

On Epimorphisms in some Categories of Infinite-Dimensional Lie Groups

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Abstract. Let X be a smooth compact connected manifold. Let $G = \text{Diff } X$ be the group of diffeomorphisms of X , equipped with the C^∞ -topology, and let H be the stabilizer of some point in X . Then the inclusion $H \rightarrow G$, which is a morphism of two regular Fréchet-Lie groups, is an epimorphism in the category of smooth Lie groups modelled on complete locally convex spaces. At the same time, in the latter category, epimorphisms between finite dimensional Lie groups have dense range. We also prove that if G is a Banach-Lie group and H is a proper closed subgroup, the inclusion $H \rightarrow G$ is not an epimorphism in the category of Hausdorff topological groups.

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1. Introduction

A morphism $f : X \rightarrow Y$ in a category \mathbf{K} is an *epimorphism* if for any object Z the mapping $f^* : \text{Mor}(Y, Z) \rightarrow \text{Mor}(X, Z)$, defined by $f^*(g) = gf$, is injective.

It is known that epimorphisms in the category of Hausdorff topological groups need not have dense range [20, 21, 22]. This result had in its time answered a long-standing open problem, see [16] for a survey of the previous state of knowledge. It was already known by that time that in the category of locally compact groups epimorphisms between finite dimensional Lie groups need not have dense range (Kallman, unpublished, see [16], Remark 2.10), however, epimorphisms between locally compact groups in the wider category of all topological groups must have a dense range ([16], Thm. 2.9).

Those results prompt us to investigate intermediate categories of – possibly infinite-dimensional – Lie groups, more precisely, group objects in the category of C^∞ manifolds modelled on complete locally convex spaces (where we follow the approach by Neeb [14]). We call such groups *locally convex Lie groups* and denote the category of such groups by **LCLG**. We show that within this, very wide, category there exist epimorphisms even between regular Fréchet-Lie groups without dense range. A *Fréchet-Lie group*, or an *FL-group*, is a locally convex group modelled on a Fréchet space (= a complete metrizable locally convex space). At the same time, we strengthen Nummela’s result above by observing that already in the category of

regular Fréchet-Lie groups (viewed generally as the best-behaved class after that of Banach-Lie groups, see [17]) an epimorphism between connected finite dimensional Lie groups must have a dense range.

If \mathbf{K} is the category of sets, or of groups, or of abelian groups, then epimorphisms in \mathbf{K} are precisely surjective morphisms. If \mathbf{K} is the category of abelian Hausdorff topological groups or of Hausdorff topological vector spaces, then a morphism $f : X \rightarrow Y$ is an epimorphism if and only if $f(X)$ is dense in Y . K. H. Hofmann asked in the 1960s whether the same is true for the category \mathbf{TG} of all Hausdorff topological groups (not necessarily abelian). The answer is negative. For example, let X be a compact connected topological manifold, let $G = \text{Homeo } X$ be the group of all self-homeomorphisms of X , equipped with the compact-open topology, and let $H = \{f \in G : f(p) = p\}$ be the stability subgroup at some point $p \in X$. Then the inclusion $H \rightarrow G$ is an epimorphism in \mathbf{TG} [21].

Now let X be a compact smooth manifold, and let $G = \text{Diff } X$ be the group of all diffeomorphisms of X . Then G has a natural structure of a regular Fréchet-Lie group (where regularity implies in particular the existence of an exponential map, although we will never use it), see [13, 10]; [17], Ch. VI, §2; [14], Ex. II.3.14. The corresponding topology on G , the C^∞ -topology, is the topology of uniform convergence of all derivatives. Let $p \in X$, and let $H = \{f \in G : f(p) = p\}$ be the stability subgroup at p . The group H also has a natural regular Fréchet-Lie group structure, see [17], p. 145, Th. 4.5. In view of the result cited above, it is natural to ask if the inclusion $H \rightarrow G$ is an epimorphism in the category of the regular Fréchet-Lie groups, or indeed in the wider category of all locally convex Lie groups. Our main result answers this question in the positive.

Theorem 1.1. *Let X be a compact connected smooth manifold, let $G = \text{Diff } X$, and let $H \subset G$ be the stability subgroup at some point $p \in X$. Then the inclusion $H \rightarrow G$ is an epimorphism in the category \mathbf{LCLG} of Lie groups modelled on complete locally convex spaces.*

Since H is a proper closed subgroup of G , it follows that epimorphisms in the category of locally convex Lie groups need not have dense range, even between regular Fréchet-Lie groups.

At the same time, the inclusion $H \rightarrow G$ of Theorem 1.1 is not an epimorphism in the category \mathbf{TG} (Section 2, Remark 2). In other words, any two Lie group morphisms $f, g : G \rightarrow K$ to a locally convex Lie group K that agree on H must be equal, but there exist distinct \mathbf{TG} -morphisms $f, g : G \rightarrow K$ to a Hausdorff topological group K that agree on H .

In Section 2 we prove a criterion for a morphism to be an epimorphism in \mathbf{TG} (Theorem 2.1) and observe that Theorem 1.1 follows from the following assertion:

(*) *Use the notation of Theorem 1.1. Every G -equivariant smooth map from X to a locally convex Lie G -group is constant.*

The notion of a locally convex G -group is defined in the beginning of Section 2. It is a locally convex group equipped with a smooth action of G . We prove the assertion (*) in Section 3.

