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Reductions for Branching Coefficients

Nicolas Ressayre

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Abstract. Let G be a connected reductive subgroup of a complex connected reductive group $\widehat{G}$. The branching problem consists in decomposing irreducible $\widehat{G}$-representations as sums of irreducible G-representations. The appearing multiplicities are parameterized by the pairs $(\nu, \hat\nu)$ of dominant weights for G and $\widehat{G}$ respectively. The support $\mathrm{LR}(G, \widehat{G})$ of these decompositions is a finitely generated semigroup of such pairs of weights. The cone $\mathcal{LR}(G, \widehat{G})$ generated by $\mathrm{LR}(G, \widehat{G})$ is convex polyhedral and the explicit list of inequalities characterizing it is known. There are the inequalities stating that ν and $\hat\nu$ are dominant and those giving faces containing regular weights (called regular faces), that are parameterized by cohomological conditions.
In this paper, we describe the multiplicities corresponding to the pairs $(\nu, \hat\nu)$ belonging to any regular face of $\mathcal{LR}(G, \widehat{G})$. More precisely, we prove that such a multiplicity is equal to a similar multiplicity for strict Levi subgroups of G and $\widehat{G}$. This generalizes, unifies and simplifies, by different methods, results obtained by Brion, Derksen-Weyman, Roth, and others.

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1. Introduction

Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. The branching problem consists in decomposing irreducible $\hat{G}$-representations as sums of irreducible G-representations.

Fix maximal tori $T \subset \widehat{T}$ and Borel subgroups $B \supset T$ and $\widehat{B} \supset \widehat{T}$ of G and $\widehat{G}$ respectively. Let $X(T)$ (resp. $X(T)^+$) denote the set of characters (resp. dominant characters) of T. For $\nu \in X(T)^+$, $V_\nu(G)$ (or shortly V_ν) denotes the irreducible representation of highest weight ν. Similarly, we use notation $X(\hat{T})$, $X(\hat{T})^+$, $V_{\hat\nu}$ relatively to $\hat{G}$. For any G-representation V, the subspace of G-fixed vectors is denoted by V^G. For $\nu \in X(T)^+$ and $\hat\nu \in X(\hat{T})^+$, set

$$c_{\nu\,\hat\nu}(G, \hat{G}) = \dim(V_\nu \otimes V_{\hat\nu})^G. \tag{1}$$

Sometimes we shortly write $c_{\nu\,\hat\nu}$ for $c_{\nu\,\hat\nu}(G, \hat{G})$. The branching is encoded by these dimensions since, for any $\hat\nu \in X(\widehat{T})^+$

$$V_{\hat\nu} = \bigoplus_{\nu \in X(T)^+} (V_\nu^*)^{\oplus c_{\nu\,\hat\nu}}, \tag{2}$$

where V_ν^* denotes the dual representation of V_ν.

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The set $\mathrm{LR}(G,\widehat{G})$ of pairs $(\nu,\hat\nu)$ in $X(T)^+\times X(\hat T)^+$ such that $c_{\nu\,\hat\nu}\neq 0$ is known to be a finitely generated subsemigroup of the free abelian group $X(T)\times X(\hat T)$ (see [6]). Consider the convex cone $\mathcal{LR}(G,\widehat{G})$ generated by $\mathrm{LR}(G,\widehat{G})$ in $(X(T)\times X(\hat T))\otimes_{\mathbb{Z}}\mathbb{Q}$. It is a closed convex polyhedral cone in $(X(T)\times X(\hat T))\otimes_{\mathbb{Z}}\mathbb{Q}$.

Let us shortly recall the description of $\mathcal{LR}(G,\widehat{G})$ from [21]. For the diagonal embedding of G in $\widehat{G}=G\times G$, the branching problem is that of tensor product decomposition. In this case, the description we are recalling is due to Belkale-Kumar [2].

Consider the natural pairing $\langle\cdot\,,\,\cdot\rangle$ between the one parameter subgroups and the characters of the torus T. Let W denote the Weyl group of T. If τ is a one parameter subgroup of T, we denote by W_τ the stabilizer of τ for the natural action of the Weyl group. Let $P(\tau)$ denote the usual parabolic subgroup of G associated to τ (see definition (6)). The cohomology group $\mathrm{H}^*(G/P(\tau),\mathbb{Z})$ is freely generated by the Schubert classes σ_w parameterized by the elements $w\in W/W_\tau$. In [2], Belkale-Kumar defined an associative product $\odot_0$ on $\mathrm{H}^*(G/P(\tau),\mathbb{Z})$.

Observing that τ is also a one parameter subgroup of $\widehat{T}$, we consider $\langle\cdot,\cdot\rangle$, $\widehat{W}$, $\widehat{W}_\tau$, $\widehat{G}/\widehat{P}(\tau)$, $\sigma_{\hat w}$ as above but with $\widehat{G}$ in place of G. Consider also the canonical G-equivariant immersion $\iota\,:\,G/P(\tau)\longrightarrow\widehat{G}/\widehat{P}(\tau)$. In [24], a corresponding morphism $\iota^{\odot_0}\,:\,\mathrm{H}^*(\widehat{G}/\widehat{P}(\tau),\mathbb{Z})\longrightarrow\mathrm{H}^*(G/P(\tau),\mathbb{Z})$ respecting the product $\odot_0$ is constructed.

Let $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ denote the Lie algebras of G and $\widehat{G}$ respectively. Consider the set $\mathrm{Wt}_T(\hat{\mathfrak{g}}/\mathfrak{g})$ of nontrivial weights of T acting on $\hat{\mathfrak{g}}/\mathfrak{g}$. An indivisible dominant one parameter subgroup τ of T is said to be *adapted* if $\{\chi\in\mathrm{Wt}_T(\hat{\mathfrak{g}}/\mathfrak{g})\,:\,\langle\tau,\chi\rangle=0\}$ spans an hyperplane of $X(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. Let $\{\tau_1,\dots,\tau_n\}$ be the set of adapted one parameter subgroups.

