Visible Actions and Criteria for Multiplicity-Freeness of Representations of Heisenberg Groups

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Abstract.   A visible action on a complex manifold is a holomorphic action that admits a J-transversal totally real submanifold S. It is said to be strongly visible if there exists an orbit-preserving anti-holomorphic diffeomorphism σ such that $\sigma|_S = \mathrm{id}_S$. Let G be the Heisenberg group and H a non-trivial connected closed subgroup of G. We prove that any complex homogeneous space $D = G^{\mathbb{C}}/H^{\mathbb{C}}$ admits a strongly visible L-action, where L stands for a connected closed subgroup of G explicitly constructed through a co-exponential basis of H in G. This leads in turn that G itself acts strongly visibly on D. The proof is carried out by finding explicitly an orbit-preserving anti-holomorphic diffeomorphism and a totally real submanifold S, for which the dimension depends upon the dimensions of G and H. As a direct application, our geometric results provide a proof of various multiplicity-free theorems on continuous representations on the space of holomorphic sections on D. Moreover, we also generate as a consequence, a geometric criterion for a quasi-regular representation of G to be multiplicity-free.

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1.   Introduction

This paper investigates a classification problem on (strongly) visible actions on complex homogeneous spaces of Heisenberg groups.

The notion of (strongly) visible actions on complex manifolds has been introduced by T. Kobayashi [3, 4]. A holomorphic action of a Lie group L on a connected complex manifold D is called *strongly visible* ([4, Definition 3.3.1]) if there exist a real submanifold S in D and an anti-holomorphic diffeomorphism σ on D such that the conditions (V.0)–(S.2) are satisfied:

$$D' := L \cdot S \text{ is a non-empty open set in } D, \tag{V.0}$$

$$\sigma|_S = \mathrm{id}_S, \tag{S.1}$$

$$\sigma \text{ preserves each } L\text{-orbit in } D'. \tag{S.2}$$

A real submanifold S satisfying (V.0) and (S.1) is called a *slice* for the strongly visible L-action on D. A slice S satisfying (S.1) is totally real, which means that

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$T_x S \cap J_x(T_x S) = \{0\}$ for any $x \in S$, where J is a complex structure on D (see [4, Remark 3.3.2]). Further, a strongly visible action is visible [4, Definition 3.1.1] in the sense that (V.0) with $J_x(T_x S) \subset T_x(L \cdot x)$ for any $x \in S$ (see [4, Theorem 4]). We allow that a slice S meets every L-orbit in D' twice or more. Namely, S is not necessary a cross-section of L-orbits in D'.

Recently, strongly visible actions has been studied in various settings, such as Hermitian symmetric spaces [6], flag varieties [7, 17], complex linear spaces [13], nilpotent orbits in complex simple Lie algebras [15], and reductive non-symmetric spherical homogeneous spaces (for example, [14]), which are in connection with multiplicity-free representations of reductive Lie groups. Under these circumstances, a kind of decomposition theorems of reductive Lie groups and reductive homogeneous spaces, Cartan decomposition and Iwasawa decomposition for instance, play a crucial role to explicitly build the corresponding slices. However, strongly visible actions of nilpotent and solvable Lie groups are not well-known. Indeed, in contrast to the case of reductive Lie groups, there is no analogue of decomposition theorems for nilpotent and solvable Lie groups.

Let us bring up our problems as follows. Let G be a connected and simply connected nilpotent (resp. solvable) Lie group and H a non-trivial connected closed subgroup of G. We denote by $G^{\mathbb{C}}$ and $H^{\mathbb{C}}$ the complexifications of G and H, respectively. We say that $G^{\mathbb{C}}/H^{\mathbb{C}}$ is a complex nilpotent (resp. solvable) homogeneous space. Then, G naturally acts on the complex manifold $G^{\mathbb{C}}/H^{\mathbb{C}}$ holomorphically. We consider the following problems:

Problem 1.1.   Find pairs (G, H) such that the G-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ is strongly visible.

For a strongly visible G-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$, we can take a slice S and an anti-holomorphic diffeomorphism σ satisfying (V.0)–(S.2). Then, the dimension of S is not greater than $\dim G/H$ because S is a totally real submanifold in $G^{\mathbb{C}}/H^{\mathbb{C}}$, whereas, this is at least the codimension of generic G-orbits in $G^{\mathbb{C}}/H^{\mathbb{C}}$ (cf. [4, Lemma 3.2.1]). In this sense, we say that a slice S is smallest if $\dim S$ coincides with the codimension of generic orbits. Then, we have:

Problem 1.2.   Construct explicitly a small slice S for a strongly visible G-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$.

The present paper solves Problems 1.1 and 1.2 in the setting of the Heisenberg group. First, we prove:

Theorem 1.3.   *Let G be the Heisenberg group and H a non-trivial connected closed subgroup of G. Then, there exists a connected closed subgroup L of G such that the L-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ is strongly visible.*

Indeed, we specify 'smallest' L and give concrete descriptions of a 'smallest' slice S and an anti-holomorphic diffeomorphism σ, respectively, for the L-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ in the proof of Theorem 1.3 (see also Table 1 for our choice of L, S and σ for H).

The strongly visible action of a small group ensures that of a large one, namely, we obtain:

Corollary 1.4.   *For any non-trivial connected closed subgroup H, the G-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ is strongly visible.*

Hence, Corollary 1.4 gives an answer to Problem 1.1 in the case of the Heisenberg group.

We also focus attention on the relationship between strongly visible actions on nilpotent (resp. solvable) homogeneous spaces and the multiplicity-freeness of some related representations. Originally, the notion of (strongly) visible actions has been introduced as a geometric condition of the propagation theory of multiplicity-freeness property [9] (see also [3, 4] and Fact 8.1). This allows to generate a unified explanation of multiplicity-freeness property for representations which have been found independently. In fact, this new line of investigation was initiated in [8] for unitary highest weight representations of scalar type. Moreover, if we find a strongly visible action on a complex manifold, we expect to get various multiplicity-free representations.

Indeed, our main results given in Theorem 1.3 and Corollary 1.4 yield a geometric explanation of multiplicity-freeness property of the continuous representation on the space of holomorphic functions due to the propagation theory, which we will explain in Theorem 8.2.

We also seek another geometric criterion for the quasi-regular representation π_H (see (29) for definition) to be multiplicity-free. To state our geometric criterion, we give a setup as follows: Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of the Heisenberg group G and a proper connected closed subgroup H, respectively. We put $d := \dim G - \dim H$ (then $0 < d < \dim G$) and take a co-exponential basis $\{X_1, \ldots, X_d\}$ to $\mathfrak{h}$ in $\mathfrak{g}$. We set a complementary subspace $\mathfrak{q}$ of $\mathfrak{h}$ in $\mathfrak{g}$ as $\mathfrak{q} = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_d\}$ and $Q := \langle \exp \mathfrak{q} \rangle$ as a closed subgroup of G generated by $\exp \mathfrak{q}$. Then, we prove:

Theorem 1.5 (see Theorem 8.3 for details). *For a non-trivial connected closed subgroup H of the Heisenberg group G, the following conditions are equivalent:*

(i) *The quasi-regular representation π_H of G is multiplicity-free.*

(ii) *The Q-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ is strongly visible.*

In order to verify the multiplicity-freeness of π_H, we shall apply Corwin–Greenleaf formula [2] and calculate the multiplicity of each associated isotypic component of G occurring in the irreducible decomposition of π_H (see Fact 8.5).

Concerning the relationship between visible actions, the Corwin–Greenleaf multiplicity function and the Kirillov orbit method, we refer to [10] in a case where (G, H) is a semisimple symmetric pair of holomorphic type.

In view of our proof of Theorem 1.5, we find out a deep relationship between the strongly visible Q-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ and the fact that π_H is multiplicity-free.

Theorem 1.6 (cf. Theorem 8.15). *One can construct a slice S for the strongly visible Q-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ of dimension equals the rank of the support for multiplicity-free π_H (see (48) for definition).*

It is noteworthy to mention here that Theorem 1.6 gives an evidence of [5, Conjecture 3.2] affirmatively in the complex Heisenberg homogeneous spaces as same as linear actions and nilpotent orbits in complex simple Lie algebras (see [13, 16]).

This paper is organized as follows. In Section 2, we fix a general setup for the study of strongly visible actions on complex Heisenberg homogeneous spaces and prepare two anti-holomorphic diffeomorphisms on them. We next explain our strategy of the proof for Theorem 1.3, which is based on Lemma 2.17. Accordingly, we give a

proof of Theorem 1.3 for each closed subgroup H in Sections 3–6. In particular, we provide a closed subgroup L and a slice S for the L-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ satisfying (V.0)–(S.2) explicitly. In Section 7, we accomplish the proofs of Theorem 1.3 and Corollary 1.4.

Section 8 is devoted to present new results on multiplicity-free representations as applications of our main theorems. In Section 8.1, we record a brief summary on the propagation theory of multiplicity-freeness property [9] and find multiplicity-free representations as results of Theorem 1.3 and Corollary 1.4 via this theory. In Section 8.2, we present Theorem 8.3 which covers the results of Theorem 1.5. In Sections 8.3–8.5, we give a proof of Theorem 8.3. In Section 8.6, we establish a result concerning the construction of specific slices for strongly visible actions on Heiseberg homogeneous spaces (cf. Theorem 8.15).

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2. Preliminaries

2.1. Heisenberg Lie algebra and its subalgebras

Let $\mathfrak{g}$ be the $(2n+1)$-dimensional Heisenberg Lie algebra. One can take a basis

$$\mathcal{B} \equiv \mathcal{B}_{(n,n,1)} := \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \tag{1}$$

of $\mathfrak{g}$ such that the following relations hold for $1 \leq i, j \leq n$:

$$[X_i, Y_j] = \delta_{ij} Z, \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_j, Z] = 0. \tag{2}$$

Here, δ_{ij} equals one if $i = j$ and otherwise zero. Then, the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ coincides with $\mathbb{R}Z$. Further, the bracket relations (2) show that $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\}$, from which $\mathfrak{g}$ is a two-step nilpotent Lie algebra.

We define subsets $\mathcal{B}_{(p,q,\varepsilon)}$ and $\mathcal{B}_{(m,0,\varepsilon)}$ of the basis $\mathcal{B}$ for $1 \leq p, q \leq n$, $\varepsilon = 0, 1$ and $1 \leq m \leq n$ by

$$\mathcal{B}_{(p,q,\varepsilon)} := \{X_1, \ldots, X_p, Y_1, \ldots, Y_q, \varepsilon Z\}, \quad \mathcal{B}_{(m,0,\varepsilon)} := \{X_1, \ldots, X_m, \varepsilon Z\}$$

and $\mathcal{B}_{(0,0,1)} := \{Z\}$. For a subset $\mathcal{B}_{(k,\ell,\varepsilon)} \subset \mathcal{B}$, we denote by $\mathrm{span}_{\mathbb{R}} \mathcal{B}_{(k,\ell,\varepsilon)}$ the subspace of $\mathfrak{g}$ with basis $\mathcal{B}_{(k,\ell,\varepsilon)}$ and set

$$\mathfrak{h}_{(k,\ell,\varepsilon)} := \mathrm{span}_{\mathbb{R}} \mathcal{B}_{(k,\ell,\varepsilon)}. \tag{3}$$

By definition, we also write $\mathfrak{h}_{(m,0,1)} = \mathfrak{h}_{(m,0,0)} + \mathfrak{z}(\mathfrak{g})$.

The Lie algebra $\mathfrak{h}_{(p,q,1)}$ is isomorphic to the direct sum of the Heisenberg Lie algebra of dimension $2p+1$ and an abelian subalgebra of dimension $q-p$ if $p < q$, and $\mathfrak{h}_{(p,p,1)}$ is itself the Heisenberg Lie algebra of dimension $2p+1$. On the other hand, $\mathfrak{h}_{(m,0,1)}$ and $\mathfrak{h}_{(m,0,0)}$ are abelian subalgebras for any $1 \leq m \leq n$.

We know from [1, Proposition 3.1] that a subalgebra of the Heisenberg Lie algebra is characterized as follows:

Lemma 2.1. *A non-trivial subalgebra of the Heisenberg Lie algebra $\mathfrak{g}$ is isomorphic to one of the following subalgebras:*

(i) $\mathfrak{h}_{(0,0,1)} = \mathfrak{z}(\mathfrak{g}) = \mathbb{R}Z$,

(ii) $\mathfrak{h}_{(p,q,1)}$ *for some* $1 \le p \le q \le n$, $(p,q) \neq (n,n)$,

(iii) $\mathfrak{h}_{(m,0,1)}$ *for some* $1 \le m \le n$,

(iv) $\mathfrak{h}_{(m,0,0)}$ *for some* $1 \le m \le n$.

Next, let us take a non-trivial subalgebra $\mathfrak{h}$ of the Heisenberg Lie algebra $\mathfrak{g}$ with basis $\mathcal{B}_{\mathfrak{h}}$ as a subset of $\mathcal{B}$. We set $(\mathcal{B}_{\mathfrak{h}})^c := \mathcal{B} - \mathcal{B}_{\mathfrak{h}}$ and $\mathfrak{q} \equiv C(\mathfrak{h}) := \mathrm{span}_{\mathbb{R}}(\mathcal{B}_{\mathfrak{h}})^c$. Then, $\mathfrak{q}$ is a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$, namely, $\mathfrak{h} \cap \mathfrak{q} = \{0\}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. In particular, we set

$$\mathfrak{q}_{(k,\ell,\varepsilon)} = \mathrm{span}_{\mathbb{R}}(\mathcal{B}_{(k,\ell,\varepsilon)})^c. \tag{4}$$

For each subalgebra given in Lemma 2.1, we have

Lemma 2.2. (i) $(\mathcal{B}_{(0,0,1)})^c = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$.

(ii) *If* $1 \le p \le q < n$, *then* $(\mathcal{B}_{(p,q,1)})^c = \{X_{p+1}, \ldots, X_n, Y_{q+1}, \ldots, Y_n\}$, *and if* $1 \le p < q = n$, *then* $(\mathcal{B}_{(p,n,1)})^c = \{X_{p+1}, \ldots, X_n\}$.

(iii) *If* $1 \le m < n$, *then* $(\mathcal{B}_{(m,0,1)})^c = \{X_{m+1}, \ldots, X_n, Y_1, \ldots, Y_n\}$, *and if* $m = n$, *then* $(\mathcal{B}_{(n,0,1)})^c = \{Y_1, \ldots, Y_n\}$.

2.2. The Heisenberg group and its closed subgroups

Throughout this paper, let $G = \mathbf{H}_n$ denote the connected and simply connected Heisenberg group of dimension $2n + 1$ with Heisenberg Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism, from which we obtain $G = \exp \mathfrak{g}$. In particular, G has a structure which admits a coordinates system of exponential type.

For convenience, we shall use the notation as follows. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and N be a positive integer. We write $\boldsymbol{x} := (x_1, \ldots, x_N)$, $\boldsymbol{y} := (y_1, \ldots, y_N) \in \mathbb{K}^N$. Now, we define a symmetric bilinear form on $\mathbb{K}^N$ by

$$\mathbb{K}^N \times \mathbb{K}^N \to \mathbb{K}, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto (\boldsymbol{x}|\boldsymbol{y}) := x_1 y_1 + \cdots + x_N y_N. \tag{5}$$

We set $$g(\boldsymbol{x}, \boldsymbol{y}, z) := e^{x_1 X_1} \cdots e^{x_n X_n} e^{y_1 Y_1} \cdots e^{y_n Y_n} e^{zZ}. \tag{6}$$

Then, we have $$G = \{g(\boldsymbol{x}, \boldsymbol{y}, z) : \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, z \in \mathbb{R}\}.$$

We observe the multiple of $e^{x_i X_i}$ and $e^{y_j Y_j}$ in order to express that of two elements of G.

Lemma 2.3. $e^{y_j Y_j} e^{x_i X_i} = e^{x_i X_i} e^{y_j Y_j} e^{-\delta_{ij} x_i y_j Z}$ *for* $1 \le i, j \le n$.

Proof. First, let us compute

$$e^{-x_i X_i} e^{y_j Y_j} e^{x_i X_i} = e^{y_j \, \mathrm{Ad}(e^{-x_i X_i}) Y_j}. \tag{7}$$

As $[X_i, Y_j] = \delta_{ij} Z$ and $[X_i, [X_i, Y_j]] = 0$, $\mathrm{Ad}(e^{-x_i X_i}) Y_j$ is given by

$$\mathrm{Ad}(e^{-x_i X_i}) Y_j = e^{\mathrm{ad}(-x_i X_i)} Y_j = \sum_{k=0}^{\infty} \frac{(-x_i)^k}{k!} (\mathrm{ad}\, X_i)^k Y_j = Y_j - \delta_{ij} x_i Z.$$

Thus, the right-hand side of (7) equals $e^{y_j(Y_j - \delta_{ij} x_i Z)}$. Moreover, this is $e^{y_j Y_j} e^{-\delta_{ij} x_i y_j Z}$ because $[Y_i, Z] = 0$. Hence, we obtain

$$e^{y_j Y_j} e^{x_i X_i} = e^{x_i X_i}(e^{-x_i X_i} e^{y_j Y_j} e^{x_i X_i}) = e^{x_i X_i} e^{y_j Y_j} e^{-\delta_{ij} x_i y_j Z}.$$

Hence, Lemma 2.3 has been proved. ∎

Lemma 2.4. *For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t} \in \mathbb{K}^n$ and $z, u \in \mathbb{K}$, one has*

$$g(\boldsymbol{x}, \boldsymbol{y}, z) \cdot g(\boldsymbol{s}, \boldsymbol{t}, u) = g(\boldsymbol{x} + \boldsymbol{s}, \boldsymbol{y} + \boldsymbol{t}, z + u - (\boldsymbol{y}|\boldsymbol{s})),$$

and $g(\boldsymbol{x}, \boldsymbol{y}, z)^{-1} = g(-\boldsymbol{x}, -\boldsymbol{y}, -z - (\boldsymbol{y}|\boldsymbol{x}))$.

Proof. By Lemma 2.3, we observe

$$(e^{y_1 Y_1} \cdots e^{y_n Y_n})(e^{s_1 X_1} \cdots e^{s_n X_n})$$
$$= (e^{y_1 Y_1} e^{s_1 X_1})(e^{y_2 Y_2} e^{s_2 X_2}) \cdots (e^{y_n Y_n} e^{s_n X_n})$$
$$= (e^{s_1 X_1} e^{y_1 Y_1} e^{-y_1 s_1 Z})(e^{s_2 X_2} e^{y_2 Y_2} e^{-y_2 s_2 Z}) \cdots (e^{s_n X_n} e^{y_n Y_n} e^{-y_n s_n Z})$$
$$= (e^{s_1 X_1} \cdots e^{s_n X_n})(e^{y_1 Y_1} \cdots e^{y_n Y_n}) e^{-(y_1 s_1 + \cdots + y_n s_n) Z}.$$

Hence, Lemma 2.4 follows from the above equality. ∎

Lemma 2.4 implies that we have a Lie group isomorphism

$$\iota : G \to \mathbb{R}^{2n} \ltimes \mathbb{R}, \quad g(\boldsymbol{x}, \boldsymbol{y}, z) \mapsto (\boldsymbol{x}, \boldsymbol{y}, z). \tag{8}$$

Next, any connected closed subgroup H of G is realized as $H = \exp \mathfrak{h}$ of some subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Hence, the following lemma is an immediate consequence of Lemma 2.1.

