A Non-Abelian Tensor Product of Lie Crossed Modules

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Abstract.   The notions of non-abelian tensor and exterior products in the category of Lie crossed modules are introduced and investigated. Also, their relationships with the Whitehead's universal quadratic and the homology of Lie crossed modules are established.

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1.   Introduction

All Lie algebras are considered over a fixed field Λ *and* [,] *denotes the Lie bracket.*

Lie crossed modules are algebraic objects equivalent to simplicial Lie algebras with the associated Moore complex of length 1 (see, for instance, [7]). They were used by Lavendhomme and Roisin [12], as adequate coefficients of a non-abelian cohomology of T-algebras, when T is a theory of K-Lie algebras, by Kassel and Loday [11] to study the cyclic homology of associative algebras, and by Guin [9] and Ellis [8] to give and describe the non-abelian homology of Lie algebras. As Lie crossed modules are a generalization of Lie algebras, it is also of interest to seek results to the Lie algebra theory in the category of Lie crossed modules. Accordingly, many concepts of Lie algebras have been generalized to Lie crossed modules. For example, Casas and Ladra in [4] developed perfect covers theory in Lie algebras to Lie crossed modules, and in [5] defined the actor of a Lie crossed module, an object which plays a similar role to derivations of a Lie algebra. Also, Casas, Inassaridze, and Ladra [2] introduced the homology theory of Lie crossed modules, which is a generalization of the Eilenberg-MacLane homology of Lie algebras, and gave a Hopf formula for the second integral homology of a Lie crossed module. Furthermore, associated with a central extension of Lie crossed modules, they obtained a five-term exact sequence connecting the two first homologies of Lie crossed modules (see also [1, 3]). In [6] and [15], the theories of capability and stem covers of Lie crossed modules have been developed analogous to that for Lie algebras, and some very interesting connections between these notions with the second homology and the Ganea map of Lie crossed modules are given. Recently, the second and third authors in [14] presented the notions of the non-abelian tensor and exterior products of two ideal crossed submodules of a given Lie crossed module, and extended some

classical results from second homology of Lie algebras to the homology of Lie crossed modules. Especially, they gave a new description of the second homologies of Lie crossed modules as central crossed submodules of their exterior products.

In this article, using crossed squares in Lie algebras, we generalize the definitions of non-abelian tensor and exterior products for two arbitrary Lie crossed modules, and similar to the works of Ellis [8] in the Lie algebra case, we describe their relationships with the Whitehead's quadratic functor. Applications to the homology of Lie crossed modules are also obtained.

The article is divided into five section. In section 2, we review some concepts in the theory of Lie crossed modules, with special mention to the tensor product and crossed square in Lie algebras. We also give some fundamental properties on these concepts. In section 3, we define the non-abelian tensor and exterior products for Lie crossed modules and investigate some common properties between Lie crossed modules and their tensor products. Especially, we present some bounds on the nilpotency class and solvability length of the tensor product, provided such information is given at least on one of the Lie crossed modules. In section 4, we provide some relations between the products presented in this paper and a generalized version of Whitehead's universal quadratic functor of Lie algebras, which are vast generalizations of the results of Ellis [8]. In the last section, using the obtained results, we give some applications to the homology of Lie crossed modules. In particular, we get an exact sequence of eight terms associated with an extension of Lie crossed modules in the homology.

2.   Preliminaries

This section is devoted to recalling the notions and definitions needed in the rest of the paper and to fixing the notation.

Let M and P be two Lie algebras. By an *action* of P on M we mean a Λ-bilinear map $P \times M \longrightarrow M$, $(p, m) \longmapsto {}^p m$, satisfying (i) ${}^{[p,p']} m = {}^p({}^{p'} m) - {}^{p'}({}^p m)$, and (ii) ${}^p[m, m'] = [{}^p m, m'] + [m, {}^p m']$ for all $m, m' \in M$, $p, p' \in P$. For example, if P is a subalgebra of some Lie algebra L and M is an ideal in L, then Lie multiplication in L induces an action of P on M given by ${}^p m = [p, m]$. A *Lie crossed module* $\mathbf{M} = (M, P, \partial)$ is a Lie homomorphism $\partial : M \longrightarrow P$ together with an action of P on M such that (*i*) $\partial({}^p m) = [p, \partial(m)]$, (*ii*) ${}^{\partial(m)} m' = [m, m']$ for all $m, m' \in M$, $p \in P$. If M is an ideal of P, then (M, P, i) is a Lie crossed module, in which i is the inclusion map. In this way, every Lie algebra P can be regarded as Lie crossed module in the two obvious ways: $(0, P, i)$ or (P, P, id).

A *morphism of Lie crossed modules* $(\theta_1, \theta_2) : (M, P, \partial) \longrightarrow (N, Q, \sigma)$ is a pair of Lie homomorphisms $\theta_1 : M \longrightarrow N$ and $\theta_2 : P \longrightarrow Q$ such that $\sigma \theta_1 = \theta_2 \partial$ and, for all $p \in P$, $m \in M$, $\theta_1({}^p m) = {}^{\theta_2(p)} \theta_1(m)$. The corresponding category of Lie crossed modules is denoted by $\mathfrak{XLie}$.

Let $\mathbf{M} = (M, P, \partial)$ be a Lie crossed module with ideal crossed submodules $\mathbf{M_1} = (M_1, P_1, \partial)$ and $\mathbf{M_2} = (M_2, P_2, \partial)$. Then

(1) $Z(\mathbf{M}) = (M^P, Z(P) \cap st_P(M), \partial)$ is the *center* of $\mathbf{M}$, where $Z(P)$ is the center of L, $M^P = \{m \in M \mid {}^p m = 0 \text{ for all } p \in P\}$, and $st_P(M) = \{p \in P \mid {}^p m = 0 \text{ for all } m \in M\}$. The Lie crossed module $\mathbf{M}$ is *abelian* if $\mathbf{M} = Z(\mathbf{M})$, or equi-

valently, if P is abelian and P acts trivially on M. We denote the category of abelian Lie crossed module by $\mathfrak{Xvect}$.

(2) $\mathbf{M'} = ([P, M], P', \partial)$ is the *commutator crossed submodule* of $\mathbf{M}$, where $P' = [P, P]$ and $[P, M] = \langle {}^{p}m \mid p \in P, m \in M \rangle$ is the displacement subalgebra of M relative to P. The Lie crossed module $\mathbf{M}$ is *perfect* if $\mathbf{M} = \mathbf{M'}$.

(3) $[\mathbf{M_1}, \mathbf{M_2}]$ is the ideal crossed submodule $([P_2, M_1] + [P_1, M_2], [P_1, P_2], \partial)$ of $\mathbf{M}$.

(4) $\gamma_n(\mathbf{M})$ denotes the n-th term of lower central series of $\mathbf{M}$ defined recursively by $\gamma_1(\mathbf{M}) = \mathbf{M}$ and $\gamma_{n+1}(\mathbf{M}) = [\gamma_n(\mathbf{M}), \mathbf{M}]$, for $n \geq 1$. An easy induction on $n \geq 0$ indicates that $\gamma_{n+1}(\mathbf{M}) = ([_nP, M], \gamma_{n+1}(P), \partial)$, in which $[_0P, M] = M$ and inductively $[_nP, M] = [P, [_{n-1}P, M]]$. The Lie crossed module $\mathbf{M}$ is *nilpotent* of class n if $\gamma_{n+1}(\mathbf{M}) = 0$ and n is the least with this property.

(5) $\mathbf{M}^{(n)}$ denotes the n-th term of derived series of $\mathbf{M}$ defined inductively by $\mathbf{M}^{(0)} = \mathbf{M}$ and $\mathbf{M}^{(n+1)} = (\mathbf{M}^{(n)})'$, for $n \geq 0$. The Lie crossed module $\mathbf{M}$ is *solvable* of length n if $\mathbf{M}^{(n)} = 0$, where n is the least when this happens. In particular, $\mathbf{M}$ is *metabelian* if $\mathbf{M}$ is solvable of length 2.

