^{\mathrm{aff}}=\mathbb{Z}^d\rtimes S_d$, and $y_j=(0,\dots,0,1,0,\dots,0)$ denote the generators of $\mathbb{Z}^d$ in multiplicative notation.

Proposition 7.4. *Setting* $my_j = \alpha_{jE}(m)$, $my_j^{-1} = \alpha_{jF}(m)$, *defines a right* $\mathbb{C}[S_d^{\mathrm{aff}}]$*-module structure on* M*, extending its* $\mathbb{C}[S_d]$*-module structure.*

Proof. We have to check the following relations:

$$\text{(i)}\ \ y_j y_j^{-1} = 1 = y_j^{-1} y_j\,; \qquad \text{(ii)}\ \ y_j y_k = y_k y_j\,; \qquad \text{(iii)}\ \ y_{j+1} = \sigma_j y_j \sigma_j\,.$$

To prove (i) and (ii), we compute both sides of the equality

$$(\rho_d([E_0, F_0]))(m \otimes v) = \rho_d(-h_0)(m \otimes v).$$

For (i) we take v with u_{n+1} in the jth position and $u_{n+1-(d-1)}, \dots, u_{n+1-1}$ in the remaining positions, in any order.

For (ii) take v to be a tensor with u_1 in the jth place, u_{n+1} in the kth position, and distinct vectors from $\{u_2, \dots, u_n\}$ in the other positions. Note that since the central element c acts as 0 on the $\widetilde{\mathfrak{g}}$-module W, we have $\rho_d([E_0, F_0]) = \rho_d([e_0, f_0]) = \rho_d(h_0)$, $h_0 = -\theta^\vee$.

For (iii), take $v = u_{i_1} \otimes \cdots \otimes u_{i_d} \in \mathbb{E}^{\otimes d}$ with $i_j = 2$, $i_{j+1} = 1$, and the remaining i_k are distinct from $\{3, \dots, n\}$. This is possible since $d \le n$. So: v has u_2 at position j, u_1 at position $j+1$. The vector v' is obtained from v on replacing u_1 at position $j+1$ by u_{n+1}. The vector v'' is obtained from v' on replacing u_2 at position j by u_{n+1}, and u_{n+1} at position $j+1$ by u_2. The vector v''' is obtained from v on replacing u_2 at position j by u_1, and u_1 at position $j+1$ by u_2.

Now looking at $\sigma = \sigma_j = (j, j+1)$ and only at the factors at these places, we have $\rho(\sigma_j)(u_{n+1} \otimes u_2) = u_2 \otimes u_{n+1}$, and $\rho(\sigma_j)(u_2 \otimes u_1) = u_1 \otimes u_2$. Then $\rho(\sigma_j)v = v'''$, $\rho(\sigma_j)v'' = v'$,

$$\begin{aligned} m \cdot \rho(\sigma_j) y_j \rho(\sigma_j) \otimes v &= m \cdot \rho(\sigma_j) y_j \otimes v''' = (\rho_d(F_0))(m \cdot \rho(\sigma_j) \otimes v'') \\ &= (\rho_d(F_0))(m \otimes \rho(\sigma_j) v'') = (\rho_d(F_0))(m \otimes v') = m y_{j+1} \otimes v. \end{aligned}$$

Since v has distinct components, Proposition 7.1 implies $m \cdot y_{j+1} = m \cdot \rho(\sigma_j) y_j \rho(\sigma_j)$ for all $m \in M$. This completes the proof that $W \simeq \mathcal{F}(M)$ as a $\mathcal{L}\mathfrak{g}$-module. ∎

To show that $\mathcal{F}$ is an equivalence we still need to show that it is bijective on sets of morphisms. Injectivity of $\mathcal{F}$ follows from that of the Schur functor $\mathcal{S}$. For surjectivity, let $\mathbb{F} : \mathcal{F}(M) \to \mathcal{F}(M')$ be a homomorphism of $\mathcal{L}\mathfrak{g}$-modules. By Proposition 7.1, $\mathbb{F} = \mathcal{S}(f)$ for some homomorphism $f : M \to M'$ of $\mathbb{C}[S_d]$-modules. Since $\mathbb{F}$ commutes with the action of F_0 we have $(\rho_d(F_0)\mathbb{F})(m \otimes v) = (\mathbb{F}\rho_d(F_0))(m \otimes v)$, i.e.,

$$\sum_{1 \le j \le d} f(m) \cdot y_j \otimes \rho^{\otimes d}(Y_j(f_0))v = \sum_{1 \le j \le d} f(m y_j) \otimes \rho^{\otimes d}(Y_j(f_0))v$$

for all $m \in M$ and $v \in \mathbb{E}^{\otimes d}$. Choosing v suitably we deduce that $f(my_j) = f(m)y_j$ for all j $(1 \le j \le d)$. This completes the proof of the theorem 1.1. ∎

8. The functor commutes with induction

The functor $\mathcal{F}$ commutes with induction, as we proceed to explain. To simplify the exposition, we deal below with two factors. But this extends at once to the case of any finite number of factors. First we note that the natural monomorphism $S_{d_1} \times S_{d_2} \hookrightarrow S_d$, $d = d_1 + d_2$, extends to affine Weyl groups.

Proposition 8.1. *There exists a unique homomorphism* $\widetilde{\iota} = \widetilde{\iota}_{d_1,d_2} : \widetilde{A}_1 \otimes \widetilde{A}_2 \to \widetilde{A}$ *of algebras, where*

$$\widetilde{A}_i = \mathbb{C}[S_{d_i} \ltimes \mathbb{Z}^{d_i}] \quad (i = 1, 2), \qquad \widetilde{A} = \mathbb{C}[S_d \ltimes \mathbb{Z}^d],$$

with $\widetilde{\iota}(\sigma_i \otimes 1) = \sigma_i$ $(1 \le i < d_1)$, $\widetilde{\iota}(y_j \otimes 1) = y_j$ $(1 \le j \le d_1)$, *where* $\sigma_i = (i, i+1) \in S_d$, $y_j = (0, \dots, 0, 1, 0, \dots, 0)$ $(1$ *at the* j*th place*$)$, $\widetilde{\iota}(1 \otimes \sigma_i) = \sigma_{d_1+i}$ $(1 \le i < d_2)$, $\widetilde{\iota}(1 \otimes y_j) = y_{d_1+j}$ $(1 \le j \le d_2)$. *It is a monomorphism, and its restriction* $\iota = \iota_{d_1,d_2}$ *is the monomorphism* $A_1 \otimes A_2 \hookrightarrow A$, $A_i = \mathbb{C}[S_{d_i}]$ $(i = 1, 2)$, $A = \mathbb{C}[S_d]$.

Note also that there is a natural homomorphism of algebras $\mathbb{C}[S_d] \hookrightarrow \mathbb{C}[S_d \ltimes \mathbb{Z}^d]$, and the multiplication map $\mathbb{C}[y_1^{\pm 1}, \dots, y_d^{\pm 1}] \otimes \mathbb{C}[S_d] \to \mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ is an isomorphism of vector spaces.

Let M_i be a (right) $\widetilde{A}_i$-module $(i = 1, 2)$. Let $M_1 \otimes M_2$ be their (outer) tensor product; it is an $\widetilde{A}_1 \otimes \widetilde{A}_2$-module. The induced $\widetilde{A}$-module, denoted $M_1 \widetilde{\times} M_2$, is defined by

$$M_1 \widetilde{\times} M_2 = \mathrm{Ind}_{\widetilde{A}_1 \otimes \widetilde{A}_2}^{\widetilde{A}} (M_1 \otimes M_2) = (M_1 \otimes M_2) \otimes_{\widetilde{A}_1 \otimes \widetilde{A}_2} \widetilde{A}.$$

An analogous definition applies to A_i-modules, denoted by $\times$. The products $\times$, $\widetilde{\times}$ are associative up to isomorphism, and can be defined for any finite number of factors, not only two. Denote by $M_i | A_i$ the A_i-module obtained from the $\widetilde{A}_i$-module M_i by restriction.

Proposition 8.2. *Let* M_i *be a finite dimensional* $\widetilde{A}_i$*-module* $(i = 1, 2)$. *Then there is a canonical isomorphism of* A*-modules*

$$(M_1 \widetilde{\times} M_2) | A \simeq M_1 | A_1 \times M_2 | A_2.$$

Proof. The canonical map $M_1 | A_1 \times M_2 | A_2 \to (M_1 \widetilde{\times} M_2) | A$, defined by

$$(m_1 \otimes m_2) \otimes a \mapsto (m_1 \otimes m_2) \otimes a \qquad (m_i \in M_i, \quad a \in A),$$

is a well defined surjective homomorphism of A-modules. The rank of $\widetilde{A} = \mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ as an $\widetilde{A}_1 \otimes \widetilde{A}_2 = \mathbb{C}[S_{d_1} \ltimes \mathbb{Z}^{d_1}] \otimes \mathbb{C}[S_{d_2} \ltimes \mathbb{Z}^{d_2}]$-module equals that of $A = \mathbb{C}[S_d]$ as an $A_1 \otimes A_2 = \mathbb{C}[S_{d_1}] \otimes \mathbb{C}[S_{d_2}]$-module, by the comment following the previous proposition. Hence $\dim_{\mathbb{C}}(M_1 \widetilde{\times} M_2) = \dim_{\mathbb{C}}(M_1 | A_1 \times M_2 | A_2)$. ∎

The functor $\mathcal{F}$ respects the product structure of induction of modules.

Proposition 8.3. *Let* M_i *be a finite dimensional* $\widetilde{A}_i$*-module* $(i = 1, 2)$. *Then there is a canonical isomorphism of* $\mathcal{L}\mathfrak{g}$*-modules,* $\mathfrak{g} = \mathrm{sl}_n$, $\mathcal{F}(M_1 \widetilde{\times} M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$.