Lemma 2.5. *A non-trivial connected closed subgroup of the Heisenberg group G is isomorphic to one of the following closed subgroups:*

(i) $H_{(0,0,1)} := Z(G) = \exp(\mathbb{R}Z)$ *of G,*

(ii) $H_{(p,q,1)} := \exp \mathfrak{h}_{(p,q,1)}$ *for some $1 \le p \le q \le n$, $(p,q) \ne (n,n)$,*

(iii) $H_{(m,0,1)} := \exp \mathfrak{h}_{(m,0,1)}$ *for some $1 \le m \le n$,*

(iv) $H_{(m,0,0)} := \exp \mathfrak{h}_{(m,0,0)}$ *for some $1 \le m \le n$.*

We note that the closed subgroup $H_{(m,0,1)}$ in (iii) is written by

$$H_{(m,0,1)} = Z(G) H_{(m,0,0)} = H_{(m,0,0)} Z(G).$$

We put $\boldsymbol{0}_k = (0, \ldots, 0) \in \mathbb{R}^k$ $(1 \le k \le n)$. Then, each closed subgroup given in Lemma 2.5 forms as follows.

$$H_{(0,0,1)} = \{g(\boldsymbol{0}_n, \boldsymbol{0}_n, z) : z \in \mathbb{R}\},$$
$$H_{(p,q,1)} = \{g((\boldsymbol{w}_1, \boldsymbol{0}_{n-p}), (\boldsymbol{w}_2, \boldsymbol{0}_{n-q}), z) : \boldsymbol{w}_1 \in \mathbb{R}^p, \boldsymbol{w}_2 \in \mathbb{R}^q, z \in \mathbb{R}\},$$
$$H_{(m,0,1)} = \{g((\boldsymbol{w}, \boldsymbol{0}_{n-m}), \boldsymbol{0}_n, z) : \boldsymbol{w} \in \mathbb{R}^m, z \in \mathbb{R}\},$$
$$H_{(m,0,0)} = \{g((\boldsymbol{w}, \boldsymbol{0}_{n-m}), \boldsymbol{0}_n, 0) : \boldsymbol{w} \in \mathbb{R}^m\}.$$

2.3. The Lie group generated by $\exp \mathfrak{q}$

Retain the notation as in Sections 2.1 and 2.2. For a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{q} = C(\mathfrak{h})$, we denote by $\langle \exp \mathfrak{q} \rangle$ by the Lie group generated by $\exp \mathfrak{q}$. We set

(i) $Q_{(0,0,1)} := \langle \exp \mathfrak{q}_{(0,0,1)} \rangle$.

(ii) $Q_{(p,q,1)} := \langle \exp \mathfrak{q}_{(p,q,1)} \rangle$.

(iii) $Q_{(m,0,1)} := \langle \exp \mathfrak{q}_{(m,0,1)} \rangle$.

(iv) $Q_{(m,0,0)} := \langle \exp \mathfrak{q}_{(m,0,0)} \rangle$.

Lemma 2.6. *The subgroup $Q = \langle \exp \mathfrak{q} \rangle$ of G forms:*

(i) $Q_{(0,0,1)} = G$.

(ii)   (a) *If $1 \le p \le q < n$ then $Q_{(p,q,1)} = \exp(\mathfrak{q}_{(p,q,1)} + \mathbb{R}Z)$.*
        (b) *If $1 \le p < q = n$ then $Q_{(p,n,1)} = \exp \mathfrak{q}_{(p,n,1)}$.*

(iii) (a) *If $1 \le m < n$ then $Q_{(m,0,1)} = \exp(\mathfrak{q}_{(m,0,1)} + \mathbb{R}Z)$.*
        (b) *If $m = n$ then $Q_{(n,0,1)} = \exp \mathfrak{q}_{(n,0,1)}$.*

(iv) $Q_{(m,0,0)} = \exp \mathfrak{q}_{(m,0,0)}$ *for $1 \le m \le n$.*

Proof. We will prove this lemma using the bracket relations $[X_i, Y_i] = Z$ for $i = 1, 2, \ldots, n$.

First, $[\mathfrak{q}_{(0,0,1)}, \mathfrak{q}_{(0,0,1)}]$ is contained in $\mathfrak{q}_{(0,0,1)} + \mathbb{R}Z$ which coincides with the Lie algebra $\mathfrak{g}$. Hence, $Q_{(0,0,1)} = \exp \mathfrak{g} = G$.

Second, $\mathfrak{q}_{(p,q,1)} = \mathrm{span}_{\mathbb{R}}(\mathcal{B}_{(p,q,1)})^c$ contains X_n and Y_n if $q < n$. Then, the bracket $[\mathfrak{q}_{(p,q,1)}, \mathfrak{q}_{(p,q,1)}]$ lies in $\mathfrak{q}_{(p,q,1)} + \mathbb{R}Z$, from which we obtain $Q_{(p,q,1)} = \exp(\mathfrak{q}_{(p,q,1)} + \mathbb{R}Z)$. On the other hand, if $p < q = n$, then the subspace $\mathfrak{q}_{(p,n,1)} = \mathrm{span}_{\mathbb{R}}\{X_{p+1}, \ldots, X_n\}$ is an abelian subalgebra of $\mathfrak{g}$. Thus, we obtain $Q_{(p,n,1)} = \exp \mathfrak{q}_{(p,n,1)}$.

Third, $\mathfrak{q}_{(m,0,1)}$ contains X_n and Y_n. Then, we have $[\mathfrak{q}_{(m,0,1)}, \mathfrak{q}_{(m,0,1)}] \subset \mathfrak{q}_{(m,0,1)} + \mathbb{R}Z$. Hence, we obtain $Q_{(m,0,1)} = \exp(\mathfrak{q}_{(m,0,1)} + \mathbb{R}Z)$. As $\mathfrak{q}_{(n,0,1)}$ is an abelian subalgebra, we have $Q_{(n,0,1)} = \exp \mathfrak{q}_{(n,0,1)}$.

Finally, $\mathfrak{q}_{(m,0,0)}$ contains an element Z. Then, it is a subalgebra of $\mathfrak{g}$, from which $Q_{(m,0,0)} = \exp \mathfrak{q}_{(m,0,0)}$. ∎

2.4. Complex Heisenberg homogeneous spaces

Let H be a connected closed subgroup of the Heisenberg group G with Lie algebra $\mathfrak{h}$ which is one of (i)–(iv) given in Lemma 2.5. We denote by $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{q}^{\mathbb{C}}$ the complexifications of $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{q}$, respectively. Then, $\mathcal{B}, \mathcal{B}_{\mathfrak{h}}$ and $(\mathcal{B}_{\mathfrak{h}})^c$ are $\mathbb{C}$-bases of $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{q}^{\mathbb{C}}$, respectively, and we have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{q}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}} \cap \mathfrak{q}^{\mathbb{C}} = \{0\}$. We set $G^{\mathbb{C}} := \exp \mathfrak{g}^{\mathbb{C}}$ and $H^{\mathbb{C}} := \exp \mathfrak{h}^{\mathbb{C}}$. Clearly, $G^{\mathbb{C}}$ is given by

$$G^{\mathbb{C}} = \{g(\boldsymbol{x}, \boldsymbol{y}, z) : \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n, z \in \mathbb{C}\},$$

and hence it is isomorphic to $\mathbb{C}^{2n} \ltimes \mathbb{C}$ as a complex Lie group through the map $\iota : G^{\mathbb{C}} \to \mathbb{C}^{2n} \ltimes \mathbb{C}$ given by (8).

We say that $G^{\mathbb{C}}/H^{\mathbb{C}}$ is a *complex Heisenberg homogeneous space.* Clearly, this is of the form:

Lemma 2.7. *The complex homogeneous space $G^{\mathbb{C}}/H^{\mathbb{C}}$ is given by*

$$G^{\mathbb{C}}/H^{\mathbb{C}} = \{(\exp X)H^{\mathbb{C}} : X \in \mathfrak{q}^{\mathbb{C}}\}$$

and biholomorphic to $\exp \mathfrak{q}^{\mathbb{C}}$.

2.5. Equivalent holomorphic actions

In this subsection, we will introduce the notion of equivalent holomorphic actions.

Let D_1, D_2 be connected complex manifolds and K_1, K_2 be some Lie groups. Suppose that K_1 acts on D_1 and K_2 on D_2 holomorphically.

Definition 2.8. We say that the holomorphic K_1-action on D_1 is *equivalent* to the holomorphic K_2-action on D_2 by (Φ, φ) if there exists a biholomorphic diffeomorphism $\Phi : D_1 \to D_2$ and a Lie group isomorphism $\varphi : K_1 \to K_2$ such that

$$\Phi(g \cdot v) = \varphi(g) \cdot \Phi(v) \quad (v \in D_1, \; g \in K_1).$$

Let us explain that the property of the strong visibility keeps among equivalent holomorphic actions. Namely, we prove:

Proposition 2.9. *Suppose that the holomorphic K_1-action on D_1 is equivalent to the holomorphic K_2-action on D_2 by (Φ, φ). If the K_1-action on D_1 is strongly visible, then so is the K_2-action on D_2 and hence vice versa.*

Proof. By the assumption, one can take a real submanifold S_1 in D_1 such that $D_1' := K_1 \cdot S_1$ is a non-empty open set in D_1, and take an anti-holomorphic diffeomorphism σ_1 on D_1' such that $\sigma_1|_{S_1} = \mathrm{id}_{S_1}$ and σ_1 preserves each K_1-orbit in D_1'. We set $S_2 := \Phi(S_1)$ and $D_2' := K_2 \cdot S_2$. Since Φ is a diffeomorphism as smooth manifolds, S_2 is a real submanifold in D_2. Further, the set D_2' coincides with the image $\Phi(D_1')$ because

$$\Phi(D_1') = \Phi(K_1 \cdot S_1) = \varphi(K_1) \cdot \Phi(S_1) = K_2 \cdot S_2.$$

As D_1' is open in D_1, the image $\Phi(D_1') = D_2'$ is also open in D_2. Hence, S_2 satisfies the condition (V.0).

Next, we define a diffeomorphism σ_2 on D by $\sigma_2 = \Phi \circ \sigma_1 \circ \Phi^{-1}$. This is an anti-holomorphic map since Φ is holomorphic. Here, we take an element $s_2 \in S_2$. As $S_2 = \Phi(S_1)$, we write $s_2 = \Phi(s_1)$ for some $s_1 \in S_1$. Since $\sigma_1(s_1) = s_1$, we have

$$\sigma_2(s_2) = \Phi \circ \sigma_1 \circ \Phi^{-1}(\Phi(s_1)) = \Phi \circ \sigma_1(s_1) = \Phi(s_1) = s_2.$$

This implies σ_2 is the identity map on S_2, from which the condition (S.1) has been verified.

Finally, let us show that σ_2 satisfies the condition (S.2). For this, let v_2 be an element of D_2'. As $D_2' = \Phi(D_1')$, we write $v_2 = \Phi(v_1)$ for some $v_1 \in D_1'$. We recall that $\sigma_1(v_1)$ forms $\sigma_1(v_1) = g_1 \cdot v_1$ for some $g_1 \in K_1$. When we set $g_2 := \varphi(g_1) \in K_2$, we have

$$\sigma_2(v_2) = \Phi \circ \sigma_1 \circ \Phi^{-1}(\Phi(v_1)) = \Phi \circ \sigma_1(v_1) = \Phi(g_1 \cdot v_1) = g_2 \cdot v_2.$$

This means that $\sigma_2(v_2) = g_2 \cdot v_2 \in K_2 \cdot v_2$. Therefore, Proposition 2.9 is proved. ∎

Thanks to Proposition 2.9, we do not distinguish equivalent holomorphic actions in this paper.

2.6. Classification of actions on Heisenberg homogeneous spaces

In this subsection, we explain that it suffices to provide a proof of Theorem 1.3 in the case where a closed subgroup H is taken to be one of (i)–(iv) in Lemma 2.5. First, let $\mathfrak{h}_1, \mathfrak{h}_2$ be some subalgebras of the Heisenberg Lie algebra $\mathfrak{g}$ satisfying $\mathfrak{h}_1 \simeq \mathfrak{h}_2$.

Lemma 2.10. *If $\mathfrak{h}_1$ is isomorphic to $\mathfrak{h}_2$, then there exists an automorphism φ on $\mathfrak{g}$ such that the restriction of φ to $\mathfrak{h}_1$ gives rise to an isomorphism from $\mathfrak{h}_1$ to $\mathfrak{h}_2$.*

Proof. According to [1, Proposition 3.1] (see also Lemma 2.1), we may and do assume that a subalgebra $\mathfrak{h}_1$ is either $\mathfrak{h}_{(0,0,1)} = \mathbb{R}Z$, $\mathfrak{h}_{(p,q,1)} = \mathrm{span}_{\mathbb{R}} \mathcal{B}_{(p,q,1)}$, $\mathfrak{h}_{(m,0,1)} = \mathrm{span}_{\mathbb{R}} \mathcal{B}_{(m,0,1)}$ or $\mathfrak{h}_{(m,0,0)} = \mathrm{span}_{\mathbb{R}} \mathcal{B}_{(m,0,0)}$, and $\mathfrak{h}_2$ is isomorphic to $\mathfrak{h}_{(0,0,1)}$, $\mathfrak{h}_{(p,q,1)}$, $\mathfrak{h}_{(m,0,1)}$ or $\mathfrak{h}_{(m,0,0)}$.
Let us treat the case $\mathfrak{h}_2 \simeq \mathfrak{h}_{(p,q,1)}$. By [1, Proposition 3.1], there exists another basis $\mathcal{B}' = \{X_1', \ldots, X_n', Y_1', \ldots, Y_n', Z'\}$ of $\mathfrak{g}$ with relations $[X_i', Y_i'] = Z'$ $(1 \leq i \leq n)$ such that a basis of $\mathfrak{h}_2$ is taken to be the subset

$$\mathcal{B}_{\mathfrak{h}_2} = \mathcal{B}'_{(p,q,1)} := \{X_1', \ldots, X_p', Y_1', \ldots, Y_q', Z'\}$$

in $\mathcal{B}'$. Here, we define a linear transformation φ on $\mathfrak{g}$ by $\varphi(X_i) = X_i'$, $\varphi(Y_i) = Y_i'$ $(1 \leq i \leq n)$ and $\varphi(Z) = Z'$. Then, φ is an automorphism on $\mathfrak{g}$ and $\varphi : \mathfrak{h}_{(p,q,1)} \to \mathfrak{h}_2$ becomes a Lie algebra isomorphism.
Similarly, we can also prove this lemma for other cases. Thus, we omit the proof. ∎

Next, let φ be an automorphism on $\mathfrak{g}$ such that $\varphi|_{\mathfrak{h}_1} : \mathfrak{h}_1 \to \mathfrak{h}_2$ is a Lie algebra isomorphism. We extend $\varphi \in \mathrm{Aut}\,\mathfrak{g}$ to a $\mathbb{C}$-linear map on the complexification $\mathfrak{g}^{\mathbb{C}}$, namely, $\varphi(X + \sqrt{-1}Y) = \varphi(X) + \sqrt{-1}\varphi(Y)$ $(X, Y \in \mathfrak{g})$. Then, this is an automorphism on $\mathfrak{g}^{\mathbb{C}}$. Further, we can lift it to an automorphism on the complexified Heisenberg group $G^{\mathbb{C}} = \exp \mathfrak{g}^{\mathbb{C}}$, for which we still denote by the letter φ. Obviously, it is holomorphic.
By Lemma 2.10, the image $\varphi(H_1^{\mathbb{C}})$ of $H_1^{\mathbb{C}} = \exp \mathfrak{h}_1^{\mathbb{C}}$ coincides with $H_2^{\mathbb{C}} = \exp \mathfrak{h}_2^{\mathbb{C}}$, from which it gives rise to a holomorphic diffeomorphism

$$\Phi : G^{\mathbb{C}}/H_1^{\mathbb{C}} \to G^{\mathbb{C}}/H_2^{\mathbb{C}}, \quad gH_1^{\mathbb{C}} \mapsto \Phi(gH_1^{\mathbb{C}}) = \varphi(g)H_2^{\mathbb{C}}. \tag{9}$$

Retain the notation as in the proof of Lemma 2.10. Let $\mathcal{B}_{\mathfrak{h}_1} \subset \mathcal{B}$ be a basis of $\mathfrak{h}_1$ and $\mathcal{B}_{\mathfrak{h}_2} \subset \mathcal{B}'$ a basis of $\mathfrak{h}_2$. We set $\mathfrak{q}_1 = \mathrm{span}_{\mathbb{R}}(\mathcal{B} - \mathcal{B}_{\mathfrak{h}_1})$ and $\mathfrak{q}_2 = \mathrm{span}_{\mathbb{R}}(\mathcal{B}' - \mathcal{B}_{\mathfrak{h}_2})$, $Q_1 = \langle \exp \mathfrak{q}_1 \rangle$ and $Q_2 = \langle \exp \mathfrak{q}_2 \rangle$.

Lemma 2.11. *The Lie group Q_1 is isomorphic to Q_2 through φ.*

Proof. Clearly, $\varphi \in \mathrm{Aut}\,\mathfrak{g}$ gives a linear isomorphism from $\mathfrak{q}_1$ to $\mathfrak{q}_2$.
Let us see that $\varphi(Q_1)$ is contained in Q_2 as follows. Let $A_1, \ldots, A_k$ be elements of $\mathfrak{q}_1$. Then, $e^{A_1} \cdots e^{A_k}$ lies in Q_1. Since φ is an automorphism on $G^{\mathbb{C}}$, we have $\varphi(e^{A_1} \cdots e^{A_k}) = e^{\varphi(A_1)} \cdots e^{\varphi(A_k)}$. As $\varphi(A_1), \ldots, \varphi(A_k)$ are in $\mathfrak{q}_2$, $\varphi(e^{A_1} \cdots e^{A_k})$ is an element of Q_2. Hence, we obtain $\varphi(Q_1) \subset Q_2$.

On the other hand, we take an arbitrary $e^{B_1} \cdots e^{B_\ell} \in Q_2$ with $B_1, \ldots, B_\ell \in \mathfrak{q}_2$. For each $1 \leq i \leq \ell$, we set $A_i := \varphi^{-1}(B_i)$, which is an element of $\mathfrak{q}_1$. Then, we have $\varphi(e^{A_1} \cdots e^{A_\ell}) = e^{B_1} \cdots e^{B_\ell}$. This means $\varphi(Q_1) \supset Q_2$.