(6) $\mathbf{M}_{ab} = (M/[P, M], P_{ab}, \overline{\partial})$ denotes the abelianization of $\mathbf{M}$, where $P_{ab} = P/P'$ and $\overline{\partial}$ is induced by the Lie homomorphism ∂.

Let (M, P, ∂_1) and (N, P, ∂_2) be two Lie crossed modules. There are actions of M on N and of N on M given by ${}^{m}n = {}^{\partial_1(m)}n$ and ${}^{n}m = {}^{\partial_2(n)}m$. We take M (and N) to act on itself by Lie multiplication. The *non-abelian tensor product* $M \otimes N$ is defined in [8] as the Lie algebra generated by the symbols $m \otimes n$ for $m \in M$, $n \in N$, subject to the relations

(A1) $c(m \otimes n) = cm \otimes n = m \otimes cn$,

(A2) $(m + m') \otimes n = m \otimes n + m' \otimes n$,

(A3) $m \otimes (n + n') = m \otimes n + m \otimes n'$,

(A4) $[m, m'] \otimes n = m \otimes ({}^{m'}n) - m' \otimes ({}^{m}n)$,

(A5) $m \otimes [n, n'] = ({}^{n'}m) \otimes n - ({}^{n}m) \otimes n'$,

(A6) $[(m \otimes n), (m' \otimes n')] = -({}^{n}m) \otimes ({}^{m'}n')$,

for all $c \in \Lambda$, $m, m' \in M$, $n, n' \in N$. Note that the identity homomorphism $id_P : P \longrightarrow P$ is a Lie crossed module with P acting on itself by Lie multiplication, so we can always form the tensor products $P \otimes M$, $P \otimes N$ and $P \otimes P$.

Let M and N be as above. For any Lie algebra L, a function $h : M \times N \longrightarrow L$ is a *Lie pairing* if

(B1) $h(c_1m_1 + c_2m_2, n) = c_1h(m_1, n) + c_2h(m_2, n)$,

(B2) $h(m, c_1n_1 + c_2n_2) = c_1h(m, n_1) + c_2h(m, n_2)$,

(B3) $h([m, m'], n) = h(m, {}^{m'}n) - h(m', {}^{m}n)$,

(B4) $h(m, [n, n']) = h({}^{n'}m, n) - h({}^{n}m, n')$,

(B5) $h({}^{n}m, {}^{m'}n') = -[h(m, n), h(m', n')]$,

for all $c_1, c_2 \in \Lambda$, $m, m' \in M$, $n, n' \in N$. It is apparent that the function $M \times N \longrightarrow M \otimes N$, $(m, n) \longmapsto m \otimes n$, is the universal Lie pairing in the sense that any Lie pairing $h : M \times N \longrightarrow L$ determines a unique Lie homomorphism $h^* : M \otimes N \longrightarrow L$ such that $h^*(m \otimes n) = h(m, n)$.

Let $M \square N$ be the submodule of $M \otimes N$ generated by the elements $m \otimes n$ with $\partial(m) = \sigma(n)$. One easily gets that $M \square N$ lies in the centre of $M \otimes N$. Following G. Ellis in [8], the *non-abelian exterior product* $M \wedge N$ is defined to be the quotient $(M \otimes N)/(M \square N)$.

Proposition 2.1.   *With the above assumptions and notations, we have*

(i) *The maps* $\qquad \lambda : M \otimes N \longrightarrow P, \quad m \otimes n \longmapsto [\partial_1(m), \partial_2(n)]$

$$\lambda_M : M \otimes N \longrightarrow M, \quad m \otimes n \longmapsto -\,^n m$$

$$\lambda_N : M \otimes N \longrightarrow N, \quad m \otimes n \longmapsto \,^m n$$

are Lie crossed modules, in which the action of P on $M \otimes N$ is given by the equation $\,^p(m \otimes n) = \,^p m \otimes n + m \otimes \,^p n$, and M and N act on $M \otimes N$ via ∂_1 and ∂_2. Furthermore, the results holds with $\otimes$ replaced by $\wedge$.

(ii) *The functional homomorphism $\partial_1 \otimes id_N : M \otimes N \longrightarrow P \otimes N$, together with the action of $P \otimes N$ on $M \otimes N$ induced by the map $\lambda'_P : P \otimes N \longrightarrow P$, is a Lie crossed module.*

(iii) *There is an action of P on the semidirect sum $M \rtimes N$ defined by the formula $\,^p(m, n) = (\,^p m, \,^p n)$.*

(iv) *The map $\beta : M \rtimes N \longrightarrow P$ defined by $\beta(m, n) = \partial_1(m) + \partial_2(n)$ is a Lie homomorphism.*

(v) *For any $m \in M$, $n \in N$ and $x \in M \otimes N$, $\,^m x = m \otimes \lambda_N(x)$ and $\,^n x = -\lambda_M(x) \otimes n$.*

(vi) *For any $x, y \in M \otimes N$, $\partial_1 \lambda_M(x) \otimes \lambda_N(y) = -\partial_1 \lambda_M(y) \otimes \lambda_N(x)$.*

Proof.   Part (i) is proved in [8, Proposition 3] and part (ii) is routine.

(iii) The bilinearity property and the first condition obviously hold, and the second follows from the identity $\,^{(\,^p n)} m = \,^p(\,^n m) - \,^n(\,^p m)$, which is true for any $m \in M$, $n \in N$ and $p \in P$.

(iv) It easily follows from the identity $\partial_1(\,^n m) = \,^{\partial_2(n)} \partial_1(m)$, which holds for all $m \in M$, $n \in N$.

(v) Follow from part (i), and the relations $(A4)$, $(A5)$.

(vi) Assume $x = m_1 \otimes n_1$ and $y = m_2 \otimes n_2$. Then we have

$$\partial_1 \lambda_M(x) \otimes \lambda_N(y) = -\,^{n_1}\partial_1(m_1) \otimes \,^{\partial_1(m_2)} n_2 \overset{by\,(A6)}{=} \,^{n_2}\partial_1(m_2) \otimes \,^{\partial_1(m_1)} n_1$$

$$= -\partial_1 \lambda_M(y) \otimes \lambda_N(x),$$

and the proof is complete.   ■

In the context of Lie algebras, we recall from [18] the definition of Whitehead's universal quadratic functor $\Gamma(\text{-})$. For any Λ-module A, $\Gamma(A)$ is the Λ-module generated by the symbols $\gamma(a)$ with $a \in A$, subject to the relations

$$\lambda^2 \gamma(a) = \gamma(\lambda a),$$

$$\gamma(\lambda a + b) + \lambda \gamma(a) + \lambda \gamma(b) = \lambda \gamma(a + b) + \gamma(\lambda a) + \gamma(b),$$

$$\gamma(a + b + c) + \gamma(a) + \gamma(b) + \gamma(c) = \gamma(a + b) + \gamma(a + c) + \gamma(b + c),$$

for all $\lambda \in \Lambda$, $a, b, c \in A$. Note that the last two conditions imply that the map $\Delta\gamma : A \times A \longrightarrow \Gamma(A)$, taking $(a, b) \in A \times A$ to the element $\gamma(a + b) - \gamma(a) - \gamma(b) \in \Gamma(A)$, is a Λ-bilinear map, whence we get a Λ-module homomorphism $\Delta : A \otimes A \longrightarrow \Gamma(A)$.

