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Classification of Einstein Lorentzian 3-Nilpotent Lie Groups with 1-Dimensional Nondegenerate Center

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Abstract. We give a complete classification of Einstein Lorentzian 3-nilpotent simply connected Lie groups with 1-dimensional nondegenerate center.

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1. Introduction

The study of left-invariant Einstein Riemannian metrics on Lie groups is a research area that had made huge progress in the last decades (see [10, 12, 13]). However, the indefinite case remains unexplored in comparison and only few significant results had been published in this matter with many questions that are still open (see [9, 4, 3]).

In [3], the authors began an inspection of Einstein Lorentzian nilpotent Lie algebras following guidelines from previous studies of the 2-step nilpotent case (see [2] and [9]). The main Theorem of [3] states that Einstein nilpotent Lie algebras with degenerate center are exactly Ricci-flat and are obtained by a double extension process starting from a Euclidean vector space (see [3, Theorem 4.1] and [1] for the original definition of the double extension). This class of Lie algebras includes all Einstein Lorentzian nilpotent Lie algebras that are either 2-step or of dimension less than 5, in fact as a concrete application of the main Theorem, the authors were able to give a full classification of the latter.

Dimension 6 however falls outside the context of this result as the authors presented the first example in this situation of an Einstein nilpotent Lie algebra with non-degenerate center, which also happens to be 3-step nilpotent. Einstein nilpotent Lie algebras that are non Ricci-flat has been shown to exist in the Lorentzian setting (see [4]) and according to [3, Theorem 4.1] these must have non-degenerate center as well. So the study of Einstein Lorentzian nilpotent Lie algebras with nondegenerate center becomes a natural and challenging problem and the present paper can be seen as a first attempt to find a general pattern for these Lie algebras.

We start by the 3-step nilpotent case and we develop a new approach which can be used later in the general case. Let us give a brief summary of our method and state our main result.

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Let $(\mathfrak{h},[\ ,\ ])$ be a k-nilpotent Lie algebra and $\langle\ ,\ \rangle$ an Einstein Lorentzian metric on $\mathfrak{h}$ such that the center of $\mathfrak{h}$ is non-degenerate. Then $Z(\mathfrak{h})$ is non degenerate Euclidean (see [3]) and, naturally, we get the orthogonal spitting

$$\mathfrak{h} = \mathrm{Z}(\mathfrak{h}) \overset{\perp}{\oplus} \mathfrak{g}.$$

The Lie bracket on $\mathfrak{h}$ splits accordingly as $[u,v] = \omega(u,v) + [u,v]_0$ for any $u,v \in \mathfrak{g}$, where $[\ ,\ ]_0$ is a Lie bracket on $\mathfrak{g}$ and $\omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{Z}(\mathfrak{h})$ is a 2-cocycle of $(\mathfrak{g},[\ ,\ ]_0)$. It turns out that $(\mathfrak{g},[\ ,\ ]_0,\langle\ ,\ \rangle_{|\mathfrak{g}\times\mathfrak{g}})$ is a Lorentzian $(k-1)$-nilpotent Lie algebra and the Einstein equation on $\mathfrak{h}$ can be expressed entirely by means of the Lie algebra $\mathfrak{g}$ as a sort of compatibility condition between ω and the Ricci curvature $\mathrm{Ric}_{\mathfrak{g}}$ of $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}},[\ ,\ ]_0)$ (see Proposition 3.3). This shift in perspective is especially useful when the Lie algebra $\mathfrak{h}$ is 3-step nilpotent since $\mathfrak{g}$ is 2-nilpotent and, for instance, we can show that every Einstein Lorentzian 3-step nilpotent Lie algebra with non-degenerate center has positive scalar curvature (Theorem 3.8). It also gives rise to the notion of ω-quasi Einstein Lie algebras (see Definition 3.9). A careful study of ω-quasi Einstein 2-nilpotent Lie algebras leads to our main result, namely the classification of Einstein Lorentzian 3-step nilpotent Lie algebras with 1-dimensional non-degenerate center. Surprisingly enough, these are shown to only exist in dimensions 6 and 7.

Theorem 1.1. *Let $\mathfrak{h}$ be a 3-step nilpotent Lie algebra with $\dim \mathrm{Z}(\mathfrak{h}) = 1$. Let $\langle\ ,\ \rangle$ be a Lorentzian metric on $\mathfrak{h}$ such that $\mathrm{Z}(\mathfrak{h})$ is non-degenerate, then $\langle\ ,\ \rangle$ is Einstein if and only if $\mathfrak{h}$ is Ricci-flat and has one of the following forms:*

(i) $\dim \mathfrak{h} = 6$ *and $\mathfrak{h}$ is isomorphic to $\mathrm{L}_{6,19}(-1)$, i.e., $\mathfrak{h}$ has a basis $(f_i)_{i=1}^6$ such that the non vanishing Lie brackets are*

$$[f_1,f_2] = f_4, [f_1,f_3] = f_5, [f_2,f_4] = f_6, [f_3,f_5] = -f_6$$

and the metric is given by:

$$\langle\ ,\ \rangle := f_1^* \otimes f_1^* + 2f_2^* \otimes f_2^* + 2f_3^* \otimes f_3^* + 4\alpha^4 f_6^* \otimes f_6^* - 2\alpha^2 f_4^* \odot f_5^*, \quad \alpha \neq 0. \quad (1)$$

(ii) $\dim \mathfrak{h} = 7$ *and $\mathfrak{h}$ is isomorphic to the nilpotent Lie algebras $147E$ found in the classification given in* [8](p. 57)*. In precise terms, there exists a basis $\{f_i\}_{i=1}^7$ of $\mathfrak{h}$ where the non vanishing Lie bracket are given by:*

$$\begin{aligned}&[f_1,f_2] = f_5,\ [f_1,f_3] = f_6,\ [f_2,f_3] = f_4,\\ &[f_6,f_2] = (1-r)f_7,\ [f_5,f_3] = -rf_7,\ [f_4,f_1] = f_7,\end{aligned} \quad (2)$$

with $0 < r < 1$, and the metric has the form:

$$\begin{aligned}\langle\ ,\ \rangle = {}& f_1^* \otimes f_1^* + f_2^* \otimes f_2^* + f_3^* \otimes f_3^* - af_4^* \otimes f_4^*\\ &+ arf_5^* \otimes f_5^* + a(1-r)f_6^* \otimes f_6^* + a^2 f_7^* \otimes f_7^*, \quad a > 0.\end{aligned} \quad (3)$$

Outline of the paper

In Section 2 we give some preliminaries on Pseudo-Riemannian Lie algebras as well as all the notations needed for subsequent development. In Section 3, we describe an Einstein Lorentzian nilpotent Lie algebra $\mathfrak{h}$ with non-degenerate center by means of its center, a nilpotent Lorentzian Lie algebra $\mathfrak{g}$ of lower order, and a 2-cocycle $\omega \in \mathrm{Z}^2(\mathfrak{g}, \mathrm{Z}(\mathfrak{h}))$, these are called *the attributes* of $\mathfrak{h}$ (see Definition 3.1). The main result of this section is Theorem 3.8 in which we prove that any Einstein Lorentzian 3-step nilpotent Lie algebra of non-degenerate center has positive scalar curvature, at the end of the section we introduce the notion of ω*-quasi Einstein Lie algebra.*

The remainder of the document is devoted to the proof of the central results. As the reader can see, the proof of Theorem 1.1 turns out to be difficult and it is based on a sequence of Lemmas (Lemma 4.2, 4.3 and 4.4). This suggests that the complete study of Einstein Lorentzian nilpotent Lie algebras with nondegenerate center of dimension greater than 2 is a challenging mathematical problem. There are examples of such Lie algebras even in the 3-step nilpotent case (see the examples given in the end of the paper). More generally the study of left invariant Einstein pseudo-Riemannian metrics on Lie groups is far more complicated. In [4], the study of left invariant Einstein pseudo-Riemannian metrics with non vanishing scalar curvature on nilpotent Lie groups was initiated and the first examples of such metrics were given (see Example 4.6). In [7], there is a qualitative study of the set of all Einstein connections in some simple Lie groups, i.e., the Levi-Civita connections of left-invariant Einstein pseudo-Riemannian metrics.

2. Preliminaries

A *pseudo-Euclidean vector space* is a real vector space of finite dimension n endowed with a nondegenerate symmetric inner product of signature $(q, n-q) = (-\ldots-, +\ldots+)$. When the signature is $(0, n)$ (resp. $(1, n-1)$) the space is called *Euclidean* (resp. *Lorentzian*).

Let $(V, \langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space of signature $(q, n-q)$. A vector $u \in V$ is called *spacelike* if $\langle u, u\rangle > 0$, *timelike* if $\langle u, u\rangle < 0$ and *isotropic* if $\langle u, u\rangle = 0$. A family $(u_1, \ldots, u_s)$ of vectors in V is called *orthogonal* if, for $i, j = 1, \ldots, s$ and $i \neq j$, $\langle u_i, u_j\rangle = 0$. An orthonormal basis of V is an orthogonal basis $(e_1, \ldots, e_n)$ such that $\langle e_i, e_i\rangle = \pm 1$. For any endomorphism $F : V \longrightarrow V$, we denote by $F^* : V \longrightarrow V$ its adjoint with respect to $\langle\ ,\ \rangle$.

It is a well-known fact that the study of the curvature of left invariant pseudo-Riemannian metrics on Lie groups reduces to the study of its restriction to their Lie algebras. Let us recall some definitions and fix some notations. The reader can consult [2] or [3] for details.

Let $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle)$ be a *pseudo-Euclidean Lie algebra*, i.e, a Lie algebra endowed with a pseudo-Euclidean product. The *Levi-Civita product* of $\mathfrak{h}$ is the bilinear map $\mathrm{L} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ given by Koszul's formula

$$2\langle \mathrm{L}_u v, w\rangle = \langle [u,v], w\rangle + \langle [w,u], v\rangle + \langle [w,v], u\rangle. \tag{4}$$

For any $u, v \in \mathfrak{h}$, $\mathrm{L}_u : \mathfrak{h} \longrightarrow \mathfrak{h}$ is skew-symmetric and $[u,v] = \mathrm{L}_u v - \mathrm{L}_v u$. The curvature of $\mathfrak{h}$ is given by

$$K(u,v) = \mathrm{L}_{[u,v]} - [\mathrm{L}_u, \mathrm{L}_v].$$

The Ricci curvature $\mathrm{ric} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{R}$ and its Ricci operator $\mathrm{Ric} : \mathfrak{h} \longrightarrow \mathfrak{h}$ are defined by

$$\langle \mathrm{Ric}(u), v\rangle = \mathrm{ric}(u,v) = \mathrm{tr}\,(w \longrightarrow K(u,w)v)\,.$$

A pseudo-Euclidean Lie algebra is called *flat* (resp. *Ricci-flat*) if $K = 0$ (resp. $\mathrm{ric} = 0$). It is called λ-Einstein if there exists a constant $\lambda \in \mathbb{R}$ such that $\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{h}}$.