Consequently, we obtain $\varphi(Q_1) = Q_2$. Since φ is an automorphism on $G^{\mathbb{C}}$, the map $\varphi : Q_1 \to Q_2$ is a Lie group isomorphism. ∎

Proposition 2.12.   *If $\mathfrak{h}_1$ is isomorphic to $\mathfrak{h}_2$ as a Lie algebra, then the holomorphic Q_1-action on $G^{\mathbb{C}}/H_1^{\mathbb{C}}$ is equivalent to the holomorphic Q_2-action on $G^{\mathbb{C}}/H_2^{\mathbb{C}}$.*

Proof.   It follows from (9) that the following equality holds for $g \in Q_1$ and $x \in G^{\mathbb{C}}$:

$$\Phi(g \cdot xH_1^{\mathbb{C}}) = \varphi(gx)H_2^{\mathbb{C}} = \varphi(g) \cdot \varphi(x)H_2^{\mathbb{C}} = \varphi(g) \cdot \Phi(xH_1^{\mathbb{C}}).$$

By Lemma 2.11, $\varphi(g)$ is an element of Q_2 for any $g \in Q_1$. Hence, the Q_1-action on $G^{\mathbb{C}}/H_1^{\mathbb{C}}$ is equivalent to the Q_2-action on $G^{\mathbb{C}}/H_2^{\mathbb{C}}$ by (Φ, φ). ∎

Retain the setting as in Proposition 2.12 and its proof. Let L_1 be a closed subgroup of G containing Q_1 and $L_2 := \varphi(L_1) \supset \varphi(Q_1) = Q_2$. The following corollary can be proved in the same argument as the proof of Proposition 2.12.

Corollary 2.13.   *The holomorphic L_1-action on $G^{\mathbb{C}}/H_1^{\mathbb{C}}$ is equivalent to the holomorphic L_2-action on $G^{\mathbb{C}}/H_2^{\mathbb{C}}$ via (Φ, φ).*

Due to Corollary 2.13, we shall deal with closed subgroups $H_{(k,\ell,\varepsilon)}$ as in Lemma 2.5 and write the Heisenberg homogeneous space $D_{(k,\ell,\varepsilon)}$ as

$$D_{(k,\ell,\varepsilon)} := G^{\mathbb{C}}/H_{(k,\ell,\varepsilon)}^{\mathbb{C}} = \{(\exp X)H_{(k,\ell,\varepsilon)}^{\mathbb{C}} : X \in \mathfrak{q}_{(k,\ell,\varepsilon)}^{\mathbb{C}}\}. \tag{10}$$

2.7. Anti-holomorphic diffeomorphisms on complex Heisenberg homogeneous spaces

In this subsection, we will prepare two anti-holomorphic diffeomorphisms σ_1, σ_2 on the complex Heisenberg homogeneous space $D_{(k,\ell,\varepsilon)}$.

For a complex number $\alpha = \alpha_R + \sqrt{-1}\alpha_I \in \mathbb{C}$ with $\alpha_R, \alpha_I \in \mathbb{R}$, we define $\overline{\alpha} := \alpha_R - \sqrt{-1}\alpha_I$. Equivalently, $\overline{} : \mathbb{C} \to \mathbb{C}$ is the complex conjugation of $\mathbb{C}$ with respect to the real form $\mathbb{R}$. We extend it to the complex conjugation of $\mathbb{C}^n$ with respect to the real form $\mathbb{R}^n$, namely, $\mathbb{C}^n \to \mathbb{C}^n$, $\boldsymbol{x} \mapsto \overline{\boldsymbol{x}} := (\overline{x_1}, \ldots, \overline{x_n})$.

Now, we define an anti-holomorphic diffeomorphism $\widetilde{\sigma}_1$ on $G^{\mathbb{C}}$ by

$$\widetilde{\sigma}_1(g(\boldsymbol{x}, \boldsymbol{y}, z)) := g(-\overline{\boldsymbol{x}}, -\overline{\boldsymbol{y}}, \overline{z}) \quad (g(\boldsymbol{x}, \boldsymbol{y}, z) \in G^{\mathbb{C}}), \tag{11}$$

and an anti-holomorphic diffeomorphism $\widetilde{\sigma}_2$ on $G^{\mathbb{C}}$ by

$$\widetilde{\sigma}_2(g(\boldsymbol{x}, \boldsymbol{y}, z)) := g(\overline{\boldsymbol{x}}, -\overline{\boldsymbol{y}}, -\overline{z}) \quad (g(\boldsymbol{x}, \boldsymbol{y}, z) \in G^{\mathbb{C}}). \tag{12}$$

Lemma 2.14.   *$\widetilde{\sigma}_1, \widetilde{\sigma}_2$ are involutive automorphisms on $G^{\mathbb{C}}$.*

Proof.   Clearly, $\widetilde{\sigma}_i^2 = \mathrm{id}$, from which $\widetilde{\sigma}_i$ is involutive for each $i = 1, 2$. In view of Lemma 2.4, the direct computation shows

$$\widetilde{\sigma}_1(g(\boldsymbol{x}_1, \boldsymbol{y}_1, z_1) \cdot g(\boldsymbol{x}_2, \boldsymbol{y}_2, z_2))$$
$$= g(-(\overline{\boldsymbol{x}_1 + \boldsymbol{x}_2}), -(\overline{\boldsymbol{y}_1 + \boldsymbol{y}_2}), \overline{z_1 + z_2 - (\boldsymbol{y}_1|\boldsymbol{x}_2)})$$
$$= g(-\overline{\boldsymbol{x}_1} - \overline{\boldsymbol{x}_2}, -\overline{\boldsymbol{y}_1} - \overline{\boldsymbol{y}_2}, \overline{z_1} + \overline{z_2} - \overline{(\boldsymbol{y}_1|\boldsymbol{x}_2)})$$
$$= g(-\overline{\boldsymbol{x}_1} - \overline{\boldsymbol{x}_2}, -\overline{\boldsymbol{y}_1} - \overline{\boldsymbol{y}_2}, \overline{z_1} + \overline{z_2} - (-\overline{\boldsymbol{y}_1}| - \overline{\boldsymbol{x}_2}))$$
$$= g(-\overline{\boldsymbol{x}_1}, -\overline{\boldsymbol{y}_1}, \overline{z_1}) \cdot g(-\overline{\boldsymbol{x}_2}, -\overline{\boldsymbol{y}_2}, \overline{z_2})$$
$$= \widetilde{\sigma}_1(g(\boldsymbol{x}_1, \boldsymbol{y}_1, z_1)) \cdot \widetilde{\sigma}_1(g(\boldsymbol{x}_2, \boldsymbol{y}_2, z_2))$$

for any $g(\boldsymbol{s}, \boldsymbol{t}, u), g(\boldsymbol{x}, \boldsymbol{y}, z) \in G^{\mathbb{C}}$. This implies that $\widetilde{\sigma}_1$ is an automorphism on $G^{\mathbb{C}}$. Similarly, one can show that $\widetilde{\sigma}_2$ is also an automorphism on $G^{\mathbb{C}}$. ∎

Let $H_{(k,\ell,\varepsilon)}$ be a closed subgroup of G which is one of (i)–(iv) in Lemma 2.5 and $H^{\mathbb{C}}_{(k,\ell,\varepsilon)}$ the complexification of $H_{(k,\ell,\varepsilon)}$. Clearly, $\widetilde{\sigma}_i$ stablizes $H^{\mathbb{C}}_{(k,\ell,\varepsilon)}$ for each $i = 1, 2$, which gives rise to an anti-holomorphic diffeomorphism σ_i on the complex Heisenberg homogeneous space $D_{(k,\ell,\varepsilon)}$, namely,

$$\sigma_i((\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)}) := \widetilde{\sigma}_i(\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)} \quad (X \in \mathfrak{q}^{\mathbb{C}}_{(k,\ell,\varepsilon)}). \tag{13}$$

Remark 2.15. Concerning equality (13), the following equality holds:

$$\sigma_i(g(\boldsymbol{s}, \boldsymbol{t}, u) \cdot (\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)}) = \widetilde{\sigma}_i(g(\boldsymbol{s}, \boldsymbol{t}, u) \cdot \exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)}$$
$$= \widetilde{\sigma}_i(g(\boldsymbol{s}, \boldsymbol{t}, u)) \cdot \widetilde{\sigma}_i(\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)} = \widetilde{\sigma}_i(g(\boldsymbol{s}, \boldsymbol{t}, u)) \cdot \sigma_i((\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)}).$$

Hence, $\widetilde{\sigma}_i$ is compatible with σ_i in the sense of [9, Section 4.2].

2.8. Real form of complex Heisenberg homogeneous space

In the previous subsection, we have prepared the anti-holomorphic diffeomorphisms σ_1, σ_2 on $D_{(k,\ell,\varepsilon)}$ defined by (13). Then, we have a real form $M^i_{(k,\ell,\varepsilon)}$ of $D_{(k,\ell,\varepsilon)}$ corresponding to σ_i, namely,

$$M^i_{(k,\ell,\varepsilon)} := D^{\sigma_i}_{(k,\ell,\varepsilon)} = \{v \in D_{(k,\ell,\varepsilon)} : \sigma_i(v) = v\}$$

for each $i = 1, 2$. To clarify $M^i_{(k,\ell,\varepsilon)}$, we observe:

Lemma 2.16. *For an element* $v = (\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)}$ *with* $X \in \mathfrak{q}^{\mathbb{C}}_{(k,\ell,\varepsilon)}$, *the equality* $\sigma_i(v) = v$ *holds if and only if* $\widetilde{\sigma}_i(\exp X) = \exp X$ *holds.*

Proof. If $X \in \mathfrak{q}^{\mathbb{C}}_{(k,\ell,\varepsilon)}$ satisfies $\widetilde{\sigma}_i(\exp X) = \exp X$, then we have

$$\sigma_i((\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)}) = \widetilde{\sigma}_i(\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)} = (\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)}.$$

This means $\sigma_i(v) = v$.

Conversely, suppose that v satisfies $\sigma_i(v) = v$. Then, $\exp(-X)\widetilde{\sigma}_i(\exp X)$ lies in $H^{\mathbb{C}}_{(k,\ell,\varepsilon)}$. Since $\mathfrak{h}^{\mathbb{C}}_{(k,\ell,\varepsilon)} \cap \mathfrak{q}^{\mathbb{C}}_{(k,\ell,\varepsilon)} = \{0\}$, it has to be the unit element of $G^{\mathbb{C}}$. Thus, we get $\widetilde{\sigma}_i(\exp X) = \exp X$. ∎

By Lemma 2.16, our choice of real form $M^i_{(k,\ell,\varepsilon)}$ forms

$$M^i_{(k,\ell,\varepsilon)} = \{(\exp X)H^{\mathbb{C}}_{(k,\ell,\varepsilon)} : X \in \mathfrak{q}^{\mathbb{C}}_{(k,\ell,\varepsilon)}, \widetilde{\sigma}_i(\exp X) = \exp X\}. \tag{14}$$

2.9. Strategy of proof of Theorem 1.3

In this subsection, we will explain how to prove Theorem 1.3. More precisely, our proof is based on the following lemma.

Lemma 2.17. *Suppose that for the anti-holomorphic involution $\widetilde{\sigma}_i$ on $G^{\mathbb{C}}$ there exist a $\widetilde{\sigma}_i$-stable connected closed subgroup $L_{(k,\ell,\varepsilon)}$ of G containing $Q_{(k,\ell,\varepsilon)}$ and a submanifold $S_{(k,\ell,\varepsilon)}$ in $M^i_{(k,\ell,\varepsilon)}$ such that $D'_{(k,\ell,\varepsilon)} := L_{(k,\ell,\varepsilon)} \cdot S_{(k,\ell,\varepsilon)}$ is a non-empty open set in $D_{(k,\ell,\varepsilon)}$. Then, $S_{(k,\ell,\varepsilon)}$ and the anti-holomorphic diffeomorphism σ_i on $D_{(k,\ell,\varepsilon)}$ satisfy the conditions (S.1) and (S.2) for the $L_{(k,\ell,\varepsilon)}$-action on $D_{(k,\ell,\varepsilon)}$.*

Proof.   As $S_{(k,\ell,\varepsilon)} \subset M^i_{(k,\ell,\varepsilon)} = D^{\sigma_i}_{(k,\ell,\varepsilon)}$, we have $\sigma_i|_{S_{(k,\ell,\varepsilon)}} = \mathrm{id}_{S_{(k,\ell,\varepsilon)}}$, from which (S.1) has been verified.

Next, let us take an element $v = g \cdot s \in D'_{(k,\ell,\varepsilon)}$ with $g \in L_{(k,\ell,\varepsilon)}$ and $s \in S_{(k,\ell,\varepsilon)}$. Then, we have

$$\sigma_i(v) = \widetilde{\sigma}_i(g) \cdot \sigma_i(s) = \widetilde{\sigma}_i(g) \cdot s = \widetilde{\sigma}_i(g)g^{-1} \cdot v. \tag{15}$$

Since $L_{(k,\ell,\varepsilon)}$ is $\widetilde{\sigma}_i$-stable, the element $\widetilde{\sigma}_i(g)g^{-1}$ lies in $L_{(k,\ell,\varepsilon)}$. Hence, (15) implies condition (S.2). ∎

Thanks to Lemma 2.17, it is sufficient for the proof of Theorem 1.3 to find suitable $L_{(k,\ell,\varepsilon)}$ and $S_{(k,\ell,\varepsilon)}$ such that $L_{(k,\ell,\varepsilon)} \cdot S_{(k,\ell,\varepsilon)}$ is open in $D_{(k,\ell,\varepsilon)}$. Then, we will give an explicit description of such pair $(L_{(k,\ell,\varepsilon)}, S_{(k,\ell,\varepsilon)})$ for $D_{(k,\ell,\varepsilon)} = D_{(0,0,1)}$ in Section 3; for $D_{(k,\ell,\varepsilon)} = D_{(p,q,1)}$ in Section 4; for $D_{(k,\ell,\varepsilon)} = D_{(m,0,1)}$ in Section 5; and for $D_{(k,\ell,\varepsilon)} = D_{(m,0,0)}$ in Section 6.

3.   Visible actions on $G^{\mathbb{C}}/H^{\mathbb{C}}_{(0,0,1)}$

This section deals with the $Q_{(0,0,1)}$-action on $D_{(0,0,1)} = G^{\mathbb{C}}/H^{\mathbb{C}}_{(0,0,1)}$. Recall that $\mathfrak{h}_{(0,0,1)} = \mathfrak{z}(\mathfrak{g}) = \mathbb{R}Z$ is the center of $\mathfrak{g}$. Then, $H^{\mathbb{C}}_{(0,0,1)} = \exp \mathfrak{h}^{\mathbb{C}}_{(0,0,1)} = \exp \mathbb{C}Z$ and

$$\mathfrak{q}_{(0,0,1)} = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}.$$

Thus, it follows from Lemma 2.7 that $D_{(0,0,1)} := G^{\mathbb{C}}/H^{\mathbb{C}}_{(0,0,1)}$ is written as follows:

$$D_{(0,0,1)} = \{v_{(0,0,1)}(\boldsymbol{x}, \boldsymbol{y}) := g(\boldsymbol{x}, \boldsymbol{y}, 0)H^{\mathbb{C}}_{(0,0,1)} : \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n\}.$$

Now, we consider the action of $Q_{(0,0,1)}$ on $D_{(0,0,1)}$. By Lemma 2.6, $Q_{(0,0,1)}$ coincides with $G = \mathbf{H}_n$. In view of Lemma 2.4, the $Q_{(0,0,1)}$-action on $D_{(0,0,1)}$ is given by

$$g(\boldsymbol{s}, \boldsymbol{t}, u) \cdot v_{(0,0,1)}(\boldsymbol{x}, \boldsymbol{y}) = (g(\boldsymbol{s}, \boldsymbol{t}, u) \cdot g(\boldsymbol{x}, \boldsymbol{y}, 0))H^{\mathbb{C}}_{(0,0,1)}$$

$$= g(\boldsymbol{s}+\boldsymbol{x}, \boldsymbol{t}+\boldsymbol{y}, u-(\boldsymbol{t}|\boldsymbol{x}))H^{\mathbb{C}}_{(0,0,1)} = g(\boldsymbol{s}+\boldsymbol{x}, \boldsymbol{t}+\boldsymbol{y}, 0)H^{\mathbb{C}}_{(0,0,1)} = v_{(0,0,1)}(\boldsymbol{s}+\boldsymbol{x}, \boldsymbol{t}+\boldsymbol{y}).$$

Now, we take an anti-holomorphic diffeomorphism on $D_{(0,0,1)}$ as σ_1 (see (13)). By Lemma 2.16, our choice of the real form $M^1_{(0,0,1)} = D^{\sigma_1}_{(0,0,1)}$ is written by

$$M^1_{(0,0,1)} = \{v_{(0,0,1)}(\sqrt{-1}\boldsymbol{a}, \sqrt{-1}\boldsymbol{b}) : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n\} \tag{16}$$

since $\widetilde{\sigma}_1(g(\boldsymbol{x}, \boldsymbol{y}, 0)) = g(-\overline{\boldsymbol{x}}, -\overline{\boldsymbol{y}}, 0)$ for $g(\boldsymbol{x}, \boldsymbol{y}, 0) \in \exp \mathfrak{q}^{\mathbb{C}}_{(0,0,1)}$.

Proposition 3.1. *We take a connected closed subgroup $L_{(0,0,1)}$ as $Q_{(0,0,1)}$ and a submanifold $S_{(0,0,1)}$ as $M^1_{(0,0,1)}$. Then, $L_{(0,0,1)} \cdot S_{(0,0,1)}$ coincides with $D_{(0,0,1)}$.*

Proof. Let $v_{(0,0,1)}(\boldsymbol{x}, \boldsymbol{y})$ be an element of $D_{(0,0,1)}$. According to the decomposition $\mathbb{C}^n = \mathbb{R}^n + \sqrt{-1}\mathbb{R}^n$, we write $\boldsymbol{x} = \boldsymbol{x}_R + \sqrt{-1}\boldsymbol{x}_I$ for some $\boldsymbol{x}_R, \boldsymbol{x}_I \in \mathbb{R}^{2n}$ and similarly $\boldsymbol{y} = \boldsymbol{y}_R + \sqrt{-1}\boldsymbol{y}_I$ for some $\boldsymbol{y}_R, \boldsymbol{y}_I \in \mathbb{R}^{2n}$. Then, we have

$$v_{(0,0,1)}(\boldsymbol{x}, \boldsymbol{y}) = g(\boldsymbol{x}_R, \boldsymbol{y}_R, 0) \cdot v(\sqrt{-1}\boldsymbol{x}_I, \sqrt{-1}\boldsymbol{y}_I). \tag{17}$$

Since $g(\boldsymbol{x}_R, \boldsymbol{y}_R, 0) \in Q_{(0,0,1)}$ and $v_{(0,0,1)}(\sqrt{-1}\boldsymbol{x}_I, \sqrt{-1}\boldsymbol{y}_I) \in S_{(0,0,1)}$, this implies $v_{(0,0,1)}(\boldsymbol{x}, \boldsymbol{y}) \in Q_{(0,0,1)} \cdot S_{(0,0,1)}$. Hence, Proposition 3.1 has been verified. ∎

Therefore, we conclude:

Theorem 3.2. *The $Q_{(0,0,1)}$-action on $D_{(0,0,1)} = G^{\mathbb{C}}/H^{\mathbb{C}}_{(0,0,1)}$ is strongly visible with $2n$-dimensional slice $M^1_{(0,0,1)} \simeq \exp\sqrt{-1}\mathfrak{q}_{(0,0,1)}$.*

4. Visible actions on $G^{\mathbb{C}}/H^{\mathbb{C}}_{(p,q,1)}$

This section deals with the case where a closed subgroup of G is $H_{(p,q,1)}$, where $1 \leq p \leq q \leq n$, $(p,q) \neq (n,n)$. We set $D_{(p,q,1)} := G^{\mathbb{C}}/H^{\mathbb{C}}_{(p,q,1)}$. From now, let us consider the $Q_{(p,q,1)}$-action on $D_{(p,q,1)}$.