We can now describe the list of inequalities characterising $\mathcal{LR}(G,\widehat{G})$.

Theorem 1.1. *We assume that no nonzero ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$. Then, $\mathcal{LR}(G,\widehat{G})$ has nonempty interior in $X(T\times\widehat{T})\otimes\mathbb{Q}$.*
A pair $(\nu,\hat\nu)$ in $X(T)^+\times X(\widehat{T})^+$ belongs to $\mathcal{LR}(G,\widehat{G})$ if and only if for any $i=1,\dots,n$ and for any pair $(w,\hat w)\in W/W_{\tau_i}\times\widehat{W}/\widehat{W}_{\tau_i}$ such that

$$\iota^{\odot_0}(\sigma_{\hat w})\odot_0\sigma_w=\sigma_e\in\mathrm{H}^*(G/P(\tau_i),\mathbb{Z}),$$

we have
$$\langle w\tau_i,\nu\rangle+\langle\hat w\tau_i,\hat\nu\rangle\geq 0. \tag{3}$$

[21, Theorem B] asserts that the face $\mathcal{F}$ of $\mathcal{LR}(G,\widehat{G})$ associated to any inequality (3) has codimension one and contains pairs of regular weights.

In this paper, we are interested in the faces of $\mathcal{LR}(G,\widehat{G})$ containing pairs of regular weights of any codimension. They are said to be *regular.* Theorem 1.1 parameterizes bijectively the regular faces of codimension one. [22, Theorem 7.2] parameterizes all the regular faces. Here, we only recall [22, Theorem 7.2] partially:

Theorem 1.2. *Let $\mathcal{F}$ be a regular face of $\mathcal{LR}(G,\widehat{G})$. Consider the subgroup $\Xi_{\mathcal{F}}$ of $X(T)$ consisting in $\chi\in X(T)$ such that $(\chi,0)$ belongs to the span of $\mathcal{F}$. Let S be the associated subtorus of T. Namely $S=\bigcap_{\chi\in\Xi_{\mathcal{F}}}\mathrm{Ker}\,\chi$.*

Then, there exists $\hat{w}$ in $\hat{W}$ such that the span of $\mathcal{F}$ is the set of pairs $(\nu, \hat{\nu})$ in $(X(T) \times X(\hat{T})) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

$$\nu_{|S} + \hat{w}\hat{\nu}_{|S} = 0 \in X(S) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{4}$$

and

$$\widehat{G}^S \cap \hat{w}\widehat{B}\hat{w}^{-1} = \widehat{B}^S. \tag{5}$$

In equality (4), $\nu_{|S}$ and $\hat{w}\hat{\nu}_{|S}$ denote the restrictions of ν and $\hat{w}\hat{\nu}$ to the subtorus S. In equality (5), $\widehat{G}^S$ and $\widehat{B}^S$ denote the centralizer of S in $\widehat{G}$ and $\widehat{B}$ respectively.

Assume that S and $\hat{w}$ satisfy property (5). Let $(\nu, \hat{\nu})$ be a pair of dominant weights. Observe that G^S is connected and is a Levi subgroup of G containing T (see e.g. [8, § 22.3]). Moreover, B^S is a Borel subgroup of G^S containing T and ν is dominant relatively to G^S and the pair (T, B^S). In particular, one can associate the irreducible G^S-representation $V_\nu(G^S)$ of highest weight ν.

On the other hand, $\hat{w}\hat{\nu}$ is dominant for $\widehat{G}$ relatively to $\widehat{T} \subset \hat{w}\widehat{B}\hat{w}^{-1}$. Thus, by condition (5), $\hat{w}\hat{\nu}$ is dominant for $\widehat{G}^S$ relatively to $T \subset \widehat{B}^S$, and one can consider the associated $\widehat{G}^S$-representation $V_{\hat{\nu}}(\widehat{G}^S)$. Define now the corresponding multiplicity by

$$c_{\nu\, \hat{w}\hat{\nu}}(G^S, \hat{G}^S) := \dim(V_\nu(G^S) \otimes V_{\hat{\nu}}(\widehat{G}^S))^{G^S}.$$

We can now state our reduction rule for the multiplicities.

Theorem 1.3. *Let $(\nu, \hat{\nu}) \in X(T)^+ \times X(\widehat{T})^+$ be a pair of dominant weights. Let $\mathcal{F}$ be a regular face and let S and $\hat{w}$ be given by Theorem 1.2. Assume that $(\nu, \hat{\nu})$ belongs to the span of $\mathcal{F}$ (equivalently that it satisfies condition (4)). Then*

$$c_{\nu\, \hat{\nu}}(G, \hat{G}) = c_{\nu\, \hat{w}\hat{\nu}}(G^S, \hat{G}^S).$$

Theorem 1.3 is the algebraic counterpart of the geometric Theorem 1.4 below. Let $X = G/P \times \widehat{G}/\widehat{P}$ be a flag manifold for the group $G \times \widehat{G}$. Let τ be a one-parameter subgroup of G and C be an irreducible component of the fixed point set X^τ of τ in X. Let G^τ be the centralizer of the image of τ in G. We assume that (C, τ) is a *(well) covering pair* in the sense of [21, Definition 3.2.2] (see also Definition 2.1 below).

Theorem 1.4. *Let $\mathcal{L}$ be a G-linearized line bundle on X generated by its global sections such that τ acts trivially on the restriction $\mathcal{L}_{|C}$. Then the restriction map*

$$\mathrm{H}^0(X, \mathcal{L})^G \longrightarrow \mathrm{H}^0(C, \mathcal{L}_{|C})^{G^\tau},$$

between the spaces of invariant sections of $\mathcal{L}$ and $\mathcal{L}_{|C}$ is an isomorphism.