Proposition 2.2. (i) *With the above assumptions and notations, set*

$$M \times_P N = \{(m,n) \in M \oplus N \mid \partial_1(m) = \partial_2(n)\},$$

$$\langle M, N \rangle = \{(\lambda_M(x), \lambda_N(x)) \mid x \in M \otimes N\}.$$

Then $M \times_P N$ is a subalgebra of the direct sum $M \oplus N$, $\langle M, N \rangle$ is an ideal of $M \times_P N$, and the quotient $(M \times_P N)/\langle M, N \rangle$ is abelian. Moreover, there is a natural exact sequence of Lie algebras

$$\Gamma(\frac{M \times_P N}{\langle M, N \rangle}) \xrightarrow{\psi} M \otimes N \twoheadrightarrow M \wedge N,$$

where $\psi(\gamma(\overline{(m,n)})) = m \otimes n$ for $\overline{(m,n)}$ denoting the coset of $\langle M, N \rangle$ represented by $(m,n) \in M \times_P N$.

(ii) *For any Λ-modules A and B, there is an isomorphism*

$$\Gamma(A \oplus B) \cong \Gamma(A) \oplus \Gamma(B) \oplus (A \otimes_\Lambda B),$$

where $\otimes_\Lambda$ denotes the standard tensor product of Λ-modules.

(iii) *If I is a well-ordered set and A is a free Λ-module with basis $\{e_i\}_{i \in I}$, then $\Gamma(A)$ is a free Λ-module with basis*

$$\{\gamma(e_i)\}_{i \in I} \cup \{z_{ij} = \gamma(e_i + e_j) - \gamma(e_i) - \gamma(e_j)\}_{i<j}.$$

Proof. Part (i) is proved in [8]. Parts (ii) and (iii) are proved in [18]. ∎

In the sense of [7], a *crossed square* in Lie algebras is a commutative diagram of Lie algebras

$$\begin{array}{ccc} M & \xrightarrow{\lambda_1} & T \\ {\scriptstyle\partial_1}\downarrow & {\scriptstyle\lambda_2} & \downarrow{\scriptstyle\partial} \\ P & \xrightarrow{\quad} & L \end{array} \qquad (1)$$

endowed with actions of L on M, T, P (and hence actions of T on M, P via ∂ and of P on M, T via λ_2) and a bilinear map $h_1 : T \times P \longrightarrow M$ such that the following axioms hold:

(C1) λ_1, ∂_1 preserve the actions of L

(C2) $\partial, \lambda_2, \alpha = \lambda_2\partial_1 = \partial\lambda_1$ are Lie crossed modules,

(C3) $\lambda_1 h_1(t,p) = -{}^p t, \partial_1 h_1(t,p) = {}^t p,$

(C4) $h_1(\lambda_1(m), p) = -{}^p m, h_1(t, \partial_1(m)) = {}^t m,$

(C5) $h_1([t,t'], p) = h_1(t, {}^{t'}p) - h_1(t', {}^t p),$

(C6) $h_1(t, [p,p']) = h_1({}^{p'}t, p) - h_1({}^p t, p'),$

(C7) ${}^l h_1(t,p) = h_1({}^l t, p) + h_1(t, {}^l p),$

for all $t, t' \in T$, $p, p' \in P$, $m \in M$, $l \in L$. It is obvious that for any ideal crossed submodule (M, P, ∂) of a Lie crossed module (T, L, ∂), the square

$$\begin{array}{ccc} M & \xrightarrow{i_1} & T \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ P & \xrightarrow{i_2} & L \end{array}$$

with $h_1(t,p) = -{}^p t$ is a crossed square, where i_1, i_2 are the inclusion maps.

Lemma 2.3. *Consider the crossed square* (1) *of Lie algebras. Then*

(i)   *The maps λ_1, ∂_1 are Lie crossed modules.*

(ii) $(\ker \lambda_1, \ker \lambda_2, \partial_1)$ *is a central crossed submodule of* (M, P, ∂_1).

(iii) *For any $t \in T$ and $p, p' \in P$, $\,^{p'}h_1(t, p) = h_1(\,^p t, p')$.*

Proof.   Follow immediately from the above axioms.   ∎

Consider the crossed squares

$$
\begin{array}{ccc}
M & \xrightarrow{\ \lambda_1\ } & T \\
{\scriptstyle \partial_1}\downarrow & & \downarrow{\scriptstyle \partial} \\
P & \xrightarrow[\ \lambda_2\ ]{} & L
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
N & \xrightarrow{\ \mu_1\ } & T \\
{\scriptstyle \partial_2}\downarrow & & \downarrow{\scriptstyle \partial} \\
Q & \xrightarrow[\ \mu_2\ ]{} & L
\end{array}
\qquad (2)
$$

together with bilinear functions $h_1 : T \times P \longrightarrow M$ and $h_2 : T \times Q \longrightarrow N$, respectively. Plainly, both Lie algebras appeared in the above crossed squares act on each other. Then we can form the non-abelian tensor products $M \otimes N$, $M \otimes Q$, $P \otimes N$ and $P \otimes Q$. Using the Lie crossed modules

$$
\begin{aligned}
&\lambda_M : M \otimes N \longrightarrow M \ , \quad \lambda'_M : M \otimes Q \longrightarrow M \ , \quad \lambda'_P : P \otimes N \longrightarrow P \\
&\lambda_P : P \otimes Q \longrightarrow P \ , \quad \lambda_N : M \otimes N \longrightarrow N \ , \quad \lambda'_Q : M \otimes Q \longrightarrow Q \\
&\lambda'_N : P \otimes N \longrightarrow N \ , \quad \lambda_Q : P \otimes Q \longrightarrow Q
\end{aligned}
$$

each of theses tensor products acts on the other (for instance, $x \in M \otimes N$ acts on $y \in (M \otimes Q) \cup (P \otimes N) \cup (P \otimes Q)$ by $\,^x y = \,^{\lambda_M(x)}y = \,^{\lambda_N(x)}y$). Also, by Proposition 2.1(ii), the functional homomorphisms

$$
\begin{aligned}
\partial_1 \otimes id_N : M \otimes N \longrightarrow P \otimes N &\quad , \quad id_M \otimes \partial_2 : M \otimes N \longrightarrow M \otimes Q \\
\partial_1 \otimes id_Q : M \otimes Q \longrightarrow P \otimes Q &\quad , \quad id_P \otimes \partial_2 : P \otimes N \longrightarrow P \otimes Q
\end{aligned}
$$

are Lie crossed modules. We now construct the semidirect sum $(M \otimes Q) \rtimes (P \otimes N)$ and define the following maps

$$
\begin{aligned}
&\alpha : M \otimes N \longrightarrow (M \otimes Q) \rtimes (P \otimes N), &\quad &x \longmapsto (id_M \otimes \partial_2(x), -\partial_1 \otimes id_N(x)), \\
&\beta : (M \otimes Q) \rtimes (P \otimes N) \longrightarrow P \otimes Q, &\quad &(y, z) \longmapsto \partial_1 \otimes id_Q(y) + id_P \otimes \partial_2(z).
\end{aligned}
$$

Using these assumptions and notations, we have the following consequences, which play a crucial role in our investigation.

Lemma 2.4.   (i)   *For any $x \in M \otimes N$, $x_1 \in M \otimes Q$, $x_2 \in P \otimes N$ and $y \in P \otimes Q$, we have*

(a) $\,^y x_1 = -\lambda'_M(x_1) \otimes \lambda_Q(y)$;

(b) $\,^y x_2 = \lambda_P(y) \otimes \lambda'_N(x_2)$;

(c) $\,^{x_1} x_2 = \,^{\partial_1 \otimes id_Q(x_1)} x_2 = \partial_1 \lambda'_M(x_1) \otimes \lambda'_N(x_2)$;

(d) $\,^{x_2} x_1 = \,^{id_P \otimes \partial_2(x_2)} x_1 = -\lambda'_M(x_1) \otimes \partial_2 \lambda'_N(x_2)$;

(e) $\,^x x_1 = \,^{\partial_1 \otimes id_N(x)} x_1 = \,^{id_M \otimes \partial_2(x)} x_1 = \lambda_M(x) \otimes \lambda'_Q(x_1)$.