In view of Lemma 2.2, we study separately the following dichotomous cases: a general case $1 \leq p \leq q < n$; a special case $1 \leq p < q = n$.

4.1. $Q_{(p,q,1)}$-action on $D_{(p,q,1)}$

First, we consider a general case $1 \leq p \leq q < n$. The complementary subspace $\mathfrak{q}_{(p,q,1)}$ of $\mathfrak{h}_{(p,q,1)}$ in $\mathfrak{g}$ is given by

$$\mathfrak{q}_{(p,q,1)} = \mathrm{span}_{\mathbb{R}}\{X_{p+1}, \ldots, X_n, Y_{q+1}, \ldots, Y_n\}.$$

By Lemma 2.6, the Lie group $Q_{(p,q,1)} = \langle \exp \mathfrak{q}_{(p,q,1)} \rangle$ equals $\exp(\mathfrak{q}_{(p,q,1)} + \mathbb{R}Z)$, which is of the form

$$Q_{(p,q,1)} = \{g((\boldsymbol{0}_p, \boldsymbol{s}'), (\boldsymbol{0}_q, \boldsymbol{t}'), u) : \boldsymbol{s}' \in \mathbb{R}^{n-p}, \boldsymbol{t}' \in \mathbb{R}^{n-q}, u \in \mathbb{R}\}.$$

We note that $Q_{(p,q,1)}$ is isomorphic to $\mathbb{R}^{q-p} \times \mathbf{H}_{n-q}$ as a Lie group if $p < q$ and $\mathbf{H}_{n-q}$ if $p = q$. Then the complex Heisenberg homogeneous space $D_{(p,q,1)}$ is expressed by

$$D_{(p,q,1)} = \{v_{(p,q,1)}(\boldsymbol{x}', \boldsymbol{y}') : \boldsymbol{x}' \in \mathbb{C}^{n-p}, \boldsymbol{y}' \in \mathbb{C}^{n-q}\}.$$

where we write $v_{(p,q,1)}(\boldsymbol{x}', \boldsymbol{y}') := g((\boldsymbol{0}_p, \boldsymbol{x}'), (\boldsymbol{0}_q, \boldsymbol{y}'), 0)H^{\mathbb{C}}_{(p,q,1)}$.

It follows from Lemma 2.4 that the $Q_{(p,q,1)}$-action on $D_{(p,q,1)}$ is given by

$$\begin{aligned}
&g((\boldsymbol{0}_p, \boldsymbol{s}'), (\boldsymbol{0}_q, \boldsymbol{t}'), u) \cdot v_{(p,q,1)}(\boldsymbol{x}', \boldsymbol{y}') \\
&= (g((\boldsymbol{0}_p, \boldsymbol{s}'), (\boldsymbol{0}_q, \boldsymbol{t}'), u) \cdot g((\boldsymbol{0}_p, \boldsymbol{x}'), (\boldsymbol{0}_q, \boldsymbol{y}'), 0))H^{\mathbb{C}}_{(p,q,1)} \\
&= g((\boldsymbol{0}_p, \boldsymbol{s}' + \boldsymbol{x}'), (\boldsymbol{0}_q, \boldsymbol{t}' + \boldsymbol{y}'), u - ((\boldsymbol{0}_q, \boldsymbol{t}')|(\boldsymbol{0}_p, \boldsymbol{x}')))H^{\mathbb{C}}_{(p,q,1)} \\
&= g((\boldsymbol{0}_p, \boldsymbol{s}' + \boldsymbol{x}'), (\boldsymbol{0}_q, \boldsymbol{t}' + \boldsymbol{y}'), 0)H^{\mathbb{C}}_{(p,q,1)} = v_{(p,q,1)}(\boldsymbol{s}' + \boldsymbol{x}', \boldsymbol{t}' + \boldsymbol{y}')
\end{aligned}$$

for $g((\boldsymbol{0}_p, \boldsymbol{s}'), (\boldsymbol{0}_q, \boldsymbol{t}'), u) \in Q_{(p,q,1)}$ and $v_{(p,q,1)}(\boldsymbol{x}', \boldsymbol{y}') \in D_{(p,q,1)}$.

In the case $1 \leq p < q = n$, the group $Q_{(p,n,1)}$ is given by

$$Q_{(p,n,1)} = \{g((\mathbf{0}_p, \boldsymbol{s}'), \mathbf{0}_n, 0) : \boldsymbol{s}' \in \mathbb{R}^{n-p}, u \in \mathbb{R}\}$$

and the Heisenberg homogeneous space $D_{(p,n,1)}$ by

$$D_{(p,n,1)} = \{v_{(p,n,1)}(\boldsymbol{x}') := g((\mathbf{0}_p, \boldsymbol{x}'), \mathbf{0}_n, 0)H^{\mathbb{C}}_{(p,n,1)} : \boldsymbol{x}' \in \mathbb{C}^{n-p}\}.$$

Then, the $Q_{(p,n,1)}$-action on $D_{(p,n,1)}$ is written by

$$g((\mathbf{0}_p, \boldsymbol{s}'), \mathbf{0}_n, 0) \cdot v_{(p,n,1)}(\boldsymbol{x}') = v_{(p,n,1)}(\boldsymbol{s}' + \boldsymbol{x}')$$

for $g((\mathbf{0}_p, \boldsymbol{s}'), \mathbf{0}_n, 0) \in Q_{(p,n,1)}$ and $v_{(p,n,1)}(\boldsymbol{x}') \in D_{(p,n,1)}$.

4.2. Verification of (V.0) for $D_{(p,q,1)}$

Let us take $L_{(p,q,1)}$ as $Q_{(p,q,1)}$ and an anti-holomorphic diffeomorphism on $D_{(p,q,1)}$ as σ_1. Then, we prove:

Proposition 4.1. *We take a connected closed subgroup $L_{(p,q,1)}$ as $Q_{(p,q,1)}$ and a submanifold $S_{(p,q,1)}$ in $D_{(p,q,1)}$ as $M^1_{(p,q,1)}$. Then, $L_{(p,q,1)} \cdot S_{(p,q,1)}$ coincides with $D_{(p,q,1)}$.*

First, let us assume that $1 \leq p \leq q < n$. We note that the restriction of the anti-holomorphic involution $\widetilde{\sigma}_1$ to $\exp \mathfrak{q}^{\mathbb{C}}_{(p,q,1)}$ is

$$\widetilde{\sigma}_1(g((\mathbf{0}_p, \boldsymbol{x}'), (\mathbf{0}_q, \boldsymbol{y}'), 0)) = g((\mathbf{0}_p, -\overline{\boldsymbol{x}'}), (\mathbf{0}_q, -\overline{\boldsymbol{y}'}), 0)$$

for $g((\mathbf{0}_p, \boldsymbol{x}'), (\mathbf{0}_q, \boldsymbol{y}'), 0) \in \exp \mathfrak{q}^{\mathbb{C}}_{(p,q,1)}$. It follows from Lemma 2.16 that the real form $M^1_{(p,q,1)}$ is given by

$$M^1_{(p,q,1)} = \{v_{(p,q,1)}(\sqrt{-1}\boldsymbol{a}', \sqrt{-1}\boldsymbol{b}') : \boldsymbol{a}' \in \mathbb{R}^{n-p}, \boldsymbol{b} \in \mathbb{R}^{n-q}\}. \tag{18}$$

Proof of Proposition 4.1 for $1 \leq p \leq q < n$.   Let $v_{(p,q,1)}(\boldsymbol{x}', \boldsymbol{y}')$ be an element of $D_{(p,q,1)}$. We write in the following $\boldsymbol{x}' = \boldsymbol{x}'_R + \sqrt{-1}\boldsymbol{x}'_I$ and $\boldsymbol{y}' = \boldsymbol{y}'_R + \sqrt{-1}\boldsymbol{y}'_I$ for some $\boldsymbol{x}'_R, \boldsymbol{y}'_R \in \mathbb{R}^{n-p}$ and $\boldsymbol{x}'_I, \boldsymbol{y}'_I \in \mathbb{R}^{n-q}$. Then, we have

$$v_{(p,q,1)}(\boldsymbol{x}', \boldsymbol{y}') = g((\mathbf{0}_p, \boldsymbol{x}'_R), (\mathbf{0}_q, \boldsymbol{y}'_R), 0) \cdot v_{(p,q,1)}(\sqrt{-1}\boldsymbol{x}'_I, \sqrt{-1}\boldsymbol{y}'_I).$$

As $g((\mathbf{0}_p, \boldsymbol{x}'_R), (\mathbf{0}_q, \boldsymbol{y}'_R), 0) \in Q_{(p,q,1)}$ and $v_{(p,q,1)}(\sqrt{-1}\boldsymbol{x}'_I, \sqrt{-1}\boldsymbol{y}'_I) \in S_{(p,q,1)}$, this implies that $Q_{(p,q,1)} \cdot S_{(p,q,1)} = D_{(p,q,1)}$. ∎

For a special case $1 \leq p < q = n$, the real form $M^1_{(p,n,1)}$ is

$$M^1_{(p,n,1)} = \{v_{(p,n,1)}(\sqrt{-1}\boldsymbol{a}') : \boldsymbol{a}' \in \mathbb{R}^{n-p}\}. \tag{19}$$

Proof of Proposition 4.1 for $1 \leq p < q = n$.   An element $v_{(p,n,1)}(\boldsymbol{x}')$ of $D_{(p,n,1)}$ with $\boldsymbol{x}' = \boldsymbol{x}'_R + \sqrt{-1}\boldsymbol{x}'_I$ ($\boldsymbol{x}'_R, \boldsymbol{x}'_I \in \mathbb{R}^{n-p}$) is of the form

$$v_{(p,n,1)}(\boldsymbol{x}') = g((\mathbf{0}_p, \boldsymbol{x}'_R), \mathbf{0}_n, 0) \cdot v_{(p,n,1)}(\sqrt{-1}\boldsymbol{x}'_I) \in Q_{(p,n,1)} \cdot S_{(p,n,1)}.$$

Hence, we have verified $D_{(p,n,1)} = Q_{(p,n,1)} \cdot S_{(p,n,1)}$. ∎

Hence, we conclude:

Theorem 4.2. *For $1 \leq p \leq q \leq n$, $(p,q) \neq (n,n)$, the $Q_{(p,q,1)}$-action on $D_{(p,q,1)} = G^{\mathbb{C}}/H^{\mathbb{C}}_{(p,q,1)}$ is strongly visible with $(2n-p-q)$-dimensional slice $M^1_{(p,q,1)} \simeq \exp \sqrt{-1}\mathfrak{q}_{(p,q,1)}$.*

5. Visible action on $G^{\mathbb{C}}/H^{\mathbb{C}}_{(m,0,1)}$

This section considers the Heisenberg homogeneous space $D_{(m,0,1)} = G^{\mathbb{C}}/H^{\mathbb{C}}_{(m,0,1)}$ with $1 \leq m \leq n$.

5.1. $Q_{(m,0,1)}$-action on $D_{(m,0,1)}$

Now, we let m be $1 \leq m < n$. The complementary subspace $\mathfrak{q}_{(m,0,1)}$ of $\mathfrak{h}_{(m,0,1)}$ is

$$\mathfrak{q}_{(m,0,1)} = \mathrm{span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_n, Y_1, \ldots, Y_n\}.$$

Then, $D_{(m,0,1)}$ is of the form

$$D_{(m,0,1)} = \{g((\mathbf{0}_m, \boldsymbol{x}'), \boldsymbol{y}, 0)H^{\mathbb{C}}_{(m,0,1)} : \boldsymbol{x}' \in \mathbb{C}^{n-m}, \boldsymbol{y} \in \mathbb{C}^n\}$$

where $v_{(m,0,1)}(\boldsymbol{x}', \boldsymbol{y}) := g((\mathbf{0}_m, \boldsymbol{x}'), \boldsymbol{y}, 0)H^{\mathbb{C}}_{(m,0,1)}$. Here, $\mathfrak{q}_{(m,0,1)}$ is not a subalgebra. It follows from Lemma 2.6 that the subgroup $Q_{(m,0,1)} = \langle \exp \mathfrak{q}_{(m,0,1)} \rangle$ is given by

$$Q_{(m,0,1)} = \{g((\mathbf{0}_m, \boldsymbol{s}'), \boldsymbol{t}, u) : \boldsymbol{s}' \in \mathbb{R}^{n-m}, \boldsymbol{t} \in \mathbb{R}^n, u \in \mathbb{R}\}.$$

Thus, $Q_{(m,0,1)}$ is isomorphic to $\mathbb{R}^m \times \mathbf{H}_{n-m}$ and acts on $D_{(m,0,1)}$ by

$$g((\mathbf{0}_m, \boldsymbol{s}'), \boldsymbol{t}, u) \cdot v_{(m,0,1)}(\boldsymbol{x}', \boldsymbol{y})$$
$$= g((\mathbf{0}_m, \boldsymbol{s}' + \boldsymbol{x}'), \boldsymbol{t} + \boldsymbol{y}, u - (\boldsymbol{t}|(\mathbf{0}_m, \boldsymbol{x}')))H^{\mathbb{C}}_{(m,0,1)}$$
$$= g((\mathbf{0}_m, \boldsymbol{s}' + \boldsymbol{x}'), \boldsymbol{t} + \boldsymbol{y})H^{\mathbb{C}}_{(m,0,1)} = v_{(m,0,1)}(\boldsymbol{s}' + \boldsymbol{x}', \boldsymbol{t} + \boldsymbol{y}).$$

For $m = n$, we have $\mathfrak{q}_{(n,0,1)} = \mathrm{span}_{\mathbb{R}}\{Y_1, \ldots, Y_n\}$. Then, we obtain

$$Q_{(n,0,1)} = \{g(\mathbf{0}_n, \boldsymbol{t}, 0) : \boldsymbol{t} \in \mathbb{R}^n\} \simeq \mathbb{R}^n$$

and $D_{(n,0,1)} = \{v_{(n,0,1)}(\boldsymbol{y}) := g(\mathbf{0}_n, \boldsymbol{y}, 0)H^{\mathbb{C}}_{(n,0,1)} : \boldsymbol{y} \in \mathbb{C}^n\}$. Further, $Q_{(n,0,1)}$ acts on $D_{(n,0,1)}$ by

$$g(\mathbf{0}_n, \boldsymbol{t}, 0) \cdot v_{(n,0,1)}(\boldsymbol{y}) = v_{(n,0,1)}(\boldsymbol{t} + \boldsymbol{y})$$

for $g(\mathbf{0}_n, \boldsymbol{t}, 0) \in Q_{(n,0,1)}$ and $v_{(n,0,1)}(\boldsymbol{y}) \in D_{(n,0,1)}$.

5.2. Verification of (V.0) for $D_{(m,0,1)}$

First, we consider the case $m < n$. Let us take an anti-holomorphic diffeomorphism on $D_{(m,0,1)}$ as σ_1 (see (13)). We recall that the restriction of the anti-holomorphic involution $\widetilde{\sigma}_1$ on $G^{\mathbb{C}}$ (see (11)) to $\exp \mathfrak{q}^{\mathbb{C}}_{(m,0,1)}$ is given by

$$\widetilde{\sigma}_1(g((\mathbf{0}_m, \boldsymbol{x}'), \boldsymbol{y}, 0)) = g((\mathbf{0}_m, -\overline{\boldsymbol{x}'}), -\overline{\boldsymbol{y}}, 0).$$

By (14), the real form $M^1_{(m,0,1)} = D^{\sigma_1}_{(m,0,1)}$ is given as follows:

$$M^1_{(m,0,1)} = \{v_{(m,0,1)}(\sqrt{-1}\boldsymbol{a}', \sqrt{-1}\boldsymbol{b}) : \boldsymbol{a}' \in \mathbb{R}^{n-m}, \boldsymbol{b} \in \mathbb{R}^n\}. \tag{20}$$

Proposition 5.1. *We take a connected closed subgroup $L_{(m,0,1)}$ as $Q_{(m,0,1)}$ and a submanifold $S_{(m,0,1)}$ as $M^1_{(m,0,1)}$. Then, $L_{(m,0,1)} \cdot S_{(m,0,1)}$ coincides with $D_{(m,0,1)}$.*

Proof. Clearly, $Q_{(m,0,1)} \cdot S_{(m,0,1)}$ is contained in $D_{(m,0,1)}$. Then, we see the opposite inclusion. Let us take an element $v_{(m,0,1)}(\boldsymbol{x}', \boldsymbol{y})$ in $D_{(m,0,1)}$ and write $\boldsymbol{x}' = \boldsymbol{x}'_R + \sqrt{-1}\boldsymbol{x}'_I$ for $\boldsymbol{x}'_R, \boldsymbol{x}'_I \in \mathbb{R}^{n-m}$ and $\boldsymbol{y} = \boldsymbol{y}_R + \sqrt{-1}\boldsymbol{y}_I$ for $\boldsymbol{y}_R, \boldsymbol{y}_I \in \mathbb{R}^n$.

Then, we have

$$v_{(m,0,1)}(\boldsymbol{x}', \boldsymbol{y}) = g((\mathbf{0}_m, \boldsymbol{x}'_R), \boldsymbol{y}_R, 0) \cdot v_{(m,0,1)}(\sqrt{-1}\boldsymbol{x}'_I, \sqrt{-1}\boldsymbol{y}_I).$$

This implies $v_{(m,0,1)}(\boldsymbol{x}', \boldsymbol{y}) \in L_{(m,0,1)} \cdot S_{(m,0,1)}$. Hence, we have verified the inclusion $D_{(m,0,1)} \subset Q_{(m,0,1)} \cdot S_{(m,0,1)}$. Since it is obvious that $Q_{(m,0,1)} \cdot S_{(m,0,1)} \subset D_{(m,0,1)}$, we have shown $Q_{(m,0,1)} \cdot S_{(m,0,1)} = D_{(m,0,1)}$. $\blacksquare$

For the special case $m = n$, we can show the following proposition by the same way as Proposition 5.1.

Proposition 5.2. *We take $L_{(p,n,1)}$ as $Q_{(p,n,1)}$ and $S_{(n,0,1)}$ as*
$$M^1_{(n,0,1)} = \{v_{(n,0,1)}(\sqrt{-1}\boldsymbol{b}) : \boldsymbol{b} \in \mathbb{R}^n\}.$$
Then, the set $L_{(n,0,1)} \cdot S_{(n,0,1)}$ coincides with $D_{(n,0,1)}$.