Proof. Recall the Frobenius induction-restriction reciprocity: if $j : B \to A$ is a homomorphism of unital associative algebras over a field, M is a right B-module, W a left A-module, and $W|B$ is W regarded as a left B-module via j, then there is a canonical isomorphism of vector spaces $\mathrm{Ind}_B^A(M) \otimes W \simeq M \otimes_B W|B$. It is given by $(m \otimes a) \otimes w \mapsto m \otimes aw$, $m \in M$, $a \in A$, $w \in W$. Take $A = \mathbb{C}[S_d]$, $B = A_1 \otimes A_2$, $A_i = \mathbb{C}[S_{d_i}]$, $d_1 + d_2 = d$, $j = \iota_{d_1,d_2}$, $M = M_1 \widetilde{\times} M_2$, $W = \mathbb{E}^{\otimes d}$. Note that $W \simeq V^{\otimes d_1} \otimes V^{\otimes d_2}$ as an $A_1 \otimes A_2$-module, so we get a canonical isomorphism of vector spaces

$$\mathcal{F}(M_1 \widetilde{\times} M_2) \to (M_1 \times M_2) \otimes_{A_1 \otimes A_2} (V^{\otimes d_1} \otimes V^{\otimes d_2}).$$

The right side is isomorphic to $\mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$ as a vector space. It remains to check that the resulting isomorphism $\mathcal{F}(M_1 \widetilde{\times} M_2) \to \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$ of vector spaces commutes with the action of $\mathcal{L}\mathfrak{g}$, from which the proposition follows. ■

Let $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{C}^{\times d}$, and define the evaluation map $\varepsilon_{\mathbf{a}} : \mathbb{C}[S_d \ltimes \mathbb{Z}^d] \to \mathbb{C}[S_d]$ by $\sigma_i \mapsto \sigma_i$ $(1 \le i < d)$, $y_j \mapsto a_j$ $(1 \le j \le d)$. Let $I_{\mathbf{a}}$ be the ideal generated by $y_j - a_j$ $(1 \le j \le d)$ in $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$, and $M_{\mathbf{a}}$ the quotient of $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ by $I_{\mathbf{a}}$. It is a finite dimensional $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-module. As a $\mathbb{C}[S_d]$-module it is isomorphic to the right regular representation. Thus $M_{\mathbf{a}}$ is the pullback of the right regular representation $\mathbb{C}[S_d]$ via the evaluation map $\varepsilon_{\mathbf{a}}$.

We also have the evaluation map $\mathrm{ev}_a : \mathcal{L}\mathfrak{g} \to \mathfrak{g}$ for $a \in \mathbb{C}^\times$, by $t \mapsto a$. Via this map a $\mathfrak{g}$-module W can be pulled back to an $\mathcal{L}\mathfrak{g}$-module $W_a = \mathrm{ev}_a^* W$. In particular, if $\mathbb{E} = \mathbb{C}^n$ as a $\mathfrak{g} = \mathrm{sl}_n$-module, we have the $\mathcal{L}\mathfrak{g}$-module $\mathbb{E}_a = \mathrm{ev}_a^* \mathbb{E}$.

Proposition 8.4. *Let* $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{C}^{\times d}$; $d, n \ge 1$. *There is a canonical isomorphism of* $\mathcal{L}\mathfrak{g}$*-modules* $\mathcal{F}(M_{\mathbf{a}}) \simeq \mathbb{E}_{a_1} \otimes \cdots \otimes \mathbb{E}_{a_d}$.

Proof. As a $\mathbb{C}[S_d]$-module, $M_{\mathbf{a}}$ is the right regular representation. Hence the map $\mathbb{E}^{\otimes d} \to \mathcal{S}(M_{\mathbf{a}})$, $v \mapsto 1 \otimes v$, is an isomorphism of $\mathfrak{g}$-modules. Now

$$(\rho_d(E_0))(1 \otimes v) = \sum_{1 \le j \le d} 1 \cdot y_j \otimes Y_j(e_0) v = \Big(\sum_{1 \le j \le d} a_j Y_j(e_0) \Big) v.$$

Also $\Delta_d(E_0) = \sum_{1 \le j \le d} 1^{\otimes(j-1)} \otimes \rho(t \otimes e_0) \otimes 1^{\otimes(d-j)}$ acts on $\mathbb{E}_{a_1} \otimes \cdots \otimes \mathbb{E}_{a_d}$ as

$$\sum_j 1^{\otimes(j-1)} \otimes a_j \rho(e_0) \otimes 1^{\otimes(d-j)} = \sum_j a_j Y_j(e_0).$$

The map $\mathbb{E}^{\otimes d} \to \mathcal{S}(M_{\mathbf{a}})$ also commutes with F_0, thus giving the asserted isomorphism. ■

9. Applications: irreducible representations of $\mathcal{L}\mathfrak{g}$

The irreducible representations of $S_d \ltimes \mathbb{Z}^d$ can be described by Mackey theory, see e.g. [27, Section 8.2], as follows.

Let $G = H \ltimes A$ be a group, where A is a normal commutative subgroup and H a finite subgroup, acting on A. Let $\chi : A \to \mathbb{C}^\times$ be a character (multiplicative function). Denote the stabilizer of χ in G by $A_\chi = \{g \in G; \chi(gag^{-1}) = \chi(a)$ for all $a \in A\}$. This stabilizer is a subgroup of $G = H \ltimes A$, and it contains A, hence it is of the form $A_\chi = H' \ltimes A$ for some subgroup H' of H. Then χ extends to $\chi' : H' \ltimes A \to \mathbb{C}^\times$ by $\chi'(ha) = \chi(a)$. Let ρ be an irreducible representation of H'. Define ρ' to be the composition of ρ followed by the natural projection $H' \ltimes A \twoheadrightarrow H'$. Mackey theory asserts:

Proposition 9.1. *The induced representation* $\mathrm{Ind}(\chi' \otimes \rho'; H' \ltimes A, G)$ *is irreducible. It determines uniquely the datum* (H', χ, ρ). *Each irreducible representation of* G *has this form.*

We use this with $A = \mathbb{Z}^d$, $H = S_d$. A character χ of $\mathbb{Z}^d$ is a d-tuple $\mathbf{a} = (p_1^{d_1}, \dots, p_k^{d_k}) \in \mathbb{C}^{\times d}$, where $p_i^{d_i} = (p_i, \dots, p_i) \in \mathbb{C}^{\times d_i}$. The stabilizer has the form $H' \ltimes \mathbb{Z}^d$ with $H' = S_{d_1} \times \cdots \times S_{d_k}$. So an irreducible representation of $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ is determined by $(d_1, \dots, d_k)$, $d_i \geq 1$, $d_1 + \cdots + d_k = d$, distinct $a_i \in \mathbb{C}^\times$, and irreducible representations ρ_i of S_{d_i}, $1 \leq i \leq k$.

Let us express this using evaluation maps. Define the group algebra homomorphism $\varepsilon_{d.a} : \mathbb{C}[S_d^{\text{aff}}] \to \mathbb{C}[S_d]$ that maps each $\sigma \in S_d$ to itself, and y_j to a for all j, $1 \leq j \leq d$. Then $\varepsilon_{d.a} = \varepsilon_{\mathbf{a}}$ with $\mathbf{a} = (a, \dots, a) \in \mathbb{C}^{\times d}$. If M is an irreducible $\mathbb{C}[S_d]$-module, pulling M back by $\varepsilon_{d.a}$ gives an irreducible $\mathbb{C}[S_d^{\text{aff}}]$-module $M_{\mathbf{a}} = M_{d.a} := \varepsilon_{d.a}^* M$. When $\mathbf{a} = (p_1^{d_1}, \dots, p_k^{d_k})$, $p_i^{d_i} = (p_i, \dots, p_i) \in \mathbb{C}^{\times d_i}$ and M_i are $\mathbb{C}[S_{d_i}]$-modules, we write

$$(M_1 \times \cdots \times M_k)_{\mathbf{a}} = \varepsilon_{\mathbf{a}}^*(M_1 \times \cdots \times M_k)$$
$$= \varepsilon_{d_1.p_1}^* M_1 \widetilde{\times} \dots \widetilde{\times} \varepsilon_{d_k.p_k}^* M_k = M_{1,d_1.p_1} \widetilde{\times} \dots \widetilde{\times} M_{k,d_k.p_k}.$$

In summary we deduce from Mackey's theory:

Proposition 9.2. *Every finite dimensional irreducible $\mathbb{C}[S_d^{\text{aff}}]$-module is isomorphic to a product $M_{1,d_1.p_1} \widetilde{\times} \dots \widetilde{\times} M_{k,d_k.p_k}$ of $M_{d_i.p_i}$'s, $d = d_1 + \cdots + d_k$, distinct p_i.*

The theorem permits translating this result to the context of $\mathcal{L}\mathfrak{g}$, as follows.

As above, for each $a \in \mathbb{C}^\times$ there is a Lie algebra homomorphism $\text{ev}_a : \mathcal{L}\mathfrak{g} \to \mathfrak{g}$, defined by $\text{ev}_a(P(t) \otimes x) = P(a) \otimes x$. If W is an irreducible $\mathfrak{g}$-module, its pullback by ev_a is an irreducible $\mathcal{L}\mathfrak{g}$-module W_a.

Applying the functor $\mathcal{F}$, for $a \in \mathbb{C}^\times$ and a $\mathbb{C}[S_d]$-module M we obtain

$$\mathcal{F}(M_{d.a}) = \mathcal{F}(\varepsilon_{d.a}^* M) = (\varepsilon_{d.a}^* M) \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d} = \text{ev}_a^*(M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}) = \text{ev}_a^*(\mathcal{S}(M)) = \mathcal{S}(M)_a.$$

In general we have

$$\mathcal{F}(M_{1,d_1.p_1} \widetilde{\times} \dots \widetilde{\times} M_{k,d_k.p_k}) = \mathcal{F}((M_1 \times \cdots \times M_k)_{\mathbf{a}}) = \varepsilon_{\mathbf{a}}^*(M_1 \times \cdots \times M_k) \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$$

$$= \text{ev}_{p_1}^*(M_1 \otimes_{\mathbb{C}[S_{d_1}]} \mathbb{E}^{\otimes d_1}) \otimes \cdots \otimes \text{ev}_{p_k}^*(M_k \otimes_{\mathbb{C}[S_{d_k}]} \mathbb{E}^{\otimes d_k}) = \text{ev}_{p_1}^*(\mathcal{S}(M_1)) \otimes \cdots \otimes \text{ev}_{p_k}^*(\mathcal{S}(M_k)).$$

From Theorem 1.1 we then conclude

Corollary 9.3. *Every finite dimensional irreducible representation of the loop algebra $\mathcal{L}\mathfrak{g}$, $\mathfrak{g} = \text{sl}_n$, the subquotients of whose restriction to $\mathfrak{g}$ are subrepresentations of $\mathbb{E}^{\otimes d}$, is a tensor product of evaluation representations W_{p_i} at distinct points p_i. Here $W_{p_i} = \text{ev}_{p_i}^*(\mathcal{S}(M_i))$, where M_i is an irreducible $\mathbb{C}[S_{d_i}]$-module, $d = d_1 + \cdots + d_k$.*

It is stated in [6, top of p. 314] that the finite dimensional irreducible representations of the loop group $\mathcal{L}\mathfrak{g}$, $\mathfrak{g} = \text{sl}_n$ were classified in [5] (but $\widehat{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}d$ is studied there, which forces tensoring with $\mathcal{L}$; d should be eliminated) and that it is not difficult to prove that every finite dimensional irreducible representation of $\mathcal{L}\mathfrak{g}$ is isomorphic to a tensor product of W_a's. (In fact this can be deduced from the fact that a finite dimensional representation W of $\mathcal{L}\mathfrak{g}$ defines a Lie algebra homomorphism $\rho_W : \mathcal{L}\mathfrak{g} \to \text{End}_{\mathbb{C}} W$, with kernel say I_W, and [4, Lemma 1, §2.2], that asserts that $I_W = \mathfrak{g} \otimes I'_W$, where I'_W is an ideal of the PID $\mathbb{C}[t, t^{-1}]$. Thus $I'_W = \langle (t - a_1)^{n_1} \dots (t - a_k)^{n_k} \rangle$, for distinct $a_i \in \mathbb{C}^\times$ and $n_i \geq 1$, and W is irreducible precisely when all n_i are 1.) We deduce this claim from the theorem in the corollary above, and more: the $p_i \in \mathbb{C}^\times$ are distinct.