(ii) *The Lie crossed modules $\partial_1 \otimes id_N$, $\partial_1 \otimes id_Q$ and $id_M \otimes \partial_2$ preserve the actions of P and Q.*

(iii) *If $\lambda_2 \partial_1(m) = \mu_2 \partial_2(n)$ for some $m \in M$, $n \in N$, then for any $x \in M \otimes N$,*

$$
\partial_1(m) \otimes \lambda_N(x) = -\partial_1 \lambda_M(x) \otimes n.
$$

(iv) *If $\lambda_2\partial_1(m) = \mu_2\partial_2(n)$ and $\lambda_2(p) = \mu_2(q)$ for some $m \in M$, $n \in N$, $p \in P$ and $q \in Q$, then $\partial_1({}^q m) = -{}^n p$ and $\partial_2({}^p n) = -{}^m q$.*

(v) *The maps*

$$\mu'_M : P \otimes N \longrightarrow M \quad , \quad p \otimes n \longmapsto -h_1(\mu_1(n), p)$$

$$\mu'_N : M \otimes Q \longrightarrow N \quad , \quad m \otimes q \longmapsto h_2(\lambda_1(m), q)$$

are Lie crossed modules and preserve the actions of P and Q.

(vi) *There is an action of $P \otimes Q$ on $(M \otimes Q) \rtimes (P \otimes N)$ defined by ${}^y(x_1, x_2) = ({}^y x_1, {}^y x_2)$, for all $y \in P \otimes Q$ and $(x_1, x_2) \in (M \otimes Q) \rtimes (P \otimes N)$.*

Proof. (i) We only prove the equality in point (a), the others are done in the same way. Suppose $x_1 = m \otimes q$ for some $m \in M$, $q \in Q$. Then we have

$$^y x_1 = {}^{\lambda_Q(y)} m \otimes q + m \otimes [\lambda_Q(y), q] \overset{by(A5)}{=} -\lambda'_M(x_1) \otimes \lambda_Q(y).$$

(ii) Straightforward.

(iii) Assume $x = m' \otimes n'$ for some $m' \in M$, $n' \in N$. Then using Proposition 2.1(v), we have

$$-\partial_1\lambda_M(x) \otimes n = {}^{\mu_2\partial_2(n)}(\partial_1(m') \otimes n') = {}^{\lambda_2\partial_1(m)}(\partial_1(m') \otimes n') = \partial_1(m) \otimes \lambda_N(x).$$

(iv) From the assumptions, we have ${}^q m = {}^{\mu_2(q)} m = {}^{\lambda_2(p)} m = {}^p m$ and consequently, $\partial_1({}^q m) = [p, \partial_1(m)] = -{}^{\lambda_2\partial_1(m)} p = -{}^{\mu_2\partial_2(n)} p = -{}^n p$. The second relation is proved similarly.

(v) Using the bilinearity of h_1, relations (C5)–(C6) and the fact that $\mu_1 : N \longrightarrow T$ is a Lie crossed module, one easily sees that μ'_M preserves the defining relations of the tensor product and then is a Lie homomorphism. Also, for each $m \in M$ and generators $x = p \otimes n, x' = p' \otimes n' \in P \otimes N$, we have

$$\mu'_M({}^m x) = -h_1(\mu_1(n), {}^m p) - h_1({}^m \mu_1(n), p) \overset{by(C7)}{=} -{}^m h_1(\mu_1(n), p) = {}^m \mu'_M(x),$$

$$^{\mu'_M(x')} x = [-\partial_1 h_1(\mu_1(n'), p'), p] \otimes n + p \otimes {}^{-\partial\lambda_1 h_1(\mu_1(n'), p')} n$$

$$\overset{by(C3)}{=} [-{}^{\mu_1(n')} p', p] \otimes n + p \otimes {}^{\partial(p'\mu_1(n'))} n = [-{}^{n'} p', p] \otimes n + p \otimes {}^{(p'n')} n$$

$$\overset{by(A4)}{=} -{}^{n'} p' \otimes {}^p n \overset{by(A6)}{=} {}^{x'} x.$$

Then μ'_M and similarly, μ'_N are Lie crossed modules. The rest of proof is same as the first part.

(vi) For all $y \in P \otimes Q$, $x_1 \in M \otimes Q$ and $x_2 \in P \otimes N$, we have

$$^{(^y x_2)} x_1 = {}^{\lambda'_P({}^{\lambda_P(y)} x_2)} x_1 = {}^{\lambda_P(y)}({}^{\lambda'_P(x_2)} x_1) - {}^{\lambda'_P(x_2)}({}^{\lambda_P(y)} x_1).$$

The result now follows from Proposition 2.1(iii) and its proofs. ∎

Lemma 2.5. *If $h_1(\mu_1(n), p) = 0_M$ and $h_2(\lambda_1(m), q) = 0_N$ for all $m \in M$, $n \in N$, $p \in P$, $q \in Q$, then the square*

$$
\begin{array}{ccc}
M \oplus N & \overset{\rho_1}{\longrightarrow} & T \\
{\scriptstyle \partial_1 \oplus \partial_2} \downarrow & & \downarrow {\scriptstyle \partial} \\
P \oplus Q & \overset{\rho_2}{\longrightarrow} & L
\end{array}
$$

with $h(t, (p, q)) = (h_1(t, p), h_2(t, q))$ is a crossed square, where ρ_1 and ρ_2 are Lie homomorphisms defined by $\rho_1(m, n) = \lambda_1(m) + \mu_1(n)$ and $\rho_2(p, q) = \lambda_1(p) + \mu_1(q)$, and L acts on $M \oplus N$ and $P \oplus Q$ with componentwise action.

Proof. Recall first from [5, Definition 7] that the actor of any crossed module (K, R, σ), denoted by $A(K, R, \sigma)$, is defined as the crossed module

$$(Der(R, K), Der(K, R, \sigma), \Delta).$$

Consider the morphism

$$(\gamma_1, \gamma_2) : (T, L, \partial) \longrightarrow A(M \oplus N, P \oplus Q, \partial_1 \oplus \partial_2),$$

where the Lie homomorphisms γ_1, γ_2 are defined as follows:

– $\gamma_1(t)(p, q) = (h_1(t, p), h_2(t, q))$ for all $t \in T$, $p \in P$, $q \in Q$;

– $\gamma_2(l) = (\alpha_l, \beta_l)$, $l \in L$, being $\alpha_l : M \oplus N \longrightarrow M \oplus N$, $\alpha_l(m, n) = ({}^l m, {}^l n)$ and

– $\beta_l : P \oplus Q \longrightarrow P \oplus Q$, $\beta_l(p, q) = ({}^l p, {}^l q)$ for all $m \in M$, $n \in N$, $p \in P$, $q \in Q$.

An easy verification shows that

(1) $(\rho_1, \rho_2) : (M \oplus N, P \oplus Q, \partial_1 \oplus \partial_2) \longrightarrow (T, L, \partial)$ is a morphism of Lie crossed modules.

(2) $\rho_1({}^l(m, n)) = {}^l \rho_1(m, n)$, $\rho_2({}^l(p, q)) = {}^l \rho_2(p, q)$ and $\rho_1(\gamma_1(t)(p, q)) = -{}^{\rho_2(p,q)} t$.

