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Modules over Geometric Quandles and Representations of Lie-Yamaguti Algebras

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Abstract. We study quandle modules over geometric quandles Q, i.e., quandles endowed with geometric structures. In the case Q is a regular s-manifold, we exhibit how modules over Q are related with representations of Lie-Yamaguti algebras.

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Key Words: Quandle, regular s-manifold, Lie-Yamaguti algebra, Lie triple system, representation.

1. Introduction

The notion of a quandle was introduced by Joyce and Matveev as a tool for defining invariants of knots and links [19], [28]. Further invariants were supplied by the use of quandle cohomology theory [10]. While the quandle cohomology with the constant coefficient group is already useful, the natural framework for the cohomology theory is given by quandle modules (see [1] and [18]). In addition to their usefulness in such applications, it is expected that modules are important in understanding a quandle, just as in group theory and ring theory, and so on.

Topological quandles [30], smooth quandles [17], and quandle varieties [32] were defined as quandles with the structure of a topological space, smooth manifold, and algebraic variety, respectively. We will refer to them as geometric quandles. As we will explain later, geometric quandles can also be regarded as generalizations of symmetric spaces.

For a topological quandle Q, the notion of a module over Q was defined in [12], and it readily generalizes to other geometric quandles (Definition 3.5). The objective of this paper is to study modules over a geometric quandle Q, and especially, give an infinitesimal description of certain modules in the case Q is a regular s-manifold, which is a special, but in a sense "generic," kind of geometric quandle.

1.1. Quandles, Φ-spaces, regular s-manifolds and geometric quandles

A quandle is a set Q endowed with a binary operation $\triangleright$ satisfying "Quandle Axioms" (see Definition 2.1). Roughly, the axioms say that the map

$$s_x : Q \to Q; \quad y \mapsto x \triangleright y$$

is a kind of symmetry around x, although it is not required to be involutive.

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It has already been noted by Joyce that symmetric spaces can be regarded as quandles, by taking s_x to be the geodesic involution. In fact, before the quandles attracted attention from the knot theory, Loos ([26], [27]) reinterpreted a symmetric space as a manifold with a binary operation, and this lead to the theories of regular s-manifolds and Φ-spaces, which are generalizations of symmetric spaces ([14], [22]; see also [23], [24], [35]).

A *regular s-manifold* is, in our terminology, a smooth quandle for which the element $1 - d_x s_x \in \mathrm{End}(T_x Q)$ is invertible for any $x \in Q$. Among the most important theoretical results on regular s-manifolds Q is the fact that Q can be described as a homogeneous space G/H, and that the operation $\triangleright$ comes from an automorphism Φ of G [14], [22]. Such a triplet (G, H, Φ) or G/H is called a Φ-space.

A (pseudo-)Riemannian (resp. affine) manifold which admits a compatible structure of a regular s-manifold is called a *generalized symmetric space* (resp. a generalized affine symmetric space). There has been various works on generalized (affine) symmetric spaces and s-manifolds. In [2] and [3], canonical affinor structures of classical type on generalized symmetric spaces, such as almost complex structures and f-structures, are studied. For special metrics and symplectic, complex and Kähler structures on four-dimensional generalized symmetric spaces, see [6], [7], [8], and [9]. Topological (especially homotopy-theoretic) properties of generalized symmetric spaces are examined in [21], [33], and [34]. In [5], certain generalized symmetric spaces are employed to provide new examples of homogeneous spaces G/H without a solvable discrete subgroup of G acting properly, freely and cocompactly on G/H.

Generalized symmetric spaces in the setting of symplectic, Finsler and hyperbolic geometries are studied in [4], [15] and [16]. Further generalizations of the notion of a symmetric space are treated in [31].

1.2. Modules over quandles

Just as in group theory and ring theory, it seems important to consider modules over quandles. As an example of their use, a module over a quandle Q corresponds to a certain extension of Q, providing a building block for constructing and classifying quandles. Furthermore, quandle modules provide a natural setting for the cohomology theory.

The general notion of a quandle module was introduced and studied in [1] and [18] (Definition 3.1). One feature of the definition of [18] is that we consider a disjoint union $\coprod_{x \in Q} A_x$ of different modules A_x for different $x \in Q$, while the case of [1] (and the more naive definition as modules over the associated group) corresponds to the case of the direct product $Q \times A$.

For a topological quandle Q, the notion of a module over Q was given in [12]. We can easily generalize this definition to smooth, complex analytic and algebraic settings (Definition 3.5). One variant of a module is given by a certain space $\mathcal{A} = \coprod_{x \in Q} A_x$ where A_x are additive groups. If A_x are vector spaces and the module structure respects the linear structure, we call $\mathcal{A}$ a linear quandle module. In the algebraic case, there is another variant where we take $\mathcal{A}$ to be a coherent sheaf. The fibers A_x do not have to be isomorphic to each other in general, but in the transitive case (see Definition 2.4) they are, and we can think of $\mathcal{A}$ as a fiber bundle over Q.

Later in this paper, we will focus on connected regular s-manifolds Q. Our main result gives a description of certain linear quandle modules over Q in terms of infinitesimal data.

1.3. Lie-Yamaguti algebras, infinitesimal s-manifolds and their representations

For a regular s-manifold Q and $q \in Q$, we can endow $T = T_qQ$ with the structure of an "infinitesimal s-manifold," just as the tangent space of a Lie group at the identity is a Lie algebra.

Let us describe the correspondence in more details. As we stated above, a regular s-manifold Q can be considered as a homogeneous space G/H, and this is a reductive homogeneous space, defined and studied by Nomizu [29]. Given a reductive homogeneous space, Nomizu considered certain tensors on its tangent space at a point. According to a reformulation by Yamaguti [36], T is equipped with a binary operation $*$ and a ternary operation $[\,]$ which make T a "general Lie triple system" or a Lie-Yamaguti algebra according to the terminology of [20].

The binary operation on Q furthermore defines an automorphism σ of T satisfying certain conditions, and (T, σ) is what [22] calls an infinitesimal s-manifold. It has been known that local isomorphism classes of regular s-manifolds are in one-to-one correspondence with isomorphism classes of infinitesimal s-manifolds ([22, Chapter III], see Theorem 2.15).

In [37], Yamaguti studied representations of Lie-Yamaguti algebras and their cohomology groups. A representation of a Lie-Yamaguti algebra T is given by a quadruplet $(V, \rho, \delta, \theta)$ of a vector space V, a linear map $\rho : T \to \mathrm{End}(V)$ and bilinear maps $\delta, \theta : T \times T \to \mathrm{End}(V)$ (see Definition 4.1).
In this paper, we define a representation of an infinitesimal s-manifold (T, σ) to be a pair (V, ψ) where V is a representation of T and $\psi \in \mathrm{GL}(V)$ is an invertible linear map satisfying certain conditions (Definition 4.3).

1.4. Main result

Let us state our main theorem. In the following, Q can be a smooth manifold, a complex analytic manifold or an algebraic variety over $\mathbb{C}$. A regular module over Q is a linear quandle module satisfying a "nondegeneracy" condition (see Definition 3.7), and homomorphisms are bundle maps compatible with the module structures.

Theorem 1.1 (=Theorem 5.1). *Let Q be a connected regular s-manifold and $q \in Q$ a point. Write $\mathbb{K}$ for the base field.*

(1) *For a regular quandle module $\mathcal{A}$ over Q, there is a natural structure of a regular representation of (T_qQ, d_qs_q) on $\mathcal{A}_q$.*
(2) *Given regular quandle modules $\mathcal{A}$ and $\mathcal{A}'$ over Q and a homomorphism $F : \mathcal{A} \to \mathcal{A}'$, F_q is a homomorphism of representations of (T_qQ, d_qs_q). This correspondence gives an injective map between the sets of homomorphisms. In the smooth or complex analytic case, this map is surjective if Q is simply-connected.*
(3) *Assume that Q is a connected, simply-connected smooth or complex analytic regular s-manifold. Given a regular representation V of (T_qQ, d_qs_q), there exists a regular quandle module $\mathcal{A}$ over Q such that $\mathcal{A}_q$ is isomorphic to V.*

In other words, there is a faithful functor between the following categories:

- *The category* $\mathbf{Mod}^r_{\mathbb{K}}(Q)$ *of regular quandle modules* $(\mathcal{A}, \eta, \tau)$ *over* $(Q, \triangleright)$.
- *The category* $\mathbf{Rep}^r(T_qQ, d_qs_q)$ *of regular representations* $(V, \rho, \delta, \theta, \psi)$ *of the infinitesimal* s*-manifold* $(T_qQ, *, [\ ], d_qs_q)$.

In the smooth or complex analytic case, if Q *is simply-connected, it is an equivalence.*

Here are a couple of ingredients in the proof of this theorem.

We are concerned with two categories, one of regular s-manifolds and the other of infinitesimal s-manifolds. In each of them, we have a correspondence between linear modules or representations and certain "abelian group objects." Thus a functor from the category of (pointed) regular s-manifolds to that of infinitesimal s-manifolds would associate a representation to a module. In fact, the assignment $\mathcal{T} : Q \mapsto T_qQ$ extends to a functor (Corollary 2.25), i.e., a quandle homomorphism induces a homomorphism of infinitesimal s-manifolds (Theorem 2.24), but the proof of this fact is not so straightforward. It might be possible to prove this by combining results from [14], but we will give a proof in this paper since the author could not fully understand the arguments there and [14] is not easily available.

The functor $\mathcal{T}$ is essentially surjective and faithful, and its restriction to connected and simply-connected regular s-manifolds is an equivalence (Corollary 2.25). Thus, if Q is simply-connected and V is a representation of T, we can find a simply-connected regular s-manifold $\mathcal{A}$ over Q. We can show that the fibers of $\mathcal{A} \to Q$ are isomorphic to V as manifolds. It follows that $\mathcal{A} \times_Q \mathcal{A}$ is also simply-connected, and we have an addition map $\mathcal{A} \times_Q \mathcal{A} \to \mathcal{A}$. Together with the scalar multiplication maps, this makes $\mathcal{A}$ a vector bundle over Q with a quandle structure.

1.5. Plan of the paper

In Section 2, after a short account on the definition of racks and quandles and some related notions and facts, we define geometric racks and quandles, and regular s-manifolds in particular. Then we describe how regular s-manifolds and their homomorphisms are related with Φ-spaces, Lie-Yamaguti algebras and infinitesimal s-manifolds and their homomorphisms.

In Section 3, we recall from [18] the notion of a rack or quandle module over a rack or quandle X and its relation with an abelian group object in the category of racks or quandles over X. Then we introduce rack and quandle modules over a geometric rack or quandle. A couple of interesting examples of modules will also be given. In Section 4, a corresponding theory for representations is developed.
The main theorem will be stated and proven in Section 5.

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2. Geometric quandles, Lie-Yamaguti algebras and their homomorphisms

We recall basic definitions related to quandles (cf. [19]). In this paper, we use the convention where X or Q "acts" on itself from the left.

Definition 2.1. A *rack* is a set X endowed with a binary operation $\triangleright$ satisfying the following conditions.

(1) For any $x, y \in X$, there exists a unique element $y' \in X$ such that $x \triangleright y' = y$.

(2) For any $x, y, z \in X$, $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$.

We define $s_x : X \to X$ by $s_x(y) = x \triangleright y$, which is bijective by (1). We write $x \triangleright^{-1} y$ for $s_x^{-1}(y)$. We call $(X, \triangleright)$ a *quandle* if it additionally satisfies the following condition:

(3) For any $x \in X$, $x \triangleright x = x$ holds.

Definition 2.2. Let X and X' be racks or quandles. A *homomorphism* from X to X' is a map $F : X \to X'$ such that $F(x \triangleright y) = F(x) \triangleright F(y)$ holds for any $x, y \in X$. An *isomorphism* is a homomorphism which admits an inverse homomorphism. An isomorphism onto itself is called an *automorphism*.

The following is immediate from the definitions.

Proposition 2.3. (1) *If $F : X \to X'$ is a homomorphism of racks, then $F(x \triangleright^{-1} y) = F(x) \triangleright^{-1} F(y)$ holds.*

(2) *A homomorphism is an isomorphism if and only if it is bijective.*

(3) *For any $x \in X$, the map s_x is an automorphism of X.*

Definition 2.4. Let X be a rack.

(1) The *automorphism group* $\mathrm{Aut}_\triangleright(X)$ of X is the set of all automorphisms of X.

(2) The *group of inner automorphisms* of X is the subgroup

$$\mathrm{Inn}(X) := \langle s_x \mid x \in X \rangle \ \text{ of } \ \mathrm{Aut}_\triangleright(X).$$

(3) The *group of transvections* of X is the subgroup

$$\mathrm{Tr}(X) := \{ s_{x_1}^{e_1} \cdots s_{x_k}^{e_k} \mid x_i \in X, \sum e_i = 0 \} \ \text{ of } \ \mathrm{Inn}_\triangleright(X).$$

(4) A rack X is called *transitive* if the action of $\mathrm{Inn}(X)$ on X is transitive. It is often called "connected."

Remark 2.5. (1) The condition for $F : X \to X$ to be a homomorphism can be expressed as $F s_x = s_{F(x)} F$ for every $x \in X$. In particular, for $g \in \mathrm{Aut}_\triangleright(X)$ we have $g s_x g^{-1} = s_{g(x)}$, and we see that $\mathrm{Inn}(X)$ and $\mathrm{Tr}(X)$ are normal subgroups of $\mathrm{Aut}_\triangleright(X)$.

(2) We have in particular $s_x s_y = s_{x \triangleright y} s_x$ for $x, y \in X$, and it follows that $\mathrm{Tr}(X)$ can also be described as the subgroup generated by, say, $\{ s_x s_{x_0}^{-1} \mid x \in X \}$ for a fixed x_0.

(3) If X is a quandle, then X is transitive if and only if the action of $\mathrm{Tr}(X)$ on X is transitive: If an element of X is written as $s_{x_1}^{e_1} \cdots s_{x_k}^{e_k}(x_0)$, then it can also be expressed as $s_{x_1}^{e_1} \cdots s_{x_k}^{e_k} s_{x_0}^{-\sum e_i}(x_0)$. This does not hold for a rack in general, as seen from the case where $X = \mathbb{Z}$ and $x \triangleright y = y + 1$.

There is a description of transitive quandles in terms of groups and their automorphisms.

Definition 2.6. For an element h of a group G and a normal subgroup N of G, denote by $\mathrm{conj}_N(h)$ the automorphism of N defined by $g \mapsto hgh^{-1}$.

Proposition 2.7 (Theorem 7.1 of [19]). (1) *If G is group, Φ is an automorphism of G and H is contained in the subgroup G^Φ consisting of elements fixed by Φ, then the operation $xH \rhd_\Phi yH := x\Phi(x^{-1}y)H$ on G/H is well-defined and satisfies the quandle axioms.*

(2) *For a transitive quandle Q and $q \in Q$, let G be $\mathrm{Aut}_\rhd(Q)$, $\mathrm{Inn}(Q)$ or $\mathrm{Tr}(Q)$ and $\Phi = \mathrm{conj}_G(s_q)$. Then the stabilizer G_q of q is contained in G^Φ, and the natural map*

$$G/G_q \to Q; \quad gG_q \mapsto gq$$

gives an isomorphism of quandles $(G/G_q, \rhd_\Phi)$ and Q.

Now let us consider quandles equipped with geometric structures. In the setting of algebraic geometry, a variety will mean a reduced separated algebraic scheme over an algebraically closed field k.

Definition 2.8. A *topological quandle* (resp. a *smooth quandle*, a *complex analytic quandle* or a *quandle variety* over k) is a topological space (resp. a smooth manifold, a complex analytic space or an algebraic variety over k) Q equipped with a quandle operation $\rhd$ such that $Q \times Q \to Q \times Q; (q, r) \mapsto (q, q \rhd r)$ is a homeomorphism (resp. a diffeomorphism, a biholomorphic map or a biregular map).

For topologically connected and transitive quandles, we have the following description. The smooth case is due to [17] and the algebraic case to [32]. The complex analytic case is similar to the smooth case.

Theorem 2.9. *Let Q be a connected and transitive smooth quandle (resp. complex analytic quandle or quandle variety over an algebraically closed field of characteristic 0). Then $\mathrm{Inn}(Q)$ has a natural structure of a Lie group (resp. a complex Lie group or an algebraic group with possibly countably many connected components) and $\mathrm{Tr}(Q)$ is the connected component of $\mathrm{Inn}(Q)$ at the identity element.*
For $G = \mathrm{Inn}(Q)$ or $\mathrm{Tr}(Q)$, the action of G on Q is of class C^∞ (resp. holomorphic or regular). Consequently, for any choice of $q \in Q$, there is a natural isomorphism between $(G/G_q, \rhd_{\mathrm{conj}_G(s_q)})$ and Q of smooth quandles (resp. complex analytic quandles or quandle varieties).

For a special kind of smooth quandles, called regular s-manifolds, many results including the above one have been known since 1970's.