We conclude that every irreducible finite dimensional representation W of $\mathcal{L}\mathfrak{g}$ as in the Corollay integrates to a representation of $G(\mathcal{L})$. Indeed, such W is a tensor product of evaluation representations W_{p_i} at distinct points $p_i \in \mathbb{C}^\times$. Thus we have a Lie algebra homomorphism $\rho_W : \mathcal{L}\mathfrak{g} \to \operatorname{End}_{\mathbb{C}} W$ whose kernel has the form $I_W = \mathfrak{g} \otimes I'_W$, where I'_W is the ideal in $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ generated by $(t-p_1)\dots(t-p_k)$ for distinct $p_i \in \mathbb{C}^\times$. Namely W is a representation of the finite dimensional Lie algebra $\mathfrak{g}_W = \mathcal{L}\mathfrak{g}/I_W$. Since $G = \mathrm{SL}(n)$ is simply connected, W can be lifted to the finite dimensional group

$$G_W(\mathbb{C}) = G(\mathbb{C}[t,t^{-1}]/I'_W) \simeq G(\mathbb{C}[t,t^{-1}]/(t-p_1)) \times \cdots \times G(\mathbb{C}[t,t^{-1}]/(t-p_k)),$$

whose Lie algebra is $\mathfrak{g}_W \simeq \mathfrak{g}^k$. A generalization to the context of any finite dimensional representation of the loop group of the Lie algebra of a simply connected simple group is in section 13.

10. From algebras to groups

In this section we briefly recall how to pass from the Lie algebra form of the Schur duality, of a decomposition of a $\mathbb{C}[S_d] \times \mathrm{gl}_n(\mathbb{C})$-module, to the group form: a decomposition of a $\mathbb{C}[S_d] \times \mathrm{GL}(n,\mathbb{C})$-module. A more detailed introduction is at [13, Lectures 6, 15].

Proposition 10.1. *Let F be a field of characteristic 0. Let E be a finite dimensional vector space over F. Denote by $\operatorname{End} E$ the algebra of endomorphisms of E over F. Let A, B be two subalgebras of $\operatorname{End} E$. Assume A is semisimple, and $B = \operatorname{End}_A E$ is the centralizer of A in $\operatorname{End} E$. Then* (i) $A = \operatorname{End}_B E$; (ii) *$B$ is semisimple*; (iii) *as an $A \otimes B$-module, E decomposes as $E = \oplus_{i \in I} V_i \otimes W_i$, where V_i run through the set of all irreducible representations of A (up to equivalence), and W_i are irreducible representations of B.*

(i) asserts that the centralizer of the centralizer of A is A. (iii) defines a natural bijection between irreducible representations of A and of B.

Proof. Since A is semisimple, E decomposes as $E = \oplus_{i\in I} V_i \otimes W_i$, where $W_i = \operatorname{Hom}_A(V_i, E)$. Then $A = \oplus_i \operatorname{End} V_i$. By Schur's lemma $B = \operatorname{End}_A E$ is naturally isomorphic to $\oplus_i \operatorname{End}(W_i)$. ■

Recall that the *tensor algebra TU* of a vector space U over F is $TU = \oplus_{d \ge 0} U^{\otimes d}$ with multiplication $a \cdot b = a \otimes b$ ($a \in U^{\otimes i}$, $b \in U^{\otimes j}$). A choice of a basis $v_1, \dots, v_m$ of U defines an isomorphism of TU with the free algebra $F\langle v_1, \dots, v_m\rangle$. The *symmetric algebra* $\operatorname{Sym} U$ of U is the quotient of TU by the ideal generated by $u \otimes v - v \otimes u$, u, $v \in U$. A choice of a basis $v_1, \dots, v_m$ for U defines an isomorphism of $\operatorname{Sym} U$ with the polynomial algebra $F[v_1, \dots, v_m]$. We have $\operatorname{Sym} U = \oplus_{j \ge 0} \operatorname{Sym}^j U$, $\operatorname{Sym}^j U = U^{\otimes j}/\langle u \otimes v - v \otimes u;\, u,\, v \in U\rangle$ is the vector space of the polynomials homogeneous of degree j in $v_1, \dots, v_m$. The space $\operatorname{Sym}^j U$ is an irreducible representation of $\mathrm{GL}(U)$.

Lemma 10.2. (i) *For any finite dimensional vector space U over F the space $\operatorname{Sym}^d U$ is spanned by the $u^{\otimes d} = u \otimes \cdots \otimes u$, $u \in U$.* (ii) *For any algebra A over F, the algebra $\operatorname{Sym}^d A$ is spanned by*

$$\Delta_d(a) = \sum_{1 \le j \le d} I^{\otimes(j-1)} \otimes a \otimes I^{\otimes(d-j)}, \qquad a \in A.$$

Proof. (i) The subspace of $\operatorname{Sym}^d U$ spanned by $u^{\otimes d}$, $u \in U$, is a nonzero representation of the irreducible $\operatorname{Sym}^d U$, so (i) follows.

(ii) Put $x = (x_1, \dots, x_d)$ and $H_m(x) = \sum_{1 \le j \le d} x_j^m$. The fundamental theorem of symmetric functions assert that there exists a polynomial P over $\mathbb{Q}$ with $P(H_1(x), \dots, H_d(x)) = x_1 \dots x_d$. For example, when $d = 2$,

$$H_1(x)^2 - H_2(x) = (x_1 + x_2)^2 - (x_1^2 + x_2^2) = 2x_1x_2;$$

when $d = 3$: $H_1^2 - 3H_1H_2 + 2H_3 = 6x_1x_2x_3$. Now if

$$x = (a \otimes I^{\otimes(d-1)}, \dots, I^{\otimes(j-1)} \otimes a \otimes I^{\otimes(d-j)}, \dots, I^{\otimes(d-1)} \otimes a),$$

then $H_m(x) = \Delta_d(a^m)$, and so

$$P(\Delta_d(a), \Delta_d(a^2), \dots, \Delta_d(a^d)) = a \otimes \cdots \otimes a \in T^d A = A^{\otimes d}.$$

But by (i) the $a^{\otimes d}$ span $\operatorname{Sym}^d A$, so (ii) follows. ■

We apply the double centralizer theorem with $E = \mathbb{E}^{\otimes d}$, where $\mathbb{E}$ is a finite dimensional vector space over F. Let A be the image of the group algebra $F[S_d]$ in $\operatorname{End} E$. As usual, $\operatorname{gl}(\mathbb{E})$ is $\operatorname{End} \mathbb{E}$ regarded as a Lie algebra with the bracket $[a, b] = ab - ba$. So we determine B.

Proposition 10.3. *The algebra $B = \operatorname{End}_A E$ is the image of the universal enveloping algebra $\mathfrak{U}(\operatorname{gl}(\mathbb{E}))$ under its natural action $\Delta_d(b)$ on E, thus B is generated by $\Delta_d(b)$, $b \in \operatorname{gl}(\mathbb{E})$.*

Proof. The image of $\mathfrak{U}(\operatorname{gl}(\mathbb{E}))$ commutes with A so it lies in B. By definition of Sym, $B = \operatorname{Sym}^d \operatorname{End} \mathbb{E}$. Then the proposition follows from (ii) of the lemma. ■

The algebra $A = \operatorname{Im} F[S_d]$ is semisimple by Maschke theorem, so the double centralizer theorem applies, and one obtains the following Schur duality theorem.

Proposition 10.4. (i) *The image A of $F[S_d]$ and the image B of $\mathfrak{U}(\operatorname{gl}(\mathbb{E}))$ in $\operatorname{End}_F(\mathbb{E}^{\otimes d})$ are the centralizers of each other.* (ii) *Both A and B are semisimple. In particular $\mathbb{E}^{\otimes d}$ is a semisimple $\operatorname{gl}(\mathbb{E})$-module.* (iii) *There is an $A \otimes B$-modules decomposition $\mathbb{E}^{\otimes d} = \oplus_\lambda S^\lambda \otimes V^\lambda$. The summation ranges over the partitions λ of d of length $\le n$, the S^λ are the Specht modules for S_d, and the V^λ are distinct irreducible representations of $\operatorname{gl}(\mathbb{E})$, or zero.*

To rephrase this in terms of the group $\operatorname{GL}(\mathbb{E})$, note

Proposition 10.5. *The image of $\operatorname{GL}(\mathbb{E})$ in $\operatorname{End}(\mathbb{E}^{\otimes d})$ spans B.*

Proof. Denote by B' the span in $\operatorname{End}(\mathbb{E}^{\otimes d})$ of $g^{\otimes d}$, $g \in \operatorname{GL}(\mathbb{E}) = \operatorname{Aut}(\mathbb{E})$. For any $b \in \operatorname{End}(\mathbb{E})$, B' contains $b^{\otimes d}$. Indeed, $t \cdot I + b$ is invertible for all t except for finitely many values, so $p(t) = (t \cdot I + b)^{\otimes d}$ lies in B'. Write $p(t) = p_0 + p_1 t + \cdots + p_{d-1} t^{d-1} + p_d t^d$; $p_d = I^{\otimes d} \in \operatorname{End}(\mathbb{E}^{\otimes d})$, and by induction $p^{(j)}(0) = j! p_j$ lies in $\operatorname{End}(\mathbb{E}^{\otimes d})$. In particular $p_0 = b^{\otimes d}$ lies in $\operatorname{End}(\mathbb{E}^{\otimes d})$. Now the proposition follows from (ii) of the lemma. ■

Then one concludes the group version of the Schur duality.

Corollary 10.6. *As a representation of $F[S_d] \times \operatorname{GL}(\mathbb{E})$, $\mathbb{E}^{\otimes d}$ decomposes as $\oplus_\lambda S^\lambda \otimes V^\lambda$, where $V^\lambda = \operatorname{Hom}_{F[S_d]}(S^\lambda, \mathbb{E}^{\otimes d})$ are distinct irreducible representations of $\operatorname{GL}(\mathbb{E})$, or zero.*

11. Translation to affine Lie groups

We would like to state a variant of the theorem for the affine or loop *group* $G(\mathcal{L})$, $G = \mathrm{SL}(n)$, rather than for the affine Lie algebra $\mathcal{L}\mathfrak{g}$, $\mathfrak{g} = \mathrm{sl}_n$. A tempting and natural first approach, but one which I do not know to complete, is as follows.