(3) The composition $(\gamma_1, \gamma_2) o (\rho_1, \rho_2)$ coincides with the canonical morphism

$$(M \oplus N, P \oplus Q, \partial_1 \oplus \partial_2) \longrightarrow A(M \oplus N, P \oplus Q, \partial_1 \oplus \partial_2).$$

The result now follows from [5, Theorem 30]. ∎

Proposition 2.6. (i) *The map β is a Lie homomorphism such that $\beta(Im\alpha) = 0$.*

(ii) *The image of the map α is an ideal of $(M \otimes Q) \rtimes (P \otimes N)$.*
Put $\mathrm{coker}\alpha$ to be the quotient Lie algebra of $(M \otimes Q) \rtimes (P \otimes N)$ by $Im\alpha$.

(iii) *The maps*

$$\mu_M : \mathrm{coker}\alpha \longrightarrow M \quad , \quad (x_1, x_2) + Im\alpha \longmapsto \lambda'_M(x_1) + \mu'_M(x_2)$$
$$\mu_N : \mathrm{coker}\alpha \longrightarrow N \quad , \quad (x_1, x_2) + Im\alpha \longmapsto \mu'_N(x_1) + \lambda'_N(x_2)$$

are Lie crossed modules, where μ'_M and μ'_N are defined in Lemma 2.4(v).

(iv) *The homomorphism $\delta : \mathrm{coker}\alpha \longrightarrow P \otimes Q$ induced by β, together with the action induced by Lemma 2.4(vi), is a Lie crossed module.*

(v) *If I is a subalgebra of $\mathrm{coker}\alpha$ generated by the elements*

$$(m_1 \otimes q, p \otimes n_1 + \partial_1(m_2) \otimes n_2) + Im\alpha,$$

where for $i = 1, 2$, $\lambda_1(m_i) = \mu_1(n_i)$ and $\lambda_2(p) = \mu_2(q)$, then $(I, P \square Q, \delta)$ is an ideal crossed submodule of $(\mathrm{coker}\alpha, P \otimes Q, \delta)$.

Proof. (i) Using the actions of $P \otimes N$ on $M \otimes Q$, $P \otimes Q$ and the fact that the Lie crossed module $\partial_1 \otimes id_Q$ commutes with the action of P, we have

$$\partial_1 \otimes id_Q({}^{x_2} x_1) = \partial_1 \otimes id_Q({}^{\lambda'_P(x_2)} x_1) = {}^{\lambda'_P(x_2)}(\partial_1 \otimes id_Q(x_1)) = {}^{x_2}(\partial_1 \otimes id_Q(x_1))$$

$$= {}^{id_P \otimes \partial_2(x_2)}(\partial_1 \otimes id_Q(x_1)),$$

for all $x_1 \in M \otimes Q$ and $x_2 \in P \otimes N$, so the result holds by Proposition 2.1(iv).

(ii) For any $x \in M \otimes N$, $x_1 \in M \otimes Q$ and $x_2 \in P \otimes N$,

$$[(id_M \otimes \partial_2(x), -\partial_1 \otimes id_N(x)), (x_1, x_2)]$$
$$= \left({}^{id_M \otimes \partial_2(x)}x_1 - {}^{\partial_1 \otimes id_N(x)}x_1 - {}^{x_2}(id_M \otimes \partial_2(x)), -{}^{\partial_1 \otimes id_N(x)}x_2\right)$$
$$= \left(-{}^{x_2}(id_M \otimes \partial_2(x)), {}^{x_2}(\partial_1 \otimes id_N(x))\right) \qquad \text{(by Lemma 2.4(i))}$$
$$= \left(-(id_M \otimes \partial_2({}^{\lambda'_P(x_2)}x)), (\partial_1 \otimes id_N({}^{\lambda'_P(x_2)}x))\right) \quad \text{(by Lemma 2.4(ii))},$$

and so belongs to $\mathrm{Im}\,\alpha$. Also, one easily sees that the image of $\mathrm{Im}\,\alpha$ under β is zero, proving (ii).

(iii) The proof is similar to that of Lemma 2.4(v).

(iv) Note first that $\mathrm{Im}\,\alpha$ is invariant under the action of $P \otimes Q$ and so, we have an action of $P \otimes Q$ on $\mathrm{coker}\,\alpha$. We now show that δ satisfies the conditions of Lie crossed module.

(1) For all $y \in P \otimes Q$ and $v = (x_1, x_2) + \mathrm{Im}\,\alpha \in \mathrm{coker}\,\alpha$, we have

$$\delta({}^y v) = \partial_1 \otimes id_Q({}^{\lambda_P(y)}x_1) + id_P \otimes \partial_2({}^{\lambda_P(y)}x_2)$$
$$= {}^{\lambda_P(y)}(\partial_1 \otimes id_Q(x_1) + id_P \otimes \partial_2(x_2)) = {}^y\delta(v).$$

(2) For all $v = (x_1, x_2) + \mathrm{Im}\,\alpha$, $w = (x'_1, x'_2) + \mathrm{Im}\,\alpha \in \mathrm{coker}\,\alpha$, we have

$$\delta(v) w = \left({}^{x_1}x'_1 + {}^{x_2}x'_1, {}^{x_1}x'_2 + {}^{x_2}x'_2\right) + \mathrm{Im}\,\alpha \qquad \text{(by Lemma 2.4(i))}$$
$$= \left({}^{x_1}x'_1 + {}^{x_2}x'_1 - {}^{x'_2}x_1, {}^{x_2}x'_2\right) + \left({}^{x'_2}x_1, {}^{x_1}x'_2\right) + \mathrm{Im}\,\alpha.$$

But, assuming $x_1 = m \otimes q$ and $x'_2 = p \otimes n$, one sees from Lemma 2.4(i) that

$$\left({}^{x'_2}x_1, {}^{x_1}x'_2\right) = \left({}^q m \otimes {}^p\partial_2(n), -{}^q\partial_1(m) \otimes {}^p n\right)$$
$$= \left(id_M \otimes \partial_2({}^q m \otimes {}^p n), -\partial_1 \otimes id_N({}^q m \otimes {}^p n)\right) \in \mathrm{Im}\,\alpha,$$

from which we deduce that ${}^{\delta(v)}w = {}^{(x_1, x_2)}(x'_1, x'_2) + \mathrm{Im}\,\alpha = {}^v w$, as desired.

(v We must verify the conditions of ideality. Assume

$$a = (m_1 \otimes q_1, p_1 \otimes n_1 + \partial_1(m_2) \otimes n_2) + \mathrm{Im}\,\alpha \quad \text{and} \quad y = p \otimes q$$

are arbitrary generators of I and $P \otimes Q$, respectively. Since

$$\partial_1(m_2) n_2 = {}^{\partial\lambda_1(m_2)}n_2 = {}^{\partial\mu_1(n_2)}n_2 = 0,$$
$$\lambda_1({}^{q_1}m_1) = {}^{\mu_2(q_1)}\lambda_1(m_1) = {}^{\lambda_2(p_1)}\mu_1(n_1) = \mu_1({}^{p_1}n_1),$$
$$\lambda_2(-{}^q p) = -{}^{\mu_2(q)}\lambda_2(p) = {}^{\lambda_2(p)}\mu_2(q) = \mu_2({}^p q),$$

it immediately follows that ${}^y a = ({}^{q_1}m_1 \otimes {}^p q, -{}^q p \otimes {}^{p_1}n_1 - {}^q p \otimes {}^{\partial_1(m_2)}n_2) + \mathrm{Im}\,\alpha \in I$. Therefore, the action of $P \otimes Q$ on I, and analogously, the action of $P \square Q$ on $\mathrm{coker}\,\alpha$ are contained in I. It remains to verify that $\delta(I) \subseteq P \square Q$. We have

$$\delta(a) = (\partial_1(m_1) + p_1) \otimes (\partial_2(n_1) + q_1) - \partial_1(m_1) \otimes \partial_2(n_1)$$
$$- p_1 \otimes q_1 + \partial_1(m_2) \otimes \partial_2(n_2). \qquad (3)$$

As $\lambda_2\partial_1(m_i) = \partial\lambda_1(m_i) = \partial\mu_1(n_i) = \mu_2\partial_2(n_i)$ for $i = 1, 2$, $\lambda_2(p_1) = \mu_2(q_1)$ and
$$\lambda_2(\partial_1(m_1) + p_1) = \lambda_2\partial_1(m_1) + \lambda_2(p_1) = \mu_2\partial_2(n_1) + \mu_2(q_1) = \mu_2(\partial_2(n_1) + q_1),$$

we conclude that all terms in the right-hand side of the equality (3) belongs to $P \square Q$, as required. $\blacksquare$

In the remainder of this paper, we will always assume that $\mathbf{M} = (M, P, \partial_1)$, $\mathbf{N} = (N, Q, \partial_2)$ *and* $\mathbf{T} = (T, L, \partial)$ *are Lie crossed modules which get from crossed squares* (2).