We also note that if $f: G \rightarrow H$ is a morphism between finite dimensional Lie groups which is an epimorphism in the category of locally convex groups (or already in the smaller category of regular Fréchet-Lie groups, if G is connected), then f must have dense range in H . Another new result states that any morphism $f: G \rightarrow H$ between Banach-Lie groups that is an epimorphism in the category **TG** of Hausdorff topological groups must have dense range. Even more, if H is a proper closed subgroup (not necessarily a Banach-Lie subgroup) of a Banach-Lie group G , then the inclusion $H \rightarrow G$ is not an epimorphism of Hausdorff topological groups.

Finally, the anonymous referee has kindly provided an example of an embedding of connected finite dimensional Lie groups (the subgroup of upper triangular matrices in $SL_2(\mathbb{R})$) that is an epimorphism of Banach-Lie groups.

A number of unanswered questions are collected at the end of the article.

2. Criterion for an inclusion to be an epimorphism

Let G and F be two topological groups. Let us say that F is a G -group if G acts continuously on F by automorphisms, that is, a continuous mapping $\sigma: G \times F \rightarrow F$ is given such that for every $g \in G$ the map $\sigma_g: F \rightarrow F$ defined by $\sigma_g(x) = \sigma(g, x)$ is an automorphism of F and the mapping $g \mapsto \sigma_g$ is a group homomorphism. In this situation, the semidirect product $G \rtimes_{\sigma} F$ can be defined, which is again a topological group. We shall usually write simply gx instead of $\sigma(g, x)$.

Similarly, if G and F are locally convex Lie groups (in the sense of [14]), we say that F is a *locally convex* G -group if, as above, an action $\sigma: G \times F \rightarrow F$ is given which is in addition a C^{∞} -mapping between manifolds. In this case, the semidirect product $G \rtimes_{\sigma} F$ has a natural locally convex Lie group structure. If both G and F are Fréchet-Lie groups, so is the semidirect product.

For a subset A of a group G we denote by $C_G(A)$ the centralizer of A in G , that is, the subgroup $\{x \in G : xa = ax \text{ for every } a \in A\}$.

Theorem 2.1. *Let $i: H \rightarrow G$ be a morphism in the category **TG** of Hausdorff topological groups. The following are equivalent:*

- (1) i is an epimorphism in **TG**;
- (2) $C_K(fi(H)) = C_K(f(G))$ for any morphism $f: G \rightarrow K$ in **TG**;
- (3) if F is a topological G -group and $x \in F$ is $i(H)$ -fixed (that is, $hx = x$ for every $h \in i(H)$), then x is G -fixed.

If H is a closed subgroup of G and $i: H \rightarrow G$ is the embedding, these three conditions are also equivalent to the fourth:

- (4) for every topological G -group F every G -equivariant continuous mapping $f: G/H \rightarrow F$ is constant.

Proof. Clearly a morphism $i: H \rightarrow G$ is an epimorphism in **TG** if and only if the inclusion $i(H) \rightarrow G$ is. Hence without loss of generality we may assume that H is a subgroup of G and $i: H \rightarrow G$ is the inclusion.

We show that not (1) \Rightarrow not (2). By definition, the inclusion $H \rightarrow G$ is not an epimorphism if and only if there exist a Hausdorff topological group L and two distinct morphisms $g, h: G \rightarrow L$ which agree on H . Let $C = \{1, a\}$ be a cyclic

group of order 2. The group C acts on $L \times L$ by permutations of coordinates, let $K = C \rtimes (L \times L)$ be the corresponding semidirect product. Let $j : L \times L \rightarrow K$ be the natural embedding, and let $b \in K$ be the image of $a \in C$ under the natural embedding $C \rightarrow K$. If $x, y \in L$, then $j(x, y)$ commutes with b if and only if $x = y$. Define a morphism $f : G \rightarrow K$ by $f(x) = j(g(x), h(x))$. Then b commutes with $f(H)$ but not with $f(G)$. Thus $C_K(f(H)) \neq C_K(f(G))$.

We show that not (2) \Rightarrow not (1). Suppose $f : G \rightarrow K$ is a morphism such that some $b \in K$ commutes with $f(H)$ but not with $f(G)$. Let σ be the inner automorphism of K corresponding to b . Then f and σf are distinct morphisms of G into K which agree on H . Thus the inclusion of H into G is not an epimorphism.

We show that (2) \Rightarrow (3). Let F be a G -group, and let $x \in F$ be an H -fixed element. Let K be the semidirect product of G and F , and let $i : G \rightarrow K$ and $j : F \rightarrow K$ be the canonical embeddings. If the condition (2) holds, then $C_K(i(H)) = C_K(i(G))$. Since x is H -fixed, we have $j(x) \in C_K(i(H))$. It follows that $x \in C_K(i(G))$. Hence x is G -fixed.

We show that (3) \Rightarrow (2). Let $f : G \rightarrow K$ be a continuous homomorphism, and let $x \in C_K(f(H))$. The group K can be considered as a G -group, equipped with the G -action defined by $(g, k) \mapsto f(g)kf(g)^{-1}$. The element x is H -fixed. If the condition (3) holds, then x is also G -fixed, which means that $x \in C_K(f(G))$.

Finally, the equivalence (3) \Leftrightarrow (4) is clear, since for any G -group F there is a natural one-to-one correspondence between H -fixed elements of F and G -equivariant morphisms $f : G/H \rightarrow F$, which assigns to every H -fixed element $x \in F$ the morphism $gH \mapsto gx$ of G/H to F . \blacksquare

Theorem 2.2. *Let $i : H \rightarrow G$ be a morphism in the category **LCLG** of locally convex Lie groups. The following are equivalent:*

- (1) i is an epimorphism in **LCLG**;
- (2) $C_K(fi(H)) = C_K(f(G))$ for any morphism $f : G \rightarrow K$ in **LCLG**;
- (3) if F is a locally convex Lie G -group and $x \in F$ is $i(H)$ -fixed (that is, $hx = x$ for every $h \in i(H)$), then x is G -fixed.