Let us recall that the group $\mathrm{SL}(n, \mathbb{C})$ is generated by the Chevalley-type generators

$$B = \{I + zE_{i,i+1},\, I + zE_{i+1,i};\, 1 \le i < n,\, z \in \mathbb{C}\};$$

here $E_{i,j}$ is the $n \times n$-matrix with entries 0 except on row i and column j where the entry is 1. Over $\mathcal{L} = \mathbb{C}[t, t^{-1}]$, where in fact $\mathbb{C}$ can be taken to be any field, we have:

Proposition 11.1. *For all $n > 1$ the group $\mathrm{SL}(n, \mathcal{L})$ is generated by*

$$B_t = \{I + ztE_{n,1},\, I + zE_{i,i+1},\, I + zt^{-1}E_{1,n},\, I + zE_{i+1,i};\, 1 \le i < n,\, z \in \mathbb{C}\}.$$

Proof. It suffices to show that the group $\mathrm{SL}(n, \mathcal{L})$ is equal to $E(n, \mathcal{L})$, the group generated by $\{I + pE_{i,j};\, 1 \le i \ne j \le n,\, p \in \mathcal{L}\}$, for all $n \ge 1$. Note that $E(n, \mathcal{L})$ contains the Weyl group of $\mathrm{SL}(n, \mathcal{L})$, thus the transpositions $I - E_{i,i} - E_{j,j} + E_{i,j} - E_{j,i}$, and the diagonal matrices with entries in $\mathcal{L}^\times = \mathbb{C}^\times t^{\mathbb{Z}}$. Multiplying $A \in \mathrm{SL}(n, \mathcal{L})$ on the left by elementary matrices in $E(n, \mathcal{L})$ is equivalent to performing elementary row operations on A. Denote the entries of A by $a_{i,j} \in \mathcal{L}$, $i=$ row, $j=$ column. Since $\det A = 1$, the ideal $\langle a_{i,1} \rangle$ generated by the entries on the first column of A is equal $\mathcal{L}$. Each $a_{i,1}$ has the form $t^{n_i} p_i$, $n_i \in \mathbb{Z}$, $p_i \in \mathbb{C}[t]$. The polynomial ring $\mathbb{C}[t]$ is a Euclidean domain, so performing row operations permits applying the division algorithm to each pair (p_{i_1}, p_{i_2}) (note that we use $I + yE_{i,j}$ with $y \in \mathcal{L}$), until the first column of A becomes $e_1 = {}^t(1, 0, \dots, 0)$ ($t=$ transpose). By induction on n, we may assume that A with top row and left column removed is I_{n-1}. Performing elementary operations on the right, namely on the columns, we get the top row of A to be e_1. ∎

A generalization to other algebras $\mathcal{L}$ is given by L. Vaserstein [29, Corollary 3.3], confirming a conjecture of H. Bass on stabilization in algebraic K-theory of $\mathrm{GL}(n)$ over a ring.

Corollary 11.2. *For $n > 2$, $\mathrm{SL}(n, \mathcal{L})$ is generated by B and $\mathrm{diag}(t, t^{-1}, 1, \dots, 1)$. For $n > 1$, $\mathrm{GL}(n, \mathcal{L})$ is generated by B and $\mathrm{diag}(t, 1, \dots, 1)$.*

It is tempting to expect the map e that sends each of $E = zE_{i,i+1}$, $zE_{i+1,i}$, $ztE_{n,1}$ and $zt^{-1}E_{1,n}$ to $I + E$, to extend to an isomorphism from the loop Lie algebra $\mathcal{L}\mathfrak{g}$ to the loop group $G(\mathcal{L})$, when $n > 1$, and use this to translate the theorem to the group language. However, the image of the exponential map is not polynomial, so that the image of the map e as above would not be in the loop group $G(\mathcal{L})$. See e.g. sl_2 and $[e_1, f_0] = \mathrm{diag}(t, 1/t)$.

Put $e_{i,j}(z) = I + zE_{i,j}$, $i \ne j$, $z \in \mathbb{C}$. The generators $e_{i,i+1}(z)$, $e_{i+1,i}(z)$ ($1 \le i < n$, $z \in \mathbb{C}$) of $\mathrm{SL}(n, \mathbb{C})$ generate in particular all $e_{i,j}(z)$, $i \ne j$ as commutators, and $\mathrm{SL}(n, \mathbb{C})$ has a presentation by the generators $e_{i,j}(z)$ ($1 \le i \ne j \le n$, $z \in \mathbb{C}$), subject to the Steinberg relations:

(1) $e_{i,j}(\lambda) e_{i,j}(\mu) = e_{i,j}(\lambda + \mu)$, $\lambda,\ \mu \in \mathbb{C}$;

(2) $[e_{i,j}(\lambda), e_{j,k}(\mu)] = e_{i,k}(\lambda\mu)$ when $i \ne k$;

(3) $[e_{i,j}(\lambda), e_{k,s}(\mu)] = 1$ if $i \ne s$, $j \ne k$.

Proposition 11.1 asserts that $\mathrm{SL}(n,\mathcal{L})$ can be presented by $e_{i,j}(z)$ $(i \neq j)$, $e_{i,j}(zt)$ $(i > j)$, $e_{i,j}(zt^{-1})$ $(i < j)$, subject to relations as in (1), (2), (3). A group analogue of the Theorem 1.1 would be given on defining the action of $e_{n,1}(zt)$ and $e_{1,n}(z/t)$ on $M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$, compatible with the diagonal $\mathrm{SL}(n,\mathbb{C})$-action on $\mathbb{E}^{\otimes d}$, so that the relations (e.g. $[e_{n,j}(\lambda t), e_{j,k}(\mu)] = e_{n,k}(\lambda\mu t)$ $(k \neq n)$ for all $j \neq k$, n) are held. But I do not know to do this.

The next section offers an algebro-geometric definition of the functor, in group theoretic terms, involving only finite dimensional varieties.

12. Geometric definition of the functor

Here is a sketch, explained to me by P. Deligne, of a geometric definition of the functor of Theorem 1.1 *purely in group theoretic terms*, involving only finite dimensional varieties.

Let M be a finite dimensional $\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$-module. Then $\mathbb{C}[S_d \ltimes \mathbb{Z}^d] \simeq \mathbb{C}[S_d] \ltimes \mathbb{C}[\mathbb{Z}^d]$, where from the perspective of an algebraic-geometer

$$\mathbb{C}[\mathbb{Z}^d] \simeq \mathbb{C}[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}] = \Gamma(\mathbb{G}_m^d, \mathcal{O})$$

is the affine ring of $\mathbb{G}_m^d$. So M can be viewed as the module of global sections $\Gamma(\mathcal{M}, \mathcal{O})$ of an S_d-equivariant quasi-coherent sheaf of modules $\mathcal{M}$ over $\mathbb{G}_m^d = \operatorname{Spec}\mathbb{C}[\mathbb{Z}^d] = \operatorname{Spec}\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$.

Let us briefly review this construction. If $(X, \mathcal{O}_X)$ is a ringed space, $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ a ring homomorphism from a ring R into the ring of global sections on X, M an R-module, $\pi : (X, \mathcal{O}_X) \to (\{*\}, R)$ the morphism of ringed spaces with $\pi : X \to \{*\}$ (= a point) the unique map and the ring map $\alpha : R \to \Gamma(X, \mathcal{O}_X)$, then we define $\mathcal{M} = \pi^* M$; it is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Equivalently, $\mathcal{M}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$, where the map $R \to \mathcal{O}_X(U)$ is the composition of α and the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.

This construction gives a functor from the category of R-modules M to the category of quasi-coheret sheaves $\mathcal{M}$ on X that commutes with arbitrary colimits. For every $x \in X$ we have $\mathcal{M}_x = \mathcal{O}_{X,x} \otimes_R M$ functorially in M; and for all $\mathcal{O}_X$-modules $\mathcal{G}$ we have $\mathrm{Mor}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{G}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{G}))$ where the R-module structure on $\Gamma(X, \mathcal{G})$ is obtained from the $\Gamma(X, \mathcal{O}_X)$-module structure via α. We say $\mathcal{M}$ is the sheaf associated to the module M and the ring map α, and if $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = \mathrm{id}_R$ we simply say that $\mathcal{M}$ is the *sheaf associated to the module* M. Let us quote also: if $(X, \mathcal{O}_X)$ is a ringed space, $R = \Gamma(X, \mathcal{O}_X)$, M an R-module, $\mathcal{M}$ the quasi-coherent sheaf of $\mathcal{O}_X$-modules associated to M, and $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ a morphism of ringed spaces, then $g^*\mathcal{M}$ is the sheaf associated to the $\Gamma(Y, \mathcal{O}_Y)$-module $\Gamma(Y, \mathcal{O}_Y) \otimes_R M$.

In our case we have $X = \mathbb{G}_m^d$, $R = \mathbb{C}[\mathbb{Z}^d] \simeq \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. As a quasi-coherent module over $\mathbb{G}_m^d$, $\mathcal{M}$ has finite support. Indeed, if M is a finite dimensional $\mathbb{C}[S_d] \ltimes \mathbb{C}[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$-module, it is in particular a $\mathbb{C}[t_i, t_i^{-1}]$-module, i.e. we have a homomorphism $\mathbb{C}[t_i, t_i^{-1}] \to \operatorname{End} M$, whose kernel is an ideal in $\mathbb{C}[t_i, t_i^{-1}]$, of the form $\langle \prod_j (t_i - a_{ij})^{m_{ij}} \rangle$, $a_{ij} \in \mathbb{C}^\times$. Localizing it, that is, reducing the product to one factor, and taking m to be the maximum of the m_{ij}, M defines a homomorphism $\mathbb{C}[S_d] \ltimes B_m \hookrightarrow \operatorname{End} M$, where

$$B_m = \mathbb{C}[t_1, \ldots, t_d]/\langle (t_1 - a_1)^m, (t_2 - a_2)^m, \ldots, (t_d - a_d)^m \rangle.$$

The ring B_m is local, with a nilpotent maximal ideal of dimension $md-1$ as a space over $\mathbb{C}$.

Suppose that this support is contained in the regular set of $a_i \neq a_j$ when $i \neq j$. Then it is a finite union of orbits of S_d, and M is a direct sum indexed by the orbits. Consider the case when the support is a single orbit. Even in this case, M can be quite subtle, not just "induced from a product of 1-dimensional representations", but rather extensions of such. Let the orbit be that of $\mathbf{a} = (a_1, \ldots, a_d)$, with distinct a_i. Then M is obtained as follows. One starts with a finite dimensional module over the ring B_m, with m large enough. The ring B is the tensor product of the rings $B_{i,m} = \mathbb{C}[t]/\langle (t - a_i)^m \rangle$. Then one transplant it to the other points of the orbit of $(a_1, \ldots, a_d)$, using the action of S_d on $\mathbb{G}_m^d$, and one takes the sum of these d coherent modules. Only the simple subquotients, as S_d-equivariant coherent modules with finite support on $\mathbb{G}_m^d$, are induced from a 1-dimensional representation, at $(a_1, \ldots, a_d)$. In general M is an extension of such irreducibles.

Such modules, with support in one regular orbit, should correspond to the $\mathcal{L}\mathfrak{g}$-modules all of whose simple subquotients are isomorphic to $\mathcal{L}\mathfrak{g} \to \prod_{1 \le i \le d} \mathrm{sl}_n$ (evaluation at the a_i) $\to \otimes_i \operatorname{End} \mathbb{E}$ ($\mathbb{E}$: defining representation of sl_n).