Therefore, we conclude:

Theorem 5.3. *The $Q_{(m,0,1)}$-action on $D_{(m,0,1)} = G^{\mathbb{C}}/H^{\mathbb{C}}_{(m,0,1)}$ is strongly visible with $(2n-m)$-dimensional slice $M^1_{(m,0,1)} \simeq \exp\sqrt{-1}\mathfrak{q}_{(m,0,1)}$.*

6. Visible action on $G^{\mathbb{C}}/H^{\mathbb{C}}_{(m,0,0)}$

This section deals with the case where a complex Heisenberg homogeneous space is $D_{(m,0,0)} := G^{\mathbb{C}}/H^{\mathbb{C}}_{(m,0,0)}$ for $1 \leq m \leq n$. According to Lemma 2.2 for $\mathfrak{q}_{(m,0,0)}$, we divide into two cases: $1 \leq m < n$ and $m = n$.

Let us consider a general case $1 \leq m < n$. The complementary subspace $\mathfrak{q}_{(m,0,0)}$ is given by
$$\mathfrak{q}_{(m,0,0)} = \mathrm{span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_n, Y_1, \ldots, Y_n, Z\},$$
and then $Q_{(m,0,0)} = \{g((\mathbf{0}_m, \boldsymbol{s}'), \boldsymbol{t}, u) : \boldsymbol{s}' \in \mathbb{R}^{n-m}, \boldsymbol{t} \in \mathbb{R}^n, u \in \mathbb{R}\}$.

Thus, $Q_{(m,0,0)}$ is isomorphic to $\mathbb{R}^m \times \mathbf{H}_{n-m}$ which coincides with $Q_{(m,0,1)}$ (see Section 5.1). It is noteworthy that a closed subgroup of G which we will consider is not $Q_{(m,0,0)}$ but $Q_{(m-1,0,0)}$ which contains $Q_{(m,0,0)}$. On the other hand, the $Q_{(m,0,0)}$-action on $D_{(m,0,0)}$ will be discussed in Theorem 8.13, separated from this section. If $m > 1$, then $Q_{(m-1,0,0)}$ forms

$$Q_{(m-1,0,0)} = \{g((\mathbf{0}_{m-1}, s_m, \boldsymbol{s}'), \boldsymbol{t}, u) : \boldsymbol{s}' \in \mathbb{R}^{n-m}, \boldsymbol{t} \in \mathbb{R}^n, s_m, u \in \mathbb{R}\},$$

whereas, if $m = 1$ then $Q_{(0,0,0)}$ coincides with G. Hence, the main object of this section is the $Q_{(m-1,0,0)}$-action on $D_{(m,0,0)}$.

For a special case $m = n$, the Lie algebra $\mathfrak{q}_{(n,0,0)}$ is an abelian subalgebra of maximal dimension. Then, we will consider the $Q_{(n,0,0)}$-action on $D_{(n,0,0)}$. Here, $Q_{(n,0,0)} = \{g(\mathbf{0}, \boldsymbol{t}, u) : \boldsymbol{t} \in \mathbb{R}^n, u \in \mathbb{R}\} \simeq \mathbb{R}^{n+1}$.

6.1. $Q_{(m-1,0,0)}$-action on $D_{(m,0,0)}$ for $1 \leq m < n$

First, we consider the case $1 \leq m < n$ and the $Q_{(m-1,0,0)}$-action on $D_{(m,0,0)}$. Now, we write $v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z) := g((\mathbf{0}_m, \boldsymbol{x}'), \boldsymbol{y}, z)H^{\mathbb{C}}_{(m,0,0)}$ for $\boldsymbol{x}' \in \mathbb{C}^{n-m}, \boldsymbol{y} \in \mathbb{C}^n, z \in \mathbb{C}$. Then, the complex Heisenberg homogeneous space $D_{(m,0,0)}$ is

$$D_{(m,0,0)} = \{v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z) : \boldsymbol{x}' \in \mathbb{C}^{n-m}, \boldsymbol{y} \in \mathbb{C}^n, z \in \mathbb{C}\}$$

Lemma 6.1. *The $Q_{(m-1,0,0)}$-action on $D_{(m,0,0)}$ is written by*

$$g((\mathbf{0}_{m-1}, s_m, \boldsymbol{s}'), \boldsymbol{t}, u) \cdot v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z)$$
$$= v_{(m,0,0)}(\boldsymbol{s}' + \boldsymbol{x}', \boldsymbol{t} + \boldsymbol{y}, u + z + s_m(t_m + y_m) - (\boldsymbol{t}'|\boldsymbol{x}'))$$

for $g((\mathbf{0}_{m-1}, s_m, \boldsymbol{s}'), \boldsymbol{t}, u) \in Q_{(m-1,0,0)}$ and $v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z) \in D_{(m,0,0)}$. Here, for $\boldsymbol{t} = (t_1, \ldots, t_n) \in \mathbb{C}^n$ we write $\boldsymbol{t}' = (t_{m+1}, \ldots, t_m)$.

Proof. It follows from Lemma 2.4 that

$$g((\mathbf{0}_{m-1}, s_m, \boldsymbol{s}'), \boldsymbol{t}, u) \cdot v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z)$$
$$= g((\mathbf{0}_{m-1}, s_m, \boldsymbol{s}' + \boldsymbol{x}'), \boldsymbol{t} + \boldsymbol{y}, u + z - (\boldsymbol{t}'|\boldsymbol{x}')) H_{(m,0,0)}^{\mathbb{C}}. \qquad (21)$$

The element $e^{s_m X_m}$ commutes with $e^{(s_i + x_i) X_i}$ for all i, with $e^{(t_j + y_j) Y_j}$ for all $j \neq m$ and with $e^{(u+z-(\boldsymbol{t}'|\boldsymbol{x}'))Z}$, whereas it does not coincide with $e^{(t_m + y_m) Y_m}$ because Lemma 2.3 implies

$$e^{s_m X_m} e^{(t_m + y_m) Y_m} = (e^{s_m X_m} e^{(t_m + y_m) Y_m} e^{-s_m X_m}) e^{s_m X_m}$$
$$= e^{(t_m + y_m) Y_m} e^{s_m (t_m + y_m) Z} e^{s_m X_m}.$$

Then, we obtain

$$g((\mathbf{0}_{m-1}, s_m, \boldsymbol{s}' + \boldsymbol{x}'), \boldsymbol{t} + \boldsymbol{y}, u + z - (\boldsymbol{t}'|\boldsymbol{x}'))$$
$$= g((\mathbf{0}_{m-1}, 0, \boldsymbol{s}' + \boldsymbol{x}'), \boldsymbol{t} + \boldsymbol{y}, u + z + s_m(t_m + y_m) - (\boldsymbol{t}'|\boldsymbol{x}')) \cdot e^{s_m X_m}.$$

As $e^{s_m X_m} \in H_{(m,0,0)}^{\mathbb{C}}$, we obtain

$$(21) = v_{(m,0,0)}(\boldsymbol{s}' + \boldsymbol{x}', \boldsymbol{t} + \boldsymbol{y}, u + z + s_m(t_m + y_m) - (\boldsymbol{t}'|\boldsymbol{x}')).$$

Hence, Lemma 6.1 has been proved. ∎

6.2. Verification of (V.0) for $Q_{(m-1,0,0)}$-action on $D_{(m,0,0)}$

In this subsection, we will find a submanifold $S_{(m,0,0)}$ of the real form $M_{(m,0,0)}^1 = D_{(m,0,0)}^{\sigma_1}$ explicitly such that $L_{(m,0,0)} \cdot S_{(m,0,0)}$ is open in $D_{(m,0,0)}$. Here, $M_{(m,0,0)}^1$ is of the form

$$M_{(m,0,0)}^1 = \{v_{(m,0,0)}(\sqrt{-1}\boldsymbol{a}', \sqrt{-1}\boldsymbol{b}, c) : \boldsymbol{a}' \in \mathbb{R}^{n-m}, \boldsymbol{b} \in \mathbb{R}^n, c \in \mathbb{R}\}.$$

Proposition 6.2. *We take a closed subgroup $L_{(m,0,0)}$ as $Q_{(m-1,0,0)}$ and a submanifold $S_{(m,0,0)}$ in $M_{(m,0,0)}^1$ as*

$$S_{(m,0,0)} = \{v_{(m,0,0)}(\sqrt{-1}\boldsymbol{a}', \sqrt{-1}\boldsymbol{b}, 0) \in M_{(m,0,0)}^1 : \boldsymbol{b} \in (\mathbb{R}^{\times})^n\}. \qquad (22)$$

Then, the set $L_{(m-1,0,0)} \cdot S_{(m,0,0)}$ equals

$$D'_{(m,0,0)} = \{v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z) \in D_{(m,0,0)} : \boldsymbol{y}_I \in (\mathbb{R}^{\times})^n\} \qquad (23)$$

and then is an open set in $D_{(m,0,0)}$. Here, an element $\boldsymbol{y} \in \mathbb{C}^n$ is written as $\boldsymbol{y} = \boldsymbol{y}_R + \sqrt{-1}\boldsymbol{y}_I$ for some $\boldsymbol{y}_R, \boldsymbol{y}_I \in \mathbb{R}^n$.

Proof. We take an element $v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z) \in D'_{(m,0,0)}$ and write $\boldsymbol{x}' = \boldsymbol{x}'_R + \sqrt{-1}\boldsymbol{x}'_I$ with $\boldsymbol{x}'_R, \boldsymbol{x}'_I \in \mathbb{R}^{n-m}$ and $z = z_R + \sqrt{-1}z_I$ with $z_R, z_I \in \mathbb{R}$. Further, we denote by $(y_m)_R$ the m-th component of $\boldsymbol{y}_R$, equivalently, the real part of y_m.

We notice that $(y_m)_I \neq 0$ since $\boldsymbol{y}_I \in (\mathbb{R}^\times)^n$. Here, we set

$$\boldsymbol{s}' := \boldsymbol{x}'_R, \ \boldsymbol{t} := \boldsymbol{y}_R, \ \boldsymbol{a}' := \boldsymbol{x}'_I, \ \boldsymbol{b} := \boldsymbol{y}_I,$$

$$s_m := (z_I + (\boldsymbol{y}'_R | \boldsymbol{x}'_I))(y_m)_I^{-1}, \ u := z_R - s_m(y_m)_R.$$

Then, the following equality is fulfilled:

$$v_{(m,0,0)}(\boldsymbol{x}', \boldsymbol{y}, z) = g((\boldsymbol{0}_{m-1}, s_m, \boldsymbol{s}'), \boldsymbol{t}, u) \cdot v_{(m,0,0)}(\sqrt{-1}\boldsymbol{a}', \sqrt{-1}\boldsymbol{b}, 0). \qquad (24)$$

Hence, this implies $D'_{(m,0,0)} \subset Q_{(m-1,0,0)} \cdot S_{(m,0,0)}$.

Equality (24) also shows that $Q_{(m-1,0,0)} \cdot S_{(m,0,0)} \subset D'_{(m,0,0)}$ because $\boldsymbol{y}_I = \boldsymbol{b} \in (\mathbb{R}^\times)^n$. Therefore, Proposition 6.2 has been proved. ∎

We take a submanifold $\mathfrak{q}'_{(m,0,0)}$ in $\mathfrak{q}_{(m,0,0)}$ as

$$\mathfrak{q}'_{(m,0,0)} = \sum_{i=m+1}^{n} \mathbb{R}X_i + \sum_{j=1}^{n} (\mathbb{R}^\times)Y_j, \qquad (25)$$

Then, $S_{(m,0,0)}$ is diffeomorphic to $\exp \sqrt{-1}\mathfrak{q}'_{(m,0,0)}$. As a consequence, we get:

Theorem 6.3. *For $1 \leq m \leq n$, the $Q_{(m-1,0,0)}$-action on $D_{(m,0,0)}$ is strongly visible with $(2n-m)$-dimensional slice $S_{(m,0,0)} \simeq \exp \sqrt{-1}\mathfrak{q}'_{(m,0,0)}$.*

6.3. $Q_{(n,0,0)}$-action on $D_{(n,0,0)}$

In this subsection, we consider the $Q_{(n,0,0)}$-action on $D_{(n,0,0)}$. We recall that $Q_{(n,0,0)} = \{g(\boldsymbol{0}, \boldsymbol{t}, u) : \boldsymbol{t} \in \mathbb{R}^n, u \in \mathbb{R}\}$ and

$$D_{(n,0,0)} = \{v_{(n,0,0)}(\boldsymbol{y}, z) := g(\boldsymbol{0}, \boldsymbol{y}, z)H^\mathbb{C}_{(n,0,0)} : \boldsymbol{y} \in \mathbb{C}^n, \ z \in \mathbb{C}\}.$$

Then, this action is given by

$$g(\boldsymbol{0}, \boldsymbol{t}, u) \cdot v_{(n,0,0)}(\boldsymbol{y}, z) = v_{(n,0,0)}(\boldsymbol{t} + \boldsymbol{y}, u + z)$$

for $g(\boldsymbol{0}, \boldsymbol{t}, u) \in Q_{(n,0,0)}$ and $v_{(n,0,0)}(\boldsymbol{y}, z) \in D_{(n,0,0)}$.

In this setting, we take an anti-holomorphic diffeomorphism on $D_{(n,0,0)}$ as σ_2 (see (13)). Then, the real form $M^2_{(n,0,0)} = D^{\sigma_2}_{(n,0,0)}$ is given by

$$M^2_{(n,0,0)} = \{v_{(n,0,0)}(\sqrt{-1}\boldsymbol{b}, \sqrt{-1}c) : \boldsymbol{b} \in \mathbb{R}^n, c \in \mathbb{R}\}. \qquad (26)$$

Then, we have:

Proposition 6.4. *We take $L_{(n,0,0)}$ as $Q_{(n,0,0)}$ and $S_{(n,0,0)}$ as $M^2_{(n,0,0)}$. Then, $L_{(n,0,0)} \cdot S_{(n,0,0)}$ coincides with $D_{(n,0,0)}$.*

Hence, we obtain:

Theorem 6.5. *The $Q_{(n,0,0)}$-action on $D_{(n,0,0)}$ is strongly visible with $(n+1)$-dimensional slice $M^2_{(n,0,0)} \simeq \exp \sqrt{-1}\mathfrak{q}_{(n,0,0)}$.*

7. Proof of Theorem 1.3 and proof of Corollary 1.4

In this section, we will completely give a proof of Theorem 1.3 and a proof of Corollary 1.4.

Proof of Theorem 1.3.   Two equivalent holomorphic actions keep the strong visibility (see Proposition 2.12). Then, it suffices to show Theorem 1.3 for $D_{(0,0,1)}$, $D_{(p,q,1)}$, $D_{(m,0,1)}$ and $D_{(m,0,0)}$. We have already proved the strong visibility for the $Q_{(0,0,1)}$-action on $D_{(0,0,1)}$ in Theorem 3.2; the $Q_{(p,q,1)}$-action on $D_{(p,q,1)}$ in Theorem 4.2; the $Q_{(m,0,1)}$-action on $D_{(m,0,1)}$ in Theorem 5.3; the $Q_{(m'-1,0,0)}$-action on $D_{(m',0,0)}$ in Theorem 6.3 if $1 \leq m' < n$; and the $Q_{(n,0,0)}$-action on $D_{(n,0,0)}$ in Theorem 6.5. Therefore, the proof of Theorem 1.3 has been accomplished.   ∎

Table 1 indicates our choice of a connected closed subgroup $L_{(k,\ell,\varepsilon)}$ and a slice $S_{(k,\ell,\varepsilon)}$ for the strongly visible action on $D_{(k,\ell,\varepsilon)}$ for each non-trivial connected closed subgroup $H_{(k,\ell,\varepsilon)} \subset G$. Here, the positive integer m' in the fifth line is restricted to $1 \leq m' < n$.

$H_{(k,\ell,\varepsilon)}$	$L_{(k,\ell,\varepsilon)}$	σ	$\mathfrak{s}_{(k,\ell,\varepsilon)}$	$\dim S_{(k,\ell,\varepsilon)}$	Theorem
$H_{(0,0,1)}$	$Q_{(0,0,1)}$	σ_1	$\sqrt{-1}\mathfrak{q}_{(0,0,1)}$	$2n$	3.2
$H_{(p,q,1)}$	$Q_{(p,q,1)}$	σ_1	$\sqrt{-1}\mathfrak{q}_{(p,q,1)}$	$2n-p-q$	4.2
$H_{(m,0,1)}$	$Q_{(m,0,1)}$	σ_1	$\sqrt{-1}\mathfrak{q}_{(m,0,1)}$	$2n-m$	5.3
$H_{(m',0,0)}$	$Q_{(m'-1,0,0)}$	σ_1	$\sqrt{-1}\mathfrak{q}'_{(m',0,0)}$	$2n-m'$	6.3
$H_{(n,0,0)}$	$Q_{(n,0,0)}$	σ_2	$\sqrt{-1}\mathfrak{q}_{(n,0,0)}$	$n+1$	6.5

Table 1: $L_{(k,\ell,\varepsilon)}$, σ and $S_{(k,\ell,\varepsilon)} \simeq \exp \mathfrak{s}_{(k,\ell,\varepsilon)}$

Next, we will give a proof of Corollary 1.4. More precisely, we will show that our choice of S and anti-holomorphic diffeomorphism σ for the strongly visible L-action on D satisfies three conditions (V.0)–(S.2) for the G-action on D.

Proof of Corollary 1.4.   By our proof of Theorem 1.3, we take a slice S, an anti-holomorphic involution $\widetilde{\sigma}$ and the induced anti-holomorphic diffeomorphism σ satisfying (V.0)–(S.2) for the strongly visible L-action on D.
First, we consider the subset $D' := G \cdot S$ in D. This is of the form

$$D' = G \cdot (L \cdot S) = \bigcup_{g \in G} g \cdot (L \cdot S). \tag{27}$$

Since $L \cdot S$ is open in D, so is $g \cdot (L \cdot S)$ for any $g \in G$. Hence, D' is open in D. Next, it is clear that $\sigma|_S = \mathrm{id}_S$. Further, σ preserves each G-orbit in $D' := G \cdot S$. Indeed, we take an element $v \in D'$ and write $v = g \cdot s$ for some $g \in G$ and $s \in S$ according to (27). By the same argument of (15), we have $\sigma(v) = \widetilde{\sigma}(g)g^{-1} \cdot v \in G \cdot v$. Hence, we have shown Corollary 1.4.   ∎

8.   Application to representation theory

This section presents new multiplicity-free theorems for unitary and irreducible representations of the Heisenberg group of infinite dimension (hence not characters). These theorems are built on the investigation of strongly visible actions on complex Heisenberg homogeneous spaces.

8.1. Propagation theory of multiplicity-freeness property

Originally, the notion of strongly visible actions has been introduced as a geometric point of view of the propagation theory of multiplicity-freeness property established

by T. Kobayashi (see the original papers [4, 9]). In this subsection, we explain a multiplicity-free theorem of the Heisenberg group which is gained as an application of our results on visible actions on complex Heisenberg homogeneous spaces.