The proof is quite similar to the proof of Theorem 2.1 and hence omitted. The essential point is that semidirect products exist in **LCLG**. We did not include the analogue of condition (4) of Theorem 2.1 in order to avoid possible ambiguities of the notion ‘‘locally convex Lie subgroup of a locally convex Lie group’’. However, if X is a compact smooth manifold, $G = \text{Diff } X$ and H is the stabilizer of a point $p \in X$, as in Theorem 1.1, the quotient G/H is well-defined in the category of manifolds modelled on (complete) locally convex spaces and can be identified with X . This means that for every manifold M modelled on a (complete) locally convex space there is a one-to-one correspondence between smooth maps $X \rightarrow M$ and those smooth maps $G \rightarrow M$ that are constant on cosets gH . Thus we can apply the last paragraph of the proof of Theorem 1.1 (the equivalence (3) \Leftrightarrow (4)) in this situation, and we conclude that the fact that the embedding $H \rightarrow G$ is an epimorphism in the category **LCLG** is equivalent to the following:

- (*) *If X is a compact connected smooth manifold, $G = \text{Diff } X$, and F is a locally convex Lie G -group, then every G -equivariant smooth mapping $j : X \rightarrow F$ is constant.*

Remark 2.3. (1) Let X be a compact connected manifold and $G = \text{Homeo}(X)$. Let $H \subset G$ be the stability subgroup at some point of X . Then the inclusion $H \rightarrow G$ is an epimorphism in **TG** [21]. In virtue of Theorem 2.1, this assertion is equivalent to the following:

For any topological G -group F any G -equivariant mapping $j : X \rightarrow F$ is constant.

The latter was proved in [12, Example 3.7] (for the case X is a cube or a sphere, but the general case can be proved by the same method, Theorem 3.5 of [12] applies). Thus, as noted in [18, 19], the solution of the epimorphism problem for Hausdorff topological groups obtained in [20, 21] can be deduced from Megrelishvili’s results.

We sketch the proof of the fact that $j : X \rightarrow F$ (as above) must be constant. Let $x, y \in X$. Connect x and y by a smooth arc in X , and pick some points $a_0 = x, b_0, a_1, b_1, \dots, a_n, b_n = y$ going along the arc so that we get a fine partition of the arc. If every b_{i-1} is very close to a_i , the element

$$h = j(a_0)j(b_0)^{-1}j(a_1)j(b_1)^{-1} \dots j(a_n)j(b_n)^{-1}$$

of the group F is very close to $j(x)j(y)^{-1}$. On the other hand, we can find $g \in G$ close to identity such that $g(a_i)$ is very close to $g(b_i)$ ⁽¹⁾, and then

$$gh = j(ga_0)j(gb_0)^{-1}j(ga_1)j(gb_1)^{-1} \dots j(ga_n)j(gb_n)^{-1}$$

is close to e_F , the identity of F . Thus the element $c = j(x)j(y)^{-1}$ of F has the following property: we can find h close to c and $g \in G$ close to e_G so that gh is close to e_F . It follows that $c = e_F$ and $j(x) = j(y)$.

(2) If $G = \text{Diff } X$ and $H \subset G$ are as in Theorem 1.1, the inclusion $H \rightarrow G$ is *not* an epimorphism in the category **TG**. Indeed, according to Theorem 2.1, it suffices to construct a topological vector space V , a jointly continuous linear representation of G on V , and a non-constant G -equivariant map from X to V . We can take for V the space generated by points of X , that is, the space of measures on X with a finite support. Let us describe the topology on V that suits our purposes.

Let V_0 be the hyperplane in V consisting of all measures of total mass zero. It will be sufficient to describe a locally convex topology on V_0 such that the action of G on V_0 is jointly continuous. Every metric d on X gives rise to the “transportation metric” (also known as the Kantorovich-Rubinstein metric, and under many other names, see [23]) \bar{d} on V_0 defined by

$$\bar{d}(v, 0) = \inf \left\{ \sum |c_i|d(x_i, y_i) : v = \sum c_i(x_i - y_i) \right\}, \quad c_i \in \mathbb{R}, \quad x_i, y_i \in X$$

and $\bar{d}(u, v) = \bar{d}(u - v, 0)$. Consider a smooth Riemannian metric d on X and the topology on V_0 generated by \bar{d} . If $g \in G$ is close to identity in the C^1 -topology, g is a C -Lipschitz transformation of (X, d) , where $C > 1$ is a given constant. The joint continuity of the action of G on V_0 easily follows (Proposition 6.2).

Comparing this remark with the previous one, one can conclude that if the embedding $H \rightarrow G$ in the category **TG** is an epimorphism, then the action of G on G/H is “far from being smooth” in a certain sense.

¹A neighborhood of our arc in X can be identified with a Euclidean space so that the arc corresponds to a straight line segment; in that case the existence of g is geometrically obvious.

(3) Epimorphisms in the the category **AlgGr** of linear algebraic groups over a given algebraically closed field need not have a dense range [3] (we are grateful to Ugo Bruzzo for pointing out to us this reference as well as [4], and for his help with a copy of the article). For example, if G is a connected linear algebraic group and H is a parabolic subgroup of G , then the embedding $H \rightarrow G$ is an epimorphism in **AlgGr** (this readily follows from the fact that every morphism of the projective variety G/H to an affine variety is constant). See [4] for recent advances in this topic.