3. Tensor and exterior products

In this section, using Proposition 2.6, we define the tensor and exterior products of Lie crossed modules and give some fundamental properties of them.

Definition 3.1. The *non-abelian tensor and exterior products* of Lie crossed modules $\mathbf{M}$ and $\mathbf{N}$ are defined, respectively, as

$$\mathbf{M} \otimes \mathbf{N} = (\mathrm{coker}\,\alpha, P \otimes Q, \delta),$$
$$\mathbf{M} \wedge \mathbf{N} = \frac{(\mathrm{coker}\,\alpha, P \otimes Q, \delta)}{\mathbf{M}\square\mathbf{N}} = (\frac{\mathrm{coker}\,\alpha}{I}, P \wedge Q, \bar{\delta}),$$

where $\mathbf{M}\square\mathbf{N}$ is the Lie crossed module $(I, P\square Q, \delta)$ introduced in Proposition 2.6(v). ∎

Note that the above generalize the definitions given in [14] and, moreover, if L acts trivially on M, N, P and Q in the crossed squares (2), then $\mathbf{M} \otimes \mathbf{N}$ is identical to the tensor product of abelian Lie crossed modules introduced in [6].

The following statement extends [14, Lemma 2.1] and is important in giving the next results.

Proposition 3.2. (i) *There are two Lie crossed module morphisms* (μ_M, λ_P) : $\mathbf{M} \otimes \mathbf{N} \longrightarrow \mathbf{M}$ *and* $(\mu_N, \lambda_Q) : \mathbf{M} \otimes \mathbf{N} \longrightarrow \mathbf{N}$, *where* μ_M, μ_N *are Lie crossed modules defined in Proposition* 2.6(iii).

(ii) *The Lie crossed modules* $\ker(\mu_M, \lambda_P)$ *and* $\ker(\mu_N, \lambda_Q)$ *are abelian.*

(iii) *If* ∂_1 *and* ∂_2 *are onto, then the squares*

$$
\begin{array}{ccc}
\mathrm{coker}\,\alpha & \xrightarrow{\mu_M} & M \\
\delta \downarrow & & \downarrow \partial_1 \\
P \otimes Q & \xrightarrow{\lambda_P} & P
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathrm{coker}\,\alpha & \xrightarrow{\mu_N} & N \\
\delta \downarrow & & \downarrow \partial_2 \\
P \otimes Q & \xrightarrow{\lambda_Q} & Q
\end{array}
$$

together with bilinear functions $h_1'(m, p\otimes q) = (-{}^p m \otimes q, p \otimes h_2(\lambda_1(m), q)) + Im\,\alpha$ *and* $h_2'(n, p \otimes q) = (h_1(\mu_1(n), p) \otimes q, -p \otimes {}^q n) + Im\,\alpha$, *respectively, are crossed squares.*

Proof. For all parts, we only the first one, the second one can be deduced in the same way.

(i) Consider the following commutative diagram

$$
\begin{array}{ccccc}
& & P \otimes Q & & \\
& {}^{\partial_1 \otimes id_Q}\nearrow & \downarrow \lambda_P & \nwarrow^{id_P \otimes \partial_1} & \\
M \otimes Q & & P & & P \otimes N \\
& {}_{\lambda_M'}\searrow & \uparrow \partial_1 & \nearrow_{\mu_M'} & \\
& & M & &
\end{array}
$$

where μ'_M is the Lie crossed module defined in Lemma 2.4(v). It is easy to see that $\partial_1 \mu_M = \lambda_P \delta$. We must now show that $\mu_M({}^y(x_1, x_2) + \mathrm{Im}\,\alpha) = {}^{\lambda_P(y)}\mu_M((x_1, x_2) + \mathrm{Im}\,\alpha)$ for all $x_1 \in M \otimes Q$, $x_2 \in P \otimes N$ and $y \in P \otimes Q$. But this follows from the facts that $\lambda'_M({}^y x_1) = {}^{\lambda_P(y)}\lambda'_M(x_1)$ and $\mu'_M({}^y x_2) = {}^{\lambda_P(y)}\mu'_M(x_2)$.

(ii) It is enough to note that $\ker\lambda_P$ is central (since λ_P is a Lie crossed module), and for all $x_1 \in M \otimes Q$, $x_2 \in P \otimes N$, $y \in \ker\lambda_P$, ${}^y x_i = {}^{\lambda_P(y)} x_i = 0$ for $i = 1, 2$.

(iii) We give a proof similar to that of Lemma 2.5. Consider the morphism of Lie crossed modules (μ_M, λ_P) obtained in part (i) and

$$(\gamma_1, \gamma_2) : (M, P, \partial_1) \longrightarrow A(\mathrm{coker}\,\alpha, \mathrm{P} \otimes \mathrm{Q}, \delta),$$

where the Lie homomorphisms γ_1, γ_2 are given as follows:

- $\gamma_1(m)(p \otimes q) = (-{}^p m \otimes q, p \otimes h_2(\lambda_1(m), q)) + \mathrm{Im}\,\alpha$ for all $m \in M$, $p \in P$, $q \in Q$;

- $\gamma_2(p) = (\alpha_p, \beta_p)$, $p \in P$, being $\alpha_p : \mathrm{coker}\,\alpha \longrightarrow \mathrm{coker}\,\alpha$, $\alpha_p(x) = {}^p x$ and $\beta_p : P \otimes Q \longrightarrow P \otimes Q$,

- $\beta_p(y) = {}^p y$ for all $x \in \mathrm{coker}\,\alpha$, $y \in P \otimes Q$.

In the following, we examine the conditions of [5, Theorem 2.1].

(1) For any $p \in P$, $x \in \mathrm{coker}\,\alpha$ and $y \in P \otimes Q$, we have $\mu_M({}^p x) = {}^p \mu_M(x)$ and $\lambda_P({}^p y) = {}^p \lambda_P(y)$.

(2) For any $m \in M$ and $y \in P \otimes Q$, we have $\mu_M(\gamma_1(m)(y)) = -{}^{\lambda_P(y)} m$, because, assuming $y = p \otimes q$, we have

$$\mu_M(\gamma_1(m)(y)) = {}^q({}^p m) - h_1(\mu_1 h_2(\lambda_1(m), q), p) \overset{by(C3)}{=} {}^q({}^p m) + h_1(\lambda_1({}^q m), p)$$

$$\overset{by(C4)}{=} {}^q({}^p m) - {}^p({}^q m) = -{}^{\lambda_P(y)} m.$$

(3) The composition of morphisms $(\gamma_1 \mu_M, \gamma_2 \lambda_P)$ is equal to the canonical morphism $(\zeta_1, \zeta_2) : (\mathrm{coker}\,\alpha, \mathrm{P} \otimes \mathrm{Q}, \delta) \longrightarrow A(\mathrm{coker}\,\alpha, \mathrm{P} \otimes \mathrm{Q}, \delta)$, in which the maps ζ_1, ζ_2 are defined by

- $\zeta_1(x)(y) = -{}^y x$ for all $x \in \mathrm{coker}\,\alpha$, $y \in P \otimes Q$;

- $\zeta_2(y) = (\alpha'_y, \beta'_y)$, $y \in P \otimes Q$, being $\alpha'_y : \mathrm{coker}\,\alpha \longrightarrow \mathrm{coker}\,\alpha$, $\alpha'_y(x) = {}^y x$ and $\beta'_y : P \otimes Q \longrightarrow P \otimes Q$,

- $\beta'_y(y') = {}^y y'$ for all $x \in \mathrm{coker}\,\alpha$, $y' \in P \otimes Q$.