Definition 2.10 ([22, Definition II.2]). A *regular s-manifold* is a smooth quandle Q such that $\mathrm{id}_{T_qQ} - d_qs_q \in \mathrm{End}(T_qQ)$ is invertible for any $q \in Q$.

This was based on a reinterpretation of the notion of a symmetric space by Loos as a manifold with a binary operation.

It has also been known that the tangent space of a regular s-manifold Q at a point can be endowed with the structure of a certain kind of algebra, and that Q is locally determined by this algebra.

Definition 2.11 ([37, §1]). (1) A *Lie-Yamaguti algebra* over a field k is a triplet $(T, *, [\ ])$, where T is a finite dimensional k-vector space, $* : T \times T \to T$ is a bilinear operation, and $[\ ] : T \times T \times T \to T$ is a trilinear operation, such that the following hold for any $x, y, z, v, w \in T$.

(LY1) $x * x = 0$.

(LY2) $[x, x, y] = 0$.

(LY3) $[x, y, z] + [y, z, x] + [z, x, y] + (x * y) * z + (y * z) * x + (z * x) * y = 0$.

(LY4) $[x * y, z, w] + [y * z, x, w] + [z * x, y, w] = 0$.

(LY5) $[x, y, z * w] = [x, y, z] * w + z * [x, y, w]$.

(LY6) $[x, y, [z, v, w]] = [[x, y, z], v, w] + [z, [x, y, v], w] + [z, v, [x, y, w]]$.

For $x, y \in T$, we define $D_{x,y} \in \mathrm{End}(T)$ by $D_{x,y}(z) := [x, y, z]$.

(2) A *homomorphism of Lie-Yamaguti algebras* is a linear map $f : T \to T'$ satisfying $f(x * y) = f(x) * f(y)$ and $f([x, y, z]) = [f(x), f(y), f(z)]$.

Remark 2.12. (1) The notion of a Lie-Yamaguti algebra was defined by Yamaguti ([36, Def. 2.1]), under the name "general Lie triple system", as the infinitesimal algebra of a reductive homogeneous space studied by Nomizu. Here we employ the definition from [37] which differs from [36] by a sign. The name "Lie-Yamaguti algebra" appeared in [20].

(2) In the rest of the paper, k will be $\mathbb{R}$ or $\mathbb{C}$. In particular, it is of characteristic 0 and therefore (LY1) is equivalent to $x * y = y * x$ for any $x, y \in T$ and (LY2) is equivalent to $[x, y, z] = -[y, x, z]$ for any $x, y, z \in T$. Note that $[\ ]$ works somewhat differently in the third argument.

(3) Using $D_{x,y}$, the conditions (LY2), (LY4), (LY5) and (LY6) can be expressed as follows, respectively:

(LY2)' $D_{x,x} = 0$.

(LY4)' $D_{x*y,z} + D_{y*z,x} + D_{z*x,y} = 0$.

(LY5)' $D_{x,y}(z * w) = D_{x,y}(z) * w + z * D_{x,y}(w)$, i.e., $D_{x,y}$ is a derivation for $*$.

(LY6)' $D_{x,y}[z, v, w] = [D_{x,y}(z), v, w] + [z, D_{x,y}(v), w] + [z, v, D_{x,y}(w)]$, i.e., $D_{x,y}$ is a derivation for $[\ ]$.

Definition 2.13 ([22, Definition III.20]). (1) An *infinitesimal s-manifold* is a pair (T, σ) of a Lie-Yamaguti algebra T and a linear map $\sigma : T \to T$ satisfying the following:

(ISM0) Both σ and $\mathrm{id}_T - \sigma$ are invertible.

(ISM1) $\sigma(x * y) = \sigma(x) * \sigma(y)$.

(ISM2) $\sigma([x, y, z]) = [\sigma(x), \sigma(y), \sigma(z)]$.

(ISM3) $\sigma([x, y, z]) = [x, y, \sigma(z)]$.

(2) A *homomorphism of infinitesimal s-manifolds* (T, σ) and (T', σ') is a homomorphism $f : T \to T'$ of Lie-Yamaguti algebras satisfying $f \circ \sigma = \sigma' \circ f$.

Remark 2.14. (1) The conditions (ISM1), (ISM2) and the first half of (ISM0) say that σ is an automorphism of T. The condition (ISM3) (for all $z \in T$) can be expressed as $\sigma(D_{x,y}) = D_{x,y}$, where $\sigma(f) := \sigma \circ f \circ \sigma^{-1}$ for $f \in \mathrm{End}(T)$.

(2) In [22, Definition III.20], a more geometric term "tensor" is employed rather than "operation". The notations correspond as follows:

$$\begin{aligned} S_0(x) &= \sigma(x), \\ \tilde{T}_0(x,y) &= -x * y, \\ \tilde{R}_0(x,y)(z) &= -[x,y,z]. \end{aligned}$$

The conditions (LY1), (LY2) correspond to the condition (iv) in [22, Proposition III.19] ; (LY3) to (v); (LY4) to (vi); (LY5), (LY6) and (ISM3) to (ii); (ISM0) to (i); and (ISM1) and (ISM2) to (iii).

Theorem 2.15. *If Q is a connected regular s-manifold and $q \in Q$, the following hold:*

(1) ([22, Proposition II.33]) *As a quandle, Q is transitive.*

(2) ([22, Theorem II.32, Proposition II.38]) *The group $G = \mathrm{Tr}(Q)$ can be endowed with a structure of a connected Lie group with a smooth and transitive action on Q. There is a natural isomorphism between $(G/G_q, \triangleright_{\mathrm{conj}_G(s_q)})$ and Q.*

(3) ([22, Theorem III.22]) *One can endow T_qQ with natural operations $*$ and $[\ ]$ so that (T_qQ, d_qs_q) is an infinitesimal s-manifold.*

(4) ([22, Proposition III.19]) *Regular s-manifolds Q and Q' are locally isomorphic near $q \in Q$ and $q' \in Q'$ if and only if (T_qQ, d_qs_q) and $(T_{q'}Q', d_{q'}s_{q'})$ are isomorphic.*

(5) ([22, Theorem III.25]) *For any infinitesimal s-manifold (T, σ), there exist a connected and simply-connected regular s-manifold Q and an isomorphism $(T_qQ, d_qs_q) \cong (T, \sigma)$.*

It is unique up to a unique isomorphism: If Q' also satisfies the condition with $q' \in Q'$, there is a unique isomorphism of Q onto Q' that sends q to q' and is compatible with the isomorphisms $T_qQ \cong T$ and $T_{q'}Q' \cong T$.

Remark 2.16. (1) For the uniqueness of the isomorphism in (5), see also [22, Theorem III.22, Proposition III.23, Appendix B.7].

(2) Exactly analogous statements hold in the complex analytic case. In the algebraic case (over an algebraically closed field of characteristic 0), the statements (1)–(3) hold. ■

In the next subsection, we will see that the assignment in (3) can be made into a functor: To a homomorphism (not only to an isomorphism as in (4)) of regular s-manifolds, we can associate a homomorphism of infinitesimal s-manifolds.
For that, we will need some construction from the proof of (3)–(5) of this theorem. Let us start with some more definitions.

Definition 2.17. (1) ([14]) A Φ-*space* is a triplet (G, H, Φ) satisfying the following conditions:

- G is a connected Lie group.
- Φ is an endomorphism of G.
- H is a subgroup of G^Φ containing the connected component G_0^Φ of G^Φ at the identity.

A Φ-space (G, H, Φ) is called a *regular Φ-space* if Φ is an automorphism and $\mathrm{Lie}(H)$ (which is equal to $\mathrm{Lie}(G)^{d_e\Phi}$) is equal to the *generalized* eigenspace of $d_e\Phi \in \mathrm{End}(\mathrm{Lie}(G))$ for 1.

If Φ is an automorphism, the quotient space G/H with the operation $\triangleright_\Phi$ defined in Proposition 2.7 (1) is called a *homogeneous Φ-space*. If furthermore (G, H, Φ) is a regular Φ-space, then $(G/H, \triangleright_\Phi)$ is called a *homogeneous regular Φ-space*.

(1)' A *weak Φ-space* is a triplet (G, H, Φ) satisfying the following conditions:

- G is a connected Lie group.
- Φ is an endomorphism of G.
- H is a closed subgroup contained in G^Φ.

A *transitive weak Φ-space* is a weak Φ-space where Φ is an automorphism and $(G/H, \triangleright_\Phi)$ is a transitive quandle.

If Φ is an automorphism, then $(G/H, \triangleright_\Phi)$ is called a *homogeneous weak Φ-space*. It is called a *homogeneous transitive weak Φ-space* if it is transitive as a quandle.

(2) ([14]) A *homomorphism* of (weak) Φ-spaces (G, H, Φ) to (G', H', Φ') is a homomorphism of Lie groups $G \to G'$ which maps H into H' and commutes with Φ and Φ'.

We make similar definitions in the complex analytic and algebraic categories.

Definition 2.18. (1) For a smooth quandle (resp. a complex analytic quandle or a quandle variety) Q, a *(weak) Φ-space for Q* is a pair of a (weak) Φ-space (G, H, Φ) with Φ an automorphism and an isomorphism $(G/H, \triangleright_\Phi) \to (Q, \triangleright)$.

(2) Let (G, H, Φ) and (G', H', Φ') be (weak) Φ-spaces for Q and Q' given by isomorphisms $\alpha : G/H \to Q$ and $\alpha' : G'/H' \to Q'$. For a quandle homomorphism $F : Q \to Q'$, a *homomorphism $\tilde{F} : (G, H, \Phi) \to (G', H', \Phi')$ of (weak) Φ-spaces over F* is a homomorphism of weak Φ-spaces, given by $\tilde{F} : G \to G'$, which is compatible with F: Namely, if $\pi : G \to G/H$ and $\pi' : G' \to G'/H'$ are the natural projections, then $F \circ \alpha \circ \pi = \alpha' \circ \pi' \circ \tilde{F}$.

Remark 2.19. (1) Note that, if Q is a regular s-manifold and (G, H, Φ) is a weak Φ-space for Q, then 1 is not an eigenvalue of the endomorphism induced by $d_e\Phi$ on $\mathrm{Lie}(G)/\mathrm{Lie}(H)$, and thus (G, H, Φ) is in fact a regular Φ-space.

(2) From the previous theorems, a connected transitive smooth quandle Q (or complex analytic quandle or quandle variety in characteristic 0) is a homogeneous weak Φ-space. More precisely, if we take a point $q \in Q$ and let $G = \mathrm{Tr}(Q)$, then the triple $(G, G_q, \mathrm{conj}_G(s_q))$ together with the map $G/G_q \to Q; gG_q \mapsto g \cdot q$ is a transitive weak Φ-space for Q. If Q is a connected regular s-manifold, then $(G, G_q, \mathrm{conj}_G(s_q))$ is a regular Φ-space for Q.

Definition 2.20 ([14], [22, Definition III.29])**.** (1) A *local regular s-triplet* is a triplet $(\mathfrak{g}, \mathfrak{h}, \varphi)$ satisfying the following conditions:

- $\mathfrak{g}$ is a Lie algebra.
- φ is an automorphism of $\mathfrak{g}$.
- $\mathfrak{h} = \mathfrak{g}^\varphi$, and it is also equal to the generalized eigenspace of φ for 1.

Note that we actually do not have to specify $\mathfrak{h}$.

(2) A *homomorphism* $(\mathfrak{g}, \mathfrak{h}, \varphi) \to (\mathfrak{g}', \mathfrak{h}', \varphi')$ *of local regular s-triplets* is a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{g}'$ which commutes with φ and φ' (and hence maps $\mathfrak{h}$ into $\mathfrak{h}'$).

We can associate an infinitesimal s-manifold to a local regular s-triplet, and vice versa, as follows. Restricting to "prime" local regular s-triplets, we have a one-to-one correspondence [22, Theorem III.30].

Construction 2.21. Let $(\mathfrak{g}, \mathfrak{h}, \varphi)$ be a local regular s-triplet. There is a canonical decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ as a vector space, where $\mathfrak{m}$ is the sum of the generalized eigenspaces of φ for eigenvalues different from 1. We see that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ holds: In other words, $(\mathfrak{g}, \mathfrak{m}, \mathfrak{h})$ is a reductive triple. For an element $X \in \mathfrak{g}$, let $X_{\mathfrak{m}}$ and $X_{\mathfrak{h}}$ denote its $\mathfrak{m}$-component and $\mathfrak{h}$-component, respectively. Then we have $\varphi(X_{\mathfrak{m}}) = \varphi(X)_{\mathfrak{m}}$ and $\varphi(X_{\mathfrak{h}}) = \varphi(X)_{\mathfrak{h}} = X_{\mathfrak{h}}$.

We define operations on $\mathfrak{m}$, and hence on $\mathfrak{g}/\mathfrak{h}$ via the natural identification $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$, as follows: For $x, y, z \in \mathfrak{m}$,

$$\sigma(x) := \varphi(x), \quad x * y := [x, y]_{\mathfrak{m}}, \quad [x, y, z] := [[x, y]_{\mathfrak{h}}, z].$$

(Note that $\varphi(\mathfrak{m}) \subseteq \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ hold.) Then $\mathfrak{m}$ with the operations $*$ and $[\ ]$ is a Lie-Yamaguti algebra. It is straightforward to see that σ is an automorphism on $(\mathfrak{m}, *, [\ ])$ and that $\sigma([x, y, z]) = [x, y, \sigma(z)]$ holds. The eigenvalues of φ on $\mathfrak{m}$ are all different from 1, and it follows that $\mathrm{id}_{\mathfrak{m}} - \sigma$ is invertible. Thus we have a structure of an infinitesimal s-manifold on $\mathfrak{m}$ or $\mathfrak{g}/\mathfrak{h}$.

Definition 2.22. (1) For an infinitesimal s-manifold (T, σ), a pair of a local regular s-triplet $(\mathfrak{g}, \mathfrak{h}, \varphi)$ and an isomorphism $\mathfrak{g}/\mathfrak{h} \to T$ is called a *local regular s-triplet for T*.

(2) Given local regular s-triplets $(\mathfrak{g}, \mathfrak{h}, \varphi)$ and $(\mathfrak{g}', \mathfrak{h}', \varphi')$ for T and T', respectively, and a homomorphism $f : T \to T'$ of infinitesimal s-manifolds, a *homomorphism of local regular s-triplets* $(\mathfrak{g}, \mathfrak{h}, \varphi) \to (\mathfrak{g}', \mathfrak{h}', \varphi')$ *over f* is a homomorphism of local regular s-triplets which induces f.

Construction 2.23. For a Lie-Yamaguti algebra T, a reductive triple inducing T can be constructed as follows ([29, pp. 61–62], [36, Prop. 2.1]). We write $D_{x,y} \in \mathrm{End}(T)$ for the map $z \mapsto [x, y, z]$.

- $\mathfrak{h}(T)$ is the linear subspace of $\mathrm{End}(T)$ generated by $\{D_{x,y} \mid x, y \in T\}$, which is in fact a Lie subalgebra of $\mathrm{End}(T)$.
- $\mathfrak{g}(T) := T \oplus \mathfrak{h}(T)$.
- $[(x, u), (y, v)] := (x * y + u(y) - v(x), D_{x,y} + [u, v])$.

It is straightforward to check that this defines a Lie algebra and that $(\mathfrak{g}(T), T, \mathfrak{h}(T))$ is a reductive triple, using the axioms of Lie-Yamaguti algebras. (A sign in the definition of $[\ ]$ is different from the one in [36] since we are following the conventions in [37].)

An automorphism σ of T extends to an automorphism φ_σ of $\mathfrak{g}(T)$ by the rule $\varphi_\sigma(x, f) = (\sigma(x), \sigma \circ f \circ \sigma^{-1})$. If (T, σ) is an infinitesimal s-manifold, then

$\sigma([x,y,z]) = [x,y,\sigma(z)]$ holds for any $x,y,z \in T$. This is equivalent to the equation $\sigma \circ D_{x,y} \circ \sigma^{-1} = D_{x,y}$, hence $\varphi_\sigma|_{\mathfrak{h}(T)} = \mathrm{id}_{\mathfrak{h}(T)}$ holds. Since σ does not have 1 as an eigenvalue, we see that $(\mathfrak{g}(T), \mathfrak{h}(T), \varphi_\sigma)$ is a local regular s-triplet. (This is a "prime" local regular s-triplet in the sense that $\mathfrak{h}(T)$ does not contain a proper ideal of $\mathfrak{g}(T)$ and $[T,T] \cap \mathfrak{h}(T) = \mathfrak{h}(T)$; see [22, Theorem III.30].)

It is easy to see that $(\mathfrak{g}(T), \mathfrak{h}(T), \varphi_\sigma)$ is a local regular s-triplet for T. ■

Now let Q be a connected regular s-manifold. We take a regular Φ-space (G,H,Φ) for Q, and let $q \in Q$ be the point corresponding to the residue class of $e \in G$. Then $(\mathrm{Lie}(G), \mathrm{Lie}(H), d_e\Phi)$ is a local regular s-triplet, and therefore we have a structure of an infinitesimal s-manifold on $\mathrm{Lie}(G)/\mathrm{Lie}(H) \cong T_qQ$. This is independent of the choice of a Φ-space. (In fact, there is also a differential-geometric description of the infinitesimal algebra using a canonical affine connection on Q. See [22, Chapter III].)