3. Idea of the proof of the Main Theorem

Let X be a compact connected smooth manifold, $G = \text{Diff } X$,

$$H = \{f \in G : f(p) = p\} \quad \text{for some } p \in X.$$

We want to prove the following statement:

If F is a locally convex Lie G -group and $j : X \rightarrow F$ is a G -equivariant smooth mapping, then j is constant.

That will suffice, see Theorem 2.2 and the discussion after that.

Let \mathfrak{f} be the Lie algebra of F [10], [13], [14]. Then G acts on \mathfrak{f} , and the action map $G \times \mathfrak{f} \rightarrow \mathfrak{f}$ is jointly continuous (in fact, even smooth, see Lemma 4.1). Equip the dual space \mathfrak{f}^* with the topology of uniform convergence on compact sets. If \mathfrak{f} is a Fréchet space, then \mathfrak{f}^* is σ -compact (= the union of countably many compact sets), see Lemma 4.2. In general \mathfrak{f}^* is covered by G -invariant σ -compact subspaces (Corollary 4.7). There is a natural mapping $j^* : \mathfrak{f}^* \rightarrow \Omega^1(X)$, where $\Omega^1(X)$ is the space of C^∞ -smooth differential 1-forms on X , equipped with the C^∞ -topology (= the topology of uniform convergence of all derivatives). The construction of j^* is explained below. The mapping j^* is continuous (Proposition 4.3). It is G -equivariant since j is. We shall show in the next section that $\Omega^1(X)$ has no non-zero G -invariant σ -compact subspaces (Proposition 4.5). Since \mathfrak{f}^* is covered by G -invariant σ -compact subspaces, it follows that $j^* = 0$. This means that j is constant.

If M is a locally convex manifold and V is a locally convex vector space, a *smooth V -valued differential 1-form* on M is a smooth mapping of the tangent bundle of M to V which is linear on every $T_x M$, the tangent space to M at $x \in M$. The space \mathfrak{f}^* can be identified with the space of all smooth left-invariant differential 1-forms on F , and the mapping $j^* : \mathfrak{f}^* \rightarrow \Omega^1(X)$ from the previous paragraph is just the restriction of the natural map $\Omega^1(F) \rightarrow \Omega^1(X)$.

An alternative way to describe j^* is the following. Consider the *canonical* \mathfrak{f} -valued 1-form θ on F such that $\theta(gv) = v$ for every $g \in F$ and $v \in \mathfrak{f}$, where gv denotes the tangent vector to F at g obtained from v by the left translation by g . Consider the space $\Omega^1(X, \mathfrak{f})$ of all smooth \mathfrak{f} -valued 1-forms on X . Let $\tau \in \Omega^1(X, \mathfrak{f})$ be the inverse image of θ under j . In other words, if u is a tangent vector to X at x , then $\tau(u) = \theta(j_*(u))$, where j_* is the differential of j at x . Every $h \in \mathfrak{f}^*$ induces a natural map $h_* : \Omega^1(X, \mathfrak{f}) \rightarrow \Omega^1(X, \mathbb{R}) = \Omega^1(X)$, and we have $j^*(h) = h_*(\tau)$. Indeed, both forms have equal values on any vector u tangent to X , namely, the value that equals $h(\theta(j_*(u)))$.

4. Details of the proof

It remains to prove the statements mentioned in Section 3.

Lemma 4.1. *If G and F are locally convex groups and G acts smoothly on F by automorphisms, the action of G on the Lie algebra of F is jointly continuous (in fact, smooth).*

Proof. The action of G on the Lie algebra \mathfrak{f} on F is exactly the restriction to $G \times \mathfrak{f}$ of the adjoint representation of the semidirect product $G \ltimes F$, viewed as a map $(G \times F) \times (\mathfrak{g} \times \mathfrak{f}) \rightarrow \mathfrak{g} \times \mathfrak{f}$. But the adjoint representation is a smooth mapping ([14], p. 333, Ex. II.3.9). ■

Lemma 4.2. *If E is a metrizable LCS, then the dual space E^* , equipped with the topology of uniform convergence on compact sets, is σ -compact.*

This is of course well known. We decided to include a proof, since this is a crucial point in our arguments.

Proof. Pick a countable base (U_n) of neighborhoods of zero in E . Then we have $E^* = \bigcup U_n^\circ$, where U° is the polar set, $U^\circ = \{f \in E^* : \sup\{|f(x)| : x \in U\} \leq 1\}$. Every U_n° is compact with respect to the w^* -topology (the Banach-Alaoglu Theorem). Since U_n° is equicontinuous, the w^* -topology on U_n° is the same as the topology of uniform convergence on compact sets [5, Ch. 3, page 17, Proposition 5]. ■

Proposition 4.3. *Let X be a compact smooth manifold, F a locally convex Lie group, \mathfrak{f} the Lie algebra of F , $j : X \rightarrow F$ a smooth mapping. The mapping $j^* : \mathfrak{f}^* \rightarrow \Omega^1(X)$ considered in Section 3 is continuous. (The topology on \mathfrak{f}^* is the topology of uniform convergence on compact subsets of \mathfrak{f} .)*

Proof. Pick a Riemannian metric on X . For a smooth vector field v on X denote by L_v the Lie differentiation with respect to v [11, Ch.1, §3]. A typical neighborhood of zero in $\Omega^1(X)$ looks as follows: pick smooth vector fields v_1, \dots, v_k on X , and consider those forms $\omega \in \Omega^1(X)$ for which the form $\eta = L_{v_1} \dots L_{v_k} \omega$ has a small norm, that is, $|\eta(u)| < \varepsilon$ for all unit tangent vectors u .