Before that, let us give a quick review on the propagation theory of multiplicity-freeness property, based on [9]. Let L be a Lie group and $\mathcal{V} \to D$ a L-equivariant Hermitian holomorphic vector bundle over a connected complex manifold D. We denote by $\mathcal{O}(D, \mathcal{V})$ the space of holomorphic sections of $\mathcal{V} \to D$. This carries a Fréchet topology by the uniform convergence on compact sets. Then, we naturally define a continuous representation ϖ of L on the space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections by

$$[\varpi(g)s](x) := g \cdot s(g^{-1} \cdot x)$$

for $g \in L$, $s \in \mathcal{O}(D, \mathcal{V})$ and $x \in D$. Under the setting, the propagation theory of multiplicity-freeness property is explained as follows. Here, we say that an anti-holomorphic diffeomorphism σ on D lifts to an anti-holomorphic bundle endomorphism on $\mathcal{V}$ if σ gives rise to an anti-holomorphic diffeomorphism on $\mathcal{V}$, denoted by the same letter σ, such that the restriction of σ to the fiber $\mathcal{V}_x$ at each $x \in D$ is an isometric and anti-linear map to the fiber $\mathcal{V}_{\sigma(x)}$ at $\sigma(x)$.

Fact 8.1 ([9, Theorem 4.3]). *Let $\mathcal{V} \to D$ be an L-equivariant Hermitian holomorphic vector bundle. Suppose that the following conditions are satisfied:*

(a) *The L-action on D is strongly visible. In particular, there exist a real submanifold S in D and an anti-holomorphic diffeomorphism σ on $D' := L \cdot S$ satisfying the conditions* (V.0)–(S.2) *equipped with an anti-holomorphic automorphism $\widetilde{\sigma}$ of L satisfying $\sigma(g \cdot x) = \widetilde{\sigma}(g) \cdot \sigma(x)$ $(g \in L, x \in D)$.*

(b) *For each $x \in S$, the unitary representation of the isotropy subgroup L_x on the fiber $\mathcal{V}_x$ is multiplicity-free.*

(c) *Under the condition* (b), *we write $\mathcal{V}_x = \oplus_{i=1}^{m(x)} \mathcal{V}_x^{(i)}$ for the irreducible decomposition as a representation of L_x. Then, σ lifts to an anti-holomorphic bundle endomorphism on $\mathcal{V}$ such that $\sigma(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ for each $i = 1, 2, \ldots, n(x)$ and each $x \in S$.*

Then, any unitary representation which is realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free. In particular, the continuous representation ϖ of L on $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.

This theory also plays an important role that one can give an unified explanation of the multiplicity-freeness for various kinds of representations. On the other hand, once we give an example of strongly visible actions, we expect that one can find several multiplicity-free representations.

So far, we have found the strongly visible actions on the complex Heisenberg homogeneous spaces (see Theorem 1.3). Applying Fact 8.1 to our setting, we can explain the multiplicity-freeness of a kind of representations of the Heisenberg group as follows.

Let G be the Heisenberg group and H a non-trivial connected closed subgroup of G. Due to Theorem 1.3, one can find a connected closed subgroup L of G such that the L-action on the complex Heisenberg homogeneous space $D = G^{\mathbb{C}}/H^{\mathbb{C}}$ is strongly visible. Further, it follows from Corollary 1.4 that the G-action on D is strongly visible.

Let $\mathcal{V}$ be a trivial line bundle $D \times \mathbb{C}$ over D. Then, $\mathcal{O}(D, D \times \mathbb{C})$ is naturally identified with the space $\mathcal{O}(D)$ of holomorphic functions on D via the map $\mathcal{O}(D, D \times \mathbb{C}) \to \mathcal{O}(D)$, $s \mapsto f$ which is characterized by $s(x) = (x, f(x))$ $(x \in D)$. Now, we let G act holomorphically on $\mathcal{V} = D \times \mathbb{C}$ by $g \cdot (x, z) := (g \cdot x, z)$, where $g \in G$, $x \in D$ and $z \in \mathbb{C}$. Then, $\mathcal{V} \to D$ is G-equivariant and

$$[\varpi(g)s](x) = (x, f(g^{-1} \cdot x))$$

for $s \in \mathcal{O}(\mathcal{V}, D)$, $g \in G$ and $x \in D$. Hence, we define a representation π of G on $\mathcal{O}(D)$ by

$$[\pi(g)f](x) := f(g^{-1} \cdot x) \tag{28}$$

for $g \in G$, $f \in \mathcal{O}(D)$ and $x \in D$, and then ϖ is equivalent to π.

Theorem 8.2. *Let G be the Heisenberg group and H a non-trivial connected closed subgroup of G. Then, the continuous representation $(\pi, \mathcal{O}(D))$ of G is multiplicity-free. Moreover, let L be a connected closed subgroup of G such that the L-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ is strongly visible. Then, the restriction $\pi|_L$ of π to L is still multiplicity-free.*

Proof. By the proof of Theorem 1.3, one can find a real submanifold S, an anti-holomorphic involution $\widetilde{\sigma}$ on $G^{\mathbb{C}}$ and an anti-holomorphic diffeomorphism σ on $G^{\mathbb{C}}/H^{\mathbb{C}}$ such that (V.0)–(S.2) are satisfied, $\widetilde{\sigma}$ stabilizes L and $\sigma(g \cdot x) = \widetilde{\sigma}(g) \cdot \sigma(x)$ for $g \in L$, $x \in G^{\mathbb{C}}/H^{\mathbb{C}}$ (see Remark 2.15). Thus, condition (a) of Fact 8.1 has been verified.

Let $\mathcal{V}$ be a trivial bundle $D \times \mathbb{C}$ over D. Since each fiber $\mathcal{V}_x \simeq \mathbb{C}$ $(x \in D)$ is one-dimensional, the unitary representation of L_x on $\mathcal{V}_x$ is irreducible for any $x \in S$, in particular, this is multiplicity-free. Then, condition (b) is satisfied.

The anti-holomorphic diffeomorphism σ on D lifts to an anti-holomorphic bundle endomorphism on $\mathcal{V}$ as follows:

$$\sigma(x, z) = (\sigma(x), \overline{z}) \quad (x \in D, z \in \mathbb{C}).$$

Clearly, $\sigma(\mathcal{V}_x) = \mathcal{V}_x$ for all $x \in S$, from which condition (c) holds.

Hence, it follows from Fact 8.1 that the continuous representation $(\varpi, \mathcal{O}(D, \mathcal{V}))$ of L is multiplicity-free. Therefore, π is multiplicity-free as a representation of L since $\varpi \simeq \pi$. ∎

8.2. Geometric criterion of multiplicity-freeness for associated quasi-regular representation

This section investigates the relationship between strongly visible actions on complex Heisenberg homogeneous spaces and the multiplicity-free quasi-regular representations of the Heisenberg group.

First of all, we will mention a geometric criterion of multiplicity-freeness of the quasi-regular representation π_H in Theorem 8.3, and its proof will be given in Sections 8.3 and 8.5.

Let $G = \exp \mathfrak{g}$ be the Heisenberg group and $H = \exp \mathfrak{h}$ a non-trivial connected closed subgroup of G. We set $d := \dim G - \dim H$. Then, there exists a co-exponential basis $\{W_1, \ldots, W_d\}$ to $\mathfrak{h}$ in $\mathfrak{g}$, which means that the map $H \times \mathbb{R}^d \to G$, $(h, w_1, \ldots, w_d) \mapsto h e^{w_1 W_1} \cdots e^{w_d W_d}$ is a diffeomorphism.

We set $\mathfrak{q} := \operatorname{span}_{\mathbb{R}}\{W_1, \ldots, W_d\}$ and $Q := \langle \exp \mathfrak{q} \rangle$.

Let $d\mu$ be a G-invariant measure on the homogeneous space G/H which is induced from the Lebesgue measure on $\mathfrak{q}$ via the diffeomorphism $\mathfrak{q} \simeq \exp \mathfrak{q} \simeq G/H$. We note that G-invariant measures on G/H are unique up to scalars. We denote by $L^2(G/H)$ the space of square integrable functions on G/H with respect to $d\mu$. Then, we define the quasi-regular representation π_H of G on $L^2(G/H)$ by

$$[\pi_H(g)f](xH) = f(g^{-1}xH) \quad (x, g \in G, f \in L^2(G/H)) \tag{29}$$

The next theorem is a new characterization for the unitary representation π_H of G to be multiplicity-free by the strongly visible action on the complex Heisenberg homogeneous space.

Theorem 8.3. *Let G be the Heisenberg group. For a non-trivial connected closed subgroup H of G, the following conditions are equivalent:*

(i) *The quasi-regular representation π_H of G is multiplicity-free.*

(ii) *The Q-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$ is strongly visible.*

(iii) *H is not isomorphic to $H_{(m,0,0)}$ for any $m = 1, 2, \ldots, n-1$.*

Our proof of Theorem 8.3 consists of two parts. The first part is to show (i) $\Leftrightarrow$ (iii) (see Section 8.3). For this, we describe the irreducible decomposition of the quasi-regular representation of G in terms of the coadjoint orbits and calculate the multiplicities by applying the Corwin-Greenleaf formula [2]. The second one is to show the equivalence (ii) $\Leftrightarrow$ (iii) (see Section 8.5). As we have seen in Sections 3–7, the implication (iii) $\Rightarrow$ (ii) is true. For the proof of the opposite implication, it suffices to show that the $Q_{(m,0,0)}$-action on $G^{\mathbb{C}}/H^{\mathbb{C}}_{(m,0,0)}$ is not strongly visible. To carry out, we use the propagation theory of multiplicity-freeness property.

8.3. Equivalence between (i) and (iii)

This subsection considers the irreducible decomposition of the quasi-regular representation π_H of G. It is known that π_H can be realized as the representation $\mathrm{Ind}_H^G \mathbf{1}$ induced from the trivial representation $\mathbf{1}$ of H. In general, the irreducible decomposition of $\mathrm{Ind}_H^G \chi$ induced from a unitary character χ of H can be formulated in terms of the coadjoint orbits, which is known as the Corwin–Greenleaf formula [2] (see also [12, Theorem 1.1]). Then, we shall apply the Corwin–Greenleaf formula to our setting for the proof of the equivalence (i) $\Leftrightarrow$ (iii).

First, we review the basic facts on the coadjoint orbits of the Heisenberg group G of dimension $2n + 1$. We denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Let G act on $\mathfrak{g}^*$ by the coadjoint representation, namely, $g \cdot \xi := \mathrm{Ad}^*(g)\xi$ for $g \in G$ and $\xi \in \mathfrak{g}^*$ is given by $(g \cdot \xi)(X) := \xi(\mathrm{Ad}(g^{-1})X)$ $(X \in \mathfrak{g})$. Let $\mathcal{B}$ be a basis of $\mathfrak{g}$ given by (1) and

$$\mathcal{B}^* := \{X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*, Z^*\} \tag{30}$$

the dual basis of $\mathcal{B}$. Any element $\xi \in \mathfrak{g}^*$ is written down as

$$\xi \equiv \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) = \sum_{i=1}^n \alpha_i X_i^* + \sum_{j=1}^n \beta_j Y_j^* + \gamma Z$$

for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$.

A direct computation shows that

$$g(\boldsymbol{x}, \boldsymbol{y}, z) \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) = \xi(\boldsymbol{\alpha} + \gamma\boldsymbol{y}, \boldsymbol{\beta} - \gamma\boldsymbol{x}, \gamma)$$

for $g(\boldsymbol{x}, \boldsymbol{y}, z) \in G$ and $\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) \in \mathfrak{g}^*$. Then, the coadjoint orbits of the Heisenberg group G are described explicitly as follows.

Lemma 8.4. *Let G be the Heisenberg group. For an element $\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) \in \mathfrak{g}^*$, we have:*

(1) *If $\gamma \neq 0$, then the coadjoint orbit $G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ (of dimension equals $2n$) is given by:*

$$G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) = \{\xi(\boldsymbol{x}, \boldsymbol{y}, \gamma) : \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\}.$$

(2) *The coadjoint orbit $G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0)$ equals $\{\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0)\}$.*

Hence, the subset

$$R := \{\xi(\boldsymbol{0}, \boldsymbol{0}, \gamma) : \gamma \in \mathbb{R}^\times\} \sqcup \{\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0) : \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n\} \subset \mathfrak{g}^* \qquad (31)$$

turns out to be a cross-section of the orbit space $\mathfrak{g}^*/G$.

Next, the Kirillov orbit method explains that there is a one-to-one correspondence between the unitary dual $\widehat{G}$ and the orbit space $\mathfrak{g}^*/G$. By Lemma 8.4, we write a bijection $R \to \widehat{G}$ as $\xi(\boldsymbol{0}, \boldsymbol{0}, \gamma) \mapsto \tau_{\xi(\boldsymbol{0}, \boldsymbol{0}, \gamma)} \equiv \tau_\gamma$ and $\xi(\boldsymbol{a}, \boldsymbol{b}, 0) \mapsto \tau_{\xi(\boldsymbol{a}, \boldsymbol{b}, 0)} \equiv \tau_{\boldsymbol{a}, \boldsymbol{b}}$.

We are ready to mention the Corwin–Greenleaf formula for the Heisenberg group G. The following is refereed by the statement of [12, Theorem 1.1].

Fact 8.5. *The quasi-regular representation $\pi_H = \mathrm{Ind}_H^G \mathbf{1}$ is decomposed into the direct integral of irreducible representations of G as*

$$\pi_H = \mathrm{Ind}_H^G \mathbf{1} \simeq \int_{(G \cdot \mathfrak{q}^*)/G}^{\oplus} m_{\pi_H}(\xi) \tau_\xi \, d\xi. \qquad (32)$$

Here, $d\xi$ is a G-invariant measure on $(G \cdot \mathfrak{q}^*)/G$ induced from the Lebesgue measure on $\mathfrak{q}^* \subset \mathfrak{g}^*$ via $\mathfrak{q}^* \to (G \cdot \mathfrak{q}^*)/G$, and $m_{\pi_H} : (G \cdot \mathfrak{q}^*)/G \to \mathbb{N} \cup \{\infty\}$ is the multiplicity function.

The multiplicity function $m_{\pi_H}(\xi)$ is either finite and bounded or infinite, according to the fact that either $2 \dim(H \cdot \xi) = \dim(G \cdot \xi)$ or $2 \dim(H \cdot \xi) < \dim(G \cdot \xi)$ hold for almost all ξ with respect to the measure $d\xi$. In the finite case, namely, $2 \dim(H \cdot \xi) = \dim(G \cdot \xi)$ for almost all ξ, $m_{\pi_H}(\xi)$ equals the number of H-orbits in $\mathfrak{q}^* \cap (G \cdot \xi)$.

Remark 8.6. According to these two cases, we say that π_H is of finite or of infinite multiplicities.

Our proof of the equivalence (i) $\Leftrightarrow$ (iii) is carried out by showing $m_{\pi_H}(\xi) = 1$ for any generic $\xi \in \mathfrak{q}^*$ (namely, the orbit $G \cdot \xi$ is of maximal dimension) in (32) for our setting according to Fact 8.5. Before that, we explain the following lemma:

Lemma 8.7.   *Let G be the Heisenberg group, H_1, H_2 connected closed subgroups of G and π_{H_1}, π_{H_2} the quasi-regular representations of G. Suppose that H_1 is isomorphic to H_2. Then, π_{H_1} is multiplicity-free if and only if π_{H_2} also is.*

Proof.   By Lemma 2.10, there exists a Lie group automorphism φ on $G = \exp \mathfrak{g}$ such that the restriction $\varphi|_{H_1}$ gives rise to the isomorphism from H_1 to H_2. Then, the induced map $\Phi : G/H_1 \to G/H_2$, $xH_1 \mapsto \varphi(x)H_2$ is a C^∞-diffeomorphism. Let $d\mu_2$ be a G-invariant measure on G/H_2 and $d\mu_1 := \Phi^*(d\mu_2)$ the pull-back of $d\mu_2$. Then, $d\mu_1$ is a G-invariant measure on G/H_1. Here, if $f \in L^2(G/H_2, d\mu_2)$ then $f \circ \Phi \in L^2(G/H_1, d\mu_1)$, which gives rise to the unitary isomorphism

$$\Phi^\vee : L^2(G/H_2, d\mu_2) \to L^2(G/H_1, d\mu_1), \ f \mapsto \Phi^\vee(f) := f \circ \Phi.$$

Then, we have $\pi_{H_1}(g) \circ \Phi^\vee = \Phi^\vee \circ \pi_{H_2}(\varphi(g)) \quad (\forall g \in G)$.

Hence, π_{H_1} is equivalent to $\pi_{H_2} \circ \varphi$, which is enough to conclude.   ■

Now, we are going to prove the equivalence (i) $\Leftrightarrow$ (iii).

Proof of (i) $\Leftrightarrow$ (iii) in Theorem 8.3.   Due to Lemma 8.7, it suffices to verify the equivalence when $H \equiv H_{(k,\ell,\varepsilon)}$ are $H_{(0,0,1)}$, $H_{(p,q,1)}$, $H_{(m,0,1)}$ and $H_{(m,0,0)}$. We divide our proof into two cases.

Case 1:   Let $H_{(k,\ell,\varepsilon)}$ be either $H_{(0,0,1)}, H_{(p,q,1)}$ or $H_{(m,0,1)}$. Then, the Lie algebra $\mathfrak{h}_{(k,\ell,\varepsilon)}$ contains the center $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}Z$, from which $\mathfrak{q}^*_{(k,\ell,\varepsilon)}$ is a subspace of $\mathrm{span}_{\mathbb{R}}\{X_i^*, Y_j^* : 1 \le i, j \le n\}$. By Lemma 8.4, we have $G \cdot \xi = \{\xi\}$ for any $\xi \in \mathfrak{q}^*_{(k,\ell,\varepsilon)}$. Thus, we obtain

$$(G \cdot \mathfrak{q}^*_{(k,\ell,\varepsilon)})/G = \mathfrak{q}^*_{(k,\ell,\varepsilon)}/G \simeq \mathfrak{q}^*_{(k,\ell,\varepsilon)}. \tag{33}$$

Clearly, $H \cdot \xi = \{\xi\}$. Hence, we have verified the equality $2\dim(H \cdot \xi) = \dim(G \cdot \xi)$. By Fact 8.5, the multiplicity $m_{\pi_H}(\xi)$ is finite for any $\xi \in \mathfrak{q}^*_{(k,\ell,\varepsilon)}$. Further, we have $\mathfrak{q}^*_{(k,\ell,\varepsilon)} \cap (G \cdot \xi) = \{\xi\} = H \cdot \xi$. Hence, it follows from Fact 8.5 that we conclude $m_{\pi_H}(\xi) = 1$ for any $\xi \in \mathfrak{q}^*_{(k,\ell,\varepsilon)}$. Therefore, $\pi_{H_{(k,\ell,\varepsilon)}}$ is multiplicity-free.

Case 2:   Let us consider $H_{(m,0,0)}$ for $1 \le m \le n$. The complementary subspace $\mathfrak{q}^*_{(m,0,0)}$ contains the center $\mathbb{R}Z^*$. Thus, $\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) \in \mathfrak{q}^*_{(m,0,0)}$ is generic if and only if $\gamma \ne 0$.