Conversely, if (T,σ) is an infinitesimal s-manifold, let $(\mathfrak{g}(T), \mathfrak{h}(T), \varphi_\sigma)$ be the associated local regular s-triplet, G the connected and simply-connected Lie group corresponding to $\mathfrak{g}(T)$ and H the connected Lie subgroup of G corresponding to $\mathfrak{h}(T)$. The automorphism φ_σ of $\mathfrak{g}(T)$ defines an automorphism Φ of G, and since H is a subgroup of G^Φ with the same Lie algebra, we see that H is a closed subgroup. Thus (G,H,Φ) is a regular Φ-space, and $(G/H, \triangleright_\Phi)$ is a connected and simply-connected regular s-manifold whose infinitesimal algebra is isomorphic to (T,σ).

2.1. Homomorphisms of transitive geometric quandles

Although some part of the following theorem is a combination of results in [14], we will give a detailed proof since the monograph [14] is of limited availability.

Theorem 2.24 (See [14]). *Let Q and Q' be connected regular s-manifolds and take $q \in Q$, $q' \in Q'$.*

(1) *If $F : Q \to Q'$ is a homomorphism with $F(q) = q'$, then $d_qF : T_qQ \to T_{q'}Q'$ is a homomorphism of infinitesimal s-manifolds.*
If $F_1, F_2 : Q \to Q'$ are two homomorphisms with $F_1(q) = F_2(q) = q'$ and $d_qF_1 = d_qF_2$, then $F_1 = F_2$ holds.

(2) *In the smooth or complex analytic case, if $f : T_qQ \to T_{q'}Q'$ is a homomorphism of infinitesimal s-manifolds, then there exists a unique local homomorphism of the form $F : (Q,q) \to (Q',q')$ such that $d_qF = f$.*
If Q is simply-connected, one can take a global homomorphism.

Combined with Theorem 2.15, this implies the following.

Corollary 2.25. *In the smooth or complex analytic setting, let* $\mathbf{Crsm}_*$ *be the category of pointed connected regular s-manifolds,* $\mathbf{SCrsm}_*$ *the category of pointed simply-connected and connected regular s-manifolds, and* $\mathbf{Ism}$ *the category of infinitesimal s-manifolds. The assignment $(Q,q) \mapsto T_qQ$ gives an essentially surjective faithful functor $\mathcal{T} : \mathbf{Crsm}_* \to \mathbf{Ism}$, and its restriction $\mathbf{SCrsm}_* \to \mathbf{Ism}$ is an equivalence.*

Remark 2.26. In the algebraic case, we can say that the assignment gives a faithful functor. ■

For the proof, we first study homomorphisms of transitive geometric quandles in a little more generality than is needed for the proof of the theorem.

We begin with the case of a *surjective* quandle homomorphism, although this is not logically required for the proof of the theorem. In this case, we have natural homomorphisms between the groups of inner automorphisms and transvections, respectively.

Proposition 2.27. *Let $F : Q \to Q'$ be a surjective homomorphism of quandles.*

(1) *There are naturally induced surjective homomorphisms*

$$\mathrm{Inn}(F) : \mathrm{Inn}(Q) \to \mathrm{Inn}(Q') \quad \textit{and} \quad \mathrm{Tr}(F) : \mathrm{Tr}(Q) \to \mathrm{Tr}(Q'),$$

characterized by $\mathrm{Inn}(F)(s_q) = s_{F(q)}$ *and* $\mathrm{Tr}(F)(s_q^{-1}s_r) = s_{F(q)}^{-1}s_{F(r)}$ *(and, more generally,* $\mathrm{Tr}(F)(s_{q_1}^{e_1}\cdots s_{q_k}^{e_k}) = s_{F(q_1)}^{e_1}\cdots s_{F(q_k)}^{e_k}$ *for* $\sum e_i = 0$*).*

These homomorphisms are compatible with the actions of $\mathrm{Inn}(Q)$*,* $\mathrm{Inn}(Q')$*,* $\mathrm{Tr}(Q)$ *and* $\mathrm{Tr}(Q')$ *on* Q *and* Q'*.*

(2) *If furthermore Q and Q' are connected and transitive smooth quandles (resp. complex analytic quandles or quandle varieties) and F is a C^∞-map (resp. holomorphic or regular map), then* $\mathrm{Inn}(F)$ *and* $\mathrm{Tr}(F)$ *are C^∞-maps (resp. holomorphic or regular maps).*

Consequently, if we choose $q \in Q$, let $G = \mathrm{Tr}(Q)$, $G' = \mathrm{Tr}(Q')$, $\Phi = \mathrm{conj}_G(s_q)$ and $\Phi' = \mathrm{conj}_{G'}(s_{F(q)})$ and think of (G, G_q, Φ) and $(G', G'_{F(q)}, \Phi')$ as transitive weak Φ-spaces for Q and Q', then $\mathrm{Tr}(F)$ *is a homomorphism of weak Φ-spaces over F.*

Proof. (1) The fact that F induces such a homomorphism $\mathrm{Inn}(Q) \to \mathrm{Inn}(Q')$ is proven in [11, Proposition 2.26]. This homomorphism maps $\mathrm{Tr}(Q)$ into $\mathrm{Tr}(Q')$. The surjectivity is obvious.

The statement on the compatibility of group actions follows from $F(s_q(r)) = s_{F(q)}(F(r))$, which is nothing but the condition that F is a quandle homomorphism.

(2) We prove the first assertion for inner automorphism groups. The case of the groups of transvections is similar.

From Sard's theorem, the transitivity of the action of $\mathrm{Inn}(Q)$ on Q and the compatibility of the actions, we see that F is submersive. For any $q' \in Q'$, let us take a neighborhood U' of q' and a local section $\sigma : U' \to Q$ of F. (In the algebraic case, take an étale neighborhood.) Then the action of $\mathrm{Inn}(Q)$ on Q' is locally given by the composite map

$$\mathrm{Inn}(Q) \times U' \cong \mathrm{Inn}(Q) \times \sigma(U') \hookrightarrow \mathrm{Inn}(Q) \times Q \to Q \to Q',$$

hence is of class C^∞ (resp. holomorphic or regular). This implies that $\mathrm{Inn}(F)$ is smooth, holomorphic or regular, respectively. (In the algebraic case, it is also possible to use Zariski's Main Theorem.)

The assertion of the second paragraph is a consequence of the compatibility of the actions. ∎

In the general case, we have a group homomorphism after extending the group acting on Q.

Proposition 2.28. *Let Q and Q' be quandles, $G = \mathrm{Inn}(Q)$ and $G' = \mathrm{Inn}(Q')$ (resp. $G = \mathrm{Tr}(Q)$ and $G' = \mathrm{Tr}(Q')$) and let $F : Q \to Q'$ be a quandle homomorphism.*

(1) *Let $G^\dagger := \{(g, g') \in G \times G' \mid F \circ g = g' \circ F\} \subseteq G \times G'$, and let $\pi_F : G^\dagger \to G$ and $F^\dagger : G^\dagger \to G'$ be the maps induced by the projection maps.*
Then $G^\dagger$ is a subgroup of $G \times G'$ which satisfies the following:

- *$\pi_F : G^\dagger \to G$ is surjective.*
- *$G^\dagger$ is invariant under $\mathrm{conj}_G(s_q) \times \mathrm{conj}_{G'}(s_{F(q)})$ for any $q \in Q$.*
- *If we define an action of $G^\dagger$ on Q via π_F and an action of $G^\dagger$ on Q' via $F^\dagger$, then F is a $G^\dagger$-equivariant map.*
- *For any $q \in Q$, the stabilizer subgroup ${G^\dagger}_q$ (with respect to the above action) is elementwise fixed by $\mathrm{conj}_G(s_q) \times \mathrm{conj}_{G'}(s_{F(q)})$, and is mapped into ${G'}_{F(q)}$ by $F^\dagger : G^\dagger \to G'$.*

(2) *Assume furthermore that Q and Q' are transitive. Then, for any $q \in Q$, there is a natural identification of $(G^\dagger/{G^\dagger}_q, \rhd_{\Phi^\dagger})$ with Q, where $\Phi^\dagger$ is the restriction of $\mathrm{conj}_G(s_q) \times \mathrm{conj}_{G'}(s_{F(q)})$ to $G^\dagger$. The group homomorphism $F^\dagger$ commutes with $\Phi^\dagger$ and $\mathrm{conj}_{G'}(s_{F(q)})$, and it commutes with F.*

(3) *If Q and Q' are connected and transitive smooth quandles, complex analytic quandles or quandle varieties, then $G^\dagger$ is a closed subgroup of $G \times G'$.*
In the case $G = \mathrm{Tr}(Q)$ and $G' = \mathrm{Tr}(Q')$, let $\tilde{G}$ be the connected component of $G^\dagger$ at the identity and $\tilde{\Phi}$ the restriction of $\Phi^\dagger$ to $\tilde{G}$. Then, we have a homomorphism $\tilde{F} : (\tilde{G}, \tilde{G}_q, \tilde{\Phi}) \to (G', {G'}_{F(q)}, \mathrm{conj}_{G'}(s_{F(q)}))$ of weak Φ-spaces over F.

Proof. (1) It is easy to see that $G^\dagger$ is a subgroup of $G \times G'$. From the assumption that F is a quandle homomorphism, we have $F \circ s_x = s_{F(x)} \circ F$ (resp. $F \circ s_x \circ s_q^{-1} = s_{F(x)} \circ s_{F(q)}^{-1} \circ F$) for any $x \in Q$ (and $q \in Q$), and hence $(s_x, s_{F(x)}) \in G^\dagger$ (resp. $(s_x \circ s_q^{-1}, s_{F(x)} \circ s_{F(q)}^{-1}) \in G^\dagger$). Since G is generated by s_x (resp. $s_x \circ s_q^{-1}$), we see that the projection $\pi_F : G^\dagger \to G$ is surjective.

The assertion that F is $G^\dagger$-equivariant follows from the definition of $G^\dagger$.

If we assume that $(g, g') \in G^\dagger$, i.e., $F \circ g = g' \circ F$, then we have

$$F \circ s_q \circ g \circ s_q^{-1} = s_{F(q)} \circ g' \circ s_{F(q)}^{-1} \circ F,$$

i.e., $(\mathrm{conj}_G(s_q)(g), \mathrm{conj}_{G'}(s_{F(q)})(g')) \in G^\dagger$. Thus $G^\dagger$ is invariant under $\mathrm{conj}_G(s_q) \times \mathrm{conj}_{G'}(s_{F(q)})$.

Note that in general, for an automorphism g of Q satisfying $g(q) = q$,

$$s_q \circ g \circ s_q^{-1} = s_q \circ s_{g(q)}^{-1} \circ g = g$$

holds. Thus, if $g^\dagger = (g, g') \in G^\dagger$ is in the stabilizer ${G^\dagger}_q$ of q, then we have $\mathrm{conj}_G(s_q)(g) = g$. Since F is $G^\dagger$-equivariant, $F(q)$ is fixed by $g^\dagger$, or g'. Hence $\mathrm{conj}_{G'}(s_{F(q)})(g') = g'$. Thus $g^\dagger$ is fixed by $\mathrm{conj}_G(s_q) \times \mathrm{conj}_{G'}(s_{F(q)})$ and $F^\dagger(g^\dagger) = g' \in {G'}_{F(q)}$.

(2) By (1), $\Phi^\dagger$ fixes ${G^\dagger}_q$ elementwise, and we can define a quandle $(G^\dagger/{G^\dagger}_q, \triangleright_{\Phi^\dagger})$ as in Proposition 2.7.

Since π_F is surjective and $\pi_F^{-1}(G_q) = {G^\dagger}_q$, we have in consequence a bijective map $G^\dagger/{G^\dagger}_q \to G/G_q$. Since π_F commutes with $\Phi^\dagger$ and $\mathrm{conj}_G(s_q)$, it gives a quandle isomorphism $(G^\dagger/{G^\dagger}_q, \triangleright_{\Phi^\dagger}) \cong (G/G_q, \triangleright_{\mathrm{conj}_G(s_q)}) \cong Q$.

It is easy to see that $F^\dagger$ commutes with $\Phi^\dagger$ and $\mathrm{conj}_{G'}(s_{F(q)})$. To see the compatibility with F, let $\pi : G^\dagger \to G^\dagger/{G^\dagger}_q$ and $\pi' : G' \to G'/G'_{F(q)}$ denote the projections and $\alpha : G^\dagger/{G^\dagger}_q \cong Q$ and $\alpha' : G'/G'_{F(q)} \cong Q'$ the isomorphisms determined by the actions of $G^\dagger$ and G' on Q and Q', and $g^\dagger = (g, g')$ an element of $G^\dagger$, and then we have

$$F(\alpha(\pi(g^\dagger))) = F(g \cdot q) = g' \cdot F(q) = F^\dagger(g^\dagger)F(q) = \alpha'(\pi'(F^\dagger(g^\dagger))).$$

(3) In the continuous case, for any $x \in Q$, the map

$$G \times G' \to Q' \times Q' : (g, g') \mapsto (F(g(x)), g'(F(x)))$$

is smooth, complex analytic or regular, respectively. Thus $F(g(x)) = g'(F(x))$ defines a closed subset of $G \times G'$ (since the diagonal set in $Q' \times Q'$ is closed). Taking the intersection over all $x \in Q$, we see that $G^\dagger$ is a closed subgroup.

In the case $G = \mathrm{Tr}(Q)$, it is a connected group and so the restriction of $\pi_F : G^\dagger \to G$ to the identity component $\tilde{G}$ of $G^\dagger$ is surjective as well.

It is easy to see that, when $G^\dagger$ is replaced by $\tilde{G}$, other statements in (1) and (2) are also satisfied. Thus $(\tilde{G}, \tilde{G}_q, \tilde{\Phi})$ is a weak Φ-space for Q and $\tilde{F} := F^\dagger|_{\tilde{G}}$ is a homomorphism of weak Φ-spaces over F. ■

Remark 2.29. We do not necessarily have a natural homomorphism $G \to G'$, even if Q and Q' are transitive. For inner automorphism groups, this has been pointed out in [11, §§2.7]. Here we will give an example with $G = \mathrm{Tr}(Q)$ and $G' = \mathrm{Tr}(Q')$.

Let $X_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and let Q be the conjugacy class of X_0 in $\mathrm{SL}(2, \mathbb{C})$, considered as a quandle by the conjugation operation. Then let $Q' = Q \times \mathbb{C}^2$ with

$$(X, a) \triangleright (Y, b) = (XYX^{-1}, a + X(b - a)).$$

Then Q and Q' are connected regular s-manifolds, hence transitive smooth quandles. We embed Q into Q' as the zero section $Q \times \{0\}$. Take $X_1 = -X_0 \in Q$. Then $s_{X_1} \circ s_{X_0}^{-1}$ is the identity map on Q, while on Q' we have $s_{(X_1,0)} \circ s_{(X_0,0)}^{-1}(Y, b) = (Y, -b)$. Thus the natural correspondence does not give a well-defined map.

In this case, the transvection group of Q is $\mathrm{PSL}(2, \mathbb{C})$, while the subgroup of $\mathrm{Tr}(Q')$ generated by transvections from Q is $\mathrm{SL}(2, \mathbb{C})$. ■

Next, we look at the infinitesimal counterpart. We use the notations in Construction 2.23. Recall in particular that $\mathfrak{g}(T) = T \oplus \mathfrak{h}(T)$ for a Lie-Yamaguti algebra T and therefore that an element of $\mathfrak{g}(T)$ can be written as (x, u) with $x \in T$ and $u \in \mathrm{End}(T)$.

Proposition 2.30. (1) *For a homomorphism $f : T \to T'$ of Lie-Yamaguti algebras, let*

$$\begin{aligned}\tilde{\mathfrak{g}} &:= \{((x,u),(x',u')) \in \mathfrak{g}(T) \oplus \mathfrak{g}(T') \mid f(x) = x', u' \circ f = f \circ u\},\\ \tilde{\mathfrak{m}} &:= \{((x,0),(x',0)) \in \mathfrak{g}(T) \oplus \mathfrak{g}(T') \mid f(x) = x'\},\\ \tilde{\mathfrak{h}} &:= \{((0,u),(0,u')) \in \mathfrak{g}(T) \oplus \mathfrak{g}(T') \mid u' \circ f = f \circ u\}.\end{aligned}$$

Then $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}(T) \oplus \mathfrak{g}(T')$ is a Lie subalgebra, $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{m}}, \tilde{\mathfrak{h}})$ is a reductive triple, and the projection maps $\pi_f : \tilde{\mathfrak{g}} \to \mathfrak{g}(T)$ and $\tilde{f} : \tilde{\mathfrak{g}} \to \mathfrak{g}(T')$ are homomorphisms of reductive triples. The map π_f is surjective and induces an isomorphism $\tilde{\mathfrak{m}} \cong T$ of Lie-Yamaguti algebras, and $\tilde{f}$ induces f.