In this paper, we deal with nilpotent Lie algebras and in this case the ricci curvature is given by

$$\mathrm{ric}(u,v) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) - \frac{1}{4}\mathrm{tr}(J_u \circ J_v), \tag{5}$$

where J_u is the skew-symmetric endomorphism given by $J_u(v) = \mathrm{ad}_v^* u$. Moreover, if $\mathcal{J}_1$ and $\mathcal{J}_2$ denote the symmetric endomorphisms given by

$$\langle \mathcal{J}_1 u, v\rangle = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*),\ \langle \mathcal{J}_2 u, v\rangle = -\mathrm{tr}(J_u \circ J_v) = \mathrm{tr}(J_u \circ J_v^*). \tag{6}$$

then the Ricci operator has the following expression

$$\mathrm{Ric} = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2, \tag{7}$$

The endomorphisms $\mathcal{J}_1$ and $\mathcal{J}_2$ can be expressed in a useful way. Indeed, if $(e_1, \ldots, e_n)$ is a basis of $[\mathfrak{h},\mathfrak{h}]$, then, for any $u, v \in \mathfrak{h}$, the Lie bracket can be written

$$[u,v] = \sum_{i=1}^{n} \langle J_i u, v\rangle e_i, \tag{8}$$

where $(J_1, \ldots, J_n)$ is a family of skew-symmetric endomorphisms with respect to $\langle\ ,\ \rangle$. This family will be called *Lie structure endomorphisms* associated to $(e_1, \ldots, e_n)$. The following proposition will be very useful later. See [3, Proposition 2.3] for its proof.

Proposition 2.1. *Let $(\mathfrak{h}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra, $(e_1, \ldots, e_n)$ a basis of $[\mathfrak{h},\mathfrak{h}]$ and $(J_1, \ldots, J_n)$ the corresponding structure endomorphisms. Then*

$$\mathcal{J}_1 = -\sum_{i,j=1}^{n} \langle e_i, e_j\rangle J_i \circ J_j \quad \textit{and} \quad \mathcal{J}_2 u = -\sum_{i,j=1}^{n} \langle e_i, u\rangle \mathrm{tr}(J_i \circ J_j) e_j. \tag{9}$$

In particular, $\mathrm{tr}\mathcal{J}_1 = \mathrm{tr}\mathcal{J}_2$.

3. Lorentzian nilpotent Einstein Lie algebras with nondegenerate center

In [3], we studied Lorentzian nilpotent Einstein Lie algebras with degenerate center and we gave the first example of a Lorentzian 3-step nilpotent Ricci-flat Lie algebra with nondegenerate center. We also showed that an Einstein Lorentzian nilpotent Lie algebra with non zero scalar curvature must have a nondegenerate center. A first example of such algebras was given in [4]. A 2-step nilpotent Einstein Lorentzian Lie algebra must be Ricci-flat with degenerate center so it is natural to start by studying 3-step nilpotent Einstein Lorentzian Lie algebras with nondegenerate center which, according to [3, Corollary 3.1], must be Euclidean.

Any nilpotent Lie algebra can be obtained by Skjelbred-Sund's method, namely, by an extension from a nilpotent Lie algebra of lower dimension and a 2-cocycle with values in a vector space (see [6]). We will adapt this method to our study.

Let $(\mathfrak{h},\langle\ ,\ \rangle_{\mathfrak{h}})$ be a Lorentzian k-step nilpotent Lie algebra of dimension n with nondegenerate Euclidean center $Z(\mathfrak{h})$ of dimension $p\geq 1$. Denote by $\langle\ ,\ \rangle_z$ the restriction of $\langle\ ,\ \rangle$ to $Z(\mathfrak{h})$, $\mathfrak{g}=Z(\mathfrak{h})^{\perp}$ and by $\langle\ ,\ \rangle_{\mathfrak{g}}$ the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{g}$.

We get that
$$\mathfrak{h}=\mathfrak{g}\overset{\perp}{\oplus}Z(\mathfrak{h}),$$
where $(Z(\mathfrak{h}),\langle\ ,\ \rangle_z)$ is a Euclidean vector space and $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}})$ is a Lorentzian vector space. Moreover, for any $u,v\in\mathfrak{g}$, we have
$$[u,v]=[u,v]_{\mathfrak{g}}+\omega(u,v),$$
where $[u,v]_{\mathfrak{g}}\in\mathfrak{g}$ and $\omega(u,v)\in Z(\mathfrak{h})$. The Jacobi identity applied to $[\ ,\ ]$ is easily seen equivalent to $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}})$ being a Lie algebra and $\omega:\mathfrak{g}\times\mathfrak{g}\longrightarrow Z(\mathfrak{h})$ a 2-cocycle of $\mathfrak{g}$ with respect to the trivial representation of $\mathfrak{g}$ in $Z(\mathfrak{h})$, namely, for any $u,v,w\in\mathfrak{g}$,
$$\omega([u,v]_{\mathfrak{g}},w)+\omega([v,w]_{\mathfrak{g}},u)+\omega([w,u]_{\mathfrak{g}},v)=0.$$
Moreover,
$$Z(\mathfrak{g})\cap\ker\omega=\{0\}\quad\text{and}\quad C^n(\mathfrak{h}):=[C^{n-1}(\mathfrak{h}),\mathfrak{h}]=C^n(\mathfrak{g})+\omega(C^{n-1}(\mathfrak{g}),\mathfrak{g}),\qquad(10)$$
for any $n\in\mathbb{N}$. This implies that $(\mathfrak{h},[\ ,\ ])$ is k-step nilpotent if and only if $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}})$ is $(k-1)$-step nilpotent and $C^{k-2}(\mathfrak{g})\not\subset\ker\omega$.

Definition 3.1. Let $(\mathfrak{h},[\ ,\ ],\langle\ ,\ \rangle_{\mathfrak{h}})$ be a Lorentzian nilpotent Lie algebra with nondegenerate Euclidean center. We call the triple $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}},[\ ,\ ]_{\mathfrak{g}})$, $(Z(\mathfrak{h}),\langle\ ,\ \rangle_z)$ and $\omega\in Z^2(\mathfrak{g},Z(\mathfrak{h}))$ the *attributes* of $(\mathfrak{h},[\ ,\ ],\langle\ ,\ \rangle_{\mathfrak{h}})$.

We proceed now to express the Ricci curvature of $\mathfrak{h}$ in terms of its attributes $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}},[\ ,\ ]_{\mathfrak{g}})$, $(Z(\mathfrak{h}),\langle\ ,\ \rangle_z)$ and $\omega\in Z^2(\mathfrak{g},Z(\mathfrak{h}))$. For any $u\in\mathfrak{g}$, we consider $\omega_u:\mathfrak{g}\longrightarrow Z(\mathfrak{h})$, $v\longrightarrow\omega(u,v)$, $\omega_u^*:Z(\mathfrak{h})\longrightarrow\mathfrak{g}$ its transpose given by
$$\langle\omega_u^*(x),v\rangle_{\mathfrak{g}}=\langle\omega(u,v),x\rangle_z.$$
For any $x\in Z(\mathfrak{h})$, we define $S_x:\mathfrak{g}\longrightarrow\mathfrak{g}$ by $S_x(u)=\omega_u^*(x)$.

It is clear that S_x is skew-symmetric. Recall that, for any $u\in\mathfrak{g}$, we denote by $J_u:\mathfrak{g}\longrightarrow\mathfrak{g}$ the skew-symmetric endomorphism given by $J_u(v)=\mathrm{ad}_v^*(u)$.

On the other hand, define the endomorphism $D:\mathfrak{g}\longrightarrow\mathfrak{g}$ by
$$\langle Du,v\rangle_{\mathfrak{g}}=\mathrm{tr}(\omega_u^*\circ\omega_v).\qquad(11)$$
It is clear that D is symmetric with respect to $\langle\ ,\ \rangle_{\mathfrak{g}}$. Let $(z_1,\ldots,z_p)$ be a basis of $Z(\mathfrak{h})$. There exists a unique family $(S_1,\ldots,S_p)$ of skew-symmetric endomorphisms such that, for any $u,v\in\mathfrak{g}$,
$$\omega(u,v)=\sum_{i=1}^{p}\langle S_iu,v\rangle_{\mathfrak{g}}z_i.\qquad(12)$$
This family will be called ω*-structure endomorphisms* associated to $(z_1,\ldots,z_p)$. A direct computation using (11) and (12) shows that
$$D=-\sum_{i,j}\langle z_i,z_j\rangle_zS_i\circ S_j.\qquad(13)$$
This operator has an interesting property.

Proposition 3.2. *If ω satisfies*

$$\omega(\mathrm{ad}_u^* v, w) + \omega(v, \mathrm{ad}_u^* w) = 0 \tag{14}$$

for any $u, v, w \in \mathfrak{g}$, then D is a derivation of $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g})$.

Proof. Since ω is a 2-cocycle then

$$\omega_{[u,v]_\mathfrak{g}} = \omega_u \circ \mathrm{ad}_v - \omega_v \circ \mathrm{ad}_u.$$

We also have, for any $u, v, w \in \mathfrak{g}$

$$\begin{aligned}\langle [Du, v]_\mathfrak{g}, w\rangle_\mathfrak{g} + \langle [u, Dv]_\mathfrak{g}, w\rangle_\mathfrak{g} &= -\mathrm{tr}(\omega_{\mathrm{ad}_v^* w} \circ \omega_u^*) + \mathrm{tr}(\omega_{\mathrm{ad}_u^* w} \circ \omega_v^*),\\ &= \mathrm{tr}(\omega_w \circ \mathrm{ad}_v^* \circ \omega_u^*) - \mathrm{tr}(\omega_w \circ \mathrm{ad}_u^* \circ \omega_v^*),\\ \langle D[u, v]_\mathfrak{g}, w\rangle &= \mathrm{tr}(\omega_{[u,v]_\mathfrak{g}} \circ \omega_w^*)\\ &= \mathrm{tr}(\omega_u \circ \mathrm{ad}_v \circ \omega_w^*) - \mathrm{tr}(\omega_v \circ \mathrm{ad}_u \circ \omega_w^*).\end{aligned}$$

■

Proposition 3.3. *The Ricci curvature $\mathrm{ric}_\mathfrak{h}$ of $(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_\mathfrak{h})$ is given by*

$$\begin{aligned}\mathrm{ric}_\mathfrak{h}(u, v) &= \mathrm{ric}_\mathfrak{g}(u, v) - \frac{1}{2}\mathrm{tr}(\omega_u^* \circ \omega_v), \quad u, v \in \mathfrak{g},\\ \mathrm{ric}_\mathfrak{h}(x, y) &= -\frac{1}{4}\mathrm{tr}(S_x \circ S_y), \quad x, y \in Z(\mathfrak{h}),\\ \mathrm{ric}_\mathfrak{h}(u, x) &= -\frac{1}{4}\mathrm{tr}(J_u \circ S_x), \quad x \in Z(\mathfrak{h}), u \in \mathfrak{g},\end{aligned}$$

where $\mathrm{ric}_\mathfrak{g}$ is the Ricci curvature of $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g}, \langle\ ,\ \rangle_\mathfrak{g})$.