Suppose vector fields v_1, \dots, v_k and $\varepsilon > 0$ are given. We must prove the following: if $h \in \mathfrak{f}^*$ is small on a certain compact set K and $\omega = j^*(h)$, then $|L_{v_1} \dots L_{v_k} \omega(u)| < \varepsilon$ for all unit tangent vectors u . Consider the space $\Omega^1(X, \mathfrak{f})$ of all smooth \mathfrak{f} -valued 1-forms on X . If v is a smooth vector field on X , the Lie differentiation L_v is well-defined as a map from $\Omega^1(X, \mathfrak{f})$ to itself.

Consider the forms $\theta \in \Omega^1(F, \mathfrak{f})$ and $\tau \in \Omega^1(X, \mathfrak{f})$ introduced in the previous section: θ is the canonical \mathfrak{f} -valued left-invariant 1-form on F , and $\tau = j^*(\theta)$. Denote by K the compact subset of \mathfrak{f} consisting of all vectors of the form $L_{v_1} \dots L_{v_k} \tau(u)$, where u runs over all unit tangent vectors to X . If h is ε -small on K , $\omega = j^*(h) = h_*(\tau)$, and u is a unit tangent vector to X , then $L_{v_1} \dots L_{v_k} \omega(u) = h_*(L_{v_1} \dots L_{v_k} \tau)(u)$, since h_* commutes with every Lie differentiation L_v . The last expression belongs to $h(K)$ and hence is ε -small. Thus j^* is continuous. ■

A topological space is *Polish* if it is homeomorphic to a complete separable metric space. A topological space is Polish if and only if it is Čech-complete and has a countable base (see [9], Th.4.3.26). If a topological group with a countable

base is locally Čech-complete (that is, admits a Čech-complete neighborhood of the identity), then it is Čech-complete ([2], Proposition 4.3.17), hence Polish. In particular, every FL-group with a countable base is Polish.

Recall that a set in a topological space is *meagre* if it is contained in the union of countably many nowhere dense sets. If a space Y is completely metrizable (that is, admits a compatible complete metric), then no meagre subset of Y contains interior points, in virtue of the Baire category theorem.

Lemma 4.4. *Suppose that a topological group G acts continuously on a Hausdorff space Y . Suppose that for every neighbourhood U of unity in G and for every $y \in Y$ the closure of the set Uy in Y is not compact. If G is not meagre in itself (in particular, if G is Polish), then Y contains no non-empty G -invariant σ -compact subsets.*

Proof. Let $y \in A = \bigcup_{n=0}^{\infty} K_n \subset Y$, where each K_n is compact. We must show that the orbit Gy is not contained in A . Let $F_n = \{g \in G : gy \in K_n\}$. Then each F_n is closed in G and has no interior points. Since G is not meagre, there exists a $g \in G \setminus \bigcup_n F_n$, and we have $gy \notin A$. ■

Proposition 4.5. *If X is a compact manifold and $G = \text{Diff } X$, the space $\Omega^1(X)$ contains no non-zero G -invariant σ -compact subspaces.*

Proof. (2) We apply Lemma 4.4 to the group $G = \text{Diff } X$ and the space $Y = \Omega^1(X) \setminus \{0\}$. Since G is Polish, it suffices to check that for any neighbourhood U of the identity in G , any non-zero 1-form $\omega \in \Omega^1 = \Omega^1(X)$ and any compact subset $K \subset \Omega^1$ we have $U\omega \not\subset K$. We'll confine ourselves to the case when X is a circle. Let $\omega = f(\theta) d\theta$. Assume that U consists of all diffeomorphisms g of X such that the first k derivatives of g are close to those of the identity map. As the $(k+1)$ -th derivative of $g \in U$ can be arbitrary large, the differential forms $g^*(\omega) = f(g(\theta))g'(\theta) d\theta$, where g runs over U , cannot all lie in a compact set. ■

If G is a Polish group that acts on an LCS V by linear transformations so that the action law $G \times V \rightarrow V$ is jointly continuous, we say that V is a G -module.

Proposition 4.6. *If G is a Polish group and V is a G -module, the topology of V is generated by G -morphisms of V to metrizable G -modules.*

In more detail, the assertion is that for every neighborhood U of zero in V there exist a G -morphism $p : V \rightarrow V'$ to a metrizable G -module V' and a neighborhood U' of zero in V' such that $p^{-1}(U') \subset U$.

Proof. Let U be a given neighborhood of zero in V . For every $g \in G$ find a neighborhood W_g of g in G and a neighborhood U_g of zero in V so that $W_g U_g \subset U$. There is a countable collection of W_g 's that cover G ; consider the corresponding U_g 's. With each of these U_g do the same as we did with U . We get a larger countable collection of neighborhoods of zero. Proceed in a similar manner. In countably many steps we obtain a countable collection \mathcal{B} of neighborhoods of zero such that for every $O \in \mathcal{B}$ and $g \in G$ there exist $O' \in \mathcal{B}$ and a neighborhood W of g in G such that $WO' \subset O$. We can throw in finite intersections and images

²The referee pointed out that the tools of [8] can be used to prove this proposition.