If f is surjective, then π_f is an isomorphism and there is a Lie algebra homomorphism $\tilde{f} : \mathfrak{g}(T) \to \mathfrak{g}(T')$ which restricts to f.

(2) *Let $f : (T,\sigma) \to (T',\sigma')$ be a homomorphism of infinitesimal s-manifolds. Then $\tilde{\mathfrak{g}}$ defined as above is invariant under $\varphi_\sigma \times \varphi_{\sigma'}$. If $\tilde{\varphi}$ denotes the restriction of $\varphi_\sigma \times \varphi_{\sigma'}$ to $\tilde{\mathfrak{g}}$, then $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\varphi})$ is a local regular s-triplet for (T,σ), and $\tilde{f}$ is a homomorphism of local regular s-triplets over f.*

Proof. (1) It is clear that $\tilde{\mathfrak{g}}, \tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{h}}$ are linear subspaces and that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{m}} \oplus \tilde{\mathfrak{h}}$. So, we look at the following three types of commutators.
For $((x,0),(x',0)),((y,0),(y',0)) \in \tilde{\mathfrak{m}}$, we have $f(x) = x'$ and $f(y) = y'$, and

$$[((x,0),(x',0)),((y,0),(y',0))] = ((x * y, D_{x,y}),(x' * y', D_{x',y'})).$$

Since f is a homomorphism, we have $f(x * y) = x' * y'$. Also, for any $z \in T$,

$$D_{f(x),f(y)}(f(z)) = [f(x), f(y), f(z)] = f([x,y,z]) = f(D_{x,y}(z)),$$

hence $D_{f(x),f(y)} \circ f = f \circ D_{x,y}$. Thus $((x * y, D_{x,y}),(x' * y', D_{x',y'}))$ belongs to $\tilde{\mathfrak{g}}$.
If $((0,u),(0,u')),((0,v),(0,v')) \in \tilde{\mathfrak{h}}$, then $u' \circ f = f \circ u$ and $v' \circ f = f \circ v$. We have

$$[((0,u),(0,u')),((0,v),(0,v'))] = ((0,[u,v]),(0,[u',v'])),$$

and $[u',v'] \circ f = u' \circ v' \circ f - v' \circ u' \circ f = f \circ u \circ v - f \circ v \circ u = f \circ [u,v]$. Thus $\tilde{\mathfrak{h}}$ is closed under the Lie bracket.
If $((0,u),(0,u')) \in \tilde{\mathfrak{h}}$ and $((x,0),(x',0)) \in \tilde{\mathfrak{m}}$, then $f(x) = x'$ and $u' \circ f = f \circ u$, and

$$[((0,u),(0,u')),((x,0),(x',0))] = ((u(x),0),(u'(x'),0)),$$

which belongs to $\tilde{\mathfrak{m}}$ since $f(u(x)) = u'(f(x)) = u'(x')$. Thus $[\tilde{\mathfrak{h}}, \tilde{\mathfrak{m}}] \subseteq \tilde{\mathfrak{m}}$, and this concludes the proof of the statement that $\tilde{\mathfrak{g}}$ is a Lie subalgebra and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{m}}, \tilde{\mathfrak{h}})$ is a reductive triple.

Obviously, the projections are Lie algebra homomorphisms, and map $\tilde{\mathfrak{m}}$ into T and T' and $\tilde{\mathfrak{h}}$ into $\mathfrak{h}(T)$ and $\mathfrak{h}(T')$. Hence they are homomorphisms of reductive triples.

To show the surjectivity of π_f, it suffices to show that $(x,0)$ and $(0,D_{x,y})$ are in the image for any $x,y \in T$. The former is clear from $((x,0),(f(x),0)) \in \tilde{\mathfrak{m}}$. For the latter, we already saw that $D_{f(x),f(y)} \circ f = f \circ D_{x,y}$, which means that $((0,D_{x,y}),(0,D_{f(x),f(y)})) \in \tilde{\mathfrak{h}}$, and therefore that $(0,D_{x,y}) \in \operatorname{Im} \pi_f$.

It is clear that $\tilde{\mathfrak{m}}$ maps to T bijectively. Since π_f is a homomorphism of reductive triples, the induced map $\tilde{\mathfrak{m}} \to T$ is an isomorphism of Lie-Yamaguti algebras.

Similarly, the map $\tilde{\mathfrak{m}} \to T'$ induced by $\tilde{f}$ is a homomorphism of Lie-Yamaguti algebras which can be identified with f.

Finally, if f is surjective, then the condition $u' \circ f = f \circ u$ determines u' uniquely for a given u, and π_f is an isomorphism.

(2) The assumption means that $f \circ \sigma = \sigma' \circ f$. Then, for $((x,0),(x',0)) \in \tilde{\mathfrak{m}}$ we have $f(x) = x'$ and

$$(\varphi_\sigma \times \varphi_{\sigma'})((x,0),(x',0)) = ((\sigma(x),0),(\sigma'(x'),0)),$$

and from $f(\sigma(x)) = \sigma'(f(x)) = \sigma'(x')$ we see that $\tilde{\mathfrak{m}}$ is stable under $\varphi_\sigma \times \varphi_{\sigma'}$.

For $((0,u),(0,u')) \in \tilde{\mathfrak{h}}$, $u \in \mathfrak{h}(T)$ and $u' \in \mathfrak{h}(T')$ imply $\varphi_\sigma(u) = u$ and $\varphi_{\sigma'}(u') = u'$. Thus $((0,u),(0,u'))$ is fixed by $\varphi_\sigma \times \varphi_{\sigma'}$, and in particular $\tilde{\mathfrak{h}}$ is stable under $\varphi_\sigma \times \varphi_{\sigma'}$.

Thus $\varphi_\sigma \times \varphi_{\sigma'}$ restricts to an automorphism $\tilde{\varphi}$ on $\tilde{\mathfrak{g}}$, and $\tilde{\mathfrak{h}}$ is elementwise fixed by φ. Since $\tilde{\varphi}|_{\tilde{\mathfrak{m}}}$ can be identified with σ under the identification of $\tilde{\mathfrak{m}}$ with T, it does not have 1 as an eigenvalue. Thus $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\varphi})$ is a local regular s-triplet.

Since π_f commutes with $\tilde{\varphi}$ and φ_σ, it is a homomorphism of local regular s-triplets, and therefore $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\varphi})$ is a local regular s-triplet for (T, σ).

Finally, $\tilde{f}$ commutes with $\tilde{\varphi}$ and $\varphi_{\sigma'}$, and induces f on $\tilde{\mathfrak{m}} \cong T$, so it is a homomorphism of local regular s-triplets over f. ■

Now let us prove Theorem 2.24.

Proof. (1) Let

- $G = \mathrm{Tr}(Q)$, $\Phi = \mathrm{conj}_G(s_q)$, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(G_q)$, $\varphi = d_e\Phi$ and
- $G' = \mathrm{Tr}(Q')$, $\Phi' = \mathrm{conj}_{G'}(s_{q'})$, $\mathfrak{g}' = \mathrm{Lie}(G')$, $\mathfrak{h}' = \mathrm{Lie}(G'_{q'})$, $\varphi' = d_e\Phi'$.

By Proposition 2.28 (3) and Remark 2.19 (1), we have regular Φ-spaces $(\tilde{G}, \tilde{G}_q, \tilde{\Phi})$ and $(G', G'_{q'}, \Phi')$ for Q and Q' and a homomorphism $\tilde{F} : \tilde{G} \to G'$ of regular Φ-spaces over F. Let $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{G})$, $\tilde{\mathfrak{h}} = \mathrm{Lie}(\tilde{G}_q)$ and $\tilde{\varphi} = d_e\tilde{\Phi}$. Then we have a homomorphism $d_e\tilde{F} : (\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\varphi}) \to (\mathfrak{g}', \mathfrak{h}', \varphi')$ of local regular s-triplets. This induces a homomorphism $\overline{d_e\tilde{F}} : \tilde{\mathfrak{g}}/\tilde{\mathfrak{h}} \to \mathfrak{g}'/\mathfrak{h}'$ of infinitesimal s-manifolds. The domain and the target can be identified with T_qQ and $T_{q'}Q'$ as infinitesimal s-manifolds, and the map $\overline{d_e\tilde{F}}$ with d_qF. Thus d_qF is a homomorphism of infinitesimal s-manifolds.

Let $\mathfrak{m}$, $\tilde{\mathfrak{m}}$ and $\mathfrak{m}'$ denote the sum of eigenspaces of $d_e\Phi$, $d_e\tilde{\Phi}$ and $d_e\Phi'$ for the eigenvalues different from 1. Then $d_e\pi_F$ restricts to an isomorphism $\alpha_F : \tilde{\mathfrak{m}} \to \mathfrak{m}$ since $d_e\pi_F$ commutes with $d_e\tilde{\Phi}$ and $d_e\Phi$. Similarly, $d_e\tilde{F}$ restricts to a map $\beta_F : \tilde{\mathfrak{m}} \to \mathfrak{m}'$. Since the map $\mathfrak{g}/\mathfrak{h} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{h}} \to \mathfrak{g}'/\mathfrak{h}'$ induced by $d_e\pi_F$ and $d_e\tilde{F}$ can be identified with d_qF, we may think of $\beta_F \circ \alpha_F^{-1}$ as d_qF. For a small neighborhood U of 0 in $\mathfrak{m}$, $\exp(v), \exp(\alpha_F^{-1}(v))$ and $\exp(d_qF(v))$ converge for any $v \in U$ and the map $U \to Q; v \mapsto \exp(v)q$ is an open embedding. Then, if $x = \exp(v)q$,

$$\begin{aligned} F(x) &= F(\exp(v)q) = F(\exp(\alpha_F^{-1}(v))q) \\ &= \tilde{F}(\exp(\alpha_F^{-1}(v))) \cdot F(q) = \exp(\beta_F(\alpha_F^{-1}(v)))q' \\ &= \exp((d_qF)(v))q'. \end{aligned}$$

Thus $F(x)$ depends only on $(d_qF)(v)$.

If $F_1, F_2 : Q \to Q'$ are quandle homomorphisms, these arguments show that

$$\{x \in Q \mid F_1(x) = F_2(x), d_x F_1 = d_x F_2\}$$

is open. It is obviously closed. Thus, if $F_1(q) = F_2(q)$ and $d_q F_1 = d_q F_2$, it is the whole Q, i.e., $F_1 = F_2$.

(2) Since the universal cover of a regular s-manifold is again a regular s-manifold, it suffices to show the assertion for simply-connected Q and Q'.

We write $(T, \sigma) = (T_q Q, d_q s_q)$ and $(T', \sigma') = (T_{q'} Q', d_{q'} s_{q'})$. By Proposition 2.30 (2) there is a local regular s-triplet $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\varphi})$ for (T, σ) and a homomorphism $\tilde{f} : (\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\varphi}) \to (\mathfrak{g}(T'), \mathfrak{h}(T'), \varphi_{\sigma'})$ of local regular s-triplets over f.

Let $\tilde{G}$ and G' be the connected and simply-connected Lie groups corresponding to $\tilde{\mathfrak{g}}$ and $\mathfrak{g}(T')$, and $\tilde{\Phi}$ and Φ' their automorphisms corresponding to $\tilde{\varphi}$ and $\varphi_{\sigma'}$. Then $\tilde{G}^{\tilde{\Phi}}$ and $(G')^{\Phi'}$ are closed. Let $\tilde{H}$ and H' be their connected components at the identity elements. By the uniqueness part of Theorem 2.15 (5), $(\tilde{G}, \tilde{H}, \tilde{\Phi})$ and (G', H', Φ') are regular Φ-spaces for Q and Q', respectively. More precisely, $\tilde{G}/\tilde{H} \to Q$ can be chosen so that the induced map $T_q Q = T \cong T_{e\tilde{H}}(\tilde{G}/\tilde{H}) \to T_q Q$ is the identity map, and similarly for Q'. Let $\tilde{F} : \tilde{G} \to G'$ be the Lie group homomorphism corresponding to $\tilde{f}$. Then it is a homomorphism of regular Φ-spaces, hence induces a homomorphism $F : Q \to Q'$ of regular s-manifolds, which satisfies $d_q F = f$. ∎

2.2. Ideals of a Lie-Yamaguti algebra and subalgebras of the associated Lie algebra

We will later use the notion of an ideal of a Lie-Yamaguti algebra and some related facts.

Definition 2.31 ([37, §1]). Let T be a Lie-Yamaguti algebra.

(1) A *subalgebra* of T is a vector subspace $U \subseteq T$ satisfying $U * U \subseteq U$ and $[U, U, U] \subseteq U$.

(2) An *ideal* of T is a vector subspace $U \subseteq T$ satisfying $U * T \subseteq U$ and $[U, T, T] \subseteq U$. (It follows that $T * U, [T, U, T], [T, T, U]$ are also contained in U.)

(3) An ideal U of T is an *abelian ideal* if $U * U = 0$ and $[T, U, U] = 0$. (It follows that $[U, T, U] = [U, U, T] = 0$.)

(4) If V is another Lie-Yamaguti algebra, an *extension* of T by V is an exact sequence $0 \to V \to T^* \to T \to 0$ of vector spaces where T^* is endowed with a structure of Lie-Yamaguti algebra and the mappings are homomorphisms. Note that the image of V is an ideal of T^*.

The kernel of a homomorphism of Lie-Yamaguti algebras is an ideal, and the quotient space of a Lie-Yamaguti algebras by an ideal is a Lie-Yamaguti algebra in a natural way.

Lemma 2.32. *For subspaces U_1, U_2 of T, let us denote by $\mathfrak{h}(U_1, U_2)$ the subspace of $\mathfrak{h}(T)$ generated by $\{D_{x,y} \mid x \in U_1, y \in U_2\}$.*

(1) *If U is an ideal of T, then $U \oplus \mathfrak{h}(U, T)$ is an ideal of $\mathfrak{g}(T) = T \oplus \mathfrak{h}(T)$ and $U \oplus \mathfrak{h}(U, U)$ is an ideal of $U \oplus \mathfrak{h}(U, T)$. Also, $\mathfrak{h}(U, T)$ and $\mathfrak{h}(U, U)$ are ideals of $\mathfrak{h}(T)$, and hence are Lie subalgebras.*

(2) *If* $f : T \to T'$ *is a surjective homomorphism of Lie-Yamaguti algebras and* $U = \mathrm{Ker}(f)$*, then* $U \oplus \mathfrak{h}(U,T)$ *is contained in the kernel of the natural map* $\tilde{f} : \mathfrak{g}(T) \to \mathfrak{g}(T')$.

(3) *If* U *is an abelian ideal of* T*, then* $\mathfrak{h}(U,U) = 0$ *and* $U \oplus \{0\}$ *is an abelian ideal of* $U \oplus \mathfrak{h}(U,T)$.

Proof. (1) This is [37, §1, Proposition]. The proof is straightforward except that we use the identity $[D_{x,y}, D_{z,v}] = D_{[x,y,z],v} + D_{z,[x,y,v]}$, which is equivalent to (LY6).

(2) For $(x,u) \in T \oplus \mathfrak{h}(T)$, we have $\tilde{f}(x,u) = (f(x), u')$ with $u' \circ f = f \circ u$. Thus, if $x \in U$, $\tilde{f}(x,0) = (0,0)$ holds. We have only to show $u' = 0$ for $u = D_{x,y}$ with $x \in U$ and $y \in T$. For $z \in T$, we have $u'(f(z)) = f(u(z)) = f([x,y,z])$, and this is 0 since $[x,y,z] \in U = \mathrm{Ker}(f)$. From the surjectivity of f, we conclude that $u' = 0$.

(3) For any $x, y \in U$, it follows from the definition of an abelian ideal that $D_{x,y} = 0$, and hence $\mathfrak{h}(U,U) = 0$. Thus U is an ideal of $U \oplus \mathfrak{h}(U,T)$ by (1). For $x, y \in U$, we have $[(x,0),(y,0)] = (x * y, D_{x,y}) = 0$, and so U is an abelian ideal of $U \oplus \mathfrak{h}(U,T)$. ■

3. Quandle modules over geometric quandles

As a candidate for the notion of modules over a rack or a quandle, one can consider an invertible action of Q on an additive group M, denoted by $x \triangleright m$, and impose the condition $x \triangleright (y \triangleright m) = (x \triangleright y) \triangleright (x \triangleright m)$. This is equivalent to considering modules over the associated group $\mathrm{As}(X)$, which is defined as the group generated by $\{g_x \mid x \in X\}$ subject to the condition $g_{x \triangleright y} = g_x g_y g_x^{-1}$ (see [13]). A more general and natural notion of quandle modules and rack modules was introduced in [1] and [18].