Proof. According to (5), for any $a, b \in \mathfrak{h}$,

$$\mathrm{ric}_\mathfrak{h}(a, b) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_a^\mathfrak{h} \circ (\mathrm{ad}_b^\mathfrak{h})^*) - \frac{1}{4}\mathrm{tr}(J_a^\mathfrak{h} \circ J_b^h),$$

where $\mathrm{ad}_a^h : \mathfrak{h} \longrightarrow \mathfrak{h}$, $b \mapsto [a, b]$ and $J_a^\mathfrak{h} : \mathfrak{h} \longrightarrow \mathfrak{h}$, $b \mapsto (\mathrm{ad}_b^\mathfrak{h})^*(a)$. The desired formula will be a consequence of this one and the following relations. For any $u \in \mathfrak{g}$, $x \in Z(\mathfrak{h})$, with respect to the splitting $\mathfrak{h} = \mathfrak{g} \oplus Z(\mathfrak{h})$, we have

$$\mathrm{ad}_u^\mathfrak{h} = \begin{pmatrix} \mathrm{ad}_u^\mathfrak{g} & 0 \\ \omega_u & 0 \end{pmatrix}, \; J_u^\mathfrak{h} = \begin{pmatrix} J_u^\mathfrak{g} & 0 \\ 0 & 0 \end{pmatrix}, \; J_x^\mathfrak{h} = \begin{pmatrix} S_x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}_x^\mathfrak{h} = 0.$$

■

Corollary 3.4. *$(\mathfrak{h}, [\ ,\ ], \langle\ ,\ \rangle_\mathfrak{h})$ is λ-Einstein if and only if for any $u, v \in \mathfrak{g}$ and $x, y \in Z(\mathfrak{h})$,*

$$\begin{aligned}\mathrm{ric}_\mathfrak{g}(u, v) = \lambda\langle u, v\rangle_\mathfrak{g} + \frac{1}{2}\mathrm{tr}(\omega_u^* \circ \omega_v),\ \mathrm{tr}(J_u \circ S_x) = 0\\ \text{and}\ \ \mathrm{tr}(S_x \circ S_y) = -4\lambda\langle x, y\rangle_z.\end{aligned} \tag{15}$$

Let us derive some consequences of Proposition 3.3 and Corollary 3.4. In what follows $\mathfrak{h}$ will be an Einstein Lorentzian nilpotent Lie algebra with nondegenerate center, we denote $[\ ,\ ]_\mathfrak{h}$ its Lie bracket, $\langle\ ,\ \rangle_\mathfrak{h}$ its Lorentzian product and $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g}, \langle\ ,\ \rangle_\mathfrak{g})$, $(Z(\mathfrak{h}), \langle\ ,\ \rangle_z)$ and $\omega \in Z^2(\mathfrak{g}, Z(\mathfrak{h}))$ its attributes.

Recall that a pseudo-Euclidean Lie algebra $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is called Ricci-soliton if there exists a constant $\lambda \in \mathbb{R}$ and a derivation D of $\mathfrak{g}$ such that $\mathrm{Ric}_\mathfrak{g} = \lambda \mathrm{Id}_\mathfrak{g} + D$. By combining Corollary 3.4 and Proposition 3.2 we get the following result.

Proposition 3.5. *Let $\mathfrak{h}$ be a Einstein Lorentzian nilpotent Lie algebra with Euclidean nondegenerate center. If ω satisfies* (14) *then $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\langle\ ,\ \rangle_{\mathfrak{g}})$ is Ricci-soliton.*

Proposition 3.6. *Let $\mathfrak{h}$ be a λ-Einstein Lorentzian nilpotent Lie algebra with non-degenerate center. If $\lambda\neq 0$ then the cohomology class of the attribute ω is non trivial. In particular, $H^2(\mathfrak{g},Z(\mathfrak{h}))\neq\{0\}$.*

Proof. Suppose that there exists $\alpha\in\mathfrak{g}$ such that, for any $u,v\in\mathfrak{g}$, $\omega(u,v)=-\alpha([u,v]_{\mathfrak{g}})$. Fix an orthonormal basis $\{e_1,\ldots,e_n\}$ of $\mathfrak{g}$ with $\langle e_1,e_1\rangle=-1$. For any $x\in Z(\mathfrak{h})$, we have:

$$\begin{aligned}\mathrm{tr}(S_x^2)&=\langle S_x(e_1),S_x(e_1)\rangle_{\mathfrak{g}}-\sum_{i=2}^n\langle S_x(e_i),S_x(e_i)\rangle_{\mathfrak{g}}\\&=\langle\omega_{e_1}^*(x),S_x(e_1)\rangle_{\mathfrak{g}}-\sum_{i=2}^n\langle\omega_{e_i}^*(x),S_x(e_i)\rangle_{\mathfrak{g}}\\&=-\langle\mathrm{ad}_{e_1}^*\circ\alpha^*(x),S_x(e_1)\rangle_{\mathfrak{g}}+\sum_{i=2}^n\langle\mathrm{ad}_{e_i}^*\circ\alpha^*(x),S_x(e_i)\rangle_{\mathfrak{g}}\\&=-\langle J_{\alpha^*(x)}(e_1),S_x(e_1)\rangle_{\mathfrak{g}}+\sum_{i=2}^n\langle J_{\alpha^*(x)}(e_i),S_x(e_i)\rangle_{\mathfrak{g}}\\&=-\mathrm{tr}(J_{\alpha^*(x)}\circ S_x).\end{aligned}$$

By virtue of Corollary 3.4, we get that $\lambda\langle x,x\rangle_z=0$ for any $x\in Z(\mathfrak{h})$ and hence $\lambda=0$. ■

Proposition 3.7. *Let $\mathfrak{h}$ be a λ-Einstein Lorentzian nilpotent Lie algebra with non-degenerate center. Then $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}$ is a non-degenerate Lorentzian subspace of $\mathfrak{g}$. Moreover, if $\mathfrak{h}$ is 3-step nilpotent and $\lambda\geq 0$ then $\mathrm{Z}(\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}$.*

Proof. According to [3, Corollary 3.3], $[\mathfrak{h},\mathfrak{h}]$ is nondegenerate Lorentzian and, one can easily see that $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}^{\perp}=[\mathfrak{h},\mathfrak{h}]^{\perp}\cap\mathfrak{g}$. Therefore $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}^{\perp}$ is nondegenerate Euclidean and hence $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}$ is nondegenerate Lorentzian.

Suppose now that $\mathfrak{h}$ is 3-step nilpotent. Then $\mathfrak{g}$ is 2-step nilpotent and therefore $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}\subset\mathrm{Z}(\mathfrak{g})$. Let $x\in\mathrm{Z}(\mathfrak{g})\cap[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}^{\perp}$. Since $\mathrm{ad}_x=0$ and $J_x=0$, by virtue of (5) , $\mathrm{Ric}_{\mathfrak{g}}(x)=0$. If $\lambda\geq 0$, the first equation of system (15) gives that:

$$0\leq\lambda\langle x,x\rangle=-\frac{1}{2}\mathrm{tr}(\omega_x^*\circ\omega_x)=Q.$$

Since ω is a 2-cocycle, $\omega(Z(\mathfrak{g}),[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}})=0$ and hence

$$Q=-\frac{1}{2}\sum_{i=1}^m\langle\omega(x,f_i),\omega(x,f_i)\rangle\leq 0$$

where $\{f_1,\ldots,f_m\}$ is an orthonormal basis of $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}^{\perp}$. It follows that $x\in\mathrm{Z}(\mathfrak{g})\cap\ker\omega$ and hence $x=0$ by virtue of (10). Thus $\mathrm{Z}(\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}$. ■

Theorem 3.8. *Let $\mathfrak{h}$ be a λ-Einstein Lorentzian 3-step nilpotent Lie algebra with nondegenerate center. Then $\lambda \geq 0$.*

Proof. According to (15), since $\mathfrak{h}$ is λ-Einstein then

$$\mathrm{Ric}_{\mathfrak{g}} = \lambda \mathrm{Id}_{\mathfrak{g}} + \frac{1}{2}D \quad \text{and} \quad \mathrm{tr}(S_x \circ S_y) = -4\lambda \langle x, y\rangle_z, \tag{16}$$

for any $x, y \in Z(\mathfrak{h})$. On the other hand, by virtue of Proposition 3.7, $[\mathfrak{g},\mathfrak{g}]$ is nondegenerate Lorentzian and hence $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus [\mathfrak{g},\mathfrak{g}]^\perp$. We choose an orthonormal basis $\mathbb{B}_0 = (e_1, \ldots, e_s)$ of $[\mathfrak{g},\mathfrak{g}]$ with $\langle e_1, e_1\rangle_{\mathfrak{g}} = -1$ and an orthonormal basis $\mathbb{B}_1 = (z_1, \ldots, z_p)$ of $Z(\mathfrak{h})$ and we consider the Lie structure endomorphisms $(J_1, \ldots, J_s)$ associated to $\mathbb{B}_0$ and given by (8) and $(S_1, \ldots, S_p)$ the ω-structure endomorphisms associated to $\mathbb{B}_1$ and given by (12).