Fix now $\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) \in \mathfrak{q}^*_{(m,0,0)}$ with $\gamma \ne 0$. Then, it follows from Lemma 8.4 that $\dim(G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)) = 2n$. On the other hand, the coadjoint $H_{(m,0,0)}$-orbit through $\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ is given as follows: If $m < n$ then

$$H_{(m,0,0)} \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) = \{\xi(\boldsymbol{\alpha}, (\boldsymbol{y}', \boldsymbol{\beta}''), \gamma) : \boldsymbol{y}' \in \mathbb{R}^m\}$$

where we shall write $\boldsymbol{\beta} = (\boldsymbol{\beta}', \boldsymbol{\beta}'') \in \mathbb{R}^n$ for some $\boldsymbol{\beta}' \in \mathbb{R}^m$ and $\boldsymbol{\beta}'' \in \mathbb{R}^{n-m}$ and $(\boldsymbol{y}', \boldsymbol{\beta}'') \in \mathbb{R}^n$; and if $m = n$ then

$$H_{(n,0,0)} \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) = \{\xi(\boldsymbol{\alpha}, \boldsymbol{y}, \gamma) : \boldsymbol{y} \in \mathbb{R}^n\}. \tag{34}$$

These assert that $\dim(H_{(m,0,0)} \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)) = m \quad (1 \le m \le n)$.

Hence, the equality $2\dim(H_{(m,0,0)} \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)) = \dim(G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma))$ holds if and only if $m = n$. It follows from Fact 8.5 that the quasi-regular representation

$\pi_{(m,0,0)} \equiv \pi_{H_{(m,0,0)}}$ is not multiplicity-free if $m < n$, in particular, $m_{\pi_{(m,0,0)}}(\xi) = \infty$ for any generic element $\xi \in \mathfrak{q}^*_{(m,0,0)}$.

Finally, let us show that $\pi_{(n,0,0)}$ is multiplicity-free. We recall

$$\mathfrak{q}^*_{(n,0,0)} = \{\xi(\mathbf{0}, \boldsymbol{\beta}', \gamma') : \boldsymbol{\beta}' \in \mathbb{R}^n, \gamma' \in \mathbb{R}\}$$

and $G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) = \{\xi(\boldsymbol{x}, \boldsymbol{y}, \gamma) : \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\}$ for generic $\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) \in \mathfrak{q}^*_{(n,0,0)}$. Then,

$$\mathfrak{q}^*_{(n,0,0)} \cap (G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)) = \{\xi(\mathbf{0}, \boldsymbol{y}, \gamma) : \boldsymbol{y} \in \mathbb{R}^n\}. \tag{35}$$

In view of the description (34) of the $H_{(n,0,0)}$-orbit, the right-hand side of (35) coincides with $H_{(m,0,0)} \cdot \xi(\mathbf{0}, \mathbf{0}, \gamma)$. Thus, $\mathfrak{q}^*_{(n,0,0)} \cap (G \cdot \xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma))$ itself is a $H_{(n,0,0)}$-orbit. By Fact 8.5, $m_{\pi_{(n,0,0)}}(\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)) = 1$ for any generic element $\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) \in \mathfrak{q}^*_{(n,0,0)}$. Therefore, $\pi_{(n,0,0)}$ is multiplicity-free.

As a consequence, the equivalence (i) $\Leftrightarrow$ (iii) has been completely proved. ∎

8.4. Realization of $(\pi_{(m,0,0)}, L^2(G/H_{(m,0,0)}))$ in $(\pi, \mathcal{O}(D_{(m,0,0)}))$

In the study of the multiplicity-freeness problem of the unitary representation on the Hilbert space $L^2(X)$ of the square integrable functions on a smooth manifold X, Kobayashi first introduced in the proof of [4, Corollary 22] an idea using the heat kernel that gives a realization of $L^2(X)$ into the space $\mathcal{O}(Y)$ of holomorphic functions on a complex manifold Y where the theory of strongly visible actions can be applied.

Following this idea, we provides a realization of the quasi-regular representation $\pi_{(m,0,0)} \equiv \pi_{H_{(m,0,0)}}$ on $L^2(G/H_{(m,0,0)})$ of $Q_{(m,0,0)}$ through the continuous representation π on $\mathcal{O}(D_{(m,0,0)})$ in the setting where $1 \leq m < n$. Namely, we find a continuous and injective operator from $L^2(G/H_{(m,0,0)})$ to $\mathcal{O}(D_{(m,0,0)})$ which intertwines $\pi_{(m,0,0)}$ and π. The aim of this subsection is Theorem 8.11.

Recall from Section 6 that the connected closed subgroup $Q_{(m,0,0)} \subset G$ is given by $Q_{(m,0,0)} = \exp \mathfrak{q}_{(m,0,0)} = \{g((\mathbf{0}_m, \boldsymbol{s}'), \boldsymbol{t}, u) : \boldsymbol{s}' \in \mathbb{R}^{n-m}, \boldsymbol{t} \in \mathbb{R}^n, u \in \mathbb{R}\}$. This is isomorphic to $\mathbb{R}^m \times \mathbf{H}_{n-m}$ as a Lie group through the Lie group isomorphism

$$\mathbb{R}^m \times \mathbf{H}_{n-m} \xrightarrow{\sim} Q_{(m,0,0)}, \quad (\boldsymbol{y}'', g(\boldsymbol{x}', \boldsymbol{y}', z)) \mapsto g((\mathbf{0}_m, \boldsymbol{x}'), (\boldsymbol{y}'', \boldsymbol{y}), z).$$

Then, let $\mathbb{R}^m \times \mathbf{H}_{n-m}$ act on $D_{(m,0,0)} = G^{\mathbb{C}}/H^{\mathbb{C}}_{(m,0,0)}$ holomorphically via this isomorphism. Similarly, the complexification $Q^{\mathbb{C}}_{(m,0,0)} = \exp \mathfrak{q}^{\mathbb{C}}_{(m,0,0)}$ is isomorphic to $\mathbb{C}^m \times \mathbf{H}^{\mathbb{C}}_{n-m}$ as a complex Lie group. Combining this isomorphism with Lemma 2.7, we get a biholomorphic map

$$\Psi : \mathbb{C}^m \times \mathbf{H}^{\mathbb{C}}_{n-m} \xrightarrow{\sim} D_{(m,0,0)}, \quad (\boldsymbol{y}'', g(\boldsymbol{x}', \boldsymbol{y}', z)) \mapsto g((\mathbf{0}_m, \boldsymbol{x}'), (\boldsymbol{y}'', \boldsymbol{y}), z)H^{\mathbb{C}}_{(m,0,0)}.$$

In particular, the holomorphic $(\mathbb{R}^m \times \mathbf{H}_{n-m})$-action on $\mathbb{C}^m \times \mathbf{H}^{\mathbb{C}}_{n-m}$ is equivalent to that on $D_{(m,0,0)}$ via Ψ (see Definition 2.8).

We also define a continuous representation ρ of $\mathbb{R}^m \times \mathbf{H}_{n-m}$ on $\mathcal{O}(\mathbb{C}^m \times \mathbf{H}^{\mathbb{C}}_{n-m})$, the space of holomorphic functions on $\mathbb{C}^m \times \mathbf{H}^{\mathbb{C}}_{n-m}$, by

$$[\rho(\boldsymbol{y}, g)f](\boldsymbol{v}, x) := f(\boldsymbol{v} - \boldsymbol{y}, g^{-1}x) \tag{36}$$

for $(\boldsymbol{y}, g) \in \mathbb{R}^m \times \mathbf{H}_{n-m}$, $f \in \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})$ and $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. Then, $(\rho, \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}))$ is equivalent to $(\pi, \mathcal{O}(D_{(m,0,0)}))$ (see (28) for definition) via the $(\mathbb{R}^m \times \mathbf{H}_{n-m})$-intertwining operator

$$\Psi^{\vee} : \mathcal{O}(D_{(m,0,0)}) \xrightarrow{\sim} \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}), \quad f \mapsto \Psi^{\vee}(f) := f \circ \Psi.$$

We set $M_{(m,0,0)} := G/H_{(m,0,0)}$ and consider the restriction of the quasi-regular representation $\pi_{(m,0,0)}$ of G to $Q_{(m,0,0)} \simeq \mathbb{R}^m \times \mathbf{H}_{n-m}$. This is equivalent to the regular representation ρ of $\mathbb{R}^m \times \mathbf{H}_{n-m}$ on $L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$, defined by the same as (36), via the intertwining operator

$$\Psi_{\mathbb{R}}^{\vee} : L^2(M_{(m,0,0)}) \xrightarrow{\sim} L^2(\mathbb{R}^m \times \mathbf{H}_{n-m}), \quad f \mapsto f \circ \Psi|_{\mathbb{R}^m \times \mathbf{H}_{n-m}}.$$

It is our concern here is to construct a continuous and injective operator from $L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$ to $\mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})$. The key ingredient of our construction is to use the heat kernel on $\mathbb{R}^m$ and that on $\mathbf{H}_{n-m}$. Then, let us give a short summary on them, which is based on the literature [11, 18].

The heat kernel on $\mathbb{R}^m$ is the fundamental solution of the heat equation $\partial_t u(\boldsymbol{v}, t) = \Delta_{\mathbb{R}^m} u(\boldsymbol{v}, t)$ on $\mathbb{R}^m \times \mathbb{R}_+$ where $\Delta_{\mathbb{R}^m}$ designates the Laplace operator on $\mathbb{R}^m$ and $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$, which is given for $\boldsymbol{v} \in \mathbb{R}^m$ by

$$h_{\mathbb{R}^m}(\boldsymbol{v}) \equiv (h_{\mathbb{R}^m})_t(\boldsymbol{v}) = (4\pi t)^{-\frac{m}{2}} e^{-\frac{1}{4t}(\boldsymbol{v}|\boldsymbol{v})}.$$

Since $h_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}_+$ is analytic, this can be extended holomorphically to $\mathbb{C}^m$, still denoted by $h_{\mathbb{R}^m}$. For $f \in L^2(\mathbb{R}^m)$, we define a function $B_{\mathbb{R}^m} f \equiv (B_{\mathbb{R}^m})_t f$ on $\mathbb{C}^m$ by

$$(B_{\mathbb{R}^m} f)(\boldsymbol{v}) := \int_{\mathbb{R}^m} f(\boldsymbol{s}) h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s}) \, d\boldsymbol{s} \quad (\boldsymbol{v} \in \mathbb{C}^m) \tag{37}$$

where $d\boldsymbol{s}$ is the Lebesgue measure on $\mathbb{R}^m$. Indeed, the right-hand side of (37) is absolutely convergent for any $\boldsymbol{v} \in \mathbb{C}^m$, and $B_{\mathbb{R}^m}$ defines a map from $L^2(\mathbb{R}^m)$ to $\mathcal{O}(\mathbb{C}^m)$. It is well-known:

Fact 8.8. *The map $B_{\mathbb{R}^m} : L^2(\mathbb{R}^m) \to \mathcal{O}(\mathbb{C}^m)$ is a continuous and injective $\mathbb{R}^m$-intertwining operator.*

The operator $B_{\mathbb{R}^m}$ is called the hert kernel transform on $\mathbb{R}^m$ and is also known as the Segal–Bargmann transform.

The heat kernel transform has been studied for the Heisenberg group. Let $h_{\mathbf{H}_{n-m}} \equiv (h_{\mathbf{H}_{n-m}})_t$ be the heat kernel on $\mathbf{H}_{n-m}$, which is the fundamental solution of the heat equation $\partial_t u(g, t) = \Delta_{\mathbf{H}_{n-m}} u(g, t)$ on $\mathbf{H}_{n-m} \times \mathbb{R}_+$ where $\Delta_{\mathbf{H}_{n-m}}$ designates the Laplace operator on $\mathbf{H}_{n-m}$. An explicit description of $h_{\mathbf{H}_{n-m}}$ is referred to [18, Theorem 2.8.1] or [11, (2.2.1)]. Similarly to the case of $\mathbb{R}^m$, $h_{\mathbf{H}_{n-m}}$ is an analytic function on $\mathbf{H}_{n-m}$, and then can be extended holomorphically to the complexification $\mathbf{H}_{n-m}^{\mathbb{C}}$. For $f \in L^2(\mathbf{H}_{n-m})$, we define a function $B_{\mathbf{H}_{n-m}} f \equiv (B_{\mathbf{H}_{n-m}})_t f$ on $\mathbf{H}_{n-m}^{\mathbb{C}}$ by

$$(B_{\mathbf{H}_{n-m}} f)(g) := \int_{\mathbf{H}_{n-m}} f(k) h_{\mathbf{H}_{n-m}}(k^{-1} g) \, dk \quad (g \in \mathbf{H}_{n-m}^{\mathbb{C}}) \tag{38}$$

where dk is a Haar measure on $\mathbf{H}_{n-m}$. Then, $B_{\mathbf{H}_{n-m}}$ defines a map from $L^2(\mathbf{H}_{n-m})$ to $\mathcal{O}(\mathbf{H}_{n-m}^{\mathbb{C}})$ and is called the heat kernel transform on $\mathbf{H}_{n-m}$. We know:

Fact 8.9 (cf. [11, Section 3.1]).   *The map $B_{\mathbf{H}_{n-m}} : L^2(\mathbf{H}_{n-m}) \to \mathcal{O}(\mathbf{H}_{n-m}^{\mathbb{C}})$ is a continuous and injective $\mathbf{H}_{n-m}$-intertwining operator.*

We are ready to construct our operator from $L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$ to $\mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})$. In the following, we always fix a positive number t. For $f \in L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$, we define a function Bf on $\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$ by

$$(Bf)(\boldsymbol{v}, x) := \int_{\mathbb{R}^m \times \mathbf{H}_{n-m}} f(\boldsymbol{s}, k) h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s}) h_{\mathbf{H}_{n-m}}(k^{-1}x) \, d\boldsymbol{s}dk \qquad (39)$$

for $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. We verify:

Lemma 8.10.   *The right-hand side of (39) is absolutely convergent for any $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$.*

Proof.   As $h_{\mathbb{R}^m}(\boldsymbol{v}) = (4\pi t)^{-m/2} e^{-(\boldsymbol{v}|\boldsymbol{v})/4t}$ ($\boldsymbol{v} \in \mathbb{C}^m$), a direct computation shows

$$\int_{\mathbb{R}^m} |h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s})|^2 \, d\boldsymbol{s} = C_{\boldsymbol{v}} \int_{\mathbb{R}^m} |h_{\mathbb{R}^m}(\boldsymbol{s})|^2 \, d\boldsymbol{s} < \infty \qquad (40)$$

for any $\boldsymbol{v} \in \mathbb{C}^m$ where $C_{\boldsymbol{v}} := e^{(\boldsymbol{v}_I|\boldsymbol{v}_I)/2t}$ under the expression $\boldsymbol{v} = \boldsymbol{v}_R + \sqrt{-1}\boldsymbol{v}_I$ for some $\boldsymbol{v}_R, \boldsymbol{v}_I \in \mathbb{R}^m$. On the other hand, [11, (3.1.1)] explains

$$\int_{\mathbf{H}_{n-m}} |h_{\mathbf{H}_{n-m}}(k^{-1}x)|^2 \, dk < \infty \qquad (41)$$

for any $x \in \mathbf{H}_{n-m}^{\mathbb{C}}$. Then, we obtain

$$
\begin{aligned}
c(\boldsymbol{v}, x) &:= \left(\int_{\mathbb{R}^m \times \mathbf{H}_{n-m}} |h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s}) h_{\mathbf{H}_{n-m}}(k^{-1}x)|^2 \, d\boldsymbol{s}dk \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^m} |h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s})|^2 \, d\boldsymbol{s} \right)^{\frac{1}{2}} \left(\int_{\mathbf{H}_{n-m}} |h_{\mathbf{H}_{n-m}}(k^{-1}x)|^2 \, dk \right)^{\frac{1}{2}} < \infty \qquad (42)
\end{aligned}
$$

for any $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. Using Hölder's inequality, we get

$$\int_{\mathbb{R}^m \times \mathbf{H}_{n-m}} |f(\boldsymbol{s}, k) h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s}) h_{\mathbf{H}_{n-m}}(k^{-1}x)| \, d\boldsymbol{s}dk \leq c(\boldsymbol{v}, x) \|f\| < \infty \qquad (43)$$

for any $f \in L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$ and any $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. Here, $\|\cdot\|$ stands for the norm on $L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$. ∎

By definition (39), Bf is a holomorphic function on $\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$, and then we get a map B from $L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$ to $\mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})$.
In what follows, we prove:

Theorem 8.11.   *The map $B : L^2(\mathbb{R}^m \times \mathbf{H}_{n-m}) \to \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})$ is a continuous and injective $(\mathbb{R}^m \times \mathbf{H}_{n-m})$-intertwining operator.*

Proof.   We divide our proof into three steps as follows.

Step 1: We show first that B is a continuous map. We recall in this respect that $\mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})$ carries a Fréchet topology by the uniform convergence on compact sets in $\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. Namely, for a compact set U in $\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$, a semi-norm $|f|_U$ for $f \in \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})$ is defined by $|f|_U := \max_{(\boldsymbol{v},x) \in U} |f(\boldsymbol{v}, x)|$.

Let us take any $f, f_0 \in L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$. The inequality (43) implies

$$|(Bf - Bf_0)(\boldsymbol{v}, x)| \leq c(\boldsymbol{v}, x)\|f - f_0\|$$

for $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. Then, we estimate

$$|Bf - Bf_0|_U = \max_{(\boldsymbol{v},x) \in U} |(Bf - Bf_0)(\boldsymbol{v}, x)| \leq \left(\max_{(\boldsymbol{v},x) \in U} c(\boldsymbol{v}, x)\right) \|f - f_0\| \qquad (44)$$

for any compact set U in $\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. Here, the property (40) on the heat kernel $h_{\mathbb{R}^m}$ on $\mathbb{R}^m$ and [11, (3.1.1)] on the heat kernel $h_{\mathbf{H}_{n-m}}$ on $\mathbf{H}_{n-m}$ show

$$\max_{(\boldsymbol{v},x) \in U} c(\boldsymbol{v}, x) \leq \left(\max_{(\boldsymbol{v},x) \in U} C_{\boldsymbol{v}} \int_{\mathbb{R}^m} |h_{\mathbb{R}^m}(\boldsymbol{s})|^2 \, d\boldsymbol{s}\right)^{\frac{1}{2}} \left(\max_{(\boldsymbol{v},x) \in U} \int_{\mathbf{H}_{n-m}} |h_{\mathbf{H}_{n-m}}(k^{-1}x)|^2 \, dk\right)^{\frac{1}{2}}$$
$$< \infty.$$

Hence, (44) implies that B is continuous.