Several particular cases of Theorems 1.3 or 1.4 were known before. If $G = T$ is a maximal torus of $G = \mathrm{SL}_n(\mathbb{C})$, King-Tollu-Toumazet [9, Theorem 5.8] proved Theorem 1.3 using hive model for the Kostka coefficients. For $G = \mathrm{SL}_n(\mathbb{C})$ diagonally embedded in $\widehat{G} = G \times G$, the multiplicities $c_{\nu\,\hat{\nu}}(G, \hat{G})$ are the Littlewood-Richardson coefficients (LR-coefficients for short). Since the Levi subgroups of G have type A, Theorem 1.3 expresses some of these coefficients as products of smaller LR-coefficients. This case was proved independently by Derksen-Weyman in [5, Theorem 7.4] and King-Tollu-Toumazet in [10, Theorem 1.4].

In [5], the representations of quivers are the general context and the proof of [10] is combinatorial using hive model. More generally, M. Roth obtained these results for any tensor product decomposition (that is, if G is diagonally embedded in G^s). M. Roth uses both the Borel-Weil theorem and Lie algebra cohomology. If ν is regular then Theorem 1.4 can be obtained combining [4, Theorem 3] and [21]. Similar reductions can be found in [3, 15, 17]. Our main result could also be obtained from Meinrenken-Sjamaar's work on quantification commutes with reduction [16]. Nevertheless, the proof would be indirect and based on the subtil Meinrenken-Sjamaar's work on singular symplectic reductions.

Our proof is new and quite simple. It starts with the geometric realization of the multiplicities given by the Borel-Weil theorem. The Zariski Main Theorem (also used in [5]) is used to translate the problem in that of extending invariant sections of line bundles defined on some dense open subset to a normal projective variety. Then, by normality (of the Schubert varieties), it remains to prove that the considered sections have no pole. This last property is shown by a quite explicit local computation. In [26], our local computation is replaced by a computation of Lie algebra cohomology. In particular, normality of Schubert varieties that is essential in our proof is not used [26]. [4] and [5] make use of more algebraic arguments.

In [23], we extend our method to flag ind-varieties to obtain a generalization of Theorem 1.3 for Kac-Moody groups. Some explicit computations have to be replaced by more abstract arguments.

In Section 4, Theorem 1.4 is applied to recover known results in representation theory.

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2. Proof of Theorem 1.4

Recall that G is a connected reductive subgroup of a complex connected reductive group $\widehat{G}$. Fix now two parabolic subgroups P and $\widehat{P}$ of G and $\widehat{G}$ respectively. Consider the variety $X = G/P \times \widehat{G}/\widehat{P}$ endowed with the diagonal G-action.

Let τ be a one-parameter subgroup of G. Consider the centralizer G^τ of τ in G and the parabolic subgroup (see [18])

$$P(\tau) = \left\{ g \in G \ :\ \lim_{t \to 0} \tau(t).g.\tau(t)^{-1} \text{ exists in } G \right\}. \tag{6}$$

Let C be an irreducible component of the fixed point set X^τ of τ in X. Let C^+ denote the associated Białynicki-Birula cell:

$$C^+ = \{x \in X \ :\ \lim_{t \to 0} \tau(t)x \text{ belongs to } C\}. \tag{7}$$

Keep in mind that C is G^τ-stable and closed in X, whereas C^+ is $P(\tau)$-stable and locally closed in X. Consider the locally closed subvariety Y of $G/P(\tau) \times X$ defined by

$$Y = \{(gP(\tau)/P(\tau), x) \ :\ g^{-1}x \in C^+\}.$$

The morphism $\pi : G \times C^+ \longrightarrow Y$, $(g, x) \longmapsto (gP(\tau)/P(\tau), gx)$ identifies Y with the quotient of $G \times C^+$ by the $P(\tau)$ -action given by $p.(g, x) = (gp^{-1}, px)$.

The variety Y is denoted by $G \times_{P(\tau)} C^+$. Set $[g:x] = \pi(g,x)$ and consider the G-equivariant map

$$\eta \;:\; G \times_{P(\tau)} C^+ \longrightarrow X, \quad [g:x] \longmapsto gx.$$

Similarly, we define $G \times_{P(\tau)} \overline{C^+}$ and

$$\overline{\eta} \;:\; G \times_{P(\tau)} \overline{C^+} \longrightarrow X, \quad [g:x] \longmapsto gx,$$

where $\overline{C^+}$ denotes the closure of C^+ in X. Recall from [21] the notion of well covering pair.

Definition 2.1. The pair (C,τ) is said to be *covering* if η is birational. It is said to be *well covering* if there exists a $P(\tau)$-stable open subset Ω of C^+ intersecting C such that η induces an isomorphism from $G \times_{P(\tau)} \Omega$ onto an open subset of X.

Proof of Theorem 1.4. Note that $G \times_{P(\tau)} \overline{C^+}$ is closed in $G/P \times X$; in particular, it is projective and $\overline{\eta}$ is proper. Also, the pair (C,τ) being covering the map $\overline{\eta}$ is birational. Since X is normal, the Zariski main theorem implies that the fibers of $\overline{\eta}$ are connected. Moreover, $\overline{\eta}$ induces a G-equivariant isomorphism (see e.g. [20, Chap IV, Corollary 5])

$$\mathrm{H}^0(X,\mathcal{L}) \simeq \mathrm{H}^0(G \times_{P(\tau)} \overline{C^+}, \overline{\eta}^*(\mathcal{L})).$$

In particular,
$$\mathrm{H}^0(X,\mathcal{L})^G \simeq \mathrm{H}^0(G \times_{P(\tau)} \overline{C^+}, \overline{\eta}^*(\mathcal{L}))^G.$$