Since $\mathfrak{g}$ is 2-step nilpotent then $[\mathfrak{g},\mathfrak{g}] \subset Z(\mathfrak{g})$, hence for any $i = 1, \ldots, s$ we have $J_i([\mathfrak{g},\mathfrak{g}]) = 0$. Moreover, J_i being skew-symmetric leaves $[\mathfrak{g},\mathfrak{g}]^\perp$ invariant and we shall denote its restriction to $[\mathfrak{g},\mathfrak{g}]^\perp$ by J_i as well. On the other hand, since ω is a 2-cocycle then $\omega(Z(\mathfrak{g}), [\mathfrak{g},\mathfrak{g}]) = 0$ and hence, by virtue of (12), for any $i = 1, \ldots, p$, $S_i([\mathfrak{g},\mathfrak{g}]) \subset [\mathfrak{g},\mathfrak{g}]^\perp$, we denote $B_i : [\mathfrak{g},\mathfrak{g}] \longrightarrow [\mathfrak{g},\mathfrak{g}]^\perp$ the resulting linear map. Since S_i is skew-symmetric, then for any $u \in [\mathfrak{g},\mathfrak{g}]^\perp$, $S_i u = -B_i^* u + D_i u$ where $D_i : [\mathfrak{g},\mathfrak{g}]^\perp \longrightarrow [\mathfrak{g},\mathfrak{g}]^\perp$ is skew-symmetric. By using (7), (9) and (13), we get that (16) is equivalent to

$$\begin{cases} -\frac{1}{2}J_1^2 + \frac{1}{2}\sum_{i=2}^{s} J_i^2 + \frac{1}{2}\sum_{i=1}^{p}(D_i^2 - B_i B_i^*) = \lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]^\perp}. \\ \sum_{i,j=1}^{s} \langle e_i, \ . \ \rangle \mathrm{tr}(J_i \circ J_j) e_j + 2\sum_{i=1}^{p} B_i^* B_i = -4\lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]}. \\ \mathrm{tr}(D_i D_j) - 2\mathrm{tr}(B_i^* B_j) = -4\lambda \delta_{ij}, \ i, j = 1, \ldots, p. \end{cases} \tag{17}$$

By taking the trace of the first two equations and using the third one we obtain that:

$$\sum_{i=1}^{p} \mathrm{tr}(D_i^2) = -4(2s + m + 3p)\lambda, \quad m = \dim[\mathfrak{g},\mathfrak{g}]^\perp.$$

But $[\mathfrak{g},\mathfrak{g}]^\perp$ is a Euclidean vector space and $D_i : [\mathfrak{g},\mathfrak{g}]^\perp \longrightarrow [\mathfrak{g},\mathfrak{g}]^\perp$ is skew-symmetric and hence $\mathrm{tr}(D_i^2) \leq 0$ which completes the proof. ∎

To sum up the results of this section, we reduced the study of Einstein Lorentzian k-step nilpotent Lie algebras to the study of a class of Lorentzian $(k-1)$-step nilpotent Lie algebras endowed with a 2-cocycle with values in a Euclidean vector space which in some cases can be Ricci-soliton. It is natural to give a name to this class of Lie algebras.

Definition 3.9. A pseudo-Euclidean Lie algebra $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle_{\mathfrak{g}})$ will be called *ω-quasi Einstein of type p* if there exists $\lambda \in \mathbb{R}$ and a 2-cocycle ω with values in a Euclidean vector space $(V, \langle\ ,\ \rangle_z)$ of dimension p such that $\ker\omega \cap Z(\mathfrak{g}) = \{0\}$ and

$$\mathrm{Ric}_{\mathfrak{g}} = \lambda \mathrm{Id}_{\mathfrak{g}} + \frac{1}{2}D, \quad \mathrm{tr}(S_x \circ S_y) = -4\lambda\langle x, y\rangle_z$$

where $S_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ denotes the ω-structure endomorphism corresponding to $x \in V$ and D is given by

$$\langle Du, v\rangle_{\mathfrak{g}} = \mathrm{tr}(\omega_u^* \circ \omega_v)$$

and $\omega_u : \mathfrak{g} \longrightarrow V,\ v \mapsto \omega(u, v)$.

4. ω-quasi Einstein Lorentzian 2-step nilpotent Lie algebras of type 1

In this section, having in mind Proposition 3.7 and Theorem 3.8, we give a complete description of ω-quasi Einstein Lorentzian 2-step nilpotent Lie algebras of type 1 with nondegenerate Lorentzian derived ideal and Einstein constant $\lambda \geq 0$ as an important step towards the determination of Einstein Lorentzian 3-step nilpotent Lie algebras with nondegenerate 1-dimensional center.

Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \langle\ ,\ \rangle_{\mathfrak{g}})$ be a 2-step nilpotent Lie algebra such that $Z(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is nondegenerate Lorentzian. Put $n = \dim[\mathfrak{g}, \mathfrak{g}]$ and $m = \dim[\mathfrak{g}, \mathfrak{g}]^{\perp}$.

Suppose that $\mathfrak{g}$ is ω-quasi Einstein of type 1 with Einstein constant $\lambda \geq 0$. Denote by $S : \mathfrak{g} \longrightarrow \mathfrak{g}$ the skew-symmetric endomorphism given by $\omega(u, v) = \langle Su, v\rangle_{\mathfrak{g}}$. Since ω is a 2-cocycle and $[\mathfrak{g}, \mathfrak{g}] \subset Z(\mathfrak{g})$ then $S([\mathfrak{g}, \mathfrak{g}]) \subset [\mathfrak{g}, \mathfrak{g}]^{\perp}$ leading to a linear map $B : [\mathfrak{g}, \mathfrak{g}] \longrightarrow [\mathfrak{g}, \mathfrak{g}]^{\perp}$. The condition $Z(\mathfrak{g}) \cap \ker \omega = \{0\}$ implies that B is injective. On the other hand, the skew-symmetry of S gives that, for any $u \in [\mathfrak{g}, \mathfrak{g}]^{\perp}$, $Su = -B^* u + Lu$ where L is a skew-symmetric endomorphism of $[\mathfrak{g}, \mathfrak{g}]^{\perp}$. Now consider the endomorphism D associated to ω and given by (11). According to (13), $D = -S^2$ and hence

$$Du = \begin{cases} B^*Bu - LBu \text{ if } u \in [\mathfrak{g}, \mathfrak{g}], \\ B^*Lu + BB^*u - L^2 u \text{ if } u \in [\mathfrak{g}, \mathfrak{g}]^{\perp}. \end{cases}$$

The fact that $\mathfrak{g}$ is ω-quasi Einstein is equivalent to

$$-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 - \frac{1}{2}D = \lambda \mathrm{Id}_{\mathfrak{g}}, \quad \mathrm{tr}(S^2) = -4\lambda, \tag{18}$$

where, by virtue of (7), $\mathrm{Ric}_{\mathfrak{g}} = -\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2$.

Let us proceed now to a crucial step which is not possible to perform when ω has it values in a vector space of dimension ≥ 2.

We consider the symmetric endomorphism on $[\mathfrak{g}, \mathfrak{g}]$ given by $A = B^*B$. Since B is injective and $[\mathfrak{g}, \mathfrak{g}]^{\perp}$ is nondegenerate Euclidean, we have $\langle Au, u\rangle_{\mathfrak{g}} > 0$ for any $u \in \mathfrak{g} \setminus \{0\}$. There are two types of nondiagonalizable symmetric endomorphisms on a Lorentzian vector space (see [15, p. 261-262]). Those which have an isotropic eigenvector or those which have two linearly orthogonal vectors (e, f) such that $\langle e, e\rangle = 1$, $\langle f, f\rangle = -1$, $T(e) = ae - bf$ and $T(f) = be + af$. The fact that A is positive definite prevents it to be of these types and hence A is diagonalizable in an orthonormal basis $\mathbb{B}_1 = (e_1, \ldots, e_n)$ of $[\mathfrak{g}, \mathfrak{g}]$ such that $\langle e_1, e_1\rangle_{\mathfrak{g}} = -1$. Let $(J_1, \ldots, J_n)$ be the Lie structure endomorphisms associated to $\mathbb{B}_1$. Note that the J_i vanishes on $[\mathfrak{g}, \mathfrak{g}] \subset Z(\mathfrak{g})$ and hence leaves invariant $[\mathfrak{g}, \mathfrak{g}]^{\perp}$. We denote the restriction of J_i to $[\mathfrak{g}, \mathfrak{g}]^{\perp}$ by J_i as well.

Using (7) and (9), we get that (18) is equivalent to

$$\begin{cases} -\frac{1}{2}J_1^2 + \frac{1}{2}\sum_{j=2}^{n} J_j^2 + \frac{1}{2}(L^2 - BB^*) = \lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]^\perp}, \\ -2B^*B - \sum_{i,j=1}^{n} \langle e_i, u\rangle \mathrm{tr}(J_i \circ J_j) e_j = 4\lambda \mathrm{Id}_{[\mathfrak{g},\mathfrak{g}]}, \\ \mathrm{tr}(L^2) - 2\mathrm{tr}(BB^*) = -4\lambda, \\ LB = 0. \end{cases} \tag{19}$$

Taking the trace of the first two equations and using the the third equation of (19) we get that :

$$\mathrm{tr}(L^2) = -4(2n+m+3)\lambda, \quad n = \dim[\mathfrak{g},\mathfrak{g}],\ m = \dim[\mathfrak{g},\mathfrak{g}]^\perp.$$

When $m = n$, $B : [\mathfrak{g},\mathfrak{g}] \longrightarrow [\mathfrak{g},\mathfrak{g}]^\perp$ is an isomorphism and therefore $LB = 0$ leads to $L = 0$ and by the previous equation $\lambda = 0$. We will show that this fact is still true in the general setting.

Put $\mathbb{B}_2 = (f_1, \ldots, f_n) = \left(\frac{B(e_1)}{|B(e_1)|}, \ldots, \frac{B(e_n)}{|B(e_n)|}\right)$ which is obviously an orthonormal basis of $\mathrm{Im}(B)$. Since $LB = 0$, L vanishes on $\mathrm{Im}(B)$ and leaves invariant $\mathrm{Im}(B)^\perp = \ker BB^*$. Thus $L(f_i) = 0$ and there exists an orthonormal basis

$$\mathbb{B}_3 = (g_1, h_1, \ldots, g_r, h_r, p_1, \ldots, p_s)$$

of $\ker BB^*$ such that

$$L(g_i) = \mu_i h_i,\ L(h_i) = -\mu_i g_i, L(p_j) = 0.$$

The basis $\mathbb{B}_1$ consists of eigenvectors of B^*B and hence the second relation in (19) is equivalent to

$$B^*B(e_i) = -\left(2\lambda + \frac{1}{2}\langle e_i, e_i\rangle_{\mathfrak{g}} \mathrm{tr}(J_i^2)\right) e_i,\ \mathrm{tr}(J_i \circ J_j) = 0,\ i,j = 1, \ldots, n, j \neq i.$$

On the other hand, we also have,

$$BB^*(f_i) = -\left(2\lambda + \frac{1}{2}\langle e_i, e_i\rangle_{\mathfrak{g}} \mathrm{tr}(J_i^2)\right) f_i, \quad i = 1, \ldots, n. \tag{20}$$

Summing up the above remarks, if M_i denotes the matrix of the restriction of J_i to $[\mathfrak{g},\mathfrak{g}]^\perp$ in the basis $\mathbb{B}_2 \cup \mathbb{B}_3$ then (18) implies that

$$\begin{aligned} M_1^2 - \sum_{k=2}^{n} M_k^2 &= \mathrm{Diag}\left(-\frac{1}{2}\mathrm{tr}(M_1^2), \frac{1}{2}\mathrm{tr}(M_2^2), \ldots, \frac{1}{2}\mathrm{tr}(M_n^2),\right. \\ &\quad \left. -(2\lambda + \mu_1^2), \ldots, -(2\lambda + \mu_r^2), -2\lambda, \ldots, -2\lambda\right). \end{aligned} \tag{21}$$

To study this equation, we need matrix analysis of Hermitian square matrices (see [11]). Let us recall one of the main theorems of this theory. A $m \times m$ Hermitian matrix A has real eigenvalues which can be ordered

$$\lambda_1(A) \leq \ldots \leq \lambda_m(A).$$

Theorem 4.1. [11] *Let $A, B \in \mathcal{M}_m(\mathbb{C})$ be two Hermitian matrices. Then for all $1 \leq k \leq m$:*

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_m(B).$$

Based on this theorem, the following lemma is a breakthrough in our study.