Definition 3.1. Let X be a rack. A rack module over X is a triplet

$$((A_x)_{x \in X}, (\eta_{xy})_{x,y \in X}, (\tau_{xy})_{x,y \in X}),$$

where A_x is an additive group for any $x \in X$, $\eta_{xy} : A_y \to A_{x \triangleright y}$ is an isomorphism and $\tau_{xy} : A_x \to A_{x \triangleright y}$ is a homomorphism for any $x, y \in X$, satisfying the following conditions:

(1) $\eta_{x,y \triangleright z} \eta_{yz} = \eta_{x \triangleright y, x \triangleright z} \eta_{xz}$,

(2) $\eta_{x,y \triangleright z} \tau_{yz} = \tau_{x \triangleright y, x \triangleright z} \eta_{xy}$,

(3) $\tau_{x,y \triangleright z} = \eta_{x \triangleright y, x \triangleright z} \tau_{xz} + \tau_{x \triangleright y, x \triangleright z} \tau_{xy}$.

If X is a quandle, it is called a quandle module if it further satisfies $\eta_{xx} + \tau_{xx} = \mathrm{id}_{A_x}$.

Remark 3.2. (1) To see that this in fact generalizes the notion of a quandle module as an $\mathrm{As}(X)$-module, see Example 3.15.

(2) Note that the terms in conditions (1), (2) and (3) exhaust naturally conceivable ways of going from A_z, A_y and A_x to $A_{x \triangleright (y \triangleright z)} = A_{(x \triangleright y) \triangleright (x \triangleright z)}$ using η's and τ's.

This definition is natural in that rack modules and quandle modules are in one-to-one correspondence with the abelian group objects in the categories of racks and quandles over X.

Definition 3.3. Let $\mathbf{C}$ be a category with fiber products and X an object of $\mathbf{C}$. An abelian group object in $\mathbf{C}$ over X (also called a Beck module over X) is a tuple $(A, \pi, \zeta, \alpha, \iota)$ where

- A is an object of $\mathbf{C}$ and $\pi : A \to X$ is a morphism, by which A is regarded as an object over X, and
- $\zeta : X \to A$, $\alpha : A \times_X A \to A$ and $\iota : A \to A$ are morphisms over X,

satisfying the "axiom of abelian groups:"

- $\alpha \circ (\alpha \times_X \mathrm{id}_A) = \alpha \circ (\mathrm{id}_A \times_X \alpha)$ as morphisms $A \times_X A \times_X A \to A$,
- $\alpha \circ \sigma = \alpha$, where $\sigma : A \times_X A \to A \times_X A$ is the morphism interchanging the factors,
- $\alpha \circ (\mathrm{id}_A \times_X \zeta) = \alpha \circ (\zeta \times_X \mathrm{id}_A) = \mathrm{id}_A$ as endomorphisms of $A \cong A \times_X X \cong X \times_X A$, and
- $\alpha \circ (\mathrm{id}_A, \iota) = \alpha \circ (\iota, \mathrm{id}_A) = \zeta \circ \pi$ as endomorphisms of A.

A morphism $(A, \pi, \zeta, \alpha, \iota) \to (A', \pi', \zeta', \alpha', \iota')$ is defined as a morphism $f : A \to A'$ compatible with other data, i.e., $\pi' \circ f = \pi$, $f \circ \zeta = \zeta'$, $f \circ \alpha = \alpha' \circ (f \times_X f)$ and $f \circ \iota = \iota' \circ f$.

Proposition 3.4 ([1, Theorem 2.18], [18, Theorem 2.2, Theorem 2.6])**.**
Let $A = \coprod_{x \in X} A_x$, $\pi : A \to X$ the projection, $\zeta : X \to A$ the zero section, $\alpha : A \times_X A \to A$ and $\iota : A \to A$ the fiberwise operations of taking the sum and the inverse, respectively. We write (x, a) to refer to an element $a \in A_x$ and define $(x, a) \triangleright (y, b) = (x \triangleright y, \eta_{xy}(b) + \tau_{xy}(a))$.

Then A is a rack, and π, ζ, α and ι are rack homomorphisms. If X is a quandle and $(A_x)_{x \in X}$ is a quandle module, then A is a quandle and π, ζ, α and ι are quandle homomorphisms.

Furthermore, this correspondence gives an equivalence between the category of rack (or quandle) modules over X and the category of abelian group objects in the category of racks (resp. quandles) over X.

Over topological racks and quandles, the notion of a module was defined in [12]. We give a definition in a somewhat more abstract way, for topological quandles, quandle manifolds and quandle varieties, and their counterparts for racks.
Let $(X, \mathcal{A})$ be one of the following:

- X is a topological rack and $\mathcal{A}$ is a topological space together with a continuous map $\mathcal{A} \to X$ and an addition operation $\mathcal{A} \times_X \mathcal{A} \to X$ which makes $\mathcal{A}$ a relative topological additive group.
- X is a rack manifold (of a certain class) and $\mathcal{A}$ is a vector bundle over X. (This is a special case of the above.)
- X is a rack variety and $\mathcal{A}$ is a variety over X endowed with an addition operation $\mathcal{A} \times_X \mathcal{A} \to X$ which makes $\mathcal{A}$ a relative commutative group variety.
- X is a rack variety and $\mathcal{A}$ is a coherent sheaf on X.

Let $\pi_i : X^2 \to X$ and $p_i : X^3 \to X$ denote the i-th projection maps, $p_{ij} = (p_i, p_j) : X^3 \to X^2$, $\mu : X^2 \to X$ the quandle operation and $\mu_{ij} := \mu \circ p_{ij} : X^3 \to X$.

Definition 3.5. (1) A *rack module* over X is a triplet $(\mathcal{A}, \eta, \tau)$, where $\mathcal{A}$ is as above, $\eta : \pi_2^*\mathcal{A} \to \mu^*\mathcal{A}$ is an isomorphism (in the appropriate category) over $X \times X$ and $\tau : \pi_1^*\mathcal{A} \to \mu^*\mathcal{A}$ is a homomorphism over $X \times X$, satisfying the following conditions:

(a) $(p_1, \mu_{23})^*\eta \circ p_{23}^*\eta = (\mu_{12}, \mu_{13})^*\eta \circ p_{13}^*\eta$,

(b) $(p_1, \mu_{23})^*\eta \circ p_{23}^*\tau = (\mu_{12}, \mu_{13})^*\tau \circ p_{12}^*\eta$,

(c) $(p_1, \mu_{23})^*\tau = (\mu_{12}, \mu_{13})^*\eta \circ p_{13}^*\tau + (\mu_{12}, \mu_{13})^*\tau \circ p_{12}^*\tau$.

Here in (1), for example, the domains of $p_{23}^*\eta$ and $p_{13}^*\eta$ are $p_{23}^*\pi_2^*\mathcal{A}$ and $p_{13}^*\pi_2^*\mathcal{A}$ respectively, and both can be identified with (or are equal to, depending on how the pullbacks are defined) $p_3^*\mathcal{A}$. Similarly, the codomain of $p_{23}^*\eta$ and the domain of $(p_1, \mu_{23})^*\eta$ are $p_{23}^*\mu^*\mathcal{A}$ and $(p_1, \mu_{23})^*\pi_2^*\mathcal{A}$, both of which can be identified with $\mu_{23}^*\mathcal{A}$, so the two morphisms can be composed.

(2) If X is a quandle, a rack module $(\mathcal{A}, \eta, \tau)$ is called a *quandle module* over X if it further satisfies $\Delta^*\eta + \Delta^*\tau = \mathrm{id}_{\mathcal{A}}$, where $\Delta : X \to X \times X$ is the diagonal map. Here we use the natural identifications $\Delta^*\pi_1^*\mathcal{A} \cong \Delta^*\pi_2^*\mathcal{A} \cong \Delta^*\mu^*\mathcal{A} \cong \mathcal{A}$.

In the case X is a manifold and $\mathcal{A}$ is a vector bundle or X is an algebraic variety and $\mathcal{A}$ is a coherent sheaf, we will call $(\mathcal{A}, \eta, \tau)$ a *linear* rack or quandle module.

Remark 3.6. If X is a transitive rack variety, a linear (coherent) rack module $\mathcal{A}$ over X is always locally free of constant rank, and we will identify it with the associated vector bundle.

Definition 3.7. Let Q be a connected regular s-manifold in the smooth, complex analytic or algebraic category and $(\mathcal{A}, \eta, \tau)$ a linear quandle module over Q. We say $\mathcal{A}$ is *regular* if $\mathrm{id}_{\mathcal{A}_q} - \eta_{qq} = \tau_{qq} : \mathcal{A}_q \to \mathcal{A}_q$ is invertible for any (or, equivalently, some) $q \in Q$.

Definition 3.8. Let $\mathcal{A}$ and $\mathcal{B}$ be rack (resp. quandle) modules over X.

(1) A *rack* (resp. *quandle*) *module homomorphism* $f : \mathcal{A} \to \mathcal{B}$ is a continuous additive map, a bundle map of locally constant rank, a homomorphism of $\mathcal{O}_X$-modules, etc., compatible with η and τ.

More specifically, if $\eta^{\mathcal{A}}$, $\tau^{\mathcal{A}}$, $\eta^{\mathcal{B}}$ and $\tau^{\mathcal{B}}$ denote the homomorphisms defining the module structures, we require that

$$\eta^{\mathcal{B}} \circ \pi_2^*(f) = \mu^*(f) \circ \eta^{\mathcal{A}} \quad \text{and} \quad \tau^{\mathcal{B}} \circ \pi_1^*(f) = \mu^*(f) \circ \tau^{\mathcal{A}}.$$

(2) A *rack* (resp. *quandle*) *submodule* $\mathcal{A}'$ of $\mathcal{A}$ is a subspace whose fibers are subgroups, a subbundle, a coherent subsheaf, etc., which is closed under the maps η and τ. Formally, the requirement is that η and τ maps $\pi_2^*\mathcal{A}'$ and $\pi_1^*\mathcal{A}'$ to $\mu^*\mathcal{A}'$, respectively. (In the coherent case, note that $\mu = \pi_2 \circ (\pi_1, \mu)$ is flat since (π_1, μ) is an isomorphism, and hence that $\pi_1^*\mathcal{A}'$, $\pi_2^*\mathcal{A}'$ and $\mu^*\mathcal{A}'$ can be considered as subsheaves of $\pi_1^*\mathcal{A}$, $\pi_2^*\mathcal{A}$ and $\mu^*\mathcal{A}$, respectively.)

Remark 3.9. If X is transitive, a bundle map compatible with η is automatically of constant rank.

Proposition 3.10. *Let X be a geometric rack or quandle as above.*

(1) *Rack (resp. quandle) modules over X form a category by the homomorphisms defined above.*

(2) *In the case X is a manifold and modules are vector bundles, if f is a rack (resp. quandle) module homomorphism of modules over X which is a bundle map of locally constant rank, then* $\mathrm{Ker} f$, $\mathrm{Im} f$, *and* $\mathrm{Coker} f$ *are rack (resp. quandle) modules in a natural way.*

In particular, if X is transitive, then linear rack (resp. quandle) modules form an abelian category.

(3) *Coherent rack (resp. quandle) modules over a rack (resp. quandle) variety form an abelian category.*

Proof. (1), (2) Straightforward.

(3) We can show that the sheaf-theoretic kernel, image and cokernel of a rack homomorphism are again rack modules in a natural way, using the fact that μ (as well as π_1 and π_2) is flat as remarked above.

Take $\mathcal{K} = \mathrm{Ker} f$, for example. Then, from the exact sequence $0 \to \mathcal{K} \to \mathcal{A} \xrightarrow{f} \mathcal{B}$, we obtain the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \pi_2^*\mathcal{K} \arrow[r] & \pi_2^*\mathcal{A} \arrow[r, "\pi_2^* f"] \arrow[d, "\eta^{\mathcal{A}}", "\cong"'] & \pi_2^*\mathcal{B} \arrow[d, "\eta^{\mathcal{B}}", "\cong"'] \\
0 \arrow[r] & \mu^*\mathcal{K} \arrow[r] & \mu^*\mathcal{A} \arrow[r, "\mu^* f"] & \mu^*\mathcal{B}
\end{tikzcd}$$

Thus there is a unique isomorphism $\eta^{\mathcal{K}} : \pi_2^*\mathcal{K} \to \mu^*\mathcal{K}$ making the whole diagram commutative. We can find $\tau^{\mathcal{K}}$ in a similar way, and the conditions for $\mathcal{K}$ to be a rack module follow from those for $\mathcal{A}$. ■

Remark 3.11. Unlike the case of group representations, we do not seem to have tensor products and duals.

We state the correspondence of modules and abelian group objects in the case of linear modules over geometric quandles.

Definition 3.12. Let Q be a smooth quandle, complex analytic quandle or a quandle variety, and let $\mathbb{K}$ denote the base field.

(1) The category of linear quandle modules over Q is denoted by $\mathbf{Mod}_{\mathbb{K}}(Q)$.

(2) We define the category $\mathbf{Vec}(\mathbf{Quandle}/Q)$ of vector bundles in the category of geometric quandles (in each setting) over Q as follows.

- An object is a pair $(\mathcal{A}, \triangleright_{\mathcal{A}})$, where
  - $\mathcal{A}$ is endowed with a structure of a vector bundle over Q,
  - $\triangleright_{\mathcal{A}}$ is a binary operation on $\mathcal{A}$ which makes $\mathcal{A}$ a smooth quandle, complex analytic quandle or quandle variety,

  such that the projection map $\mathcal{A} \to Q$, the zero section $Q \to \mathcal{A}$, the fiberwise addition $\mathcal{A} \times_Q \mathcal{A} \to \mathcal{A}$, the fiberwise inversion $\mathcal{A} \to \mathcal{A}$ and the scalar multiplications $\mathcal{A} \to \mathcal{A}$ are homomorphism of geometric quandles.
- A morphism is a linear bundle map which is also a quandle homomorphism.

Remark 3.13. The definition of the latter category is not entirely category-theoretic. We can easily see that $\mathcal{A}$ as in (2) is an abelian group object in the category of geometric quandles over Q, and furthermore is a $\mathbb{K}$-vector space object. However, in order to obtain an equivalence as in the following proposition, there has to be some kind of continuity condition on the scalar (especially when we extend to the case where the base field is of positive characteristic).

A category theoretic formulation using a "field object" is possible, but we will skip it for brevity.

Proposition 3.14. *Let Q be a transitive smooth quandle, complex analytic quandle or quandle variety and let $\mathbb{K}$ denote the base field. Then there is an equivalence of the categories $\mathbf{Mod}_{\mathbb{K}}(Q)$ and $\mathbf{Vec}(\mathbf{Quandle}/Q)$. An analogous statement holds for racks.*

Proof. In the algebraic case, it follows from the transitivity of Q that a quandle module $\mathcal{A}$ over Q is a locally free sheaf. Therefore, we can think of $\mathcal{A}$ as a vector bundle in any of the settings. Also note that homomorphisms are always of constant rank.

For $x, y \in Q$, we write $\mathcal{A}_x$ for the fiber of $\mathcal{A}$ at x and η_{xy} for the map $\mathcal{A}_y \to \mathcal{A}_{x \triangleright y}$ induced by η. We also write (x, a) for an element $a \in \mathcal{A}_x$.

Given an object $(\mathcal{A}, \eta, \tau)$ of $\mathbf{Mod}_{\mathbb{K}}(Q)$, the correspondence of Proposition 3.4 gives a quandle structure by $(x, a) \triangleright_{\mathcal{A}} (y, b) = (x \triangleright y, \eta_{xy}(b) + \tau_{xy}(a))$. This obviously gives a smooth, complex analytic or regular map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. The map $(\triangleright_{\mathcal{A}}, \pi_2) : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ is an isomorphism, since its inverse is given by $((x, a), (y, b)) \mapsto (x \triangleright^{-1} y, \eta_{x, x \triangleright^{-1} y}(b - \tau_{x, x \triangleright^{-1} y}(a))$. Thus $(\mathcal{A}, \triangleright_{\mathcal{A}})$ is a smooth quandle, etc.

The projection, zero section, addition, and inversion are quandle homomorphisms by Proposition 3.4, and the scalar multiplications are quandle homomorphisms from the above expression for $\triangleright_{\mathcal{A}}$ and linearity of η and τ. They are smooth maps etc., so we have an object of $\mathbf{Vec}(\mathbf{Quandle}/Q)$.

In both categories, a morphism is a (linear) bundle map. A bundle map f is a morphism in $\mathbf{Mod}_{\mathbb{K}}(Q)$ if and only if it is a homomorphism of modules over Q as a discrete quandle. By Proposition 3.4, this condition is equivalent to the condition that f is a (discrete) quandle homomorphism compatible with the projection to Q, addition, etc. Thus we have a fully faithful functor $F : \mathbf{Mod}_{\mathbb{K}}(Q) \to \mathbf{Vec}(\mathbf{Quandle}/Q)$.