To see this, we first prove that $\gamma_1 \mu_M = \zeta_1$. But this is equivalent to show that $\gamma_1(\mu_M(x))(y) = -{}^y x$ for all $x \in \mathrm{coker}\,\alpha$ and $y \in P \otimes Q$. Without loss of generality we may assume that $x = (x_1, x_2) + \mathrm{Im}\,\alpha \in \mathrm{coker}\,\alpha$, where $x_1 = m' \otimes q'$ and $x_2 = p' \otimes n'$, and $y = p \otimes q$. Since ∂_1 and ∂_2 are surjective, there exist elements $m, m_1 \in M$ and $n, n_1 \in N$ such that $p = \partial_1(m)$, $p' = \partial_1(m_1)$, $q = \partial_2(n)$ and $q' = \partial_2(n_1)$. So, we have

$$\gamma_1(\mu_M(x))(y) \overset{by(C4)}{=} (-{}^m \mu_M(x) \otimes \partial_2(n), \partial_1(m) \otimes {}^{\mu_M(x)} n) + \mathrm{Im}\,\alpha$$

$$\overset{by(A4)}{=} (-{}^m \mu_M(x) \otimes \partial_2(n), [\partial_1(m), \partial_1(\mu_M(x))] \otimes n + \partial_1(\mu_M(x)) \otimes {}^{\partial_1(m)} n) + \mathrm{Im}\,\alpha$$

$$= (0, \partial_1(\mu_M(x)) \otimes {}^{\partial_1(m)} n) + \mathrm{Im}\,\alpha$$

$$= (0, \partial_1(\lambda'_M(x_1)) \otimes {}^{\partial_1(m)} n + \partial_1(\mu'_M(x_2)) \otimes {}^{\partial_1(m)} n) + \mathrm{Im}\,\alpha$$

We claim that $\partial_1(\mu'_M(x_2)) \otimes {}^{\partial_1(m)}n = -{}^yx_2$ and $({}^yx_1, \partial_1(\lambda'_M(x_1)) \otimes {}^{\partial_1(m)}n) \in \mathrm{Im}\,\alpha$, because

$$\partial_1(\mu'_M(x_2)) \otimes {}^{\partial_1(m)}n \stackrel{by(C4)}{=} -\partial_1({}^{\mu_1(n')}m_1) \otimes {}^{\partial_1(m)}n = -{}^{n'}\partial_1(m_1) \otimes {}^{\partial_1(m)}n$$

$$\stackrel{by(A6)}{=} {}^n\partial_1(m) \otimes {}^{\partial_1(m_1)}n' = -{}^yx_2,$$

and

$$({}^yx_1, \partial_1(\lambda'_M(x_1)) \otimes {}^{\partial_1(m)}n) = ({}^{n_1}m' \otimes \partial_2({}^mn), -\partial_1({}^{n_1}m') \otimes {}^mn) \in \mathrm{Im}\,\alpha.$$

It is thus deduced that $\gamma_1(\mu_M(x))(y) = (-{}^yx_1, -{}^yx_2) + \mathrm{Im}\,\alpha = -{}^yx$. We now show that $\gamma_2\lambda_P = \zeta_2$, or equivalently, that $\alpha_{\lambda_P(y)} = \alpha'_y$ and $\beta_{\lambda_P(y)} = \beta'_y$, for all $y \in P \otimes Q$. But using the definitions of the maps involved, we immediately obtain the equalities. Thus, in view of [5, Theorem 2.1], the result holds and the proof is complete. ∎

In Proposition 3.2(i), we denote the images of the morphisms (μ_M, λ_P) and (μ_N, λ_Q) by $[\mathbf{M}, \mathbf{N}]^{\mathbf{M}}$ and $[\mathbf{N}, \mathbf{M}]^{\mathbf{N}}$, respectively. Note that when $\mathbf{M} = \mathbf{N}$, we have $[\mathbf{M}, \mathbf{N}]^{\mathbf{M}} = \mathbf{M}'$. In the following, we prove that the tensor product $\mathbf{M} \otimes \mathbf{N}$ is nilpotent or solvable if such information is given on one of the above crossed submodules. Also, we present some bounds for the nilpotency class and solvability length of $\mathbf{M} \otimes \mathbf{N}$ in terms of the nilpotency class and solvability length of $[\mathbf{M}, \mathbf{N}]^{\mathbf{M}}$ (or $[\mathbf{N}, \mathbf{M}]^{\mathbf{N}}$).

Theorem 3.3. (i) *If $[\mathbf{M}, \mathbf{N}]^{\mathbf{M}}$ is solvable of length m, then $\mathbf{M} \otimes \mathbf{N}$ is solvable of length m or $m + 1$, and $[\mathbf{N}, \mathbf{M}]^{\mathbf{N}}$ is solvable of length at most $m + 1$.*

(ii) *If $\mathbf{M}$ and $\mathbf{N}$ are simply connected and $[\mathbf{M}, \mathbf{N}]^{\mathbf{M}}$ is nilpotent of class c, then $\mathbf{M} \otimes \mathbf{N}$ is nilpotent of class c or $c + 1$, and $[\mathbf{N}, \mathbf{M}]^{\mathbf{N}}$ is nilpotent of class at most $c + 1$.*

(iii) *If $[\mathbf{M}, \mathbf{N}]^{\mathbf{M}}$ is abelian, then $\mathbf{M} \otimes \mathbf{N}$ is metabelian.*

Proof. According to Proposition 3.2(i)–(ii), we have the following abelian extensions

$$\ker(\mu_M, \lambda_P) \rightarrowtail \mathbf{M} \otimes \mathbf{N} \twoheadrightarrow [\mathbf{M}, \mathbf{N}]^{\mathbf{M}},$$

$$\ker(\mu_N, \lambda_Q) \rightarrowtail \mathbf{M} \otimes \mathbf{N} \twoheadrightarrow [\mathbf{N}, \mathbf{M}]^{\mathbf{N}}.$$

Moreover, with the assumptions of part (ii), these extensions are central, thanks to Proposition 3.2(iii) and Lemma 2.3(ii). Now, the required results easily follow from the above facts. ∎

If we regard a Lie algebra as a Lie crossed module, Theorem 3.3 reduces to [17, Theorem 2.1].

As an immediate consequence of the above theorem, we have

Corollary 3.4. (i) *If $\mathbf{M}$ is solvable of length m, then $\mathbf{M} \otimes \mathbf{M}$ is solvable of length $m - 1$ or m.*

(ii) *If $\mathbf{M}$ is simply connected and $\mathbf{M}'$ is nilpotent of class c, then $\mathbf{M} \otimes \mathbf{M}$ is nilpotent of class c or $c + 1$.*

(iii) *If $\mathbf{M}$ is metabelian, then so is $\mathbf{M} \otimes \mathbf{M}$.*

The following provides a result similar to Corollary 3.4(i) for nilpotent Lie crossed modules.