Lemma 4.2. *Let $M_1, \dots, M_n$ be a family of skew-symmetric $m \times m$ matrices with $2 \leq n \leq m$ and let $(v_1, \dots, v_{m-n})$ be a family of nonpositive real numbers such that:*

$$M_1^2 - \sum_{l=2}^{n} M_l^2 = \mathrm{Diag}\left(-\frac{1}{2}\mathrm{tr}(M_1^2), \frac{1}{2}\mathrm{tr}(M_2^2), \dots, \frac{1}{2}\mathrm{tr}(M_n^2), v_1, \dots, v_{m-n}\right). \tag{22}$$

Then $$(v_1, \dots, v_{m-n}) = (0, \dots, 0), \ \lambda_1\left(\sum_{l=2}^{n} M_l^2\right) = \sum_{l=2}^{n} \lambda_1(M_l^2).$$

Moreover, for any $i \in \{2, \dots, n\}$, $\mathrm{rank}(M_i) \leq 2$.

Proof. Denote by M the right-hand side of equation (22). By taking the trace of (22) we get:

$$\mathrm{tr}(M_1^2) - \sum_{l=2}^{n} \mathrm{tr}(M_l^2) = \frac{2}{3}\sum_{i=1}^{m-n} v_i \leq 0. \tag{23}$$

For $i = 1, \dots, n$, M_i^2 is the square of a skew-symmetric matrix so its eigenvalues are real nonpositive and satisfies

$$\lambda_{2k-1}(M_i^2) = \lambda_{2k}(M_i^2), \quad k \in \left\{1, \dots, \left[\frac{m}{2}\right]\right\}. \tag{24}$$

It is clear that $-\frac{1}{2}\mathrm{tr}(M_1^2)$ is the only nonnegative eigenvalue of M and therefore $\lambda_m(M) = -\frac{1}{2}\mathrm{tr}(M_1^2)$. Theorem 4.1 applied to (22) gives that:

$$\underbrace{\lambda_m(M) + \lambda_1\left(\sum_{l=2}^{n} M_l^2\right)}_{a} \leq \lambda_m(M_1^2) \leq \underbrace{\lambda_m(M) + \lambda_m\left(\sum_{l=2}^{n} M_l^2\right)}_{b}. \tag{25}$$

and

$$\underbrace{\lambda_{m-1}(M) + \lambda_1\left(\sum_{l=2}^{n} M_l^2\right)}_{c} \leq \lambda_{m-1}(M_1^2) \leq \underbrace{\lambda_{m-1}(M) + \lambda_m\left(\sum_{l=2}^{n} M_l^2\right)}_{d}. \tag{26}$$

Suppose that m is odd. In this case $\lambda_m(M_1^2) = 0$ and, by applying Theorem 4.1 inductively and using (24), we get that:

$$\frac{1}{2}\sum_{l=2}^{n} \mathrm{tr}(M_l^2) \leq \frac{1}{2}\sum_{l=2}^{n}(\lambda_1(M_l^2) + \lambda_2(M_l^2)) = \sum_{l=2}^{n} \lambda_1(M_l^2) \leq \lambda_1\left(\sum_{l=2}^{n} M_l^2\right).$$

As a consequence of this inequality and the fact that $\lambda_m(M) = -\frac{1}{2}\mathrm{tr}(M_1^2)$, we get

$$-\frac{1}{2}\mathrm{tr}(M_1^2) + \frac{1}{2}\sum_{l=2}^{n} \mathrm{tr}(M_l^2) \leq \lambda_m(M) + \lambda_1\left(\sum_{l=2}^{n} M_l^2\right) \overset{(25)}{\leq} \lambda_m(M_1^2) \leq 0,$$

This combined with (23) gives that $(v_1,\dots,v_{m-n})=(0,\dots,0)$. Suppose now that m is even. In this case, $\lambda_{m-1}(M_1^2)=\lambda_m(M_1^2)$ and it follows from (25) and (26) that $[a,b]\cap[c,d]\neq\emptyset$. But, we have obviously that $c\leq a$ and $d\leq b$ therefore $a\leq d$.

Thus
$$\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\leq\lambda_{m-1}(M)+\lambda_m\left(\sum_{l=2}^{n}M_l^2\right). \tag{27}$$

Since $\lambda_m(M)=-\frac{1}{2}\mathrm{tr}(M_1^2)$ then by using (23) we get that
$$\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)=-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right).$$

On other hand, by applying Theorem 4.1 once more, we get $\lambda_m(\sum_{l=2}^{n}M_l^2)\leq\sum_{l=2}^{n}\lambda_m(M_l^2)\leq 0$, moreover $\lambda_{m-1}(M)\leq 0$, so (27) implies that:
$$-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\leq 0, \tag{28}$$

Theorem 4.1 also implies that $\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\geq\sum_{l=2}^{n}\lambda_1(M_l^2)$ and hence
$$\begin{aligned}
&-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\\
&\qquad\geq-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\sum_{l=2}^{n}\lambda_1\left(M_l^2\right)\\
&\qquad\geq-\frac{1}{2}\sum_{l=2}^{n}\sum_{k=1}^{m}\lambda_k(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\sum_{l=2}^{n}\lambda_1(M_l^2)\\
&\qquad\overset{(24)}{\geq}-\sum_{l=2}^{n}\sum_{k=1}^{[\frac{m}{2}]}\lambda_{2k-1}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i+\sum_{l=2}^{n}\lambda_1(M_l^2)\\
&\qquad\geq-\sum_{l=2}^{n}\sum_{k=2}^{[\frac{m}{2}]}\lambda_{2k-1}(M_l^2)-\frac{1}{3}\sum_{i=1}^{m-n}v_i\geq 0.
\end{aligned}$$

Again we get that $v_i=0$ for all $1\leq i\leq m-n$.

To conclude, without any assumption on m, equation (25) gives
$$\begin{aligned}
0\geq\lambda_m(M)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)&=-\frac{1}{2}\mathrm{tr}(M_1^2)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)\\
&=-\frac{1}{2}\sum_{l=2}^{n}\mathrm{tr}(M_l^2)+\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)=\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)-\frac{1}{2}\sum_{l=2}^{n}\sum_{k=1}^{m}\lambda_k(M_l^2)\\
&=\lambda_1\left(\sum_{l=2}^{n}M_l^2\right)-\sum_{l=2}^{n}\lambda_1(M_l^2)-\frac{1}{2}\sum_{l=2}^{n}\sum_{k=3}^{m}\lambda_k(M_l^2)\geq 0
\end{aligned}$$

which means that $\lambda_1(\sum_{l=2}^n M_l^2) = \sum_{l=2}^n \lambda_1(M_l^2)$ and $\lambda_k(M_l^2) = 0$ for all $3 \le k \le m$ and $2 \le l \le n$. This completes the proof. ∎

If we apply this lemma to our study, we get that $\lambda = 0$, $L = 0$ and $(J_2, \dots, J_n)$ have rank 2 and satisfy $\lambda_1(\sum_{i=2}^n J_i^2) = \sum_{i=2}^n \lambda_1(J_i^2)$. The following lemma will give us a precise description of the endomorphisms $(J_2, \dots, J_n)$.

Lemma 4.3. *Let V be an m-dimensional Euclidean vector space and assume that $K_1, \dots, K_n : V \longrightarrow V$ is a family of skew-symmetric endomorphisms with $n < m$. Assume that* $\mathrm{rank}(K_i) = 2$, $\mathrm{tr}(K_i \circ K_j) = 0$ *for all $i \neq j$ and that*

$$\lambda_1(K) = \sum_{i=1}^n \lambda_1\left(K_i^2\right) \quad \textit{with} \quad K := \sum_{i=1}^n K_i^2.$$

Then we can find an orthonormal basis $\{u_0, \dots, u_n, v_1, \dots, v_{m-n-1}\}$ such that for all $1 \le i, j \le n$, $1 \le l \le m-n-1$:

$$K_i(u_0) = \alpha_i u_i,\ K_i(u_j) = -\delta_{ij}\alpha_i u_0 \quad \textit{and} \quad K_i(v_l) = 0.$$

Proof. Consider $E := \ker(K - \lambda_1(K)\mathrm{Id}_V)$ and for $i = 1, \dots, n$, denote $E_i := \mathrm{Im}(K_i)$. Note that E_i is a plan and there exists a $\alpha_i \in \mathbb{R} \setminus \{0\}$ such that for any $u \in E_i$, $K_i^2(u) = -\alpha_i^2 u$ and $\lambda_1(K_i^2) = -\alpha_i^2$. We claim that $E \subset \bigcap_{i=1}^n E_i$. Indeed, let $u \in E$ and for each $i = 1, \dots, n$ choose an orthonormal basis (e_i, f_i) of E_i and write

$$u = \langle u, e_i\rangle e_i + \langle u, f_i\rangle f_i + v_i \quad \text{and} \quad v_i \in E_i^\perp.$$

Since $\lambda_1(K) = -\alpha_1^2 - \dots - \alpha_n^2$, we get

$$-\sum_{i=1}^n \alpha_i^2 \langle u, u\rangle = \langle K^2(u), u\rangle = \sum_{i=1}^n \langle K_i^2(u), u\rangle.$$

But $K_i^2(u) = -\alpha_i^2(\langle u, e_i\rangle e_i + \langle u, f_i\rangle f_i)$ and hence

$$\langle K_i^2(u), u\rangle = -\alpha_i^2\left(\langle u, e_i\rangle^2 + \langle u, f_i\rangle^2\right).$$

So
$$0 = \sum_{i=1}^n \alpha_i^2(\langle u, u\rangle - \langle u, e_i\rangle^2 - \langle u, f_i\rangle^2) = \sum_{i=1}^n \alpha_i^2 \langle v_i, v_i\rangle = 0.$$

Thus $v_1, \dots, v_n = 0$ and the claim follows.

Choose $u_0 \in E$ such that $\langle u_0, u_0\rangle = 1$. Then clearly $(u_0, K_i(u_0))$ is an orthogonal basis of E_i. Complete this basis to get an orthonormal basis $(u_0, u_i, f_1, \dots, f_{m-2})$ of V with $u_i = \frac{1}{|K_i(u_0)|}K_i(u_0)$. We have $K_i(f_k) = 0$ for $k = 1, \dots, m-2$ and hence for $i, j \in \{1, \dots, n\}$ with $i \neq j$

$$\begin{aligned} 0 = \mathrm{tr}(K_i \circ K_j) &= -\langle K_j(u_0), K_i(u_0)\rangle - \langle K_j(u_i), K_i(u_i)\rangle \\ &= -\langle K_j(u_0), K_i(u_0)\rangle + \frac{\alpha_i^2}{|K_i(u_0)|}\langle K_j(u_i), u_0\rangle \\ &= -\left(1 + \frac{\alpha_i^2}{|K_i(u_0)|^2}\right)\langle K_j(u_0), K_i(u_0)\rangle. \end{aligned}$$

So the family $(u_0, K_1(u_0), \dots, K_n(u_0))$ is orthogonal, we orthonormalize it and complete it to get the desired basis. ∎

The relevance of the following lemma will appear later.