Given an object $(\mathcal{A}, \triangleright_{\mathcal{A}})$, by Proposition 3.4 we may write

$$(x, a) \triangleright_{\mathcal{A}} (y, b) = (x \triangleright y, \eta_{xy}(b) + \tau_{xy}(a))$$

where η_{xy} is an invertible additive map and τ_{xy} is an additive map for $x, y \in Q$. From the smoothness etc. of $\triangleright_{\mathcal{A}}$ it follows that η_{xy} and τ_{xy} form bundle maps η and τ on $Q \times Q$. They are $\mathbb{K}$-linear from the condition that scalar multiplications are homomorphisms for $\triangleright_{\mathcal{A}}$. Thus we have a quandle module over Q. It is easy to see that this module is sent to $(\mathcal{A}, \triangleright_{\mathcal{A}})$, and so the functor F is an equivalence. ∎

Example 3.15. Let Q be a topological quandle, smooth quandle, complex analytic quandle or a quandle variety, G a topological, Lie, complex Lie or algebraic group, V a representation of G and $\varphi : Q \to \mathrm{Conj}(G)$ a quandle homomorphism.

(Here our convention for the conjugation quandle is such that $g \triangleright h = ghg^{-1}$.) We take $\mathcal{A} = Q \times V$, $\eta_{xy} = \varphi(x)$ and $\tau_{xy} = 1 - \varphi(x \triangleright y)$. Then $(\mathcal{A}, \eta, \tau)$ is a quandle module over Q. In the discrete setting, we may take G to be the associated group $\mathrm{As}(Q)$. (See the paragraph before Def. 2.23 of [AG].)

In particular, if $G = \langle g \rangle$ is a cyclic group of infinite order, $f(x) = g$ for any $x \in Q$ and V is a vector space (or an additive group) with a G-action, then we obtain a quandle module by $\eta_{xy}(a) = ga$ and $\tau_{xy}(a) = (1-g)a$ ([AG, Example 2.20]).

The quandle module structure above is the one given by the natural correspondences

$$\begin{aligned} & \{\text{representations of } G\} \\ \leftrightarrow\ & \{\text{extensions of } G \text{ which are “vector-spaces over } G\text{”}\} \\ \rightarrow\ & \{\text{extensions of } Q \text{ which are “vector-spaces over } Q\text{”}\} \\ \leftrightarrow\ & \{\text{modules over } Q\}. \end{aligned}$$

The first one is given by associating to a representation V the group $G \times V$ with the multiplication

$$(g, a)(h, b) = (gh, a + gb).$$

Then the conjugation is given by

$$(g, a) \triangleright (h, b) = (g, a)(h, b)(g^{-1}, -g^{-1}a) = (g \triangleright h, gb + (1 - g \triangleright h)a).$$

The pullback of this quandle module over G by φ is exactly the quandle module $(\mathcal{A}, \eta, \tau)$ under the correspondence of Proposition 3.4.

Example 3.16. In the situation of the previous example, there is another natural module structure given by $\eta'_{xy} = \varphi(x)$ and $\tau'_{xy} = 1 - \varphi(x)$. In this case, the corresponding quandle structure on $\mathcal{A}$ is given by

$$(x, a) \triangleright (y, b) = (x \triangleright y, a + \varphi(x)(b - a)).$$

There is a homomorphism $f : (Q \times V, \eta', \tau') \to (Q \times V, \eta, \tau)$ given by the assignment $(x, a) \mapsto (x, (1 - \varphi(x))(a))$, and the two modules are isomorphic if $1 - \varphi(x)$ is invertible for any $x \in Q$. If not, the module structures are not necessarily isomorphic, as we will now see.

For $\alpha \neq 0, 1$, let $Q = \{X \in \mathrm{GL}(2, \mathbb{C}) \mid \det X = \alpha, \mathrm{Tr}(X) = 1 + \alpha\}$, i.e., the conjugacy class of $X_0 := \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$. It is a regular s-manifold by the operation $X \triangleright Y = XYX^{-1}$. We remark that Q is isomorphic to

$$\{X \in \mathrm{SL}(2, \mathbb{C}) \mid \mathrm{Tr}(X) = \sqrt{\alpha} + 1/\sqrt{\alpha}\}$$

with the conjugation operation. The standard representation of $\mathrm{SL}(2, \mathbb{C})$ gives rise to two quandle module structures (η, τ) and (η', τ') on $\mathcal{A} := Q \times \mathbb{C}^2$. We write the associated quandle operations on $\mathcal{A}$ by $\triangleright$ and $\triangleright'$. It is easy to show that $(\mathcal{A}, \triangleright)$ can be identified with the inverse image of Q by $\mathrm{Conj}(\mathrm{Aff}(2, \mathbb{C})) \to \mathrm{Conj}(\mathrm{GL}(2, \mathbb{C}))$.

We show that the two module structures are not isomorphic. First note that there is a section $Q \to \mathcal{A}$, given by

$$s(X) = (\alpha I - X) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which satisfies $(X, s(X)) \triangleright' (Y, 0) = (X \triangleright Y, 0)$ for any X and Y. If there were an isomorphism $(\mathcal{A}, \triangleright') \xrightarrow{\sim} (\mathcal{A}, \triangleright)$, given by $f_X \in \mathrm{GL}(2, \mathbb{C})$ for each $X \in Q$, then $t(X) := f_X(s(X))$ would satisfy

$$(X, t(X)) \triangleright (Y, 0) = (X \triangleright Y, 0), \text{ i.e., } (I - XYX^{-1})t(X) = 0,$$

for any $X, Y \in Q$. Since $s(X_0) \neq 0$, we would have $a := t(X_0) \neq 0$. On the other hand, applying the above condition with $X = Y = X_0$, we see that $X_0 a = a$. Using this, the above condition with $X = X_0$ implies $Ya = a$ for any $Y \in Q$, but this is a contradiction since the eigenspace of Y for the eigenvalue 1 can be any 1-dimensional subspace of $\mathbb{C}^2$ as Y moves.

Example 3.17. Here we will give two examples of quandle modules over a quandle variety which are not locally free.

Let k be a field and let $Q = \{X \in \mathrm{SL}(2, k) \mid \mathrm{Tr}(Q) = 2\}$ be the quandle variety given by the operation $X \triangleright Y = XYX^{-1}$. We endow $\mathcal{A} := Q \times k^2$ with the quandle module structure associated to the standard representation of $\mathrm{SL}(2, k)$ as in Example 3.15. Denote by $\mathfrak{m}$ the maximal ideal of the affine coordinate ring of Q corresponding to the identity matrix I. Then, for each positive integer n, $\mathfrak{m}^n\mathcal{A}$ is a quandle submodule of $\mathcal{A}$. By Proposition 3.10 (3), we have quandle modules $\mathcal{A}/\mathfrak{m}^n\mathcal{A}$ supported on $\{I\}$.

For example, $\mathcal{A}/\mathfrak{m}\mathcal{A}$ is a skyscraper sheaf at I with fiber k^2. The map η_{XY} is X if $Y = I$ and 0 otherwise, and τ_{XY} is always 0.

4. Representations and extensions of Lie-Yamaguti algebras and infinitesimal s-manifolds

The following notion of a representation of a Lie-Yamaguti algebra was defined in [37, §6].

Definition 4.1. A *representation of a Lie-Yamaguti algebra* T is a quadruplet $(V, \rho, \delta, \theta)$, where V is a vector space, $\rho : T \to \mathrm{End}(V)$ is a linear map and $D, \theta : T \times T \to \mathrm{End}(V)$ are bilinear maps, such that the following hold for any $x, y, z, w \in T$.

(RLY1) $\delta(x, y) + \theta(x, y) - \theta(y, x) = [\rho(x), \rho(y)] - \rho(x * y)$.

(RLY2) $\theta(x, y * z) - \rho(y)\theta(x, z) + \rho(z)\theta(x, y) = 0$.

(RLY3) $\theta(x * y, z) - \theta(x, z)\rho(y) + \theta(y, z)\rho(x) = 0$.

(RLY4) $\theta(z, w)\theta(x, y) - \theta(y, w)\theta(x, z) - \theta(x, [y, z, w]) + \delta(y, z)\theta(x, w) = 0$.

(RLY5) $[\delta(x, y), \rho(z)] = \rho([x, y, z])$.

(RLY6) $[\delta(x, y), \theta(z, w)] = \theta([x, y, z], w) + \theta(z, [x, y, w])$.

A *homomorphism* of representations of Lie-Yamaguti algebras is a linear map compatible with ρ, δ and θ.

Remark 4.2. (1) As pointed out in [37], we may do without δ if we take (RLY1) as the definition of δ, and the following holds:

(RLY7) $\delta(x * y, z) + \delta(y * z, x) + \delta(z * x, y) = 0$.

(2) We also have

(RLY8) $[\delta(x,y),\delta(z,w)] = \delta([x,y,z],w) + \delta(z,[x,y,w])$.

To show this, we use (RLY1) to write $\delta(z,w)$ in terms of θ and ρ, use (RLY6), rewrite $[\delta(x,y),[\rho(z),\rho(w)]]$ by Jacobi identity in $\mathrm{End}(V)$ and then apply (RLY5) and (LY5).

Definition 4.3. A *representation of an infinitesimal s-manifold* (T,σ) is a data $(V,\rho,\delta,\theta,\psi)$ where (V,ρ,δ,θ) is a representation of T and $\psi \in \mathrm{End}(V)$ is an invertible linear transformation satisfying the following for any $x,y \in T$.

(RISM1) $\rho(\sigma(x)) = \psi \circ \rho(x) \circ \psi^{-1}$.

(RISM2) $\theta(x,\sigma(y)) = \psi \circ \theta(x,y)$, $\theta(\sigma(x),y) = \theta(x,y) \circ \psi^{-1}$.

(RISM3) $\delta(x,y) = \psi \circ \delta(x,y) \circ \psi^{-1}$.

A representation (V,ψ) is called *regular* if V is finite dimensional and $1-\psi$ is invertible.

A *homomorphism* of representations of an infinitesimal s-manifold is defined as a homomorphism of representations of Lie-Yamaguti algebras commuting with ψ.

Remark 4.4. We may replace one of the equalities in (RISM2) by

(RISM4) $\theta(\sigma(x),\sigma(y)) = \psi \circ \theta(x,y) \circ \psi^{-1}$.

From (RLY1), (RISM1), (RISM4) and (ISM1) we also have

(RISM5) $\delta(\sigma(x),\sigma(y)) = \psi \circ \delta(x,y) \circ \psi^{-1}$.

Yamaguti's definition of a representation was based on Eilenberg's principle of correspondence between representations of algebras (defined by binary operations and equational relations) and their extensions. He proved, following Chevalley-Eilenberg, a correspondence between extensions of Lie-Yamaguti algebras by abelian ideals and pairs of a representation and an element of a certain cohomology group [37, §7].

Here, we give the correspondence in the case of split extensions, in a form suited for our purpose, as well as a proof for the reader's convenience.

By a Lie-Yamaguti algebra over T, we will mean a pair of a Lie-Yamaguti algebra T^* and a homomorphism $T^* \to T$. Note that, if T_1 and T_2 are Lie-Yamaguti algebras over T, so is $T_1 \times_T T_2$ in a natural way.

Proposition 4.5. (1) *For a Lie-Yamaguti algebra T and a vector space V, the following data are equivalent. Here, the map $T \oplus V \to T$ will always be the standard projection. In* (iii)*, V is regarded as a Lie-Yamaguti algebra with trivial products.*

(i) *A Lie-Yamaguti algebra structure on $T \oplus V$ over T for which $T \subseteq T \oplus V$ is a subalgebra and V is an abelian ideal.*

(i)' *A Lie-Yamaguti algebra structure on $T \oplus V$ over T of the form*

$$(x_1,v_1) * (x_2,v_2) = (x_1 * x_2, A(x_1)v_2 + B(x_2)v_1),$$
$$[(x_1,v_1),(x_2,v_2),(x_3,v_3)] = ([x_1,x_2,x_3], E(x_2,x_3)v_1 + F(x_3,x_1)v_2 + G(x_1,x_2)v_3),$$

where $A, B : T \to \mathrm{End}(V)$ *are linear and* $E, F, G : T \times T \to \mathrm{End}(V)$ *are bilinear. We may also require that* $B = -A, F(y,x) = -E(x,y)$ *and* $G(x,x) = 0$.

(ii) *A Lie-Yamaguti algebra structure on* $T \oplus V$ *over* T *satisfying the following.*

(a) $\alpha = (\mathrm{id}_T, +) : T \oplus V \oplus V \to T \oplus V$ *is a homomorphism, where* $T \oplus V \oplus V$ *is identified with* $(T \oplus V) \times_T (T \oplus V)$.

(b) *For any scalar* λ, $\mu_\lambda = (\mathrm{id}_T, \lambda) : T \oplus V \to T \oplus V; (x,v) \mapsto (x, \lambda v)$ *is a homomorphism.*

(ii)' *A Lie-Yamaguti algebra structure on* $T \oplus V$ *over* T *satisfying* (ii)(a).

(iii) *An isomorphism class of the following data:*
An extension $0 \to V \to \tilde{T} \xrightarrow{\pi} T \to 0$ *of Lie-Yamaguti algebras, Lie-Yamaguti algebra homomorphisms* $\alpha : \tilde{T} \times_T \tilde{T} \to \tilde{T}$, $\zeta : T \to \tilde{T}$, *and* $\iota : \tilde{T} \to \tilde{T}$ *over* T *such that* $(\tilde{T}, \pi, \zeta, \alpha, \iota)$ *forms an abelian group object over* T.

(iv) *A representation of* T *on* V.

The correspondence of (i)' *and* (iv) *is given by* $\rho = A$, $\delta = G$ *and* $\theta = E$.

(2) *Let* (T, σ) *be an infinitesimal* s*-manifold,* $(V, \rho, \delta, \theta)$ *a representation of* T *and* $\psi \in \mathrm{GL}(V)$. *Then* $(T \oplus V, \sigma \times \psi)$ *is an infinitesimal* s*-manifold if and only if* (V, ψ) *is a regular representation of* (T, σ), *and if so, the projection map* $T \oplus V \to T$, *the zero section* $T \to T \oplus V$, α *and* μ_λ *in* (1)(ii) *are homomorphisms of infinitesimal* s*-manifolds.*

Proof. (1) We first show that (i)–(ii)' are equivalent. From any of these, using the condition that the projection $T \oplus V \to T$ preserves the product $*$, it follows that we can write

$$(x_1, v_1) * (x_2, v_2) = (x_1 * x_2, A(x_1)v_2 + B(x_2)v_1 + C_0(x_1, x_2) + C_2(v_1, v_2)),$$

where A and B are linear with values in $\mathrm{End}(V)$ and C_0 and C_2 are bilinear with values in V. Similarly, the first component of $[(x_1, v_1), (x_2, v_2), (x_3, v_3)]$ is $[x_1, x_2, x_3]$ and the second component can be written as

$$\begin{aligned} &E(x_2, x_3)v_1 + F(x_3, x_1)v_2 + G(x_1, x_2)v_3 + D_0(x_1, x_2, x_3) \\ &\qquad + D_2^{(1)}(x_1, v_2, v_3) + D_2^{(2)}(v_1, x_2, v_3) + D_2^{(3)}(v_1, v_2, x_3) + D_3(v_1, v_2, v_3), \end{aligned}$$

where $E, F, G : T \times T \to \mathrm{End}(V)$ are bilinear and D_0, $D_2^{(j)}$ and D_3 are trilinear with values in V.

Let us assume (i). Then $C_0 = 0$ since T is a closed under $*$ and $C_2 = 0$ since $*$ is zero on V. Similarly, the second component of the triple product is also linear in v_1, v_2, v_3. In fact, from the assumption that T is closed under $[\ ]$, it follows that $D_0 = 0$; from $[V, V, V] = 0$ we have $D_3 = 0$; from $[T, V, V] = 0$ we have $D_2^{(1)} = 0$, and so on. Thus the operations are as in (i)'. From (LY1) for $T \oplus V$, $B = -A$ follows. From (LY2) for $T \oplus V$, $F(y,x) = -E(x,y)$ and $G(x,x) = 0$ follow.

Conversely, assuming (i)', it is easy to see that T is a subalgebra and V is an abelian ideal. (For the latter, note that V is an ideal since it is the kernel of $T \oplus V \to T$.) Thus (i) and (i)' are equivalent.

From (i)', it is straightforward to show that (ii) and (ii)' hold. For example, we have

$$\begin{aligned}\mu_\lambda(x_1,v_1)*\mu_\lambda(x_2,v_2) &= (x_1,\lambda v_1)*(x_2,\lambda v_2)\\ &= (x_1*x_2, A(x_1)(\lambda v_2)+B(x_2)(\lambda v_1))\\ &= \mu_\lambda((x_1,v_1)*(x_2,v_2)).\end{aligned}$$

Let us assume (ii)' and show that the second components of the products $(x_1,v_1)*(x_2,v_2)$ and $[(x_1,v_1),(x_2,v_2),(x_3,v_3)]$ are again linear in v_i. To show the latter, say, we look at the second components of the both sides of the relation

$$\alpha([(x_1,v_1,w_1),(x_2,v_2,w_2),(x_3,v_3,w_3)]) = [(x_1,v_1+w_1),(x_2,v_2+w_2),(x_3,v_3+w_3)].$$

For the left hand side, it is the sum of the second components of the two products $[(x_1,v_1),(x_2,v_2),(x_3,v_3)]$ and $[(x_1,w_1),(x_2,w_2),(x_3,w_3)]$. By setting $v_i=w_i=0$, we obtain $2D_0(x_1,x_2,x_3)=D_0(x_1,x_2,x_3)$, i.e., $D_0\equiv 0$. By setting $x_i=0$ and $w_1=v_1, v_2=v_3=0$, we have $D_3\equiv 0$. Then $x_2=x_3=0$, $w_2=v_2, w_3=0$ gives $D_2^{(1)}\equiv 0$, and so on. Thus (i)' holds, and we have shown that (i)–(ii)' are equivalent.