under homotheties with coefficient $1/2$. We assume that all neighborhoods in \mathcal{B} are convex and symmetric. In this way we can achieve that \mathcal{B} is a filter base, and the filter \mathcal{F} generated by \mathcal{B} is the filter of neighborhoods for some pseudometrizable locally convex topology \mathcal{T} on V that is coarser than the original topology \mathcal{T}_0 . By our construction, the map $G \times V \rightarrow V$ is continuous with respect to \mathcal{T} at every point of the form $(g, 0)$. We claim that it is continuous everywhere. Indeed, let $(g, x) \in G \times V$ and $O \in \mathcal{B}$. Pick a neighborhood W of g and $O' \in \mathcal{B}$ so that $WO' \subset O$. Shrinking W if necessary, we may assume that $Wx \subset x + O$ (we just use the \mathcal{T}_0 -continuity of the action). Then $W(x + O') \subset Wx + WO' \subset x + O + O'$, and the \mathcal{T} -continuity of the action follows. We take for V' the metrizable space associated with the pseudometrizable space (V, \mathcal{T}) . ■

Corollary 4.7. *Suppose G is a Polish group and V is a G -module. Equip V^* with the topology of uniform convergence on compact sets. The space V^* is covered by G -invariant σ -compact subspaces (which may be non-closed).*

Proof. If $p : V \rightarrow W$ is a G -morphism to a metrizable G -module W , the image of $p^* : W^* \rightarrow V^*$ is σ -compact (Lemma 4.2) and G -invariant. According to Proposition 4.6, W^* is covered by subspaces of this sort. ■

5. The case of finite dimensional Lie groups

Theorem 5.1. *Let $f : G \rightarrow H$ be a Lie group morphism between two finite dimensional Lie groups. Assume that f is an epimorphism in the category of locally convex Lie groups (regular Fréchet-Lie groups, if G has a countable base). Then f has dense range.*

Proof. The Lie group H acts smoothly by left translations on the space $C^\infty(H)$ equipped with the topology of compact convergence with all derivatives ([15], proof of Prop. 4.6). That is, the action mapping $\lambda : H \times C^\infty(H) \rightarrow C^\infty(H)$, $\lambda_g(f)(x) = f(g^{-1}x)$, is smooth. The complete locally convex space $C^\infty(H)$ is an abelian Lie group.

The morphism f induces a homomorphism $f_0 : G/G_0 \rightarrow H/H_0$ between discrete groups, where G_0 and H_0 denote the connected components. This f_0 is in particular an epimorphism in the category of discrete groups (Lie groups of dimension zero), so is onto by an argument of Kurosh, Livshits and Shul'geifer, see [16], top of p. 156 (or Theorem 4.3.1 in [19]). We conclude: the right homogeneous space $\overline{f(G)} \backslash H$ is connected. Find a smooth function $h_1 : \overline{f(G)} \backslash H \rightarrow \mathbb{R}$ achieving its maximum exactly on the right coset $\overline{f(G)}$. The composition of h_1 with the right quotient map $H \rightarrow \overline{f(G)} \backslash H$ gives an element h of $C^\infty(H)$ whose stabilizer under the action λ is exactly $\overline{f(G)}$. According to Theorem 2.2 (equivalence of (1) and (3)), h is H -invariant, that is, constant. It follows that $\overline{f(G)} = H$.

If G has a countable base, that is, has at most countably many connected components, the above argument (using only discrete groups, which are Fréchet-Lie, indeed zero-dimensional Lie) shows that H has the same property, therefore $C^\infty(H)$ is a Fréchet space. ■

Note that in the category of connected finite-dimensional Lie groups epimorphisms need not have a dense range. The existence of epimorphically embedded complex

algebraic groups, like the subgroup of upper triangular matrices in $GL(n, \mathbb{C})$ [3], implies the analogous fact for finite-dimensional complex Lie algebras, and the case of finite-dimensional real Lie algebras follows by complexification. See the next section (Thm. 6.3) for a stronger result: some of those embeddings are epimorphisms even in the category of Banach-Lie groups.

6. The case of Banach-Lie groups

We consider *Banach-Lie* groups, or BL-groups, as defined in [6] under the name of (real) Lie groups. We are going to prove the following: if a morphism $f : G \rightarrow H$ between BL-groups is an epimorphism in **TG**, then f has a dense range. More generally:

Theorem 6.1. *Let G be a BL group, H a proper closed subgroup. Then the inclusion $H \rightarrow G$ is not an epimorphism in **TG**.*

The argument is the same as in Remark 2, Section 2. We consider a certain metric d on the space $P = G/H$ and its extension \bar{d} over the space V_0 of measures on P with a finite support and total mass zero. We check that the action of G on V_0 is jointly continuous. If $V = V_0 \oplus \mathbb{R}$ is the space of all measures on P with a finite support, there is a natural G -map from P to V , and we invoke Theorem 2.1 (equivalence of (1) and (4)) to conclude that the inclusion $H \rightarrow G$ is not an epimorphism in **TG**.

Proposition 6.2. *Suppose a topological group G continuously acts on a metric space (M, d) by Lipschitz transformations. Suppose there is a neighborhood U of the unity in G and a constant $C > 0$ such that for every $g \in U$ the g -shift $\sigma_g : M \rightarrow M$ is C -Lipschitz. Let V be the space of measures on M with a finite support, V_0 the hyperplane of measures of total mass zero. Equip V_0 with the Kantorovich-Rubinstein metric \bar{d} . Then the action of G on V_0 is jointly continuous.*

This is essentially a version of Megrelshvili's result [12, Theorem 4.4]. If G is generated by U (this happens, for example, if G is connected), we can drop the assumption that G acts by Lipschitz transformations, as this will follow from the condition that σ_g is Lipschitz for every $g \in U$.