This abelian category of representations is the tensor product (a non-obvious notion) of the abelian categories of $\mathrm{SL}(n, \mathbb{C}[[t - a_i]])$-representations, factoring through $\mathrm{SL}(n, B_{i,m})$, with simple subquotients only the defining representation $\mathbb{E} = \mathbb{C}^n$ of the quotient $\mathrm{SL}(n, \mathbb{C})$. This corresponds to the fact that the abelian category of B_m-modules is the tensor product of the abelian categories of the $B_{i,m} = \mathbb{C}[t]/\langle (t-a_i)^m \rangle$-modules.

In general, an irreducible S_d-equivariant coherent module over $\mathbb{G}_m^d$ is obtained as follows. Start with a point $\mathbf{a} = (p_1, \ldots, p_1, p_2, \ldots, p_k) = (p_1^{d_1}, p_2^{d_2}, \ldots, p_k^{d_k})$ of $\mathbb{G}_m^d$ whose coordinates are $p_i \in \mathbb{G}_m$, occurring d_i times, the p_i being distinct, $1 \le i \le k$. Take irreducible representations M_i of the S_{d_i}. The tensor product of the M_i, put at $\mathbf{a}$, is equivariant for the action of the product of the S_{d_i}, that is the stabilizer of $\mathbf{a}$. Now, induce to S_d to get a representation whose support is the S_d-orbit of $\mathbf{a}$ in $\mathbb{G}_m^d$.

Applying the functor, one gets the tensor product of the representations (evaluation at p_i), followed by the representation of SL corresponding to the representation M_i of S_{d_i}.

As for $\mathcal{L}G$, let us review its definition. Let

$$A = \Gamma(\mathrm{SL}(n), \mathcal{O}) = \mathbb{C}[X_{ij}]/\langle \det(X_{ij}) = 1 \rangle$$

be the affine ring of $\mathrm{SL}(n)$, so $\mathrm{SL}(n) = \operatorname{Spec} A$. Let R be a $\mathbb{C}$-algebra, thus we have a homomorphism $i_R : \mathbb{C} \hookrightarrow R$. Put $S = \operatorname{Spec} R$. Then we have a morphism of bundles, that is a commutative square with top

$$S \times G \times \mathbb{G}_m = \operatorname{Spec} A \otimes R[t, t^{-1}] \overset{i_R^*}{\to} G \times \mathbb{G}_m = \operatorname{Spec} A \times \operatorname{Spec} \mathbb{C}[t, t^{-1}] = \operatorname{Spec} A[t, t^{-1}]$$

with arrows going down to

$$S \times \mathbb{G}_m = \operatorname{Spec} R \times \operatorname{Spec} \mathbb{C}[t, t^{-1}] = \operatorname{Spec} R[t, t^{-1}] \quad \overset{i_R^*}{\to} \quad \mathbb{G}_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}].$$

The loop group $\mathcal{L}G$ is the functor $\mathcal{L}G(R) = \Gamma(i_R^*(G \times \mathbb{G}_m), \mathcal{O}) = G(R[t, t^{-1}])$, where again $i_R^*(G \times \mathbb{G}_m) = S \times G \times \mathbb{G}_m = \operatorname{Spec} A \otimes R[t, t^{-1}]$.

Evaluation at $a \in \mathbb{G}_m$, thus the morphism $\mathrm{ev}_a : \mathcal{L}G \to G$ of functors $\mathcal{L}G(R) = G(R[t, t^{-1}]) \to G(R)$ by $t \mapsto a$, defines from the standard G-module $\mathbb{E} = \mathbb{A}^n$ a representation $\mathbb{E}_a = \mathrm{ev}_a^* \mathbb{E}$ of $\mathcal{L}G$ by $\mathcal{L}G \overset{\mathrm{ev}_a}{\to} G \hookrightarrow \operatorname{Aut} \mathbb{E}$, thus a bundle $\mathbb{E} \times \mathbb{G}_m \to \mathbb{G}_m$. As a morphism of functors it is

$$(\mathcal{L}G)(R) = G(R[t, t^{-1}]) \overset{\mathrm{ev}_a}{\to} G(R) \hookrightarrow \operatorname{Aut}_R(\mathbb{E} \otimes R).$$

Now using the morphisms $\mathbb{G}_m^d \overset{\mathrm{pr}_i}{\to} \mathbb{G}_m$, $\mathbf{a} = (a_1, \ldots, a_d) \mapsto a_i$ we define a bundle $\mathbb{E}^{\otimes d} \times \mathbb{G}_m^d \to \mathbb{G}_m^d$ by $\mathbb{E}_{\mathbf{a}}^{\otimes d} = \otimes_i \mathrm{pr}_i^*(\mathbb{E}_{a_i})$ over $\mathbf{a} \in \mathbb{G}_m^d$, that is S_d-equivariant. A point $g \in G(\mathcal{L})$ acts on $v_1 \otimes \cdots \otimes v_d \in \mathbb{E}_{\mathbf{a}}^d$ by $(\mathrm{ev}_{a_1}(g), \ldots, \mathrm{ev}_{a_d}(g))$.

In this notation, the functor is $\mathcal{F} : M = \Gamma(\mathcal{M}, \mathcal{O}) \mapsto \Gamma((\mathcal{M} \otimes_{\mathbb{G}_m^d} (\mathbb{E}^{\otimes d} \times \mathbb{G}_m^d)), \mathcal{O})_{\mathbb{C}[S_d]}$. It maps a $\mathbb{C}[S_d] \ltimes \mathbb{C}[\mathbb{Z}^d]$-module M to a representation W of $\mathcal{L}G$, $\mathcal{L}G(R) \to \operatorname{Aut}_R(W \otimes R)$ as a functor. From the scheme theoretic definition of the tangent space at the origin we get a representation of the loop Lie algebra

$$\mathfrak{g} \otimes \mathcal{L} = (T_e G)(\mathcal{L}) = \ker[G(\mathcal{L}[\varepsilon]/(\varepsilon^2)) \to G(\mathcal{L})],$$

thus $\mathcal{L}\mathfrak{g} = \ker[\mathcal{L}G(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \to \mathcal{L}G(\mathbb{C})]$, that occurs in Theorem 1.1.

It is simpler to deal with finite dimensional schemes. So we note that a finite dimensional representation W of $\mathcal{L}\mathfrak{g}$ is a homomorphism $\mathcal{L}\mathfrak{g} \to \operatorname{End} W$, hence it factors via $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/\langle \prod_i (t - a_i)^{m_i} \rangle$. Indeed, let I_W be its kernel. By [4, Lemma 1, §2.2], $I_W = \mathfrak{g} \otimes I'_W$, where I'_W is an ideal of the PID $\mathbb{C}[t, t^{-1}]$. Thus $I'_W = \langle (t - a_1)^{n_1} \ldots (t - a_k)^{n_k} \rangle$, for distinct $a_i \in \mathbb{C}^\times$ and $n_i \geq 1$. Localizing we get that there is only one factor in the product, and we take a fixed m, say the maximum of the m_i. Then $\mathcal{L}\mathfrak{g}$ acts on the product

$$\mathcal{R}_m = \otimes_{1 \leq i \leq d} (\mathbb{E} \otimes \mathbb{C}[t_i]/\langle (t_i - a_i)^m \rangle) = \mathbb{E}^{\otimes d} \otimes B_m,$$

$B_m = \mathbb{C}[t_1, \ldots, t_d]/\langle (t_1 - a_1)^m, \ldots, (t_d - a_d)^m \rangle$ as above, and the functor can be defined to be

$$\mathcal{F} : M = \Gamma(\mathcal{M}, \mathcal{O}) \mapsto \Gamma(\mathcal{M} \otimes_{B_m} (\mathbb{E}^{\otimes d} \otimes B_m), \mathcal{O})_{\mathbb{C}[S_d]}.$$

Here all varieties are finite dimensional. To get the unrestricted functor we take the direct limit over m on both sides.

13. Lifting finite dimensional modules to the group

From representation theoretic perspective, Shrawan Kumar pointed out to me that any finite dimensional $\mathcal{L}\mathfrak{g}$-module can be lifted to the loop group, using a Lemma of [4].

Proposition 13.1. *Any finite dimensional $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathcal{L}$-module, $\mathcal{L} = \mathbb{C}[t, t^{-1}]$, integrates to give a representation of $\mathcal{L}G = G(\mathcal{L})$, where $G = \mathrm{SL}(n)$, $\mathfrak{g} = \mathrm{sl}_n$.*

Proof. Let W be a finite dimensional representation of $\mathcal{L}\mathfrak{g}$. Thus we have a Lie algebra homomorphism

$$\rho_W : \mathcal{L}\mathfrak{g} \to \mathrm{End}_{\mathbb{C}} W.$$

Let I_W be its kernel. By [4, Lemma 1, §2.2], $I_W = \mathfrak{g} \otimes I'_W$, where I'_W is an ideal of the PID $\mathbb{C}[t, t^{-1}]$. Thus $I'_W = \langle (t-a_1)^{n_1} \dots (t-a_k)^{n_k} \rangle$, for distinct $a_i \in \mathbb{C}^\times$ and $n_i \geq 1$.

So we can think of W as a representation of a finite dimensional Lie algebra $\mathfrak{g}_W := \mathcal{L}\mathfrak{g}/I_W$. Consider the finite dimensional algebraic group $G_W := G(\mathbb{C}[t, t^{-1}]/I'_W)$. Its Lie algebra is $\mathfrak{g}_W$.

According to the first lemma below, G_W is a simply connected group. Using this, by a general result the representation W of $\mathfrak{g}_W$ integrates to a holomorphic representation of G_W.

Now the sub Lie algebra $\mathfrak{r} := (\mathfrak{g}_\theta \otimes t^{-1}) \oplus (\mathfrak{g}_{-\theta} \otimes t) \oplus \mathbb{C}(-\theta^\vee)$ of $\mathfrak{g}_W$ (where θ is the highest root of $\mathfrak{g}$, $\mathfrak{g}_\theta$ is the highest weight space, $\mathfrak{g}_{-\theta}$ the lowest weight space, and the bilinear form is normalized by $(\theta, \theta) = 2$) generated by $E_0 = t \otimes e_0$, $e_0 \in \mathfrak{g}_{-\theta}$ and $F_0 = t^{-1} \otimes f_0$, $f_0 \in \mathfrak{g}_\theta$ with $(e_0, f_0) = 1$, and $H_0 = -\theta^\vee = [E_0, F_0]$, is isomorphic to $\mathrm{sl}(2, \mathbb{C})$, by

$$E_0 \mapsto X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad F_0 \mapsto Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad H_0 = [E_0, F_0] \mapsto H = [X, Y] = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

As $\mathfrak{g}$ and $\mathfrak{r}$ generate $\mathcal{L}\mathfrak{g}$, the images of $\mathfrak{g}$ and $\mathfrak{r}$ in $\mathfrak{g}_W$ generate the Lie algebra $\mathfrak{g}_W$. The $\mathfrak{g}$ and $\mathfrak{r}$-actions on W, being locally finite, integrate to algebraic actions on W of the corresponding groups. See the second lemma below. Then the action of G_W on W is algebraic, not just holomorphic.