Now let us show that (i)–(ii)' are equivalent to (iii). Given a Lie-Yamaguti algebra structure on $T\oplus V$ satisfying (ii), let $\tilde{T}:=T\oplus V$, π the projection, α the morphism (ii)(a), $\zeta: T\to T\oplus V; x\mapsto (x,0)$ and $\iota: T\oplus T\to T\oplus V; (x,v)\mapsto (x,-v)$. Using the linearity of the operations in v, it is easy to see that these are Lie-Yamaguti homomorphisms and that $(\tilde{T},\pi,\zeta,\alpha,\iota)$ satisfies the conditions of (iii).

Conversely, let an extension $\tilde{T}$ and π,ζ,α,ι as in (iii) be given. The section ζ gives a natural isomorphism $f: T\oplus V\to\tilde{T}; (x,v)\mapsto \zeta(x)+v$ as vector spaces. Applying $\alpha\circ(\zeta\times_T \mathrm{id}_{\tilde{T}})=\mathrm{id}_{\tilde{T}}$ to $(0,v)\in T\times_T\tilde{T}$, $v\in V$, we have $\alpha(0,v)=v$ for $v\in V$. By symmetry we have $\alpha(v,0)=v$. Applying the same equality to $(x,\zeta(x))\in T\times_T\tilde{T}$, we have $\alpha(\zeta(x),\zeta(x))=\zeta(x)$. From the linearity of α, we have

$$\begin{aligned}\alpha(f(x,v),f(x,w)) &= \alpha(\zeta(x)+v,\zeta(x)+w)\\ &= \alpha(\zeta(x),\zeta(x))+\alpha(v,0)+\alpha(0,w)\\ &= \zeta(x)+v+w=f(x,v+w),\end{aligned}$$

i.e., α in (iii) pulls back to α in (ii)(a). By taking the pullback of the Lie-Yamaguti algebra structure, we are in the situation of (ii)'. Obviously, the result only depends on the isomorphism class of the extension, ζ and α.

Let us show that these correspondences are inverse to each other. It is obvious that if we start from (ii) we have $f=\mathrm{id}_{T\oplus V}$ and obtain the original data. Conversely, if we start from (iii), the isomorphism $T\oplus V\cong\tilde{T}$ is constructed so that the extension $0\to V\to\tilde{T}\to T\to 0$ corresponds to the standard extension

$$0\to V\to T\oplus V\to T\to 0$$

and α in (iii) corresponds to α in (ii). Thus it suffices to show that ζ and ι in (iii) are uniquely determined, where $\tilde{T}=T\oplus V$ is constructed from data as in (ii).

In this case $\zeta: T\to T\oplus V$ is given as $x\mapsto (x,g(x))$. Then the group axiom $\alpha\circ(\zeta\times_T\mathrm{id}_{\tilde{T}})=\mathrm{id}_{\tilde{T}}$ applied to $(x,\zeta(x))$ gives $(x,g(x)+g(x))=(x,g(x))$, i.e., $g\equiv 0$. Similarly, $\alpha\circ(\iota,\mathrm{id}_{\tilde{T}})=\mathrm{id}_{\tilde{T}}$ applied to $((x,v),\iota(x,v))$ gives $\iota(x,v)=(x,-v)$.

Equivalence with (iv): For the operations in the form of (i)', let us show that the following conditions are equivalent:

(a) For A, B, E, F and G, the associated operations make $T \oplus V$ a Lie-Yamaguti algebra (over T, satisfying (i)–(ii)').

(b) (V, A, G, E) is a representation of T, and B and F are given by $B = -A$ and $F(y, x) = -E(x, y)$.

In fact, the condition $(x, a) * (x, a) = 0$, which is the skew-symmetry condition (LY1) for $T \oplus V$, is readily seen to be equivalent to $B = -A$. The condition (LY2) for $T \oplus V$, $[(x, a), (x, a), (y, b)] = 0$, is equivalent to $F(y, x) = -E(x, y)$ and $G(x, x) = 0$. Note that (b) implies $G(x, x) = 0$ by virtue of (RLY1).
Thus, assuming either (a) or (b), we have

$$(x, a) * (y, b) = (x * y, A(x)b - A(y)a)$$

and $$[(x, a), (y, b), (z, c)] = ([x, y, z], E(y, z)a - E(x, z)b + G(x, y)c),$$

with G skew-symmetric. We will therefore assume these conditions in the rest of the proof. By

$$((x, a) * (y, b)) * (z, c) = ((x * y) * z, A(x * y)c + A(z)A(y)a - A(z)A(x)b),$$

the second component of the cyclic sum $[(x, a), (y, b), (z, c)] + ((x, a) * (y, b)) * (z, c)$ is

$$\begin{aligned} & (G(y, z) + E(y, z) - E(z, y) - [A(y), A(z)] + A(y * z))\, a \\ + \quad & (G(z, x) + E(z, x) - E(x, z) - [A(z), A(x)] + A(z * x))\, b \\ + \quad & (G(x, y) + E(x, y) - E(y, x) - [A(x), A(y)] + A(x * y))\, c. \end{aligned}$$

This is identically 0 if and only if $G(x, y) + E(x, y) - E(y, x) = [A(x), A(y)] - A(x * y)$ holds for any x and y. Thus we have shown that (RLY1) for (V, A, G, E) is equivalent to (LY3) for $T \oplus V$.
The second component of $[(x, a) * (y, b), (z, c), (w, d)]$ is

$$-E(z, w)A(y)a + E(z, w)A(x)b - E(x * y, w)c + G(x * y, z)d,$$

and if we take the cyclic sum in the first 3 arguments, the coefficient of a is

$$-E(y * z, w) - E(z, w)A(y) + E(y, w)A(z),$$

those of b and c are cyclic permutation of this, and that of d is

$$G(x * y, z) + G(y * z, x) + G(z * x, y).$$

Thus (LY4) holds for $T \oplus V$ if and only if (RLY3) and (RLY7) for (V, A, G, E) hold.
The second components of $[(x, a), (y, b), (z, c) * (w, d)]$, $[(x, a), (y, b), (z, c)] * (w, d)$ and $(z, c) * [(x, a), (y, b), (w, d)]$ are

$$\begin{aligned} & E(y, z * w)a - E(x, z * w)b + G(x, y)(A(z)d - A(w)c), \\ & A([x, y, z])d - A(w)(E(y, z)a - E(x, z)b + G(x, y)c), \\ & A(z)(E(y, w)a - E(x, w)b + G(x, y)d) - A([x, y, w])c. \end{aligned}$$

Thus (LY5) for $T \oplus V$ holds if and only if

$$\begin{aligned} E(y, z * w) &= -A(w)E(y,z) + A(z)E(y,w), \\ -E(x, z * w) &= A(w)E(x,z) - A(z)E(x,w), \\ -G(x,y)A(w) &= -A(w)G(x,y) - A([x,y,w]) \end{aligned}$$

and

$$G(x,y)A(z) = A([x,y,z]) + A(z)G(x,y)$$

hold. The first two equalities correspond to (RLY2), and the other two to (RLY5). Finally, we calculate

$$\begin{aligned} &[(x,a),(y,b),[(z,c),(v,d),(w,e)]] - [[(x,a),(y,b),(z,c)],(v,d),(w,e)] \\ &\quad - [(z,c),[(x,a),(y,b),(v,d)],(w,e)]] - [(z,c),(v,d),[(x,a),(y,b),(w,e)]]. \end{aligned}$$

Then we see that (LY6) for $T \oplus V$ is equivalent to the following:

$$\begin{aligned} E(y,[z,v,w]) - E(v,w)E(y,z) + E(z,w)E(y,v) - G(z,v)E(y,w) &= 0, \\ -E(x,[z,v,w]) + E(v,w)E(x,z) - E(z,w)E(x,v) + G(z,v)E(x,w) &= 0, \\ G(x,y)E(v,w) - E(v,w)G(x,y) - E([x,y,v],w) - E(v,[x,y,w]) &= 0, \\ -G(x,y)E(z,w) + E([x,y,z],w) + E(z,w)G(x,y) + E(z,[x,y,w]) &= 0, \\ G(x,y)G(z,v) - G([x,y,z],v) - G(z,[x,y,v]) - G(z,v)G(x,y) &= 0. \end{aligned}$$

The first two equalities are equivalent to (RLY4), the third and forth to (RLY6), and the last to (RLY8).

(2) Let $(x,a),(y,b)$ and (z,c) be elements of $T \oplus V$. Write $\tilde{\sigma}$ for $\sigma \times \psi$. We have

$$\tilde{\sigma}((x,a) * (y,b)) = (\sigma(x * y), \psi(\rho(x)b) - \psi(\rho(y)a)),$$

while

$$\tilde{\sigma}(x,a) * \tilde{\sigma}(y,b) = (\sigma(x) * \sigma(y), \rho(\sigma(x))\psi(b) - \rho(\sigma(y))\psi(a)),$$

hence (ISM1) for $T \oplus V$ holds if and only if (RISM1) holds. We have

$$\tilde{\sigma}([(x,a),(y,b),(z,c)]) = (\sigma([x,y,z]), \psi(\theta(y,z)a) - \psi(\theta(x,z)b) + \psi(\delta(x,y)c))$$

and

$$[\tilde{\sigma}(x,a), \tilde{\sigma}(y,b), \tilde{\sigma}(z,c)] =$$

$$= ([\sigma(x), \sigma(y), \sigma(z)], \theta(\sigma(y), \sigma(z))\psi(a) - \theta(\sigma(x), \sigma(z))\psi(b) + \delta(\sigma(x), \sigma(y))\psi(c)),$$

hence (ISM2) for $T \oplus V$ holds if and only if (RISM4) and (RISM5) hold.
Finally, we have

$$[(x,a),(y,b),\tilde{\sigma}(z,c)] = ([x,y,\sigma(z)], \theta(y,\sigma(z))a - \theta(x,\sigma(z))b + \delta(x,y)\psi(c)),$$

hence (ISM3) for $T \oplus V$ is equivalent to (RISM3) and the first equality of (RISM2) for (V,ψ). Note that the first equality of (RISM2) and (RISM4) imply (RISM2). ■

Proposition 4.6. *Let (T,σ) be an infinitesimal s-manifold and (V_1,ψ_1) and (V_2,ψ_2) its regular representations.*

(1) *A linear map $f : V_1 \to V_2$ is a homomorphism of representations if and only if $\mathrm{id}_T \times f : T \oplus V_1 \to T \oplus V_2$ is a homomorphism of infinitesimal s-manifolds.*

(2) *If $\tilde{f} : T \oplus V_1 \to T \oplus V_2$ is a homomorphism of infinitesimal s-manifolds over T compatible with α, then it is given as $\tilde{f} = \mathrm{id}_T \times f$ for some f.*

Proof. (1) This is straightforward from the description of the structures of infinitesimal s-manifolds on $T \oplus V_1$ and $T \oplus V_2$ (Proposition 4.5 (1) (i)', (iv), (2)).

(2) Since $\tilde{f}$ is a homomorphism over T, it restricts to $f : V_1 \to V_2$.
If we write $\tilde{f}(x,0) = (x, g(x))$, then $\tilde{f}(\alpha((x,0),(x,0))) = \alpha(\tilde{f}(x,0), \tilde{f}(x,0))$ can be rewritten as $(x, g(x)) = (x, g(x) + g(x))$. Hence we have $g(x) = 0$ and the assertion follows. ∎

Corollary 4.7. *The correspondence of Proposition 4.5 gives an equivalence of the following categories:*

(a) *The category $\mathbf{Rep}^r(T,\sigma)$ of regular representations $(V,\rho,\delta,\theta,\psi)$ of (T,σ).*

(b) *The category $\mathbf{Ab}(\mathbf{Ism}/(T,\sigma))$ of abelian group objects in $\mathbf{Ism}$ over (T,σ): Objects are $((\tilde{T},\tilde{\sigma}),\pi,\zeta,\alpha,\iota)$, where*

- *$(\tilde{T},\tilde{\sigma})$ is an infinitesimal s-manifold,*
- *$\pi : \tilde{T} \to T$ is a homomorphism of infinitesimal s-manifolds, by which $\tilde{T}$ is regarded as an infinitesimal s-manifold over T, and*
- *$\zeta : T \to \tilde{T}$, $\alpha : \tilde{T} \times_T \tilde{T} \to \tilde{T}$ and $\iota : \tilde{T} \to \tilde{T}$ are homomorphisms over T,*

satisfying the "axiom of abelian groups" over T.
The morphisms are homomorphisms of infinitesimal s-manifolds over T compatible with other data.

Proof. From (a), Proposition 4.5 (iv) $\Rightarrow$ (iii) gives an object of (b).
Proposition 4.5 (iii) $\Rightarrow$ (iv) gives a correspondence in the other direction.
By the previous proposition, these correspondences give an equivalence of categories. ∎

5. Main theorem

Now we state and prove the main theorem. For a vector bundle (or a locally free sheaf) $\mathcal{A}$ over Q and a point $q \in Q$, $\mathcal{A}_q$ will denote the fiber at q, and if $F : \mathcal{A} \to \mathcal{A}'$ is a linear bundle map, F_q denotes the linear map $\mathcal{A}_q \to \mathcal{A}'_q$ between the fibers.

Theorem 5.1. *Let Q be a connected regular s-manifold and $q \in Q$ a point. Write $\mathbb{K}$ for the base field.*

(1) *For a regular quandle module $\mathcal{A}$ over Q, there is a natural structure of a regular representation of (T_qQ, d_qs_q) on $\mathcal{A}_q$.*

(2) *Given regular quandle modules $\mathcal{A}$ and $\mathcal{A}'$ over Q and a homomorphism $F : \mathcal{A} \to \mathcal{A}'$, F_q is a homomorphism of representations of (T_qQ, d_qs_q). This correspondence gives an injective map between the sets of homomorphisms. In the smooth or complex analytic case, this map is surjective if Q is simply-connected.*

(3) *Assume that Q is a connected, simply-connected smooth or complex analytic regular s-manifold. Given a regular representation V of (T_qQ, d_qs_q), there exists a regular quandle module $\mathcal{A}$ over Q such that $\mathcal{A}_q$ is isomorphic to V.*

In other words, there is a faithful functor between the following categories:

- *The category $\mathbf{Mod}^r_{\mathbb{K}}(Q)$ of regular quandle modules $(\mathcal{A}, \eta, \tau)$ over $(Q, \triangleright)$.*
- *The category $\mathbf{Rep}^r(T_qQ, d_qs_q)$ of regular representations $(V, \rho, \delta, \theta, \psi)$ of the infinitesimal s-manifold $(T_qQ, *, [\ ], d_qs_q)$.*

In the smooth or complex analytic case, if Q is simply-connected, it is an equivalence.

Proof. Let us write (T, σ) for (T_qQ, d_qs_q).

(1) Recall that $\mathcal{A}$ is a vector bundle. By Proposition 3.14, the total space $\mathcal{A}$ and $\mathcal{A} \times_Q \mathcal{A}$ can be considered as a smooth quandle, etc., such that the projection $\Pi : \mathcal{A} \to Q$, the zero section $Z : Q \to \mathcal{A}$, the fiberwise addition $A : \mathcal{A} \times_Q \mathcal{A} \to \mathcal{A}$ and the fiberwise inversion $I : \mathcal{A} \to \mathcal{A}$ are homomorphisms of smooth quandles, etc. Note that $\mathcal{A} \times_Q \mathcal{A}$ can also be considered as the quandle associated to the direct sum quandle module $\mathcal{A} \oplus \mathcal{A}$.

We write (q, a) for a point $a \in \mathcal{A}_q$ and (q, a, b) for $(a, b) \in (\mathcal{A} \times_Q \mathcal{A})_q$ and V for $\mathcal{A}_q$. Then we have natural identifications

$$T_{(q,0)}\mathcal{A} \cong T \oplus V, \qquad T_{(q,0,0)}(\mathcal{A} \times_Q \mathcal{A}) \cong T \oplus V \oplus V$$

as vector spaces, by identifying T with the tangent spaces to the zero sections. The linear maps $d_{(q,0)}s_{(q,0)}$ and $d_{(q,0,0)}s_{(q,0,0)}$ can be identified with (d_qs_q, η_{qq}) and $(d_qs_q, \eta_{qq}, \eta_{qq})$. By the definition of a regular quandle module, $1 - \eta_{qq}$ is invertible, and so is $1 - \eta_{xx}$ for any $x \in Q$ by the transitivity of Q. It follows that the action of $\mathcal{A}_x$ on $\mathcal{A}_x$ from the left is transitive. Thus the differentials of $d_{(x,a)}s_{(x,a)}$ are conjugates of $d_{(q,0)}s_{(q,0)}$ for any $a \in \mathcal{A}_x$, and $\mathcal{A}$ is a regular s-manifold. One can prove that $\mathcal{A} \times_Q \mathcal{A}$ is a regular s-manifold in the same way.