Proof. If $\bar{d}(v, 0) < \varepsilon$, write $v = \sum c_i(x_i - y_i)$ so that $\sum |c_i|d(x_i, y_i) < \varepsilon$. If $g \in U$, we have $\bar{d}(gv, 0) \leq \sum |c_i|d(gx_i, gy_i) < C\varepsilon$. It follows that the action $G \times V_0 \rightarrow V_0$ is jointly continuous at $(1_G, 0)$. Similarly, the Lipschitz condition for σ_g implies that the action is separately continuous. We conclude that the action is jointly continuous at every point (g_0, v_0) : if h is close to 1_G and $v \in V_0$ is small, we have $hg_0(v_0 + v) - g_0v_0 = (hg_0v_0 - g_0v_0) + hg_0v$, where both summands are small. ■

A *norm* on a group G is a function $p : G \rightarrow \mathbb{R}_+$ such that:

- (1) $p(1_G) = 0$ and $p(x) > 0$ if $x \neq 1_G$;
- (2) $p(x) = p(x^{-1})$;
- (3) $p(xy) \leq p(x) + p(y)$.

If G is a metrizable group, its topology is generated by a certain norm p , in the sense that the sets of the form $\{x \in G : p(x) < \varepsilon\}$ constitute a base at 1_G . (Note that not every norm generates a group topology.)

If H is a closed subgroup of G , we can define a compatible metric d on G/H by

$$d(a, b) = \inf\{p(g) : ga = b\}.$$

We now prove Theorem 6.1. First we consider the case when G is a connected BL-group. Denote by \mathfrak{g} its Lie algebra. We assume that a norm on \mathfrak{g} is given such that \mathfrak{g} is a Banach space and $\|[X, Y]\| \leq \|X\| \cdot \|Y\|$. The *exponential length* norm on G is defined by

$$p(g) = \inf\{\|X_1\| + \dots + \|X_n\| : g = e^{X_1} \dots e^{X_n}\}.$$

This norm is compatible with the topology of G [1, Proposition 3.2]. Note that we may assume that all vectors X_i in the definition above are short, that is, belong to a given neighborhood of zero in \mathfrak{g} : otherwise replace a vector X by k vectors X/k . This will not change the sum of norms or the product $e^{X_1} \dots e^{X_n}$.

Equip G/H with the corresponding metric d defined as above:

$$d(a, b) = \inf\{\|X_1\| + \dots + \|X_n\| : e^{X_1} \dots e^{X_n}a = b\}.$$

Pick a small ε , and let $U = \{e^X : \|X\| < \varepsilon\} \subset G$ be the corresponding neighborhood of 1_G . If $\|Y\| < \varepsilon$, $g = e^Y \in U$ and $a, b \in P$, then

$$\begin{aligned} d(ga, gb) &= \inf\{\|X_1\| + \dots + \|X_n\| : e^{X_1} \dots e^{X_n}ga = gb\} \\ &= \inf\{\|X_1\| + \dots + \|X_n\| : e^{-Y}e^{X_1} \dots e^{X_n}e^Y a = b\}. \end{aligned} \tag{1}$$

Pick short vectors X'_1, \dots, X'_n so that $e^{X'_1} \dots e^{X'_n}a = b$ and

$$\|X'_1\| + \dots + \|X'_n\| < 2d(a, b). \tag{2}$$

Pick short vectors X_i so that $e^{X_i} = e^Y e^{X'_i} e^{-Y}$. Then $e^{-Y}e^{X_1} \dots e^{X_n}e^Y a = b$, and hence, according to the formula (1),

$$d(ga, gb) \leq \|X_1\| + \dots + \|X_n\|. \tag{3}$$

We have $X_i = H(Y, H(X'_i, -Y)) = X'_i + [Y, X'_i] +$ terms of higher degree, where H is the Hausdorff series [6, Ch.2, §6,7]. If ε is small enough, we have $\|X_i\| \leq 2\|X'_i\|$, therefore, from Eq. (2) and (3), $d(ga, gb) < 4d(a, b)$.

By Proposition 6.2, the action of G on V is jointly continuous. This proves Theorem 6.1 in the case G is connected.

If we drop the assumption that G is connected, the argument with connected components used in the previous section shows that we can assume that H meets all connected components of G . In that case $G/H = G_0/(G_0 \cap H)$, and we have seen above that there is metric on the manifold G/H such that G_0 acts on this manifold by Lipschitz transformations. Actually, the whole group G acts by Lipschitz transformations. This can be deduced from the following observation: every automorphism of the connected Lie group G_0 is Lipschitz, if G is equipped with the exponential length metric as above. To see that the latter statement is true, note that the tangent automorphism σ of the Lie algebra \mathfrak{g} is such that $C^{-1}\|X\| \leq \|\sigma X\| \leq C\|X\|$ for some constant $C > 0$.

An early version of the present paper contained the following questions: do epimorphisms $f: G \rightarrow H$ in the category of BL-groups have a dense range, in particular if G and H are (connected) finite-dimensional Lie groups? The anonymous referee provided the following counterexample to all three questions.