We embed $\overline{C^+}$ in $G \times_{P(\tau)} \overline{C^+}$, by $x \longmapsto [e:x]$. Note that the composition of the immersion of $\overline{C^+}$ in $G \times_{P(\tau)} \overline{C^+}$ with $\overline{\eta}$ is the inclusion map from $\overline{C^+}$ to X. In particular, $\overline{\eta}^*(\mathcal{L})_{|\overline{C^+}} = \mathcal{L}_{|\overline{C^+}}$ and the restriction map induces the following isomorphism (see for example [21, Lemma 4]):

$$\mathrm{H}^0(G \times_{P(\tau)} \overline{C^+}, \overline{\eta}^*(\mathcal{L}))^G \simeq \mathrm{H}^0(\overline{C^+}, \mathcal{L}_{|\overline{C^+}})^{P(\tau)}.$$

Since once more, the composition of the immersion of $\overline{C^+}$ in $G \times_{P(\tau)} \overline{C^+}$ with $\overline{\eta}$ is the immersion of $\overline{C^+}$ in X, we just proved that the restriction map induces the following isomorphism

$$\mathrm{H}^0(X,\mathcal{L})^G \simeq \mathrm{H}^0(\overline{C^+}, \mathcal{L}_{|\overline{C^+}})^{P(\tau)}. \tag{8}$$

On the other hand, since τ acts trivially on $\mathcal{L}_{|C}$, [21, Lemma 5] proves that the restriction map induces the following isomorphism

$$\mathrm{H}^0(C^+, \mathcal{L}_{|C^+})^{P(\tau)} \simeq \mathrm{H}^0(C, \mathcal{L}_{|C})^{G^\tau}. \tag{9}$$

With isomorphisms (8) and (9), to prove the theorem, it remains to prove that the restriction map induces the following isomorphism

$$\mathrm{H}^0(\overline{C^+}, \mathcal{L}_{|\overline{C^+}})^{P(\tau)} \simeq \mathrm{H}^0(C^+, \mathcal{L}_{|C^+})^{P(\tau)};$$

that is, that any regular $P(\tau)$-invariant section of $\mathcal{L}$ on C^+ extends to $\overline{C^+}$.

Note that τ is also a one-parameter subgroup of $\widehat{G}$ and that $\widehat{P}(\tau)$ is well-defined. Fix a maximal torus T of G containing the image of τ and a maximal torus $\widehat{T}$ of $\widehat{G}$ containing T. Note that P and $\widehat{P}$ have not really been fixed up to now; we have only considered the $G\times\widehat{G}$-variety X. In other words, we can change P and $\widehat{P}$ by conjugated subgroups of G and $\widehat{G}$ respectively. Fix a $T\times\widehat{T}$-fixed point x_0 in C, and denote by $P\times\widehat{P}$ its stabilizer in $G\times\widehat{G}$. Hence $x_0=(P/P,\widehat{P}/\widehat{P})$.

Applying [25, Theorem A] in the $(G\times\widehat{G})$-homogeneous space X, one gets that $C=G^\tau P/P\times\widehat{G}^\tau\widehat{P}/\widehat{P}$. Furthermore, the Bruhat decomposition coincides with Białynicki-Birula decomposition (see e.g. [21, Lemma 12]) and we have furthermore $C^+=P(\tau)P/P\times\widehat{P}(\tau)\widehat{P}/\widehat{P}$. In particular, $\overline{C^+}$ is a product of Schubert varieties and is normal.

Fix a $P(\tau)$-invariant regular section σ of $\mathcal{L}_{|C^+}$. Think about σ as a rational section of $\mathcal{L}_{|\overline{C^+}}$. By normality of $\overline{C^+}$, to prove that σ is regular on $\overline{C^+}$, it is sufficient to prove that σ has no pole. The section σ being regular on C^+, the poles could only occur along the codimension one irreducible components of $\overline{C^+}-C^+$. Fix such a component D. We are going to compute the vanishing order of σ along D by a quite explicit computation in a neighborhood of D in $\overline{C^+}$, and prove that it is nonnegative.

Given a root β of (G,T), s_β denotes the associated reflection in the Weyl group. The Bruhat order $\preccurlyeq$ is generated by the covering relations: for any root β and any $w\in W$, $s_\beta w\preccurlyeq w$ if $l(s_\beta w)=l(w)-1$. In particular, any divisor of $\overline{P(\tau)P/P}-P(\tau)P/P$ is the closure of $P(\tau)s_\beta P/P$ for some root β. Hence, the divisor D is the closure of either $P(\tau)s_\beta P/P\times\widehat{P}(\tau)\widehat{P}/\widehat{P}$ (for some root β of G) or of $P(\tau)P/P\times\widehat{P}(\tau)s_{\hat\beta}\widehat{P}/\widehat{P}$ (for some root $\hat\beta$ of $\widehat{G}$). Consider the first case. The second one works similarly.

Recall the construction of a neighborhood in $\overline{C^+}$ of $y=(s_\beta P/P,\widehat{P}/\widehat{P})$, a point in D, that is isomorphic to an affine space. The T-stable line δ in G/P containing P/P and $s_\beta P/P$ is contained in $\overline{C^+}$. Consider the unipotent radical $P^{u,-}$ of the parabolic subgroup of G containing T and opposite to P. Similarly, define $\widehat{P}^{u,-}$. Setting $U_y=P(\tau)\cap s_\beta P^{u,-}s_\beta$ and $\hat U_y=\widehat{P}(\tau)\cap\widehat{P}^{u,-}$, the group $U_y\times\hat U_y$ stabilizes $\overline{C^+}$. Thus the map

$$\begin{array}{rccc}\theta: & U_y\times\hat U_y\times(\delta-\{P/P\}) & \longrightarrow & \overline{C^+}\\ & (u,\hat u,x) & \longmapsto & (ux,\hat u\widehat{P}/\widehat{P}).\end{array}$$

is well-defined and T-equivariant when the T-action on $U_y\times\hat U_y\times(\delta-\{P/P\})$ is defined by $t.(u,\hat u,x)=(tut^{-1},t\hat ut^{-1},tx)$. Moreover, θ is an open immersion of which Ω denotes the image.