Pulling back the action of G_W on W to $\mathcal{L}G$ we get the proposition. ∎

Remark 13.2. (1) In fact the proposition holds for any simply connected simple group, essentially by the same proof. (2) When W is irreducible, all exponents n_i in I'_W are equal 1, as observed at the end of section 9.

Lemma 13.3. *The group G_W of the proposition is simply connected.*

Proof. Note that $\mathbb{C}[t, t^{-1}]/I'_W \simeq \mathbb{C}[t, t^{-1}]/\langle (t-a_1)^{n_1} \rangle \times \dots \times \mathbb{C}[t, t^{-1}]/\langle (t-a_k)^{n_k} \rangle$, since the $(t-a_i)^{n_i}$ are coprime. Thus

$$G_W \simeq G(\mathbb{C}[t, t^{-1}]/(t-a_1)^{n_1}) \times \dots \times G(\mathbb{C}[t, t^{-1}]/(t-a_k)^{n_k}).$$

It suffices to prove that $G(\mathbb{C}[t, t^{-1}]/(t-a)^m)$ is simply connected for any $a \in \mathbb{C}^\times$ and $m \geq 1$. If $m = 1$ then $G(\mathbb{C}[t, t^{-1}]/\langle t-a \rangle) \simeq G(\mathbb{C}) = \mathrm{SL}(n, \mathbb{C})$ is simply connected. We now assume by induction that $G(\mathbb{C}[t, t^{-1}]/(t-a)^m)$ is simply connected, and prove the same for $m+1$, $m \geq 1$. We have the short exact sequence

$$0 \to K_m \to G(\mathbb{C}[t, t^{-1}]/(t-a)^{m+1}) \to G(\mathbb{C}[t, t^{-1}]/(t-a)^m) \to 0,$$

where $K_m = \{I + (t-a)^m A;\ A \in M(n, \mathbb{C}),\ \mathrm{tr}\, A = 0\}$. In particular K_m is a vector space, hence simply connected. By induction and the long exact homotopy sequence, we get that $G(\mathbb{C}[t, t^{-1}]/(t-a)^{m+1})$ is simply connected, as required. ∎

Lemma 13.4. *The action of $G(\mathbb{C}[t, t^{-1}]/(t-a)^m)$ on W is algebraic.*

Proof. Consider the split exact sequence

$$1 \to K \to G(\mathbb{C}[t, t^{-1}]/(t-a)^m) \to G(\mathbb{C}[t, t^{-1}]/\langle t-a \rangle) \to 1.$$

The kernel K is a unipotent group. Then $\mathrm{Lie}(K)$ is a nilpotent Lie algebra and so $\rho_W : \mathrm{Lie}(K) \to \mathrm{End}_{\mathbb{C}}(W)$ has its image in the Lie algebra of the strictly upper triangular matrices in some basis of W. On this nilpotent Lie algebra, exp is an algebraic map. Hence the representation of K on W is algebraic. So is the action of the Lie group G. Hence so is the action of $G(\mathbb{C}[t,t^{-1}]/\langle (t-a)^m \rangle)$ on W. ∎

14. Polynomial representations

For completeness and clarification we review here the definition of a polynomial representation – the subject matter of this work – of $\mathrm{GL}(n)$ as an algebraic group. Thus we view $\mathrm{GL}(n)$ as an algebraic group over a field F (F could be taken to be a ring, e.g., $\mathbb{Z}$), namely as the spectrum $\mathrm{Spec}\, A_n$ of the ring

$$A_n = F[X_{i,j}\,(1 \le i,\, j \le n), x]/(x \det X = 1),$$

where $\det X = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{1 \le i \le n} X_{i,\sigma i}$ is the determinant of the matrix $X = (X_{i,j})$. In other words, the coordinate ring, or the ring of global sections $\Gamma(\mathrm{GL}(n), \mathcal{O})$, is A_n. An m-dimensional representation (ρ, W) of $\mathrm{GL}(n)$, thus a group homomorphism $\rho : \mathrm{GL}(n) \to \mathrm{Aut}\, W = \mathrm{GL}(m)$ is called (algebraic or) *rational* if ρ is a morphism of algebraic groups, namely the pullback map of rings $\rho^* : A_m \to \mathrm{Mor}(\mathrm{GL}(n), \mathbb{A}^1)$ defined by $(\rho^*(h))(g) = h(\rho(g))$ $(h \in A_m,\ g \in \mathrm{GL}(n))$ has its image in A_n.

Denote by $Y_{r,s}$ the coordinate functions on $\mathrm{GL}(m)$. If ρ is rational, then by definition

$$\rho^*(Y_{r,s}) \in A_n = F[X_{i,j}\,(1 \le i,\, j \le n),\, (\det X)^{-1}] \qquad (1 \le r,\, s \le m).$$

This $\rho^*(Y_{r,s})$ is the coefficient function $f_{r,s} \in F[X_{i,j}]$ defined by $\rho(g) w_s = \sum_r f_{r,s}(g) w_r$ where $w_1, \dots, w_m$ is a basis of W, as $(\rho^*(Y_{r,s}))(g) = Y_{r,s}(\rho(g))$. Then *polynomial* representations are those rational representations ρ with

$$\rho^*(F[Y_{r,s};\, 1 \le r,\, s \le m]) \subset F[X_{i,j};\, 1 \le i,\, j \le n] = \Gamma(M_n, \mathcal{O}) \subset A_n = \Gamma(\mathrm{GL}(n), \mathcal{O}),$$

where M_n denotes the ring of $n \times n$ matrices.

A polynomial representation of $\mathrm{GL}(n)$ is then a rational representation of $\mathrm{GL}(n)$ that extends to a representation of the algebraic monoid M_n, $(M_n, \cdot) \to (\mathrm{End}\, W, \cdot)$. As an example of the use of this language, we prove

Proposition 14.1. (a) *If* $\rho : \mathrm{GL}(n) \to \mathrm{GL}(m)$ *is a rational representation then there exists* $r \in \mathbb{Z}$ *with* $\det \rho(g) = (\det g)^r$ *for all* $g \in \mathrm{GL}(n)$. (b) *The polynomial* $\det X \in F[X_{i,j}]$ *is irreducible.* (c) *Let* $A = F[x_1, \dots, x_k]$ *be a polynomial ring in k variables. Suppose* $f \in A$ *is irreducible. Then* $A[f^{-1}]^\times = F^\times \cdot f^{\mathbb{Z}}$.

Proof. (b) Suppose $\det X = fg$ where $f,\ g \in F[X_{i,j}]$. For a fixed pair (i,j), $\det X$ has degree 1 as a polynomial in $X_{i,j}$. Hence $X_{i,j}$ occurs in only one of f or g. Let S be the set of pairs (i,j) such that $X_{i,j}$ occurs in f, T the analogous set for g. If (i,i) appears in S, and (i,j) in T, then $X_{i,i}X_{i,j}$ appears in a monomial in $fg = \det X$, contrary to the definition of $\det X$ recalled above. Hence S or T is empty, so f or g is a unit, in $F^\times$.

(c) If g is invertible in $A[f^{-1}]$, then $g^{-1} = \sum_{0 \le i \le r} f^{-i} h_i$ for some polynomials h_i in $A = F[x_1, \dots, x_k]$. Take s large enough to have $gf^s \in F[x_1, \dots, x_k]$. Multiplying $g^{-1} = \dots$ by gf^{r+s} we get $f^{r+s} = gf^s \sum_{0 \le i \le r} f^{r-i} h_i$, an equality in the UFD A, whose only units are the elements of $F^\times$. As f is irreducible, we get that $gf^s = af^t$ for some $t \in \mathbb{Z}$ and $a \in F$, done.

(a) The pullback $\rho^*(\det Y)$ is an invertible element of $\Gamma(\mathrm{GL}(m), \mathcal{O})$. By (b) and (c), $\rho^*(\det Y) = a(\det X)^r$ for some $r \in \mathbb{Z}$ and $a \in F^\times$. Applying both sides to $g \in \mathrm{GL}(n)$ we get $\det \rho(g) = a(\det g)^r$ for all g. Evaluating at $g = I$ we see that $a = 1$. ■

Example 14.2. Let E be the two dimensional F-vector space with basis e_1, e_2. The symmetric square $\mathrm{Sym}^2 E$ is a 3-dimensional representation of $\mathrm{GL}(2, F)$. In the basis e_1^2, e_1e_2, e_2^2 of $\mathrm{Sym}^2 E$, the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, F)$ acts on $\mathrm{Sym}^2 E$ as $M = \left(\begin{smallmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{smallmatrix}\right)$. In particular the representation $\mathrm{Sym}^2 E$ is polynomial, with coefficients $f_{1,1} = X_{1,1}^2$, $f_{1,2} = X_{1,1}X_{1,2}$, $f_{1,3} = X_{1,2}^2$, etc. The proposition implies in this case that $\det M = (ad - bc)^3$, a joy to compute explicitly.

15. Extensions, and cases of $d = 0, 1, n$

In this section we study extensions in the categories related by the functor $\mathcal{F}$. In the first subsection we just use the definition, or elementary means. This is motivated by correspondence from P. Deligne which uses the Hochschild-Serre spectral sequence, and is described in the second subsection. In particular we show

Proposition 15.1. *The equivalence of Theorem* 1.1 *extends to the case of* $d = 0$, *and does not extend to the case of* $d = n$.