Theorem 6.3. *Let B be the subgroup of upper-triangular matrices in $SL_2(\mathbb{R})$. Then the embedding $B \rightarrow SL_2(\mathbb{R})$ is an epimorphism in the category of BL-groups.*

Proof. It suffices to prove the analogous fact for Lie algebras which reduces to the following. Let \mathfrak{g} be a Banach-Lie algebra. Consider \mathfrak{sl}_2 -triples (h, e, f_1) and (h, e, f_2) in \mathfrak{g} , that is,

$$[h, e] = 2e, \quad [h, f_j] = -2f_j, \quad [e, f_j] = h.$$

We must prove that $f_1 = f_2$. Assume the contrary: $f = f_1 - f_2 \neq 0$. Put $\mathfrak{s} = \text{span}\{e, h, f_1\}$, $v_n = (\text{ad } f_1)^n f$, and $V = \text{span}\{v_0, v_1, \dots\}$. Since $[e, f] = 0$ and $[h, f] = -2f$, it is easy to see that V is an \mathfrak{s} -module such that each v_n is an $\text{ad } h$ -eigenvector with the eigenvalue $-2 - 2n$. As $\text{ad } h$ is a bounded operator, the vectors v_n are zero for n large enough. We therefore obtain a finite-dimensional representation of \mathfrak{sl}_2 in which h has negative spectrum. That is impossible: the spectrum of h in a finite-dimensional representation of \mathfrak{sl}_2 is symmetric with respect to zero [7, Ch. 8, § 1]. ■

7. Open questions

Let $f: G \rightarrow H$ be a morphism between two groups from a category marking a row, and suppose f is an epimorphism in a category marking a column. Does f necessarily have a dense range?

Here is a summary of what we know.

	connected f.d. Lie	f.-d. Lie	B-Lie	regular F-Lie	LC Lie	top.
connected f.-d. Lie	×	×	×	✓	✓	✓
f.-d. Lie		×	×	?	✓	✓
Banach-Lie			×	?	?	✓
Fréchet-Lie				×	×	?
loc. conv. Lie					×	?
top. groups						×

As we mentioned in Section 6, the three crosses appearing in the BL-column are due to the anonymous referee. We cordially thank the referee for this valuable contribution.

We have seen that whether or not the inclusion $i: H \rightarrow G$ of a proper closed subgroup is an epimorphism in a suitable category depends on the dynamical system $(G, G/H)$. For example, if it is smooth (or, more generally, Lipschitz, see Proposition

6.2), the inclusion i is not an epimorphism of Hausdorff groups. In order for i to be an epimorphism in the category \mathbf{TG} , the action of G on G/H must be “sufficiently mixing”. Formalizing this criterion could be an interesting task.

References

- [1] H. Ando, M. Doucha, Y. Matsuzawa: *Large scale geometry of Banach-Lie groups*, arXiv: 2011.10376 (2020).
- [2] A. Arhangel'skii, M. Tkachenko: *Topological Groups and Related Structures*, Atlantis Studies in Mathematics 1, Atlantis Press, Paris; World Scientific, Hackensack (2008).
- [3] F. Bien, A. Borel: *Sous-groupes épimorphiques de groupes algébriques linéaires I,II*, C. R. Acad. Sci. Paris Sér. I 315 (1992) 649–653, 1341–1346.
- [4] M. Brion: *Epimorphic subgroups of algebraic groups*, Math. Res. Letters 24/6 (2017) 1649–1665.
- [5] N. Bourbaki: *Espaces Vectoriels Topologiques*, Springer, Berlin (2007).
- [6] N. Bourbaki: *Lie Groups and Lie Algebras, Chapters 1–3*, Springer, Berlin (1989).
- [7] N. Bourbaki: *Lie Groups and Lie Algebras, Chapters 7–9*, Springer, Berlin (2005).
- [8] H. Glöckner: *Solutions to open problems in Neeb's recent survey on infinite-dimensional Lie groups*, Geom. Dedicata 135 (2008) 71–86.
- [9] R. Engelking: *General Topology*, 2nd ed., Sigma Series in Pure Mathematics 6, Heldermann Verlag, Berlin (1989).
- [10] R. S. Hamilton: *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7/1 (1982) 65–222.
- [11] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry, Volume 1*, Interscience Publishers, New York (1963).
- [12] M. G. Megrelishvili: *Free topological G -groups*, New Zealand J. Math. 25 (1996) 59–72.
- [13] J. Milnor: *Remarks on infinite-dimensional Lie groups*, in: *Relativité, Groupes et Topologie II*, North-Holland, Amsterdam (1984) 1007–1057.
- [14] K.-H. Neeb: *Towards a Lie theory of locally convex groups*, Jap. J. Math. 1 (2006) 291–468.
- [15] K.-H. Neeb: *On differentiable vectors for representations of infinite dimensional Lie groups*, J. Funct. Analysis 259 (2010) 2814–2855.
- [16] E. C. Nummela: *On epimorphisms of topological groups*, General Topology Appl. 9 (1978) 155–167.
- [17] H. Omori: *Infinite-Dimensional Lie Groups*, Translations of Mathematical Monographs 158, American Mathematical Society, Providence (1997).
- [18] V. Pestov: *Epimorphisms of Hausdorff groups by way of topological dynamics*, New Zealand J. Math. 26 (1997) 257–262.
- [19] V. Pestov: *Topological groups: Where to from here?*, Topology Proceedings 24 (1999) 421–502.
- [20] V. Uspenskij: *The solution of the epimorphism problem for Hausdorff topological groups*, J. Lie Theory 3 (1993) 69–70.
- [21] V. Uspenskij: *The epimorphism problem for Hausdorff topological groups*, Topology Appl. 57 (1994) 287–294.

- [22] V. Uspenskij: *Epimorphisms of topological groups and Z -sets in the Hilbert cube*, in: *Symposia Gaussiana*, Proc. 2nd Gauss Symposium, Munich, 1993; Conference A: Mathematics and Theoretical Physics, M. Behara, R. Fritsch, R. G. Lintz (eds.), de Gruyter, Berlin (1995) 733-738.
- [23] C. Villani: *Optimal Transport. Old and New*, Grundlehren der Mathematischen Wissenschaften 338, Springer, Berlin (2009).

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