Step 2: Let us take any $f \in L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$. For fixed $k \in \mathbf{H}_{n-m}$, we write $f_k(\boldsymbol{s}) := f(\boldsymbol{s}, k)$ ($\boldsymbol{s} \in \mathbb{R}^m$). Then, f_k is a function on $\mathbb{R}^m$ and belongs to $L^2(\mathbb{R}^m)$. For $\boldsymbol{v} \in \mathbb{C}^m$, we define a function $\varphi_f(\boldsymbol{v})$ on $\mathbf{H}_{n-m}$ by

$$\varphi_f(\boldsymbol{v})(k) := \int_{\mathbb{R}^m} f_k(\boldsymbol{s}) h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s}) \, d\boldsymbol{s} \quad (k \in \mathbf{H}_{n-m}).$$

Then, $\varphi_f(\boldsymbol{v})$ lies in $L^2(\mathbf{H}_{n-m})$ for any $\boldsymbol{v} \in \mathbb{C}^m$ because

$$\int_{\mathbf{H}_{n-m}} |\varphi_f(\boldsymbol{v})(k)|^2 \, dk \leq \int_{\mathbf{H}_{n-m}} \left(\int_{\mathbb{R}^m} |f(\boldsymbol{s}, k)|^2 \, d\boldsymbol{s}\right) \left(\int_{\mathbb{R}^m} |h_{\mathbb{R}^m}(\boldsymbol{v} - \boldsymbol{s})|^2 \, d\boldsymbol{s}\right) dk$$
$$= C_{\boldsymbol{v}} \left(\int_{\mathbb{R}^m} |h_{\mathbb{R}^m}(\boldsymbol{s})|^2 \, d\boldsymbol{s}\right) \|f\|^2 < \infty.$$

Let us prove that B is injective. Suppose that $f \in L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$ satisfies $Bf = 0$, namely, $(Bf)(\boldsymbol{v}, x) = 0$ for any $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. In view of (38), $(Bf)(\boldsymbol{v}, x)$ is expressed as

$$(Bf)(\boldsymbol{v}, x) = \int_{\mathbf{H}_{n-m}} \varphi_f(\boldsymbol{v})(k) h_{\mathbf{H}_{n-m}}(k^{-1}x) \, dk = [B_{\mathbf{H}_{n-m}} \varphi_f(\boldsymbol{v})](x).$$

Then, the assumption gives rise to the relation $[B_{\mathbf{H}_{n-m}} \varphi_f(\boldsymbol{v})](x) = 0$ for any $(\boldsymbol{v}, x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}$. Thanks to Fact 8.9, the function $\varphi_f(\boldsymbol{v}) \in L^2(\mathbf{H}_{n-m})$ must be zero for any $\boldsymbol{v} \in \mathbb{C}^m$, in particular, $\varphi_f(\boldsymbol{v})(k) = 0$ for any $\boldsymbol{v} \in \mathbb{C}^m$ and any $k \in \mathbf{H}_{n-m}$. Moreover, it follows from (37) that $\varphi_f(\boldsymbol{v})(k)$ coincides with $(B_{\mathbb{R}^m} f_k)(\boldsymbol{v})$. Then, we get $(B_{\mathbb{R}^m} f_k)(\boldsymbol{v}) = 0$ for any $\boldsymbol{v} \in \mathbb{C}^m$ and any $k \in \mathbf{H}_{n-m}$. By Fact 8.8, this means $f_k = 0$ for any $k \in \mathbf{H}_{n-m}$, in particular, $f_k(\boldsymbol{s}) = 0$ for any $(\boldsymbol{s}, k) \in \mathbb{R}^m \times \mathbf{H}_{n-m}$. In conclusion, we have proved $f = 0$ since $f_k(\boldsymbol{s}) = f(\boldsymbol{s}, k)$, from which B is injective.

Step 3: Finally, let us verify $B(\rho(\boldsymbol{y},g)f) = \rho(\boldsymbol{y},g)(Bf)$ for any $(\boldsymbol{y},g) \in \mathbb{R}^m \times \mathbf{H}_{n-m}$ and any $f \in L^2(\mathbb{R}^m \times \mathbf{H}_{n-m})$ (see (36) for definition). Indeed, we obtain

$$
\begin{aligned}
B(\rho(\boldsymbol{y},g)f)(\boldsymbol{v},x) &= \int_{\mathbb{R}^m \times \mathbf{H}_{n-m}} [\rho(\boldsymbol{y},g)f](\boldsymbol{s},k) h_{\mathbb{R}^m}(\boldsymbol{v}-\boldsymbol{s}) h_{\mathbf{H}_{n-m}}(k^{-1}x)\, d\boldsymbol{s}dk \\
&= \int_{\mathbb{R}^m \times \mathbf{H}_{n-m}} f(\boldsymbol{s}-\boldsymbol{y}, g^{-1}k) h_{\mathbb{R}^m}(\boldsymbol{v}-\boldsymbol{s}) h_{\mathbf{H}_{n-m}}(k^{-1}x)\, d\boldsymbol{s}dk \\
&= \int_{\mathbb{R}^m \times \mathbf{H}_{n-m}} f(\boldsymbol{s},k) h_{\mathbb{R}^m}(\boldsymbol{v}-(\boldsymbol{s}+\boldsymbol{y})) h_{\mathbf{H}_{n-m}}(((gk)^{-1}x))\, d\boldsymbol{s}dk \\
&= \int_{\mathbb{R}^m \times \mathbf{H}_{n-m}} f(\boldsymbol{s},k) h_{\mathbb{R}^m}((\boldsymbol{v}-\boldsymbol{y})-\boldsymbol{s}) h_{\mathbf{H}_{n-m}}(k^{-1}(g^{-1}x))\, d\boldsymbol{s}dk \\
&= (Bf)(\boldsymbol{v}-\boldsymbol{y}, g^{-1}x) = \rho(\boldsymbol{y},g)(Bf)(\boldsymbol{v},x) \quad ((\boldsymbol{v},x) \in \mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}).
\end{aligned}
$$

Consequently, Theorem 8.11 follows from Steps 1–3. ∎

8.5. Equivalence between (ii) and (iii)

In this subsection, we prove the equivalence between (ii) and (iii). By Proposition 2.12, it suffices to consider $H = H_{(0,0,1)}$, $H_{(p,q,1)}$, $H_{(m,0,1)}$ and $H_{(m,0,0)}$. Here, we have already proved that condition (ii) holds if $H = H_{(0,0,1)}$ in Theorem 3.2; if $H = H_{(p,q,1)}$ in Theorem 4.2; if $H = H_{(m,0,1)}$ in Theorem 5.3; and if $H = H_{(n,0,0)}$ in Theorem 6.5. Then, let us consider the remaining case, namely, the $Q_{(m,0,0)}$-action on $D_{(m,0,0)} = G^{\mathbb{C}}/H_{(m,0,0)}^{\mathbb{C}}$ for $1 \le m < n$.

Before that, let us focus on the regular representation $(\rho, L^2(\mathbb{R}^m \times \mathbf{H}_{n-m}))$. It follows from (i) ⇔ (iii) (see Section 8.3) that the quasi-regular representation $(\pi_{(m,0,0)}, L^2(M_{(m,0,0)}))$ is not multiplicity-free as a representation of $G = \mathbf{H}_n$. In particular, we know that the multiplicity $m_{\pi_{(m,0,0)}}(\xi(\mathbf{0},\mathbf{0},\gamma))$ of $\tau_{\xi(\mathbf{0},\mathbf{0},\gamma)} \in \widehat{G}$ is infinite for any $\xi(\mathbf{0},\mathbf{0},\gamma) \in \mathfrak{q}_{(m,0,0)}^*$ with $\gamma \ne 0$. Thus, $\pi_{(m,0,0)}$ is of infinite multiplicities as a representation of $Q_{(m,0,0)}$, and then, of $\mathbb{R}^m \times \mathbf{H}_{n-m}$ under the identification $Q_{(m,0,0)} \simeq \mathbb{R}^m \times \mathbf{H}_{n-m}$. Note that $\pi_{(m,0,0)}$ is equivalent to ρ by the $(\mathbb{R}^m \times \mathbf{H}_{n-m})$-intertwining operator $\Psi_{\mathbb{R}}^{\vee}$. Hence we get:

Lemma 8.12.   *The regular representation $(\rho, L^2(\mathbb{R}^m \times \mathbf{H}_{n-m}))$ of $\mathbb{R}^m \times \mathbf{H}_{n-m}$ is of infinite multiplicities.*

After this preparation we prove:

Theorem 8.13.   *If $m < n$, then the $Q_{(m,0,0)}$-action on $D_{(m,0,0)}$ is not strongly visible.*

Proof.   Suppose the $Q_{(m,0,0)}$-action on $D_{(m,0,0)}$ is strongly visible. By Fact 8.1 (see also Theorem 8.2), the representation $(\pi, \mathcal{O}(D_{(m,0,0)}))$ of $Q_{(m,0,0)}$ is multiplicity-free. This implies that $(\rho, \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}))$ turns out to be multiplicity-free as a representation of $\mathbb{R}^m \times \mathbf{H}_{n-m} \simeq Q_{(m,0,0)}$ via the intertwining operator $\Psi^{\vee}$. This means that any unitary representation of $\mathbb{R}^m \times \mathbf{H}_{n-m}$ realized in $(\rho, \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}))$ in the sense of [9, Definition 2.1] is multiplicity-free (see [4, Definition 1.5.3]). Here, Theorem 8.11 asserts that the unitary representation $(\rho, L^2(\mathbb{R}^m \times \mathbf{H}_{n-m}))$ is realized in $(\rho, \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}}))$ via the operator B (see (39) for definition). Then,

$(\rho, L^2(\mathbb{R}^m \times \mathbf{H}_{n-m}))$ must be multiplicity-free as a representation of $\mathbb{R}^m \times \mathbf{H}_{n-m}$, which contradicts Lemma 8.12.

Therefore, the $Q_{(m,0,0)}$-action on $D_{(m,0,0)}$ is not strongly visible. ∎

$$
\begin{array}{ccc}
L^2(M_{(m,0,0)}) & \dashrightarrow & \mathcal{O}(D_{(m,0,0)}) \\
\Psi_{\mathbb{R}}^\vee \downarrow & & \downarrow \Psi^\vee \\
L^2(\mathbb{R}^m \times \mathbf{H}_{n-m}) & \xrightarrow{B} & \mathcal{O}(\mathbb{C}^m \times \mathbf{H}_{n-m}^{\mathbb{C}})
\end{array}
$$

Figure 1: $(\mathbb{R}^m \times \mathbf{H}_{n-m})$-intertwining operator

Proof of (ii) ⇔ (iii) in Theorem 8.3. As mentioned above, we have already shown the implication (iii) ⇒ (ii). The opposite is also true by Theorem 8.13. Consequently, the proof of Theorem 8.3 has been accomplished. ∎

8.6. Slice for Q-action on D and the support of π_H

We end this paper by considering the multiplicity-free irreducible decomposition (32) of the quasi-regular representation π_H of G. In accordance with Lemma 8.7 and the equivalence (i) ⇔ (iii) in Theorem 8.3, we may always consider non-trivial connected closed subgroups H as $H_{(0,0,1)}$, $H_{(p,q,1)}$, $H_{(m,0,1)}$ and $H_{(n,0,0)}$.

As we have seen in Fact 8.5, the irreducible decomposition of π_H can be described in terms of the coadjoint orbits. More precisely, the irreducible representations of G occurring in (32) are parameterized by the G-orbit space $(G \cdot \mathfrak{q}^*)/G$ of the subset $G \cdot \mathfrak{q}^*$ of $\mathfrak{g}^*$. Then, we focus on $(G \cdot \mathfrak{q}^*)/G$ below.

For a cross-section R of the coadjoint orbit space $\mathfrak{g}^*/G$, the set $R \cap (G \cdot \mathfrak{q}^*)$ becomes a cross-section of $(G \cdot \mathfrak{q}^*)/G$. In Lemma 8.4, we have taken R as

$$
R = \{\xi(\mathbf{0}, \mathbf{0}, \gamma) : \gamma \in \mathbb{R}^\times\} \sqcup \{\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0) : \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n\}. \tag{31}
$$

Since the multiplicity $m_{\pi_H}(\xi)$ of $\tau_\xi \in \widehat{G}$ corresponding to $\xi \in (G \cdot \mathfrak{q}^*)/G$ does not depend on the choice of the cross-section (see Fact 8.5), we may fix R as (31) in our argument below. Now, we choose

$$
R(\mathfrak{q}^*) := R \cap (G \cdot \mathfrak{q}^*) \tag{45}
$$

as a cross-section of $(G \cdot \mathfrak{q}^*)/G$.

Lemma 8.14. *The set $R(\mathfrak{q}^*)$ coincides with $R \cap \mathfrak{q}^*$.*

Proof. Let $H \equiv H_{(k,\ell,\varepsilon)}$ be $H_{(0,0,1)}$, $H_{(p,q,1)}$ or $H_{(m,0,1)}$. We have seen in (33) that $G \cdot \mathfrak{q}_{(k,\ell,\varepsilon)}^* = \mathfrak{q}_{(k,\ell,\varepsilon)}^*$, from which we obtain $R(\mathfrak{q}_{(k,\ell,\varepsilon)}^*) = R \cap \mathfrak{q}_{(k,\ell,\varepsilon)}^*$. Since $\mathfrak{q}_{(k,\ell,\varepsilon)}^*$ is a subspace of $\mathrm{span}_{\mathbb{R}}\{X_i^*, Y_j^* : 1 \le i, j \le n\}$, $R(\mathfrak{q}_{(k,\ell,\varepsilon)}^*)$ is given by

$$
R(\mathfrak{q}_{(k,\ell,\varepsilon)}^*) = \{\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, 0) : \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n\} \cap \mathfrak{q}_{(k,\ell,\varepsilon)}^* = \mathfrak{q}_{(k,\ell,\varepsilon)}^*. \tag{46}
$$

Let H be $H_{(n,0,0)}$. Then, $\mathfrak{q}_{(n,0,0)}^* = \{\xi(\mathbf{0}, \boldsymbol{\beta}, \gamma) : \boldsymbol{\beta} \in \mathbb{R}^n, \gamma \in \mathbb{R}\}$. In view of Lemma 8.4, $G \cdot \mathfrak{q}_{(n,0,0)}^*$ is expressed as

$$
G \cdot \mathfrak{q}_{(n,0,0)}^* = \{\xi(\boldsymbol{x}, \boldsymbol{y}, \gamma) : \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \gamma \in \mathbb{R}^\times\} \sqcup \{\xi(\mathbf{0}, \boldsymbol{\beta}, 0) : \boldsymbol{\beta} \in \mathbb{R}^n\},
$$

from which the orbit space is written as

$$
(G \cdot \mathfrak{q}_{(n,0,0)}^*)/G = \{G \cdot \xi(\mathbf{0}, \mathbf{0}, \gamma) : \gamma \in \mathbb{R}^\times\} \sqcup \{G \cdot \xi(\mathbf{0}, \boldsymbol{y}, 0) : \boldsymbol{y} \in \mathbb{R}^n\}.
$$

Hence, $R(\mathfrak{q}^*_{(n,0,0)})$ is of the form

$$R(\mathfrak{q}^*_{(n,0,0)}) = \{\xi(\mathbf{0},\mathbf{0},\gamma) : \gamma \in \mathbb{R}^\times\} \sqcup \{\xi(\mathbf{0},\boldsymbol{\beta},0) : \boldsymbol{\beta} \in \mathbb{R}^n\}, \qquad (47)$$

which coincides with $R \cap \mathfrak{q}^*_{(n,0,0)}$.   ∎

Lemma 8.14 explains that $R(\mathfrak{q}^*)$ is a subset of the real vector space $\mathfrak{q}^*$. Then, there exists a minimal finite subset $\mathcal{B}(R(\mathfrak{q}^*)) = \{\xi_1,\ldots,\xi_r\}$ in $\mathfrak{q}^*$ such that $\mathcal{B}(R(\mathfrak{q}^*))$ is linearly independent in $\mathfrak{q}^*$ and arbitrary element of $R(\mathfrak{q}^*)$ is written as the linear combination of $\mathcal{B}(R(\mathfrak{q}^*))$. We say that

$$\mathrm{Supp}(\pi_H) := \{\tau_\xi \in \widehat{G} : \xi \in R(\mathfrak{q}^*), m_{\pi_H}(\xi) \neq 0\}$$

is the *support* of the irreducible representations of G occurring in (32), and that the number

$$\mathrm{rank}(\mathrm{Supp}(\pi_H)) := |\mathcal{B}(R(\mathfrak{q}^*))| = r \qquad (48)$$

is the *rank* of $\mathrm{Supp}(\pi_H)$.

Theorem 8.15.   *Let G be the Heisenberg group and H a non-trivial connected closed subgroup of G. Suppose that the quasi-regular representation π_H of G is multiplicity-free. Then we have:*

(1) $\mathrm{rank}(\mathrm{Supp}(\pi_H)) = \dim G/H$.

(2) *One can find a slice S of dimension $\mathrm{rank}(\mathrm{Supp}(\pi_H))$ for the strongly visible Q-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$.*

Proof.   Let $H \equiv H_{(k,\ell,\varepsilon)}$ be one of $H_{(0,0,1)}$, $H_{(p,q,1)}$ or $H_{(m,0,1)}$. Then, we have $R(\mathfrak{q}^*_{(k,\ell,\varepsilon)}) = \mathfrak{q}^*_{(k,\ell,\varepsilon)}$ (see (46)). This means that $\mathcal{B}(R(\mathfrak{q}^*_{(k,\ell,\varepsilon)}))$ is nothing but a basis of $\mathfrak{q}^*_{(k,\ell,\varepsilon)}$. Thus, we obtain

$$\mathrm{rank}(\mathrm{Supp}(\pi_{H_{(k,\ell,\varepsilon)}})) = \dim \mathfrak{q}_{(k,\ell,\varepsilon)} = \dim G/H_{(k,\ell,\varepsilon)}.$$

On the other hand, Table 1 asserts that our choice of slice $S_{(k,\ell,\varepsilon)}$ for the strongly visible $Q_{(k,\ell,\varepsilon)}$-action on $G^{\mathbb{C}}/H^{\mathbb{C}}_{(k,\ell,\varepsilon)}$ satisfies

$$\dim S_{(k,\ell,\varepsilon)} = \dim \sqrt{-1}\mathfrak{q}_{(k,\ell,\varepsilon)} = \dim G/H_{(k,\ell,\varepsilon)}.$$

Hence, we get $\mathrm{rank}(\mathrm{Supp}(\pi_{H_{(k,\ell,\varepsilon)}})) = \dim G/H_{(k,\ell,\varepsilon)} = \dim S_{(k,\ell,\varepsilon)}$.

Next, let H be $H_{(n,0,0)}$. Since $R(\mathfrak{q}^*_{(n,0,0)})$ is given by (47), we can take $\mathcal{B}(R(\mathfrak{q}^*_{(n,0,0)}))$ as $\{Y_1^*,\ldots,Y_n^*,Z^*\}$. Clearly, this is also a basis of $\mathfrak{q}^*_{(n,0,0)}$, from which

$$\mathrm{rank}(\mathrm{Supp}(\pi_{H_{(n,0,0)}})) = \dim \mathfrak{q}^*_{(n,0,0)} = \dim G/H_{(n,0,0)}.$$

Further, it follows from Theorem 6.5 that the $Q_{(n,0,0)}$-action on $G^{\mathbb{C}}/H^{\mathbb{C}}_{(n,0,0)}$ is strongly visible with slice $S_{(n,0,0)} \simeq \exp\sqrt{-1}\mathfrak{q}_{(n,0,0)}$. Thus, we have $\dim S_{(n,0,0)} = \dim \sqrt{-1}\mathfrak{q}_{(n,0,0)} = \dim G/H_{(n,0,0)}$. Hence, we find out that $\mathrm{rank}(\mathrm{Supp}(\pi_{H_{(n,0,0)}})) = \dim G/H_{(n,0,0)} = \dim S_{(n,0,0)}$, which achieves the proof of the theorem.   ∎

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