By Theorem 2.15 (3), $T_{(q,0)}\mathcal{A}$ and $T_{(q,0,0)}(\mathcal{A} \times_Q \mathcal{A})$ are infinitesimal s-manifolds in a natural way. The latter can be identified with the fiber product $T_{(q,0)}\mathcal{A} \times_T T_{(q,0)}\mathcal{A}$ as an infinitesimal s-manifold. In fact, the projection maps $P_1, P_2 : \mathcal{A} \times_Q \mathcal{A} \to \mathcal{A}$ induces such an isomorphism $(d_{(q,0,0)}P_1, d_{(q,0,0)}P_2)$ of vector spaces, and since P_1 and P_2 are quandle homomorphisms, it is a homomorphism of infinitesimal s-manifolds by Theorem 2.24 (1).

The differential of Π, Z, A and I are homomorphisms of infinitesimal s-manifolds by Theorem 2.24 (1), and by Corollary 4.7 we can define a structure of a regular representation of (T, d_qs_q) on (V, η_{qq}). Or, more concretely, Proposition 4.5 (ii)' $\Rightarrow$ (iv) gives the correspondence.

(2) By applying Theorem 2.24 (1), the homomorphism F induces a homomorphism $T \oplus \mathcal{A}_q \to T \oplus \mathcal{A}'_q$ of infinitesimal s-manifolds, which is the identity on T and maps $\mathcal{A}_q$ to $\mathcal{A}'_q$. By Proposition 4.6 (1), the map $\mathcal{A}_q \to \mathcal{A}'_q$ is a homomorphism of representations of (T, σ).

The injectivity of the correspondence follows from Theorem 2.24 (1).

Assume that Q is simply-connected. Then $\mathcal{A}$ is also simply-connected. By Corollary 4.7, a homomorphism $f : \mathcal{A}_q \to \mathcal{A}'_q$ of representations of (T, σ) gives rise to a homomorphism $\mathrm{id}_T \times f : T \oplus \mathcal{A}_q \to T \oplus \mathcal{A}'_q$ of infinitesimal s-manifolds compatible with

the addition maps on the two algebras. It is also compatible with μ_λ in Proposition 4.5. Following Theorem 2.24 (2), we have a quandle homomorphism $F : \mathcal{A} \to \mathcal{A}'$. Since the projection maps, the addition maps and the scalar multiplications for $\mathcal{A}$ and $\mathcal{A}'$ are quandle homomorphisms whose differentials are the projection maps, α and μ_λ on $T \oplus \mathcal{A}_q$ and $T \oplus \mathcal{A}'_q$, it follows from Theorem 2.24 (1) that F commutes with the projection maps, the addition maps and scalar multiplication maps. Thus it is a homomorphism of linear quandle modules by Proposition 3.14.

(3) For a regular representation (V, ψ) of (T, σ), we define a structure of an infinitesimal s-manifold on $T_1 := T \oplus V$ according to Proposition 4.5. The direct sum $V \oplus V$ gives rise to an infinitesimal s-manifold $T_2 := T \oplus V \oplus V$, and from the two projection maps $V \oplus V \to V$ we obtain linear maps $p_1, p_2 : T_2 \to T_1$, which are homomorphisms of infinitesimal s-manifolds over T by Proposition 4.6. It follows that the natural identification $T_2 \cong T_1 \times_T T_1$ is an isomorphism of infinitesimal s-manifolds. (This can also be proven by an explicit calculation.)

Let $\mathcal{A}$ and $\mathcal{A}_2$ be connected, simply-connected regular s-manifolds endowed with points q_1 and q_2 and isomorphisms $T_{q_1}\mathcal{A} \cong T_1$ and $T_{q_2}\mathcal{A}_2 \cong T_2$, which exist by Theorem 2.15 (5). Later in the proof, we will see that $\mathcal{A}_2 \cong \mathcal{A} \times_Q \mathcal{A}$. We regard T_1 and T_2 as infinitesimal s-manifolds over T by the projection maps $\pi : T_1 \to T$ and $\pi_2 : T_2 \to T$. By Proposition 4.5 (1), (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv), we have the following homomorphisms over T: The zero section $\zeta : T \to T_1$, the addition $\alpha : T_2 \to T_1$, the inversion $\iota : T_1 \to T_1$ and scalar multiplications $\mu_\lambda : T_1 \to T_1$ as in Proposition 4.5 (1)(ii). By Theorem 2.24 (2), the projection maps are differentials of quandle morphisms $\Pi : \mathcal{A} \to Q$, $\Pi_2 : \mathcal{A}_2 \to Q$, by which we regard $\mathcal{A}$ and $\mathcal{A}_2$ as quandles over Q. The two projections $p_1, p_2 : T_2 \to T_1$ and ζ, α and μ_λ correspond to homomorphisms $P_1, P_2 : \mathcal{A}_2 \to \mathcal{A}$, $Z : Q \to \mathcal{A}$, $A : \mathcal{A}_2 \to \mathcal{A}$, $I : \mathcal{A} \to \mathcal{A}$ and $M_\lambda : \mathcal{A} \to \mathcal{A}$ of quandles over Q.

Now we will show that $\mathcal{A}$ is a vector bundle over Q with the fiber $\mathcal{A}_q := \Pi^{-1}(q)$ isomorphic to V. Let G and G_1 be connected, simply-connected Lie groups whose Lie algebras are isomorphic to $\mathfrak{g}(T)$ and $\mathfrak{g}(T_1)$, respectively. Since $\pi : T_1 \to T$ is surjective, we have a Lie algebra homomorphism $\tilde{\pi} : \mathfrak{g}(T_1) \to \mathfrak{g}(T)$ extending π by Proposition 2.30 (1). Let $\tilde{\Pi} : G_1 \to G$ be the corresponding homomorphism. Since $V \subseteq T \oplus V = T_1$ is an abelian ideal, $\tilde{V} := V \oplus \mathfrak{h}(V, T_1)$ is an ideal of $\mathfrak{g}(T_1)$ and V is an abelian ideal of $\tilde{V}$ by Lemma 2.32. Thus we have connected normal Lie subgroups $G_{\tilde{V}} \lhd G_1$ and $G_V \lhd G_{\tilde{V}}$ with $T_e G_{\tilde{V}} = \tilde{V}$ and $T_e G_V = V$.

Claim 5.2. *The $G_{\tilde{V}}$-orbits in $\mathcal{A}$ are fibers of Π.*

Proof of the claim. By Lemma 2.32(2), $G_{\tilde{V}} \subseteq \mathrm{Ker}(\tilde{\Pi})$ holds. Thus, for $g \in G_{\tilde{V}}$ and $x \in \mathcal{A}$, we have $\Pi(g \cdot x) = \tilde{\Pi}(g) \cdot \Pi(x) = \Pi(x)$, i.e., the $G_{\tilde{V}}$-action preserves the fibers of Π.

Since V can be identified with the tangent space $T_{q_1}\mathcal{A}_q$ of the fiber $\mathcal{A}_q$, the orbit $G_{\tilde{V}} \cdot q_1$ contains an open subset of $\mathcal{A}_q$. By standard arguments in continuous actions, $G_{\tilde{V}} \cdot q_1$ is in fact open in $\mathcal{A}_q$. Since $G_{\tilde{V}}$ is normal in G_1, a translate of a $G_{\tilde{V}}$-orbit by an element of G_1 is again a $G_{\tilde{V}}$-orbit. By the transitivity of the G_1-action on $\mathcal{A}$, any $G_{\tilde{V}}$-orbit in $\mathcal{A}$ is a translate of $G_{\tilde{V}} \cdot q_1$, and is an open subset of a fiber of Π. It is also closed in the fiber since it is the complement of other orbits, and therefore is a connected component of a fiber. Since Q is simply-connected and $\mathcal{A}$ is connected, fibers of Π are connected, hence $G_{\tilde{V}}$-orbits are exactly fibers of Π. ■

If we regard V as a commutative Lie group, then G_V is a quotient of V by a discrete subgroup, and V acts on $\mathcal{A}$ via G_V.

Claim 5.3. *The action of V on $\mathcal{A}_q$ is simply transitive.*

Proof of the claim. Let H' be the connected Lie subgroup of $G_{\tilde{V}}$ corresponding to $\mathfrak{h}(V, T_1)$. We see that $G_{\tilde{V}}$ is generated by G_V and H', and $G_{\tilde{V}} = G_V H'$ holds since G_V is normal in $G_{\tilde{V}}$.

If H_1 denotes the connected subgroup of G_1 corresponding to $\mathfrak{h}(T_1)$, then $\mathcal{A}$ is described as G_1/H_1 and hence H_1 is the stabilizer of q_1 in G_1. From $\mathfrak{h}(V, T_1) \subseteq \mathfrak{h}(T_1)$ we see that $H' \subset H_1$ holds. Thus $\mathcal{A}_q = G_{\tilde{V}} \cdot q_1 = G_V H' \cdot q_1 = G_V \cdot q_1$ holds. In other words, $\mathcal{A}_q = V \cdot q_1$. (We remark that a general V-orbit is not necessarily a whole fiber.)

From $V \cong T_{q_1}\mathcal{A}_q$, we see that the stabilizer V_{q_1} is discrete. If $V_{q_1} \neq 0$, then we can take an element $v \in V \setminus V_{q_1}$ with $2v \in V_{q_1}$. If we consider the map $i : V \to A_q; v \mapsto v \cdot q_1$, the map M_λ lifts to the usual scalar multiplication on V, since i is the map induced from the Lie algebra homomorphism $V \to \mathfrak{g}(T_1)$, for which the scalar multiplication by λ on V and the map μ_λ on $\mathfrak{g}(T_1)$ are compatible. Thus we have

$$i(v) = i\Big(\frac{1}{2} \cdot 2v\Big) = M_{1/2}(i(2v)) = M_{1/2}(q_1) = M_{1/2}(i(0)) = i\Big(\frac{1}{2} \cdot 0\Big) = q_1,$$

a contradiction. Thus V_{q_1} is trivial, i.e., the action of V on $\mathcal{A}_q$ is simply transitive. ■

Claim 5.4. *The map $(P_1, P_2) : \mathcal{A}_2 \to \mathcal{A} \times_Q \mathcal{A}$ is an isomorphism.*

Proof of the claim. The fiber $\mathcal{A}_q$ is isomorphic to V by the previous claim. By the transitivity of $\mathcal{A}$, so is any fiber. It follows that any fiber of $\mathcal{A} \times_Q \mathcal{A}$ is isomorphic to $V \times V$, and that $\mathcal{A} \times_Q \mathcal{A}$ is simply-connected.

Both $\mathcal{A}_2$ and $\mathcal{A} \times_Q \mathcal{A}$ are connected and simply-connected regular s-manifolds with the infinitesimal algebra $T_1 \times_T T_1$, so the natural map $\mathcal{A}_2 \to \mathcal{A} \times_Q \mathcal{A}$ is an isomorphism. ■

In particular, A can be considered as a binary operation $\mathcal{A} \times_Q \mathcal{A} \to \mathcal{A}$. The data $(\mathcal{A}, \Pi, Z, A, I, \{M_\lambda\}_{\lambda \in \mathbb{K}})$ satisfies the relative version of axioms of $\mathbb{K}$-vector space over Q, and in particular defines a structure of a $\mathbb{K}$-vector space on each fiber $\mathcal{A}_x$.

Claim 5.5. *Let $g : \mathcal{A} \to \mathcal{A}$ be a homomorphism of smooth quandles or complex analytic quandles which covers a quandle homomorphism $Q \to Q$ and preserves $Z(Q)$. Then g induces linear maps between fibers, i.e., for any $a, b \in \mathcal{A}_x$ and $\lambda \in \mathbb{K}$, we have $g(A(a, b)) = A(g(a), g(b))$ and $g(M_\lambda(a)) = M_\lambda(g(a))$.*

Proof of the claim. Let us prove the first equality. The proof of the second one is similar.

Write $F_1 = g \circ A$ and $F_2 = A \circ (g \times_Q g)$. They are maps from $\mathcal{A} \times_Q \mathcal{A}$ to $\mathcal{A}$, and what we have to show is that they coincide. Since they are homomorphisms of smooth

quandles or complex analytic quandles, it is sufficient to show that $F_1(q_2) = F_2(q_2)$ and $d_{q_2}F_1 = d_{q_2}F_2$ by Theorem 2.24 (1).

Let $Z_2 = (Z, Z) : Q \to \mathcal{A} \times_Q \mathcal{A}$. Then $q_2 = (Z(q), Z(q)) = Z_2(q)$ under the identification $\mathcal{A}_2 \cong \mathcal{A} \times_Q \mathcal{A}$. Since g covers a map $\bar{g} : Q \to Q$ and preserves $Z(Q)$, it follows that $g(Z(x)) = Z(\bar{g}(x))$ for any $x \in Q$, and since $Z(x)$ and $Z(\bar{g}(x))$ are the zero elements of $\mathcal{A}_x$ and $\mathcal{A}_{\bar{g}(x)}$ for A, respectively, we have

$$\begin{aligned} F_1(Z_2(x)) &= g(A(Z(x), Z(x))) = g(Z(x)) = Z(\bar{g}(x)) \quad \text{and} \\ F_2(Z_2(x)) &= A(g(Z(x)), g(Z(x))) = A(Z(\bar{g}(x)), Z(\bar{g}(x))) = Z(\bar{g}(x)). \end{aligned}$$

Therefore $F_1|_{Z_2(Q)} = F_2|_{Z_2(Q)}$ holds. In particular, we have $F_1(q_2) = F_2(q_2)$ and $(d_{q_2}F_1)|_{T_{q_2}Z_2(Q)} = (d_{q_2}F_2)|_{T_{q_2}Z_2(Q)}$.

From the condition that Z is the "zero element" for A, i.e.,

$$A \circ (\mathrm{id}_{\mathcal{A}} \times_Q Z) = A \circ (Z \times_Q \mathrm{id}_{\mathcal{A}}) = \mathrm{id}_{\mathcal{A}}$$

as maps $\mathcal{A} \cong \mathcal{A} \times_Q Q \cong Q \times_Q \mathcal{A} \to \mathcal{A}$, it follows that the restrictions of $d_{q_2}F_1$ and $d_{q_2}F_2$ to the first and the second fiber directions are all equal to the map $(d_{q_1}g)|_V : V \to T_{Z(\bar{g}(q))}\mathcal{A}$. Thus $d_{q_2}F_1 = d_{q_2}F_2$ holds. ■

Claim 5.6. *There is a structure of a vector bundle on $\mathcal{A}$ for which the projection, the zero section, the addition map, the inversion map and the scalar multiplications are Π, Z, A, I and M_λ, respectively.*

Proof of the claim. Let $\tilde{\mathfrak{g}}'$ be the Lie algebra obtained by applying Proposition 2.30 to $\zeta : T \to T_1$. Then the corresponding Lie group G' maps surjectively to G, hence the map $a : G' \to Q; g \mapsto g \cdot q$ is surjective, and there is a natural map $G' \to G_1$ compatible with Z.

For any point $x_0 \in Q$, take a(n étale) neighborhood U in Q and a local section $s : U \to G'$ for a. Then the map $\varphi_s : U \times V \to \mathcal{A}; (x, v) \mapsto s(x)(v \cdot q_1)$ gives an isomorphism to $\Pi^{-1}(U)$ as manifolds over U.

From the explicit form of α and μ_λ, it is easy to see that the map $v \mapsto v \cdot q_1$ is an isomorphism of V and $\mathcal{A}_q$ as linear spaces. By the previous claim, φ_s is linear on each fiber. Thus, given two local sections $s, s' : U \to G'$ for a, the transition function $g : U \times V \to V$ defined by $\varphi_{s'}^{-1} \circ \varphi_s(x, v) = (x, g(x, v))$ is linear in v, hence the assertion. ■

Thus $\mathcal{A}$ is a vector bundle with a compatible quandle structure, i.e., a quandle module over Q (Proposition 3.14). It is regular since η_{xx} are conjugate to ψ.

If we apply (1) to $\mathcal{A}$, the isomorphism $T_{q_1}\mathcal{A} \cong T \oplus \mathcal{A}_q$ in (1) can be identified with the isomorphism $T_{q_1}\mathcal{A} \cong T_eG_1/T_eH_1 \cong T_1 = T \oplus V$: In the former, the first factor T is given by differentiating $Q \cong Z(Q) \subseteq \mathcal{A}$, and in the latter it is the image of ζ, which can be identified with d_qZ. In the former, $\mathcal{A}_q$ is the tangent space to the fiber, and so is equal to the kernel of $d_{q_1}\Pi$, and so corresponds to V in the latter.

Therefore the representation of (T, σ) associated to $\mathcal{A}$ by (1) is isomorphic to (V, ψ). This completes the proof of the main theorem. ■

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