But, on one hand, the curve $(\delta-\{P/P\})$ is isomorphic, as a T-variety, to $\mathbb{C}$ on which T acts linearly with weight $\gamma=\pm\beta$. Since γ is not a root of $P(\tau)$, the sign has to be fixed to satisfy $\langle\gamma,\tau\rangle<0$. On the other hand, since the group $U_y\times\hat U_y$ is unipotent, the exponential map $\exp:\mathrm{Lie}(U_y\times\hat U_y)\longrightarrow U_y\times\hat U_y$ is an isomorphism of T-varieties. We get a T-equivariant isomorphism

$$\iota:\mathrm{Lie}(U_y\times\hat U_y)\times\mathbb{C}\longrightarrow U_y\times\hat U_y\times(\delta-\{P/P\}).$$

We claim that $(\iota\circ\theta)^*(\sigma)$ extends to a regular section of $(\iota\circ\theta)^*(\mathcal{L})$.

Fix a basis $(\xi_i)_{i\in I}$ of $\mathrm{Lie}(U_y \times \hat{U}_y)^*$ consisting in $T\times\widehat{T}$-eigenvectors. The weights of $T\times\widehat{T}$ acting on this basis are the opposite (because of duality) of some roots of $P(\tau)$ or $\widehat{P}(\tau)$. In particular, τ acts with nonnegative weights on $\mathrm{Lie}(U_y\times\hat{U}_y)$. For $i\in I$, let $a_i\in\mathbb{Z}_{\geq 0}$ denote the opposite (because of duality) of the weight of τ acting on ξ_i. Let also $a=\langle\gamma,\tau\rangle<0$ be the opposite of the weight of τ acting on the coordinate ζ corresponding to $\delta-\{P/P\}$.

Note that $(\iota\circ\theta)^{-1}(D)$ is the divisor $(\zeta=0)$. Consider now, the $\mathbb{C}^*$-linearized line bundle $(\iota\circ\theta)^*(\mathcal{L})$. It is trivial as a line bundle, the Picard group of the affine space being trivial. Hence, it is isomorphic to $(\mathrm{Lie}(U_y\times\hat{U}_y)\times\mathbb{C})\times\mathbb{C}$ linearized by

$$t.(v,\tau)=(\tau(t)v,t^\mu\tau)\ \ \forall v\in \mathrm{Lie}(U_y\times\hat{U}_y)\times\mathbb{C}),\ \tau\in\mathbb{C}\text{ and } t\in\mathbb{C}^*,$$

for some integer μ. We first admit that

$$\mu\leq 0. \tag{10}$$

The section $(\iota\circ\theta)^*(\sigma)$ is a polynomial function in the variables ξ_i,ζ and ζ^{-1}; that is, a linear combination of monomials $m=\prod_i\xi_i^{j_i}.\zeta^j$ for some $j_i\in\mathbb{Z}_{\geq 0}$ and $j\in\mathbb{Z}$. The opposite of the weight of m for the action of $\mathbb{C}^*$ is $\sum_i(j_ia_j+ja)$. The fact that σ is $\tau(\mathbb{C}^*)$-invariant implies that the monomials occurring in the expression of $(\iota\circ\theta)^*(\sigma)$ satisfy

$$\sum_i(j_ia_j+ja)=\mu.$$

Hence
$$j=\frac{-1}{a}\left(\sum_i j_ia_i-\mu\right).$$

Now, inequality (10) implies that $j\geq 0$. In particular, $(\iota\circ\theta)^*(\sigma)$ extends to a regular function on $\mathrm{Lie}(U_y\times\hat{U}_y)\times\mathbb{C}$. It follows that σ is regular on Ω and has no pole along D.

It remains to prove inequality (10). Recall that δ is isomorphic to $\mathbb{P}^1$ and $\mathcal{L}_{|\delta}$ is isomorphic to $\mathcal{O}(d)$ as a line bundle for some integer d. Since $\mathcal{L}$ is semiample, d is nonnegative. The group $\mathbb{C}^*$ acts on the tangent space $T_y\delta$ of δ at the point y by the weight a. Similarly, it acts on $T_{P/P}\delta$ by the weight $-a$.

By assumption, the group $\mathbb{C}^*$ acts trivially on the fiber $\mathcal{L}_{P/P}$ (recall that P/P belongs to C). It acts on the fiber $\mathcal{L}_y$ by the weight μ. Now, the theory of $\mathbb{P}^1$ implies that:

$$d=\frac{\mu-0}{a}.$$

But, $d\geq 0$ and $a<0$. It follows that $\mu\leq 0$. ■

3. Proof of Theorem 1.3

Let T, B, $\widehat{T}$ and $\widehat{B}$ be like in the introduction. For any character ν of B, $\mathcal{L}_\nu$ denotes the G-linearized line bundle on G/B such that B acts on the fiber in $\mathcal{L}_\nu$ over B/B with the weight $-\nu$. By the Borel-Weil theorem, the line bundle $\mathcal{L}_\nu$ is generated by its global sections if and only if ν is dominant and, in this case, the G-representation $\mathrm{H}^0(G/B,\mathcal{L}_\nu)$ is isomorphic to the dual V_ν^* of the irreducible G-representation V_ν with highest weight ν.

Consider the complete flag variety $X = G/B \times \widehat{G}/\widehat{B}$ of the group $G \times \widehat{G}$. Let ν and $\hat{\nu}$ be like in Theorem 1.3. Let $\mathcal{L}$ be the exterior product on X of $\mathcal{L}_\nu$ and $\mathcal{L}_{\hat{\nu}}$. By the Borel-Weil theorem (applied to the group $G \times \widehat{G}$), we have

$$V_\nu^* \otimes V_{\hat{\nu}}^* = \mathrm{H}^0(X, \mathcal{L}).$$

In particular, $c_{\nu\,\hat{\nu}}(G, \widehat{G})$ is the dimension of $\mathrm{H}^0(X, \mathcal{L})^G$.