Before doing this, recall that the (main part of the) theorem asserts: when $1 \le d < n$ there is an equivalence $\mathcal{F}$ of categories, *from* the category $\mathrm{Rep}\,\mathbb{C}[S_d \ltimes \mathbb{Z}^d]$ of finite dimensional representations $\widetilde{M}$ of $\mathbb{C}[S_d \ltimes \mathbb{Z}^d] = \mathbb{C}[S_d] \ltimes \mathbb{C}[y_1^{\pm1}, ..., y_d^{\pm1}]$, *to* the category $\mathrm{Rep}(\mathcal{L}\mathfrak{g}, d)$ of finite dimensional representations $\widetilde{W}$ of $\mathcal{L}\mathfrak{g}$ the constituents of its restriction to $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$ are subquotients of $\mathbb{E}^{\otimes d}$. Here $\mathbb{E} = \mathbb{C}^n$ is the standard representation of $\mathfrak{g}$. It extends the Schur duality $\mathcal{S}$ that maps the finite dimensional $\mathbb{C}[S_d]$-module M to the $\mathfrak{g}$-module $W = M \otimes_{\mathbb{C}[S_d]} \mathbb{E}^{\otimes d}$, by introducing an $\mathcal{L}\mathfrak{g}$-structure on W using the action of the $y_j^{\pm1}$ on $\widetilde{M}$. The $\mathcal{S}$-duality is an equivalence of categories from the category of finite dimensional $\mathbb{C}[S_d]$-modules M to the category of the finite dimensional $\mathfrak{g}$-modules W whose constituents are subquotients of $\mathbb{E}^{\otimes d}$. Both categories are semisimple here, both M and W decompose as a direct sum of irreducibles. So it suffices to define $\mathcal{S}$ only on irreducibles, mapping the Specht module S^λ to the $\mathfrak{g}$-module V^λ, where λ ranges over the set of partitions of d (with $\le n$ parts), using the decomposition $\mathbb{E}^{\otimes d} = \oplus_\lambda S^\lambda \otimes V^\lambda$, $\{\lambda;\ \lambda \vdash d,\ \ell(\lambda) \le n\}$. In the affine case the categories of $\mathbb{C}[S_d^{\mathrm{aff}}]$ and $\mathcal{L}\mathfrak{g}$-modules $\widetilde{M}$ and $\widetilde{W}$ are no longer semisimple, there are nontrivial extensions. In this section we give an example of this, when $d = 1$, and use this to show that the functor $\mathcal{F}$ does not extend to an equivalence of categories when $d = n$, although $\mathcal{S}$ is an equivalence of categories when $1 \le d \le n$. To compensate, we show that $\mathcal{F}$ does define an equivalence when $d = 0$. In particular the theorem holds for all d $(0 \le d < n)$.

15.1. By definition. We first recall how extensions are described. Let A be an algebra over a field F. We shall later take $A = \mathfrak{U}(\mathcal{L}\mathfrak{g})$ and $\mathbb{C}[S_d^{\mathrm{aff}}]$. Let U, V, W be finite dimensional representations of A. Suppose $0 \to V \to U \to W \to 0$ is an exact sequence of representations of A. Then $U = V \oplus W$ as vector spaces.

For each $p \in A$ we can write $\rho_U(p) = \left(\begin{smallmatrix} \rho_V(p) & f(p) \\ 0 & \rho_W(p) \end{smallmatrix}\right)$, where $f : A \to \mathrm{Hom}_F(W, V)$ is linear. Now (U, ρ_U) is a representation if and only if

$$\begin{pmatrix} \rho_V(p) & f(p) \\ 0 & \rho_W(p) \end{pmatrix} \begin{pmatrix} \rho_V(q) & f(q) \\ 0 & \rho_W(q) \end{pmatrix} = \begin{pmatrix} \rho_V(pq) & f(pq) \\ 0 & \rho_W(pq) \end{pmatrix}$$

if and only if $f(pq) = \rho_V(p)f(q) + f(p)\rho_W(q)$. The vector space of such f is the *space of cocycles*, denoted $Z^1(W, V)$. For a linear map $X : W \to V$ define $dX : A \to \mathrm{Hom}_F(W, V)$ by $dX(p) = \rho_V(p)X - X\rho_W(p)$. Clearly $dX \in Z^1(W, V)$.

Also $dX = 0$ if and only if X is a homomorphism of representations (an intertwining operator). The space of these dX is the *space of coboundaries* $B^1(W, V)$. It is $\simeq \mathrm{Hom}_F(W, V)/\,\mathrm{Hom}_A(W, V)$.

If $f,\ f' \in Z^1(W, V)$ and $f' - f = dX \in B^1(W, V)$ then

$$\begin{pmatrix} \rho_V(p) & f'(p) \\ 0 & \rho_W(p) \end{pmatrix} = \begin{pmatrix} \rho_V(p) & f(p) + \rho_V(p)X - X\rho_W(p) \\ 0 & \rho_W(p) \end{pmatrix} = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} \rho_V(p) & f(p) \\ 0 & \rho_W(p) \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

so that $U \simeq U'$. Conversely, if $\phi : U \to U'$ is an isomorphism $\rho_{U'} = \phi^{-1}\rho_U\phi$ of the form $\phi = \left(\begin{smallmatrix} I_V & X \\ 0 & I_W \end{smallmatrix}\right)$ then $f' - f = dX \in B^1(W, V)$. Then the space $\mathrm{Ext}^1(W, V) = Z^1(W, V)/B^1(W, V)$ parametrizes extensions of W by V.

Assume V, W are finite dimensional irreducible representations of A. For $f \in \mathrm{Ext}^1(W, V)$ denote by $0 \to V \to U_f \to W \to 0$ the corresponding extension. If $f,\ f' \in \mathrm{Ext}^1(W, V)$ then $U_f \simeq U_{f'}$ as A-modules if and only if there exists an M with $M\left(\begin{smallmatrix} \rho_V(p) & f'(p) \\ 0 & \rho_W(p) \end{smallmatrix}\right) = \left(\begin{smallmatrix} \rho_V(p) & f(p) \\ 0 & \rho_W(p) \end{smallmatrix}\right)M$ for all $p \in A$, if and only if $f' = \alpha f$, as $M = \mathrm{diag}(I_V, \alpha I_W)$, for some $\alpha \in F^\times$. Thus we have

Proposition 15.2. *The isomorphism classes of representations on nontrivial extensions of W by V are parametrized by the projective space $\mathbb{P}\,\mathrm{Ext}^1(W, V)$. In particular every extension of W by V is trivial if and only if* $\mathrm{Ext}^1(W, V) = 0$.

When $d = 1 \le n$, $S_d = \{1\}$ and $\mathbb{C}[S_d \ltimes \mathbb{Z}^d] = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[y, y^{-1}]$. An irreducible representation of $\mathbb{C}[y^{\pm 1}]$ is given by the space $\mathbb{C}$, with y acting as multiplication by $a \in \mathbb{C}^\times$; denote this $\mathbb{C}[y^{\pm 1}]$-module by V_a. Now $\mathrm{Ext}^1(V_a, V_b) \ne \{0\}$ and *there is a nontrivial extension* $0 \to V_b \to U \to V_a \to 0$ *of* V_a *by* V_b *when* $b = a$, defined by $\rho_U(y) = \left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)$. When $b \ne a$ we have $\mathrm{Ext}^1(V_a, V_b) = \{0\}$.

Also when $d = 1 < n$, consider the $\mathcal{L}\mathfrak{g}$ side. Take $A = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$, $V = \mathbb{C}^n$ the standard representation of $\mathfrak{g}$, and V_a the $\mathcal{L}\mathfrak{g}$-module V where t acts as multiplication by $a \in \mathbb{C}^\times$. We want to compute the extensions

$$0 \to V = V_b \to U \to W = V_a \to 0$$

as $\mathcal{L}\mathfrak{g}$-modules. As a $\mathfrak{g}$-module this extension splits, so that we may assume $\rho_U(p) = \left(\begin{smallmatrix} \rho_V(p) & f(p) \\ 0 & \rho_W(p) \end{smallmatrix}\right)$ with $f(\mathfrak{g}) = 0$, $f(p + q) = f(p) + f(q)$, and $\rho_V(g)$ acts as g on V for $g \in \mathfrak{g}$. Also $\rho_{V_b}(p) = p(b)$, the value at $t = b$ of $p \in \mathcal{L}\mathfrak{g}$, and so $f(pq) = p(b)f(q) + f(p)q(a)$. If $p \in \mathfrak{g}$ then $f(pt^m) = pf(t^m)$, and by induction $f(t^k) = (\sum_{i+j=k-1} b^i a^j)f(t)$, for some $f(t) \in \mathrm{Hom}_F(W, V)$, $F = \mathbb{C}$.

Now $B^1(V_a, V_b)$ is the space of $dX(p) = p(b)X - Xp(a)$. If we take the p in $\mathfrak{g}$ we see that B^1 contains all $[X, Y]$, X, $Y \in \mathfrak{g}$, so it contains $\mathfrak{g}$. If we take

$p = tx$, we get all $dX(p) = bxX - aXx = (b-a)Xx$ if x and X commute. So $Z^1(V_a, V_b) = B^1(V_a, V_b) = \mathrm{gl}_n$ and $\mathrm{Ext}^1 = 0$ if $a \neq b$, but if $a = b$ then $Z^1(V_a, V_b) = \mathrm{gl}_n \neq B^1(V_a, V_b) = \mathrm{sl}_n$ and $\dim \mathrm{Ext}^1 = 1$, so *there is a nontrivial extension of V_a by V_a*, e.g. when $a = 1$. This shows that

Corollary 15.3. *When $d = 1 < n$ the categories related by the theorem are not semisimple.*

Corollary 15.4. *The functor $\mathcal{F}$ is an equivalence of categories when $d = 0$.*

Proof. When $d = 0$, $S_d \ltimes \mathbb{Z}^d = \{1\}$ and $\mathbb{C}[S_d \ltimes \mathbb{Z}^d] = \mathbb{C}$. The category of finite dimensional $\mathbb{C}$-modules M is the category of finite dimensional vector spaces over $\mathbb{C}$; it is semisimple by linear algebra. On the side of $\mathcal{L}\mathfrak{g}$, $\mathbb{E}^{\otimes d}$ is $\mathbb{C}$, and the $\mathcal{L}\mathfrak{g}$-modules we obtain are finite dimensional vector spaces on which $\mathfrak{g}$ acts as 0, extended to $\mathcal{L}\mathfrak{g}$ by letting t act as multiplication by $a \in \mathbb{C}^\times$. From $\left(\begin{smallmatrix} 0 & f(p) \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & f(q) \\ 0 & 0 \end{smallmatrix}\right) = \rho_U(p)\rho_U(q) = \rho_U(pq) = \left(\begin{smallmatrix} 0 & f(pq) \\ 0 & 0 \end{smallmatrix}\right)$ we see that the category of such $\mathcal{L}\mathfrak{g}$-modules is semisimple, establishing the theorem for $d = 0$. ■

Corollary 15.5. *The functor $\mathcal{F}$ is not an equivalence of categories when $d = n$.*

Proof. When $d = n$, the $\mathcal{S}$-duality relates the S_d-module S^λ with the $\mathfrak{g}$-module V^λ, $\lambda \vdash d$ (hence $\ell(\lambda) \leq n = d$). We claim it does not extend to the affine case. To see this, consider the partition $\lambda = \omega$, where $\omega = (1, \ldots, 1) = (1^d) \in \mathbb{Z}^d$. The space V^ω is $\wedge^d \mathbb{E}$, the one dimensional $\mathbb{C}$-space spanned by $v_1 \wedge \cdots \wedge v_n$, where $v_1, \ldots, v_n$ is any basis of $\mathbb{E} = \mathbb{C}^n$. Recall that $\mathrm{gl}_n(\mathbb{C})$ acts by

$$\Delta(X) \cdot v_1 \wedge \cdots \wedge v_d = \sum_{1 \leq j \leq d} v_1 \wedge \cdots \wedge Xv_j \wedge \cdots \wedge v_d,$$

so $\Delta(X)$ maps $v_1 \wedge \cdots \wedge v_d$ to itself times $\mathrm{tr}(X)$, namely $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$ acts as 0 on $V^\omega = \mathbb{C}$. We pull back V^ω to a representation V_a^ω of $\mathcal{L}\mathfrak{g}$ by letting t act as multiplication by $a \in \mathbb{C}^\times$. Then V_a^ω is still $\mathbb{C}$ with $\mathcal{L}\mathfrak{g}$ acting as 0. So we now look for extensions $0 \to V = V_b^\omega \to U \to W = V_a^\omega \to 0$ of representations of $\mathfrak{U}(\mathcal{L}\mathfrak{g})$. Then $U = \mathbb{C} \oplus \mathbb{C}$ as a vector space, and the action of $p \in \mathcal{L}\mathfrak{g}$ is $\rho_U(p) = \left(\begin{smallmatrix} \rho_V(p) & f(p) \\ 0 & \rho_W(p) \end{smallmatrix}\right)$, with $\rho_V(p)$ and $\rho_W(p)$ acting as 0. So

$$\begin{pmatrix} 0 & f(p) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & f(q) \\ 0 & 0 \end{pmatrix} = \rho_U(p)\rho_U(q) = \rho_U(pq) = \begin{pmatrix} 0 & f(pq) \\ 0 & 0 \end{pmatrix},$$

namely $f(pq) = 0$ for all p, $q \in A = \mathfrak{U}(\mathcal{L}\mathfrak{g})$, or $f = 0$, so $\mathrm{Ext}^1_{\mathfrak{U}(\mathcal{L}\mathfrak{g})}(V_a^\omega, V_b^\omega)$ is 0, *all extensions of V_a^ω by V_a^ω split.*

As noted in Corollary 15.4, this can be viewed as the case of $d = 0$ of the affine Schur duality.