Lemma 4.4. *Consider the following system of matrix equations on* $\mathbb{R}^{2k}$:

$$\begin{cases} K^2 = P^{-1}AP + A \\ \alpha K = AP - P^{-1}A \end{cases} \tag{29}$$

where K *is an invertible skew-symmetric matrix,* P *an orthogonal matrix,* $A = \mathrm{diag}(-\alpha_1^2, \ldots, -\alpha_{2k}^2)$ *with* $\alpha_i \neq 0$ *and* $\alpha = \pm\sqrt{\alpha_1^2 + \cdots + \alpha_{2k}^2}$. *Then* $k = 1$, *in which case we get that:*

$$\begin{aligned} A &= \begin{pmatrix} -\alpha_1^2 & 0 \\ 0 & -\alpha_2^2 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \epsilon\sqrt{\alpha_1^2+\alpha_2^2} \\ -\epsilon\sqrt{\alpha_1^2+\alpha_2^2} & 0 \end{pmatrix} \quad \textit{and} \\ P &= \begin{pmatrix} 0 & \mp\epsilon \\ \pm\epsilon & 0 \end{pmatrix}, \quad \epsilon = \pm 1. \end{aligned} \tag{30}$$

Proof. We prove the Lemma by contradiction and assume that (K, A, P) is a solution of (29) and $k > 1$. To get a contradiction, we prove first that K^2 and A commute and hence A and $P^{-1}AP$ commute as well.

Let $\lambda_1 < \ldots < \lambda_r < 0$ be the different eigenvalues of K^2 and $E_1, \ldots, E_r$ the corresponding vector eigenspaces. Since K is skew-symmetric invertible and $\mathrm{tr}(K^2) = -2\alpha^2$, we have

$$\mathbb{R}^{2k} = E_1 \oplus \ldots \oplus E_r, \dim E_i = 2p_i \quad \text{and} \quad 2\sum_{i=1}^{r} p_i\lambda_i = -2\alpha^2. \tag{31}$$

According to (29), $P^{-1}AP + A$ and $AP - P^{-1}A$ commutes and hence

$$A(P + P^{-1})A = P^{-1}A(P + P^{-1})AP.$$

Moreover the first equation of system (29) implies that

$$K^4 = P^{-1}A^2P + A^2 + AP^{-1}AP + P^{-1}APA$$

and the second equation of (29) along with the preceding remarks give that:

$$\begin{aligned} \alpha^2K^2 &= APAP + P^{-1}AP^{-1}A - A^2 - P^{-1}A^2P \\ &= APAP + P^{-1}AP^{-1}A + AP^{-1}AP + P^{-1}APA - K^4 \\ &= (AP + AP^{-1})AP + P^{-1}A(P^{-1}A + PA) - K^4 \\ &= A(P + P^{-1})AP + P^{-1}A(P^{-1} + P)A - K^4 \\ &= A(P + P^{-1})A(P + P^{-1}) - K^4. \end{aligned}$$

Therefore we get that $K^2(K^2 + \alpha^2\mathrm{Id}) = A(P + P^{-1})A(P + P^{-1})$ which leads to:

$$A^{-1}K^2(K^2 + \alpha^2\mathrm{Id}) = (P + P^{-1})A(P + P^{-1}). \tag{32}$$

But $P^{-1} = P^t$ and the endomorphism at the right hand side of the previous equality is symmetric. This implies that A^{-1} and therefore A commutes with $K^2(K^2+\alpha^2\mathrm{Id})$.

We show now that A commutes with K^2. If K^2 is proportional to Id this is obviously true. Suppose that K^2 has at least two distinct eigenvalues, i.e., $r \geq 2$. For any $i, j \in \{1, \ldots, r\}$ and for any $u \in E_i, v \in E_j$, we have

$$\langle AK^2(K^2 + \alpha^2 \mathrm{Id})(u), v\rangle = \lambda_i(\lambda_i + \alpha^2)\langle Au, v\rangle$$

$$= \langle K^2(K^2 + \alpha^2 \mathrm{Id})A(u), v\rangle = \langle K^2(K^2 + \alpha^2 \mathrm{Id})v, A(u)\rangle = \lambda_j(\lambda_j + \alpha^2)\langle Au, v\rangle.$$

Thus, $(\lambda_i - \lambda_j)(\lambda_i + \lambda_j + \alpha^2)\langle Au, v\rangle = 0$. But from (31), we get

$$2(\lambda_i + \lambda_j + \alpha^2) = -2(p_i - 1)\lambda_i - 2(p_j - 1)\lambda_j - 2\sum_{l \neq i, l \neq j} p_l \lambda_l \leq 0.$$

If $k > 2$ then the last two relations implies that if $i \neq j$ then $\langle A(E_i), E_j\rangle = 0$ and hence $A(E_i) = E_i$ for $i = 1, \ldots, r$. So A commutes with K^2.

If $k = 2$ then $r = 2$, $\dim E_1 = \dim E_2 = 2$ and $\lambda_1 + \lambda_2 = -\alpha^2$. From $\mathbb{R}^{2k} = E_1 \oplus E_2$ one can deduce easily that $K^2(K^2 + \alpha^2 \mathrm{Id}) = -\lambda_1\lambda_2 \mathrm{Id}$ and by replacing in (32) we get

$$A(P + P^{-1}) = -\lambda_1\lambda_2(P + P^{-1})^{-1}A^{-1}.$$

Now for any $u \in \mathbb{R}^{2k}$ we get that:

$$0 \geq \langle A(P + P^{-1})(u), (P + P^{-1})(u)\rangle$$

$$= -\lambda_1\lambda_2\langle (P + P^{-1})^{-1}A^{-1}(u), (P + P^{-1})(u)\rangle = -\lambda_1\lambda_2\langle A^{-1}(u), u\rangle \geq 0,$$

so $\langle A^{-1}(u), u\rangle = 0$ which is impossible.

In conclusion A commutes with K^2 and hence A commutes with $P^{-1}AP$ so there exists an orthonormal basis $\{v_1, \ldots, v_{2k}\}$ of $\mathbb{R}^{2k}$ in which both A and $P^{-1}AP$ are diagonal. For any $i \in \{1, \ldots, 2k\}$

$$Av_i = -\alpha_i^2 v_i \quad \text{and} \quad P^{-1}AP(v_i) = -\alpha_{\sigma(i)}^2 v_i$$

for some permutation σ of $\{1, \ldots, 2k\}$. The second equation of (29) gives that, for any $i \in \{1, \ldots, 2k\}$,

$$\alpha K(v_i) = AP(v_i) - P^{-1}A(v_i) = -\alpha_{\sigma(i)}^2 P(v_i) + \alpha_i^2 P^{-1}(v_i).$$

Thus
$$\alpha^2\langle K(v_i), K(v_i)\rangle = \alpha_{\sigma(i)}^4 + \alpha_i^4 - 2\alpha_{\sigma(i)}^2\alpha_i^2\langle P^2(v_i), v_i\rangle. \tag{33}$$

First assume that $\sigma(i) = i$ for some $i \in \{1, \ldots, 2k\}$. It follows from the first equation of (29) that $-2\alpha_i^2$ is an eigenvalue of K^2 so it must have multiplicity greater than 2 and since $k > 1$ this leads to $\mathrm{tr}(K^2) < -4\alpha_i^2$. On the other hand, equation (33) and the first equation of (29) imply

$$\alpha^2\langle K(v_i), K(v_i)\rangle = 2\alpha_i^4(1 - \langle P^2(v_i), v_i\rangle) \text{ and } -\langle K(v_i), K(v_i)\rangle = \langle K^2(v_i), v_i\rangle = -2\alpha_i^2.$$

Combining these two equations we obtain that $\alpha^2 = \alpha_i^2(1 - \langle P^2(v_i), v_i\rangle)$, and the Cauchy-Schwarz inequality $|\langle P^2(v_i), v_i\rangle| \leq \|v_i\|\ \|P^2 v_i\| = 1$ imply the inequalities $0 \leq 1 - \langle P^2(v_i), v_i\rangle \leq 2$, which in turn give that $0 \leq \alpha^2 \leq 2\alpha_i^2$. Finally using that $\mathrm{tr}(K^2) = -2\alpha^2$ we conclude that $-4\alpha_i^2 \leq \mathrm{tr}(K^2)$, and we get a contradiction. Thus $\sigma(i) \neq i$ for all $i = 1, \ldots, 2k$.

From $\displaystyle\sum_{i=1}^{2k}\langle K(v_i),K(v_i)\rangle = -\mathrm{tr}(K^2)=2\alpha^2$ and equation (33) we get:

$$2\alpha^4 = 2\sum_{i=1}^{2k}\alpha_i^4 - 2\sum_{i=1}^{2k}\alpha_{\sigma(i)}^2\alpha_i^2\langle P^2(v_i),v_i\rangle.$$

Now

$$\begin{aligned}\alpha^4-\sum_{i=1}^{2k}\alpha_i^4 &= (\alpha_1^2+\ldots+\alpha_{2k}^2)^2-\sum_{i=1}^{2k}\alpha_i^4\\ &=\sum_{i\neq j}\alpha_i^2\alpha_j^2=\sum_{i=1}^{2k}\alpha_i^2\alpha_{\sigma(i)}^2+\sum_{j\neq i,j\neq\sigma(i)}\alpha_i^2\alpha_j^2.\end{aligned}$$

So we obtain:

$$0\le\sum_{j\neq i,\sigma(i)}\alpha_i^2\alpha_j^2=-\sum_{i=1}^{2k}\alpha_i^2\alpha_{\sigma(i)}^2(\langle P^2(v_i),v_i\rangle+1)\le 0,$$

the right hand side of the previous equality is negative as a consequence of the Cauchy-Schwarz inequality $|\langle P^2(v_i),v_i\rangle|\le\|v_i\|\ \|P^2v_i\|=1$ which implies that $0\le\langle P^2(v_i),v_i\rangle+1\le 2$.