Set $C = G^S B/B \times \widehat{G}^S \hat{w} \widehat{B}/\widehat{B}$. By [22], there exists a one-parameter subgroup τ of S such that (C, τ) is well covering and $G^S = G^\tau$. Moreover, assumption (4) implies that τ acts trivially on $\mathcal{L}_{|C}$. Hence, Theorem 1.4 implies that

$$\mathrm{H}^0(X, \mathcal{L})^G \simeq \mathrm{H}^0(C, \mathcal{L}_{|C})^{G^S}.$$

Moreover, C is isomorphic to the complete flag manifold of the group $G^S \times \widehat{G}^S$. By property (5), $\mathcal{L}_{|C}$ is the line bundle $\mathcal{L}_\nu \otimes \mathcal{L}_{\hat{w}\hat{\nu}}$. Hence, the Borel-Weil theorem implies that $\mathrm{H}^0(C, \mathcal{L}_{|C})$ is isomorphic to $V_\nu(G^S)^* \otimes V_{\hat{w}\hat{\nu}}(\widehat{G}^S)^*$. In particular, $c_{\nu\,\hat{w}\hat{\nu}}(G^S, \hat{G}^S)$ is the dimension of $\mathrm{H}^0(C, \mathcal{L}_{|C})^{G^S}$. The theorem is proved. ∎

4. Examples

4.1. Tensor product decomposition

In this subsection, we consider the case when $\widehat{G} = G \times G$ and G is diagonally embedded in $\widehat{G}$. Assume that $\widehat{B} = B \times B$ and $\widehat{T} = T \times T$. Then a dominant weight $\hat{\nu}$ of $\widehat{T}$ is a pair (λ, μ) of dominant weights of T and $V_{\hat{\nu}}(G \times G) = V_\lambda(G) \otimes V_\mu(G)$. We denote by $c_{\lambda\,\mu\,\nu}(G)$ the coefficient $c_{\nu\,\hat{\nu}}(G, \widehat{G})$, in such a way that

$$V_\lambda(G) \otimes V_\mu(G) = \sum_\nu c_{\lambda\,\mu\,\nu}(G)\; V_\nu(G)^*, \tag{11}$$

and that $c_{\lambda\,\mu\,\nu}(G)$ is a tensor product multiplicity for G. With the notation of Theorem 1.3, we have $\widehat{G}^S = G^S \times G^S$. In particular, the coefficient $c_{\nu\,\hat{w}\hat{\nu}}(G^S, \hat{G}^S)$ is a tensor product multiplicity for the Levi subgroup G^S of G. Hence, Theorem 1.3 implies the main result of [26].

Consider the case when $G = \mathrm{GL}_n(\mathbb{C})$, T consists in diagonal matrices and B in upper triangular matrices. Then a dominant weight λ is a non-increasing sequence $(\lambda_1, \ldots, \lambda_n)$ of n integers and $c_{\lambda\,\mu\,\nu}(G)$ is a Littlewood-Richardson coefficient denoted by $c^n_{\lambda\,\mu\,\nu}$.

We introduce additional notation to describe the semigroup $\mathrm{LR}(G, \widehat{G})$. Let $\mathbb{G}(r, n)$ be the Grassmann variety of r-dimensional subspaces of $\mathbb{C}^n$. Let $F_\bullet\colon \{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$ be the standard flag of $\mathbb{C}^n$. Let $\mathcal{P}(r, n)$ denote the set of subsets of $\{1, \ldots, n\}$ with r elements. Fix $I = \{i_1 < \cdots < i_r\} \in \mathcal{P}(r, n)$. The Schubert variety $\Omega_I(F_\bullet)$ in $\mathbb{G}(r, n)$ is defined by

$$\Omega_I(F_\bullet) = \{L \in \mathbb{G}(r, n) \,:\, \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq r\}.$$

The Poincaré dual of the homology class of $\Omega_I(F_\bullet)$ is denoted by σ_I. The classes σ_I form a $\mathbb{Z}$-basis for the cohomology ring of $\mathbb{G}(r, n)$. The class associated to $[1; r]$ is the class of the point; it is denoted by $[\mathrm{pt}]$.

By [11], [12], and finally [1], we have the following statement.

Theorem 4.1. *Let (λ,μ,ν) be a triple of non-increasing sequences of n integers. Then, $c^n_{\lambda\,\mu\,\nu}\neq 0$ if and only if*

$$\sum_i \lambda_i+\sum_j \mu_j+\sum_k \nu_k=0 \tag{12}$$

and

$$\sum_{i\in I} \lambda_i+\sum_{j\in J} \mu_j+\sum_{k\in K} \nu_k\leq 0, \tag{13}$$

for any $r=1,\dots,n-1$, for any $(I,J,K)\in\mathcal{P}(r,n)^3$ such that

$$\sigma_I.\sigma_J.\sigma_K=[\mathrm{pt}]\in \mathrm{H}^*(\mathbb{G}(r,n),\mathbb{Z}). \tag{14}$$

Knutson, Tao, and Woodward proved in [13, Assertion (5) of Corollary in Section 6] that this statement is optimal in the following sense.

Theorem 4.2. *In Theorem 4.1, no inequality (13) can be omitted.*

In other words, each inequality (13) corresponds to a regular face $\mathcal{F}_{IJK}$ of codimension one of the cone $\mathcal{LR}(G,\widehat{G})$. For $I=\{i_1<\cdots<i_r\}\in\mathcal{P}(r,n)$ and λ a sequence of n integers, set $\lambda_I=(\lambda_{i_1},\dots,\lambda_{i_r})\in\mathbb{Z}^r$. Denote by $I^c\in\mathcal{P}(n-r,n)$ the complement of I in $\{1,\dots,n\}$. It is easy to check that Theorem 1.3 gives in this case the following statement.