The corresponding representation S^ω is the one dimensional sign representation with space $\mathbb{C}$ on which S_d acts via $\mathrm{sgn} : S_d \twoheadrightarrow \{\pm 1\}$. As a $\mathbb{C}[S_d \ltimes \mathbb{Z}^d] = \mathbb{C}[S_d] \ltimes \mathbb{C}[y_1^{\pm 1}, \ldots, y_d^{\pm 1}]$-module the y_j act as multiplication by $a \in \mathbb{C}^\times$. Denote it by S_a^ω. A nontrivial extension $0 \to S_b^\omega \to U \to S_a^\omega \to 0$ is given by $\rho_U(y_j) = \left(\begin{smallmatrix} a & a \\ 0 & a \end{smallmatrix}\right)$ for all j $(1 \leq j \leq n)$ when $a = b$. So $\mathrm{Ext}^1(S_a^\omega, S_b^\omega) \neq \{0\}$ if $a = b$, and $= \{0\}$ if $a \neq b$. We conclude that the functor $\mathcal{F}$ of the theorem is not an equivalence of categories when $d = n$. ■

The proof shows that there are nontrivial extensions of V_a^ω by itself when viewed as $\mathcal{L}\overline{\mathfrak{g}}$-module, $\overline{\mathfrak{g}} = \mathrm{gl}_n(\mathbb{C})$. In particular the passage from a finite dimensional $\mathcal{L}\mathfrak{g}$-module to a finite dimensional $\mathcal{L}\overline{\mathfrak{g}}$-module is more complicated than in the case of the finite dimensional algebras $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$ and $\overline{\mathfrak{g}} = \mathrm{gl}_n(\mathbb{C})$.

15.2. By Hochschild-Serre. Let V, W be two representations of $\mathfrak{g}$, viewed as representations renamed V_a, W_a of $\mathcal{L}\mathfrak{g}$ by $\mathrm{ev}_a : \mathcal{L}\mathfrak{g} \to \mathfrak{g}$, thus t acts as multiplication by $a \in \mathbb{C}^\times$ on V_a, W_a. We regard V_a, W_a as representations of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, with $\rho_V(p \otimes (t-a)) = 0$ for $p \in \mathcal{L}\mathfrak{g}$. If E is an extension of W_a by V_a then $\rho_E(p) = \left(\begin{smallmatrix} \rho_V(p) & f(p) \\ 0 & \rho_W(p) \end{smallmatrix}\right)$, so $\rho_E(p\otimes(t-a)) = \left(\begin{smallmatrix} 0 & f(p\otimes(t-a)) \\ 0 & 0 \end{smallmatrix}\right)$ and $\rho_E(pq \otimes (t-a)^2) = \rho_E(p \otimes (t-a))\rho_E(q \otimes (t-a)) = 0$ so the representation E of $\mathfrak{U}(\mathcal{L}\mathfrak{g})$ factors through $\mathcal{L}\mathfrak{g} \to \mathfrak{g}^+$, where $\mathfrak{g}^+ = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t-a)^2$. As a vector space this last algebra is $\mathfrak{g} \oplus \mathfrak{g}^-$ where $\mathfrak{g}^- = \mathfrak{g}(t-a)$ is an abelian Lie subalgebra of $\mathfrak{g}^+$ (as $(t-a)^2 = 0$ in $\mathfrak{g}^+$) isomorphic ($\mathfrak{g} \to \mathfrak{g}(t-a)$ by $X \mapsto X(t-a)$) to $\mathfrak{g}$. It is the kernel of the evaluation map $\mathrm{ev}_a : \mathfrak{g}^+ \to \mathfrak{g}$ at $t \mapsto a$. So we have an extension $0 \to \mathfrak{g}^- \to \mathfrak{g}^+ \to \mathfrak{g} \to 0$ of Lie algebras, where the quotient $\mathfrak{g}$ acts on the abelian subalgebra $\mathfrak{g}^- \simeq \mathfrak{g}$ by its adjoint representation, and we consider the $\mathfrak{g}^+$-modules extensions $0 \to V_a \to E \to W_a \to 0$.

For two $\mathfrak{g}^+$-modules A and B, one has

$$\mathrm{Ext}^i_{\mathfrak{g}^+}(A, B) = H^i(\mathfrak{g}^+, \mathcal{H}om_F(A, B)),$$

where $\mathcal{H}om_F(A, B)$ is $\mathrm{Hom}_F(A, B)$ viewed as a representation of $\mathfrak{g}^+$.

For an invariant subalgebra (ideal) $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}^+$, and a $\mathfrak{g}^+$-module V, we have $V^{\mathfrak{g}^+} = (V^{\mathfrak{h}^+})^{\mathfrak{g}^+/\mathfrak{h}}$. Deriving this one gets the Hochschild-Serre spectral sequence

$$H^p(\mathfrak{g}^+/\mathfrak{h}, H^q(\mathfrak{h}, V)) \Rightarrow H^{p+q}(\mathfrak{g}^+, V).$$

Taking $\mathfrak{h} = \mathfrak{g}^-$ and noting that $\mathcal{H}om_F(W_a, V_a) = \mathcal{H}om_F(W, V)_a$ is a representation of the quotient $\mathfrak{g}$, since W_a and V_a are, we get

$$H^p(\mathfrak{g}, H^q(\mathfrak{g}^-, \mathcal{H}om(W, V)_a)) \Rightarrow H^{p+q}(\mathfrak{g}^+, \mathcal{H}om_F(W_a, V_a)) = \mathrm{Ext}^{p+q}_{\mathfrak{g}^+}(W_a, V_a).$$

For a trivial representation Y (in our case $\mathcal{H}om(W, V)_a$) of a commutative Lie algebra $\mathfrak{h}$ one has $H^q(\mathfrak{h}, Y) = \wedge^q \mathfrak{h}^\vee \otimes Y$, where $\mathfrak{h}^\vee$ is the adjoint representation of $\mathfrak{h}$. As $\mathfrak{g}$ is semisimple $H^p(\mathfrak{g}, *) \neq \{0\}$ only for $p = 0$, and $H^0 =$ invariants, so the spectral sequence collapses to

$$\mathrm{Ext}^q_{\mathfrak{g}^+}(W_a, V_a) = [\wedge^q \mathfrak{g}^\vee \otimes \mathcal{H}om_F(W_a, V_a)]^{\mathfrak{g}}.$$

The $\mathfrak{g}^\vee$ inside $[\dots]$ is the adjoint representation of $\mathfrak{g}$, the $\mathcal{H}om_F(W_a, V_a)$ is a $\mathfrak{g}$-module, and the superscript $\mathfrak{g}$ means invariants under $\mathfrak{g}$. Higher Ext, as $\mathcal{L}\mathfrak{g}$-modules, no longer coincide with higher Ext as $\mathfrak{g}^+$- modules. So only $q = 1$ is relevant. Thus

$$\mathrm{Ext}^1_{\mathfrak{g}^+}(W_a, V_a) = H^0(\mathfrak{g}, H^1(\mathfrak{g}^-, \mathcal{H}om(W, V)_a)) = \mathrm{Hom}_{\mathfrak{g}}(W \otimes \mathfrak{g}, V).$$

It is one dimensional when $V = W = \mathbb{C}^n$ is the standard representation of $\mathfrak{g}$, corresponding to the case $d = 1$, just showing there are nontrivial extensions in this case.

However in the case of $d=n$, taking the partition $\omega=(1,\dots,1)$ of d, V^ω is the trivial representation of $\mathfrak{g}$, and for $V=W=V^\omega$ we obtain trivial $\mathrm{Ext}^1_{\mathfrak{g}^+}(V^\omega_a,V^\omega_a)$.

The corresponding representation S^ω of S_d is the sign representation $\mathrm{sgn}:S_d\twoheadrightarrow\{\pm1\}$, extended to $\mathbb{C}[S_d\ltimes\mathbb{Z}^d]=\mathbb{C}[S_d]\ltimes\mathbb{C}[y_1^{\pm1},\dots,y_d^{\pm1}]$ by letting the y_j act as multiplication by a; denote this extension by V^ω_a, and V^λ_a for a general λ.

Using the exact sequence $0\to\mathbb{Z}^d\to S_d\ltimes\mathbb{Z}^d\to S_d\to 0$, the same argument based on the Hochschild-Serre spectral sequence shows that the extensions of S^λ_a by S^μ_a are parametrized by $H^0(S_d,\mathrm{Hom}(\mathbb{Z}^d,\mathcal{H}om(S^\lambda_a,S^\mu_a)))$, which is the space of S_d-invariants on $\mathcal{H}om(P\otimes S^\lambda_a,S^\mu_a)$. Here P denotes the permutation representation of S_d on $\mathbb{C}^d$, which is $\mathrm{Ind}_{S_{d-1}}^{S_d}(1)$. It breaks as the direct sum of the standard representation $S^{(d-1,1)}$ of S_d and the trivial one $S^{(d)}$ ([13, top of p. 55]). In the case of $\lambda=\mu=\omega$ of interest, the S_d-invariants on $\mathcal{H}om(P\otimes S^\omega_a,S^\omega_a)=\mathcal{H}om(S^\omega_a,S^\omega_a)$ will be one dimensional, in particular there are nontrivial extensions of S^ω_a by itself, but all extensions of V^ω_a by itself split. Hence the functor $\mathcal{F}$ does not define an equivalence of categories and the theorem does not extend as stated to the case of $d=n$.

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