Thus $\sum_{j\neq i,\sigma(i)}\alpha_i^2\alpha_j^2=0$, but this contradicts the fact that A is invertible. We conclude that $k=1$ and in this case we can put:

$$A=\begin{pmatrix}-\alpha_1^2 & 0\\ 0 & -\alpha_2^2\end{pmatrix},\quad K=\begin{pmatrix}0&\beta\\-\beta&0\end{pmatrix}\quad\text{and}\quad P=\begin{pmatrix}\cos(\theta)&-\sin(\theta)\\ \sin(\theta)&\cos(\theta)\end{pmatrix}.$$

We get that system (29) is equivalent to:

$$\begin{cases}\beta^2-\alpha_1^2-(\alpha_1^2\cos^2(\theta)+\alpha_2^2\sin^2(\theta))=0\\ \beta^2-\alpha_2^2-(\alpha_1^2\sin^2(\theta)+\alpha_2^2\cos^2(\theta))=0\\ \cos\theta\sin\theta(\alpha_2^2-\alpha_1^2)=0\\ \pm\beta\sqrt{\alpha_1^2+\alpha_2^2}-(\alpha_1^2+\alpha_2^2)\sin\theta=0.\end{cases}$$

By summing over the first two equations in the previous system and replacing in the last equation we obtain that $\beta=\epsilon\sqrt{\alpha_1^2+\alpha_2^2}$, $\sin\theta=\pm\epsilon$ and $\cos\theta=0$ with $\epsilon=\pm1$, which ends the proof. ■

We are now in possession of all the necessary ingredients to characterize ω-quasi Einstein Lorentzian 2-step nilpotent Lie algebras of type 1 as a key step toward the proof of Theorem 1.1.

Theorem 4.5. *Let $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle)$ be a Lorentzian 2-step nilpotent Lie algebra and assume that $\mathrm{Z}(\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]$ is non-degenerate Lorentzian and let $\omega\in\mathrm{Z}^2(\mathfrak{g},\mathbb{R})$. Then $\mathfrak{g}$ is ω-quasi Einstein of type 1 with positive Einstein constant λ if and only if $\lambda=0$ and, up to an isomorphism, $(\mathfrak{g},[\ ,\ ],\langle\ ,\ \rangle,\omega)$ has one of the following forms:*

(1) $\dim \mathfrak{g} = 5$ *and there exists an orthonormal basis* $\{e_1, e_2, u_1, u_2, u_3\}$ *of* $\mathfrak{g}$ *with* $\langle e_1, e_1\rangle = -1$ *such that the non vanishing Lie brackets and* ω*-products are given by:*

$$[u_1, u_2] = \alpha e_2,\ [u_2, u_3] = \pm\alpha e_1,\ \omega(e_2, u_3) = \epsilon\alpha,$$

$$\omega(e_1, u_1) = \mp\epsilon\alpha, \quad \alpha \neq 0,\ \epsilon = \pm 1. \tag{34}$$

(2) $\dim \mathfrak{g} = 6$ *and there exists an orthonormal basis* $\{e_1, e_2, e_3, u_1, u_2, u_3\}$ *of* $\mathfrak{g}$ *such that* $\langle e_1, e_1\rangle = -1$ *and the non vanishing Lie brackets and* ω*-products are given by:*

$$\begin{cases} [u_1, u_2] = \alpha_2 e_2,\ [u_1, u_3] = \alpha_3 e_3,\ [u_2, u_3] = \epsilon\alpha e_1, \\ \omega(e_2, u_3) = \mp\epsilon\alpha_2,\ \omega(e_3, u_2) = \pm\epsilon\alpha_3,\ \omega(e_1, u_1) = \pm\alpha, \end{cases} \tag{35}$$

where $\alpha_2, \alpha_3 \neq 0$, $\epsilon = \pm 1$ *and* $\alpha = \sqrt{\alpha_2^2 + \alpha_3^2}$.

Proof. We keep the notations from the beginning of section 4. The endomorphisms $(J_2, \ldots, J_n)$ have been shown to satisfy the hypothesis of Lemma 4.3, therefore we can find an orthonormal basis $(u_1, u_2, \ldots, u_n, v_1, \ldots, v_{m-n})$ of $[\mathfrak{g}, \mathfrak{g}]^\perp$ and $(\alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ such that, for all $2 \leq i, j \leq n$ and all $1 \leq k \leq m-n$,

$$J_i(u_1) = \alpha_i u_i,\ J_i(u_j) = -\delta_{ij}\alpha_i u_1,\ \alpha_i \neq 0 \quad \text{and} \quad J_i(v_k) = 0.$$

Put $J = \sum_{i=2}^n J_i^2$, it is clear that for all $2 \leq i \leq n$, $1 \leq k \leq m-n$,

$$\begin{aligned} J(u_1) &= -(\alpha_2^2 + \ldots + \alpha_n^2)u_1,\ J(u_i) = -\alpha_i^2 u_i, J(v_k) = 0, \\ \mathrm{tr}(J_1^2) &= -2(\alpha_2^2 + \ldots + \alpha_n^2) \quad \text{and} \quad \mathrm{tr}(J_i^2) = -2\alpha_i^2. \end{aligned} \tag{36}$$

Consider $\mathbb{B}_2 = (f_1, \ldots, f_n) := \left(\frac{B(e_1)}{|B(e_1)|}, \ldots, \frac{B(e_n)}{|B(e_n)|}\right)$. By virtue of equation (21), we get that for any $i = 2, \ldots, n$ and any $v \in \{f_1, \ldots, f_n\}^\perp$

$$J_1^2(f_1) = J(f_1) - \frac{1}{2}\mathrm{tr}(J_1^2)f_1,\ J_1^2(f_i) = J(f_i) + \frac{1}{2}\mathrm{tr}(J_i^2)f_i \quad \text{and} \quad J_1^2(v) - J(v) = 0. \tag{37}$$

Since $\lambda_1(J) = \frac{1}{2}\mathrm{tr}(J_1^2)$, we deduce that

$$\langle J_1(f_1), J_1(f_1)\rangle = -\langle J(f_1), f_1\rangle + \lambda_1(J) \leq 0$$

and hence
$$J_1(f_1) = 0 \quad \text{and} \quad J(f_1) = \lambda_1(J)f_1.$$

But (36) shows that the multiplicity of $\lambda_1(J)$ is equal to one and hence $f_1 = \pm u_1$. Let us show that the restriction of J_1 to $f_1^\perp$ is invertible. We have from (19) that

$$J_1^2 = J - BB^*,$$

and from (20) we get that the restriction of BB^* to $f_1^\perp$ is positive. So if $u \in f_1^\perp$ and $J_1 u = 0$ we get

$$\sum_{i=2}^n \langle J_i u, J_i u\rangle + \langle BB^*(u), u\rangle = 0.$$

Therefore $u \in \cap_{i=1}^n \ker J_i = Z(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ and hence $u = 0$. It follows that $J_1 : f_1^\perp \longrightarrow f_1^\perp$ is invertible and thus m is odd. In view of the last equation of (37) and the fact that $f_1 = \pm u_1$, we obtain that $J_1^2(\{f_1,\dots,f_n\}^\perp) \subset \mathrm{span}\{u_2,\dots,u_n\}$, the preceding remark then leads to $m-n \le n-1$ thus $m \le 2n-1$.

For convenience we set $w_i := B(e_i)$ for $i = 1,\dots,n$. From (20) we get

$$\langle w_i, w_i\rangle = -\frac{1}{2}\mathrm{tr}(J_i^2) \quad \text{and} \quad \langle w_i, w_j\rangle = 0, i \neq j.$$

So
$$BB^*(x) = -(\alpha_2^2 + \cdots + \alpha_n^2)\langle x, u_1\rangle u_1 + \sum_{i=2}^{n}\langle x, w_i\rangle w_i. \tag{38}$$

The fact that B defines a 2-cocycle is equivalent to

$$\sum_{i=1}^{n}\left(\langle J_i u, v\rangle w_i + \langle w_i, u\rangle J_i v - \langle w_i, v\rangle J_i u\right) = 0, \quad u, v \in [\mathfrak{g},\mathfrak{g}]^\perp.$$

If we apply this equation to $u = u_1$ we get

$$\langle w_1, u_1\rangle J_1 v = -\sum_{i=2}^{n}\left(\alpha_i\langle u_i, v\rangle w_i - \alpha_i\langle w_i, v\rangle u_i\right).$$

From the definition of w_1 we get that $\langle w_1, u_1\rangle = \pm\sqrt{\alpha_2^2 + \cdots + \alpha_n^2}$ and therefore the previous equation gives that

$$J_1 = \pm\frac{1}{\sqrt{\alpha_2^2 + \ldots + \alpha_n^2}}\sum_{i=2}^{n}\alpha_i u_i \wedge w_i. \tag{39}$$

Actually this is equivalent to B being a 2-cocycle. The expression of BB^* given in (38) leads to

$$J_1^2 - \sum_{i=2}^{q} J_i^2 = (\alpha_2^2 + \ldots + \alpha_n^2)\langle x, u_1\rangle u_1 - \sum_{i=2}^{n}\langle ., w_i\rangle w_i. \tag{40}$$

Put $a = \pm\frac{1}{\sqrt{\alpha_2^2+\ldots+\alpha_n^2}}$. Equation (39) on the other hand gives that

$$\begin{aligned} J_1 w_l &= a\alpha_l^3 u_l - a\sum_{i=2}^{n}\alpha_i\langle u_i, w_l\rangle w_i,\\ J_1 u_l &= -a\alpha_l w_l + a\sum_{i=2}^{n}\alpha_i\langle w_i, u_l\rangle u_i,\\ J_1 v_k &= a\sum_{i=2}^{n}\alpha_i\langle w_i, v_k\rangle u_i, \end{aligned} \tag{41}$$

Now using (40) and then (41), it is straightforward to check that

$$\langle J_1^2 v_k, v_k\rangle = -\sum_{l=2}^{n}\langle w_i, v_k\rangle^2 = a\sum_{i=2}^{n}\alpha_i\langle w_i, v_k\rangle\langle J_1 u_i, v_k\rangle = -a^2\sum_{l=2}^{n}\alpha_i^2\langle w_i, v_k\rangle^2.$$

So we conclude that

$$\sum_{l=2}^{n}(1 - a^2\alpha_i^2)\langle w_i, v_k\rangle^2 = 0.$$

Thus either $n=2$ or $n\geq 3$ and $\langle w_i, v_k\rangle = 0$ for $i=1,\dots,n$ and $v_k = 1,\dots,m-n$. So we get that either $n=2$ or $n\geq 3$ and $m=n$.