Theorem 4.3. *Let (λ,μ,ν) be a triple of non-increasing sequences of n integers. Let $(I,J,K)\in\mathcal{P}(r,n)$ be such that*

$$\sigma_I.\sigma_J.\sigma_K=[\mathrm{pt}]. \tag{15}$$

If

$$\sum_{i\in I} \lambda_i+\sum_{j\in J} \mu_j+\sum_{k\in K} \nu_k=0 \tag{16}$$

then

$$c^n_{\lambda\,\mu\,\nu}=c^r_{\lambda_I\,\mu_J\,\nu_K}\cdot c^{n-r}_{\lambda_{I^c}\,\mu_{J^c}\,\nu_{K^c}}. \tag{17}$$

Theorem 4.3 has been proved independently in [10] and [5]. Note that if equation (16) does not hold then $c^r_{\lambda_I\,\mu_J\,\nu_K}=0$.

It is known that Theorem 4.1 also holds if condition (14) is replaced by

$$\sigma_I.\sigma_J.\sigma_K=d[\mathrm{pt}]\in \mathrm{H}^*(\mathbb{G}(r,n),\mathbb{Z}), \tag{18}$$

for some positive integer d. At the opposite, the following example shows that condition (15) cannot be replaced by condition (18) in Theorem 4.3.

Example 4.4. Here $n=6$, $r=3$ and $I=J=K=\{1,3,5\}$. Then $\sigma_I.\sigma_J.\sigma_K=2[\mathrm{pt}]$ and for any (λ,μ,ν) in $\mathrm{LR}(G,\widehat{G})$, the inequality

$$\textstyle\sum_{i\in I}\lambda_i+\sum_{j\in J}\mu_j+\sum_{k\in K}\nu_k\leq 0$$

holds. Consider $\lambda=\mu=\nu=(1\geq 1\geq 0\geq 0\geq -1\geq -1)$. Then $c^n_{\lambda\,\mu\,\nu}=3$. Hence (λ,μ,ν) belongs to $\mathrm{LR}(G,\widehat{G})$. Moreover,

$$\lambda_I=\mu_J=\nu_K=\lambda_{I^c}=\mu_{J^c}=\nu_{K^c}=(1\geq 0\geq -1)$$

and $\sum_{i\in I}\lambda_i+\sum_{j\in J}\mu_j+\sum_{k\in K}\nu_k=0$. But $c^r_{\lambda_I\,\mu_J\,\nu_K}=c^{n-r}_{\lambda_{I^c}\,\mu_{J^c}\,\nu_{K^c}}=2$ and $c^n_{\lambda\,\mu\,\nu}=3\neq 4=c^r_{\lambda_I\,\mu_J\,\nu_K}.c^{n-r}_{\lambda_{I^c}\,\mu_{J^c}\,\nu_{K^c}}$.

Remark 4.5. With notation of Section 2, if η is dominant, the map

$$\mathrm{H}^0(X,\mathcal{L})^G \longrightarrow \mathrm{H}^0(C,\mathcal{L})^{G^\tau}$$

is injective. When applied to $X=\mathcal{F}l(n)^3$ this observation shows that if $\sigma_I.\sigma_J.\sigma_K\neq 0$ then equality (16) implies that $c^n_{\lambda\,\mu\,\nu}\leq c^r_{\lambda_I\,\mu_J\,\nu_K}\cdot c^{n-r}_{\lambda_{I^c}\,\mu_{J^c}\,\nu_{K^c}}$ according to the Example 4.4.

4.2. Kronecker coefficients

Let $\alpha=(\alpha_1\geq\alpha_2\geq\cdots)$ be a partition. Denote by $l(\alpha)$ the number of nonzero parts of α. Set $|\alpha|=\sum_i\alpha_i$, α is said to be a partition of $|\alpha|$. Consider the symmetric group S_n acting on n letters. The irreducible representations of S_n are parameterized by the partitions of n, let $[\alpha]$ denote the representation corresponding to α. The Kronecker coefficients $k_{\alpha\,\beta\,\gamma}$, depending on three partitions α, β, and γ of the same integer n, are defined by the identity

$$[\alpha]\otimes[\beta]=\sum_\gamma k_{\alpha\,\beta\,\gamma}[\gamma]. \tag{19}$$

The following classical result of Murnaghan and Littlewood (see [19]) shows that Kronecker coefficients generalize Littlewood-Richardson coefficients. It can be obtained as a corollary of Theorem 1.4.

Corollary 4.6.

(1) *If* $k_{\alpha\,\beta\,\gamma}\neq 0$ *then* $\quad (n-\alpha_1)+(n-\beta_1)\geq n-\gamma_1.$ (20)

(2) *Assume that equality holds in formula* (20) *but not necessarily that* $k_{\alpha\,\beta\,\gamma}\neq 0$*. Define* $\bar\alpha=(\alpha_2\geq\alpha_3\geq\cdots)$ *and similarly define* $\bar\beta$ *and* $\bar\gamma$*. Then*

$$k_{\alpha\,\beta\,\gamma}=c^{\bar\gamma}_{\bar\alpha\,\bar\beta}, \tag{21}$$

where $c^{\bar\gamma}_{\bar\alpha\,\bar\beta}$ *is the Littlewood-Richardson coefficient.*

Proof. Let us first introduce some notation on the linear group. Let V be a complex finite dimensional vector space and let $\mathrm{GL}(V)$ be the corresponding linear group. If α is a partition with at most $\dim(V)$ nonzero parts, $S^\alpha V$ denotes the Schur power of V; it is the irreducible $\mathrm{GL}(V)$-representation of highest weight α. Let $\mathcal{F}l(V)$ denote the variety of complete flags of V. Given integers a_i such that $1\leq a_1<\cdots<a_s\leq\dim(V)-1$, $\mathcal{F}l(a_1,\dots,a_s;V)$ denotes the variety of flags $V_1\subset\cdots\subset V_s\subset V$ such that $\dim(V_i)=a_i$ for any i.
Let us choose integers e and f such that

$$l(\alpha)\leq e,\quad l(\beta)\leq f,\quad l(\gamma)\leq e+f-1. \tag{22}$$

Fix two complex vector spaces E and F of dimension e and f respectively. Consider the group $G=\mathrm{GL}(E)\times\mathrm{GL}(F)$. The Kronecker coefficient $k_{\alpha\,\beta\,\gamma}$ can be interpreted in terms of representations of G. Namely (see for example [14, 7]) $k_{\alpha\,\beta\,\gamma}$ is the multiplicity of $S^\alpha E\otimes S^\beta F$ in $S^\gamma(E\otimes F)$. To interpret this multiplicity geometrically, consider the variety

$$X=\mathcal{F}l(E)\times\mathcal{F}l(F)\times\mathcal{F}l(1,\dots,e+f-1;E\otimes F)$$

endowed with its natural G-action. Consider the $\mathrm{GL}(E)$-linearized line bundle $\mathcal{L}^\alpha$ on $\mathcal{F}l(E)$ such that $\mathrm{H}^0(\mathcal{F}l(E),\mathcal{L}^\alpha)=S^\alpha E$ (with usual notation, $\mathcal{L}^\alpha=\mathcal{L}_{-w_0\alpha}$, where w_0 is the longest element of the Weyl group of $\mathrm{GL}(E)$). Similarly, fix $\mathcal{L}^\beta$ on $\mathcal{F}l(F)$ such that $\mathrm{H}^0(\mathcal{F}l(F),\mathcal{L}^\beta)=S^\beta F$. Because of assumption (22), there exists a $\mathrm{GL}(E\otimes F)$-linearized line bundle $\mathcal{L}_\gamma$ on $\mathcal{F}l(1,\dots,e+f-1;E\otimes F)$ such that $\mathrm{H}^0(\mathcal{F}l(1,\dots,e+f-1;E\otimes F),\mathcal{L}_\gamma)=S^\gamma(E^*\otimes F^*)$. Defining the G-linearized line bundle $\mathcal{L}$ on X by $\mathcal{L}=\mathcal{L}^\alpha\otimes\mathcal{L}^\beta\otimes\mathcal{L}_\gamma$, we get

$$k_{\alpha\,\beta\,\gamma}=\dim(\mathrm{H}^0(X,\mathcal{L})^G). \tag{23}$$

Let H_E, H_F, l_E and l_F be hyperplanes and lines respectively in E and F such that $E=H_E\oplus l_E$ and $F=H_F\oplus l_F$. Let τ be the one-parameter subgroup of G acting on H_E and H_F with weight 1 and on l_E and l_F with weight 0. Let C_E be the set of complete flags of E whose hyperplane is H_E. Note that C_E is an irreducible component of $\mathcal{F}l(E)^\tau$. Similarly, define C_F. Let $C_{E\otimes F}$ be the set of points $V_1\subset\cdots\subset V_{e+f-1}$ in $\mathcal{F}l(1,\dots,e+f-1;E\otimes F)$ such that $V_1=l_E\otimes l_F$ and $V_{e+f-1}=(l_E\otimes l_F)\oplus(H_E\otimes l_F)\oplus(l_E\otimes H_F)$. Note that $C_{E\otimes F}$ is an irreducible component of $\mathcal{F}l(1,\dots,e+f-1;E\otimes F)^\tau$ isomorphic to $\mathcal{F}l(H_E\oplus H_F)$. Then $C=C_E\times C_F\times C_{E\otimes F}$ is an irreducible component of X^τ.

One easily checks that $C^+_{E\otimes F}$ is open in $\mathcal{F}l(1,\dots,e+f-1;E\otimes F)$ and that the pairs (C_E,τ) and (C_F,τ) are covering in $\mathcal{F}l(E)$ and $\mathcal{F}l(F)$ for the actions of $\mathrm{GL}(E)$ and $\mathrm{GL}(F)$, respectively. Thus (C,τ) is covering.

Let x be a point in C. Let $\mu^{\mathcal{L}}(x,\tau)$ be the opposite of the weight of the action of τ on the fiber of $\mathcal{L}$ over x. [21, Lemma 3] implies that if $\dim(\mathrm{H}^0(X,\mathcal{L})^G)>0$ then $\mu^{\mathcal{L}}(x,\tau)\leq 0$ which can be proved, by a direct computation, to be inequality (20) of the corollary.

Assume now that $\mu^{\mathcal{L}}(x,\tau)=0$, that is that τ acts trivially on $\mathcal{L}_{|C}$. Theorem 1.3 shows that

$$\dim(\mathrm{H}^0(X,\mathcal{L})^G)=\dim(\mathrm{H}^0(C,\mathcal{L}_{|C})^{G^\tau}).$$

Moreover, the number $\dim(\mathrm{H}^0(C,\mathcal{L}_{|C})^{G^\tau})$ is the multiplicity of the irreducible $\mathrm{GL}(H_E)\times\mathrm{GL}(H_F)$-representation $S^{\bar\alpha}H_E\otimes S^{\bar\beta}H_F$ in the $\mathrm{GL}(H_E\oplus H_F)$-representation $S^{\bar\gamma}(H_E\oplus H_F)$. By for example [14, Chapter I, 5.9], this multiplicity is precisely $c^{\bar\gamma}_{\bar\alpha\,\bar\beta}$. ∎

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Nicolas Ressayre, Institute de Mathématiques et de Modélisation, Université Montpellier 2, 34095 Montpellier, France; ressayre@math.univ-montp2.fr.

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