For $n=2$, we have $m=3$, (e_1,e_2) is an orthonormal basis of $[\mathfrak{g},\mathfrak{g}]$ with $\langle e_1,e_1\rangle = -1$, (u_1,u_2,v) an orthonormal basis of $[\mathfrak{g},\mathfrak{g}]^\perp$, $B(e_1)=au_1$, $B(e_2)=bv$,

$$J_2 = \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_1 = bu_2\wedge v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & -b & 0 \end{pmatrix} \quad \text{and} \quad a^2=b^2=\alpha^2.$$

This automatically leads to (34). For $n\geq 3$, we have $n=m=2k+1$. Recall that

$$[u,v] = \begin{cases} \displaystyle\sum_{i=1}^{n} \langle J_i(u), v\rangle e_i, & u,v\in[\mathfrak{g},\mathfrak{g}]^\perp \\ 0, & \text{otherwise,} \end{cases}$$

$$\omega(u,v) = \begin{cases} \langle B(u), v\rangle, & u\in[\mathfrak{g},\mathfrak{g}],\ v\in[\mathfrak{g},\mathfrak{g}]^\perp \\ 0, & \text{otherwise.} \end{cases}$$

From what have been shown, the only Lie brackets of $\mathfrak{g}$ that do not automatically vanish are

$$[u_1,u_i] = \langle J_i(u_1), u_i\rangle e_i = \alpha_i e_i \quad \text{and} \quad [u_i,u_j] = \langle J_1(u_i), u_j\rangle e_1 := \beta_{ij} e_1,$$

for $2\leq i,j\leq n$, moreover since J_1 is invertible on $u_1^\perp$ it follows that $K := (\beta_{ij})_{i,j}$ is a skew-symmetric invertible matrix. On the other hand, put $\hat{P}(f_i) := u_i$ for $2\leq i\leq m$ then $\hat{P} := (\hat{p}_{ij})_{i,j}$ is an orthogonal matrix and it is straightforward to see that $\langle B(e_i), u_j\rangle = \langle B(e_i), \hat{P}(f_j)\rangle = \epsilon_i \hat{p}_{ji}\alpha_i$ with $\epsilon_i = \pm 1$, it is clear that $P = (\epsilon_j \hat{p}_{ij})_{i,j}$ is an orthogonal matrix as well. Next since $f_1 = \pm u_1$ we get that:

$$\langle B(e_1), u_1\rangle = \pm\sqrt{\alpha_2^2+\cdots+\alpha_n^2}.$$

Finally in these notations notice that $J_1^2 - \sum_{i=2}^n J_i^2 = -BB^*$ is equivalent to $K^2 = P^{-1}AP + A$ with $A = \mathrm{diag}(-\alpha_2^2,\dots,-\alpha_n^2)$ and the cocycle condition $\oint\langle B([u,v]), w\rangle = 0$ is equivalent to $\pm\alpha K = AP - P^{-1}A$ where $\alpha = \sqrt{\alpha_2^2+\cdots+\alpha_n^2}$. This is exactly the situation of Lemma 4.4 and therefore $k=1$, i.e $n=m=3$ and thus $\dim\mathfrak{g}=6$, furthermore in view of (30) we get that the Lie algebra structure of $\mathfrak{g}$ is given by (35). This ends the proof. ∎

Following the discussion of section 3 we get as a consequence of the preceding Theorem that a Lorentzian 3-step nilpotent Lie algebras $(\mathfrak{h}, \langle\ ,\ \rangle)$ with non-degenerate 1-dimensional center is Einstein if and only if it is Ricci-flat and has one of the following forms:

1. Either $\dim\mathfrak{h} = 6$ in which case $\dim[\mathfrak{h},\mathfrak{h}] = \mathrm{codim}[\mathfrak{h},\mathfrak{h}] = 3$ and there exists an orthonormal basis $\{x,e_1,e_2,u_1,u_2,u_3\}$ of $\mathfrak{h}$ with $\langle e_1,e_1\rangle = -1$ such that the Lie algebra structure is given by:

$$[u_1,u_2]=\alpha e_2,\ [u_2,u_3]=\pm\alpha e_1,\ [e_2,u_3]=\alpha x,\ [e_1,u_1]=\mp\alpha x,\quad \alpha\neq 0. \tag{42}$$

$$[u_1,u_2]=\alpha e_2,\ [u_2,u_3]=\pm\alpha e_1,\ [e_2,u_3]=-\alpha x,\ [e_1,u_1]=\pm\alpha x,\quad \alpha\neq 0. \tag{43}$$

2. $\dim\mathfrak{h} = 7$ in which case $\dim[\mathfrak{h},\mathfrak{h}] = \operatorname{codim}[\mathfrak{h},\mathfrak{h}]+1 = 4$. Moreover there exists an orthonormal basis $\{x, e_1, e_2, e_3, u_1, u_2, u_3\}$ of $\mathfrak{h}$ such that $\langle e_1, e_1\rangle = -1$ and in which the Lie algebra structure is given by:

$$[u_1, u_2] = \alpha_2 e_2,\ [u_1, u_3] = \alpha_3 e_3,\ [u_2, u_3] = \epsilon\alpha e_1,$$
$$[e_2, u_3] = \mp\epsilon\alpha_2 x,\ [e_3, u_2] = \pm\epsilon\alpha_3 x,\ [e_1, u_1] = \pm\alpha x, \tag{44}$$

where $\alpha = \sqrt{\alpha_2^2 + \alpha_3^2}$.

Proof of the Main Theorem. In case 1, the Lie algebra structure $[\ ,\ ]$ of $\mathfrak{h}$ has one of the forms given by either (42) or (43). It is clear that (43) can be obtained from (42) simply by replacing u_3 with $-u_3$, for this reason it suffices to treat the case where $\mathfrak{h}$ is given by (42). Put:

$$f_1 = u_2,\ f_2 = u_3 + u_1,\ f_3 = u_3 - u_1,\ f_4 = \pm\alpha e_1 - \alpha e_2,\ f_5 = \pm\alpha e_1 + \alpha e_2,\ f_6 = 2\alpha^2 x.$$

Then we can easily see that:

$$[f_1, f_2] = f_4,\ [f_1, f_3] = f_5,\ [f_2, f_4] = f_6,\ [f_3, f_5] = -f_6,$$
$$[f_2, f_3] = [f_1, f_4] = [f_1, f_5] = [f_2, f_5] = [f_3, f_4] = [f_4, f_5] = [f_i, f_6] = 0.$$

Thus $\mathfrak{h} \simeq \mathrm{L}_{6,19}(-1)$ and the metric $\langle\ ,\ \rangle$ is represented in the basis $\{f_1, \ldots, f_6\}$ of $\mathfrak{h}$ by the expression (1). For case 2, when $\mathfrak{h}$ is given by (44) we can put:

$$f_1 := u_1,\ f_2 := u_2,\ f_3 := u_3,\ f_4 := \epsilon\sqrt{\alpha_2^2 + \alpha_3^2}e_1,$$
$$f_5 := \alpha_2 e_2,\ f_6 = \alpha_3 e_3,\ f_7 := \pm\epsilon(\alpha_2^2 + \alpha_3^2).$$

Then the Lie algebra structure of $\mathfrak{h}$ is given by (2) with $r = \frac{\alpha_2^2}{\alpha_2^2+\alpha_3^2}$. Moreover if we set $a = \alpha_2^2 + \alpha_3^2$ then we get that $\langle\ ,\ \rangle$ is given by (3). ■

We end our paper by some examples of Einstein Lorentzian nilpotent Lie algebras with non-degenerate center of dimension greater that one, the goal is to illustrate that such Lie algebras do occur even in the 3-step nilpotent case. This gives motivation for a future investigation.

Example 4.6. Let $\mathfrak{h}$ be the 8-dimensional nilpotent Lie algebra with Lie bracket $[\ ,\ ]$ given in a basis $\mathbb{B} = \{e_1, \ldots, e_8\}$ by:

$$\begin{cases} [e_1, e_2] = -4\sqrt{3}e_3,\ [e_1, e_3] = \sqrt{\frac{5}{2}}e_4,\ [e_1, e_4] = -2\sqrt{3}e_8,\ [e_1, e_5] = 3\sqrt{\frac{7}{2}}e_6, \\ [e_1, e_6] = -4\sqrt{2}e_7,\ [e_2, e_3] = -\sqrt{\frac{5}{2}}e_5,\ [e_2, e_4] = -3\sqrt{\frac{7}{2}}e_6,\ [e_2, e_5] = -2\sqrt{3}e_7, \\ [e_2, e_6] = -4\sqrt{2}e_8,\ [e_3, e_4] = -\sqrt{21}e_7,\ [e_3, e_5] = -\sqrt{21}e_8. \end{cases}$$

One can define a Lorentzian inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}$ by requiring $\mathbb{B}$ to be an orthonormal basis with $\langle e_6, e_6\rangle = -1$. Then it is easy to see that $\mathrm{Z}(\mathfrak{h}) = \operatorname{span}\{e_7, e_8\}$ hence non-degenerate with respect to $\langle\ ,\ \rangle$. Moreover a straightforward computation shows that $(\mathfrak{h}, \langle\ ,\ \rangle)$ is Einstein with nonvanishing scalar curvature. This example was first given in [4].

Example 4.7. Let $\langle\ ,\ \rangle$ be a Lorentzian metric on $\mathbb{R}^7$ and $\{e_1,\dots,e_7\}$ an orthonormal basis with respect to $\langle\ ,\ \rangle$ such that $\langle e_1,e_1\rangle=-1$. Define the Lie bracket $[\ ,\ ]$ by setting:

$$\begin{cases}[e_1,e_3]=\sqrt{2}e_7,\ [e_2,e_4]=\sqrt{2}e_7,\ [e_4,e_5]=-e_1,\ [e_4,e_6]=-e_1,\\ [e_3,e_5]=-e_2,\ [e_3,e_6]=-e_2.\end{cases}$$

Put $\mathfrak{h}:=(\mathbb{R}^{10},[\ ,\ ])$, then it is straightforward to check that $(\mathfrak{h},\langle\ ,\ \rangle)$ is a Ricci-flat 3-step nilpotent Lie algebra with $\mathrm{Z}(\mathfrak{h})=\mathrm{span}\{e_7,e_5-e_6\}$, therefore $\mathfrak{h}$ has non-degenerate center.

Example 4.8. Let $\langle\ ,\ \rangle$ be a Lorentzian metric on $\mathbb{R}^{10}$ and $\{e_1,\dots,e_{10}\}$ an orthonormal basis with respect to $\langle\ ,\ \rangle$ such that $\langle e_5,e_5\rangle=-1$. Choose $p,r\in\mathbb{R}$ such that $p,r\neq 0$ and define on $\mathbb{R}^{10}$ the Lie bracket $[\ ,\ ]$ given by:

$$\begin{cases}[e_1,e_3]=-\sqrt{p^2+r^2}e_5,\ [e_1,e_4]=-\sqrt{p^2+r^2}e_6,\ [e_2,e_4]=-\sqrt{p^2+r^2}e_5,\\ [e_2,e_3]=-\sqrt{p^2+r^2}e_6,\ [e_5,e_1]=pe_7,\ [e_5,e_2]=pe_8,\ [e_5,e_3]=re_9,\\ [e_5,e_4]=re_{10},\ [e_6,e_1]=pe_8,\ [e_6,e_2]=pe_7,\ [e_6,e_3]=re_{10},\ [e_6,e_4]=re_9.\end{cases} \tag{45}$$

Put $\mathfrak{h}:=(\mathbb{R}^{10},[\ ,\ ])$, then it is straightforward to check that $(\mathfrak{h},\langle\ ,\ \rangle)$ is a Ricci-flat 3-step nilpotent Lie algebra with $\mathrm{Z}(\mathfrak{h})=\mathrm{span}\{e_7,e_8,e_9,e_{10}\}$, therefore $\mathfrak{h}$ has non-degenerate center.

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