Theorem 3.5.   (i)   *If* $\mathbf{M}$ *is nilpotent of class* c*, then* $\mathbf{M} \otimes \mathbf{M}$ *is nilpotent of class at most* c*.*

(ii)   *If* $\mathbf{M}$ *is nilpotent of class 2, then* $\mathbf{M} \otimes \mathbf{M}$ *is abelian.*

Proof.   (i)   It follows from the assumption that $[_c P, M] = 0_M$ and $\gamma_{c+1}(P) = 0_P$. Hence $P \otimes P$ is a nilpotent Lie algebra of class to be c or $c + 1$ thanks to [17, Corollary 2.3]. We now prove that $[_c P \otimes P, \mathrm{coker}\alpha] = 0$. To do this, we will first show by induction that for any $n \geq 1$, every generator of $[_c P \otimes P, \mathrm{coker}\alpha]$ can be expressed as $(x \otimes y, y \otimes x') + \mathrm{Im}\alpha$, where $x, x' \in [_{n-1}P', [P, M]]$ and $y \in P'$. For $n = 1$, let $A = {}^c(a, b) + \mathrm{Im}\alpha$ be an arbitrary generator of $[P \otimes P, \mathrm{coker}\alpha]$, where $a \in M \otimes P$, $b \in P \otimes M$ and $c \in P \otimes P$. Considering the Lie crossed modules $\lambda_M : M \otimes P \longrightarrow M$, $\lambda_P : P \otimes P \longrightarrow P$ and using Proposition 2.1(v), we have

$$A = ({}^{\lambda_P(c)}a, {}^{\lambda_P(c)}b) + \mathrm{Im}\alpha = (-\lambda_M(a) \otimes \lambda_P(c), \lambda_P(c) \otimes \lambda_M(b)) + \mathrm{Im}\alpha,$$

where $\lambda_M(a), \lambda_M(b) \in [P, M]$ and $\lambda_P(c) \in P'$. Now, assume that the result holds for $n \geq 1$. Then any generator of $[_{n+1}P \otimes P, \mathrm{coker}\alpha]$ can be written as $A = (x \otimes y, y \otimes x') + \mathrm{Im}\alpha$, where $x, x' \in [_{n-1}P', [P, M]]$ and $y \in P'$. Applying arguments similar to the above, one can see that

$$A = ({}^y x \otimes \lambda_P(y), \lambda_P(y) \otimes {}^y x') + \mathrm{Im}\alpha$$

where ${}^y x, {}^y x' \in [_n P', [P, M]]$ and $\lambda_P(c) \in P'$. This completes the induction. Inasmuch as $[_{c-1}P', [P, M]] \subseteq [_c P, M] = 0_M$, we therefore infer that $[_c P \otimes P, \mathrm{coker}\alpha] = 0$, as desired.

(ii)   It follows from hypothesis that $[L', T] \subseteq [_2 L, T] = 0$ and $\gamma_3(L) = 0$. Hence, we have

$$[l_1 \otimes l_2, l_3 \otimes l_4] = 0 \qquad \text{and} \qquad {}^{l_1 \otimes l_2}((t_1 \otimes l_3, l_4 \otimes t_2) + \mathrm{Im}\alpha) = \mathrm{Im}\alpha,$$

for all $t_i \in T$ and $l_j \in L$. So, the Lie algebra $L \otimes L$ is abelian and acts trivially on $\mathrm{coker}\alpha$. ∎

In continuation, we show the tensor and exterior products of Lie crossed modules are distributive over certain direct sums. We begin by the following lemmas

Lemma 3.6.   *Consider the crossed squares defined by the diagrams* (2)*, and suppose that*

$$
\begin{array}{ccc}
K & \xrightarrow{\upsilon_1} & T \\
\downarrow{\scriptstyle \partial_3} & & \downarrow{\scriptstyle \partial} \\
R & \xrightarrow{\upsilon_2} & L
\end{array}
$$

is another crossed square. Also, suppose that

(i) $h_1(\mu_1(n), p) = 0$ *and* $h_2(\lambda_1(m), q) = 0$,

(ii) $m \otimes {}^n r = m \otimes {}^q r = 0_{M \otimes R}$ *and* $n \otimes {}^m r = n \otimes {}^p r = 0_{N \otimes R}$,

(iii) $p \otimes {}^q k = 0_{P \otimes K}$ *and* $q \otimes {}^p k = 0_{Q \otimes K}$,

for all $m \in M$*,* $n \in N$*,* $k \in K$*,* $p \in P$*,* $q \in Q$*,* $r \in R$*.*

Then assuming $\mathbf{M} = (M, P, \partial_1)$*,* $\mathbf{N} = (N, Q, \partial_2)$ *and* $\mathbf{K} = (K, R, \partial_3)$*, there is an isomorphism* $(\mathbf{M} \oplus \mathbf{N}) \otimes \mathbf{K} \xrightarrow{\cong} (\mathbf{M} \otimes \mathbf{K}) \oplus (\mathbf{N} \otimes \mathbf{K})$*.*

Proof. Note first that the tensor product appearing in the left hand side of the required isomorphism is defined using Lemma 2.5. By the definition of the tensor product of Lie crossed modules, we assume $\mathbf{M} \otimes \mathbf{K} = (\mathrm{coker}\alpha_1, \mathrm{P} \otimes \mathrm{R}, \delta_1)$, $\mathbf{N} \otimes \mathbf{K} = (\mathrm{coker}\alpha_2, \mathrm{Q} \otimes \mathrm{R}, \delta_2)$, $(\mathbf{M} \oplus \mathbf{N}) \otimes \mathbf{K} = (\mathrm{coker}\alpha, (\mathrm{P} \oplus \mathrm{Q}) \otimes \mathrm{R}, \delta)$, where $\alpha_1 = (id_M \otimes \partial_3, -\partial_1 \otimes id_K)$, $\alpha_2 = (id_N \otimes \partial_3, -\partial_2 \otimes id_K)$, $\alpha = (id_{M \oplus N} \otimes \partial_3, -(\partial_1 \oplus \partial_2) \otimes id_K)$. It suffices to define an isomorphism (ϕ_1, ϕ_2)

$$
\begin{array}{ccc}
\mathrm{coker}\alpha & \xrightarrow{\ \phi_1\ } & \mathrm{coker}\alpha_1 \oplus \mathrm{coker}\alpha_2 \\
\delta \downarrow & & \downarrow \delta_1 \oplus \delta_2 \\
(P \oplus Q) \otimes R & \xrightarrow{\ \phi_2\ } & (P \otimes R) \oplus (Q \otimes R).
\end{array}
\qquad (4)
$$

The second component ϕ_2 is the isomorphism given in [8, Lemma 7], which is defined on generators by $\phi_2((p,q) \otimes r) = (p \otimes r, q \otimes r)$. We now construct ϕ_1, which will be induced on $\mathrm{coker}\alpha$ by a Lie homomorphism $\langle \theta_1, \theta_2 \rangle$

$$
\begin{array}{ccc}
((M \oplus N) \otimes R) \rtimes (P \oplus Q) \otimes K & \xrightarrow{\ nat.\ } & \mathrm{coker}\alpha. \\
\langle \theta_1, \theta_2 \rangle \downarrow & \nearrow \phi_1 & \\
\mathrm{coker}\alpha_1 \oplus \mathrm{coker}\alpha_2 & &
\end{array}
$$

Let us define θ_1 and θ_2 on generators as follows:

$$
\theta_1((m,n) \otimes r) = ((m \otimes r, 0) + \mathrm{Im}\alpha_1, (n \otimes r, 0) + \mathrm{Im}\alpha_2),
$$

$$
\theta_2((p,q) \otimes k) = ((0, p \otimes k) + \mathrm{Im}\alpha_1, (0, q \otimes k) + \mathrm{Im}\alpha_2).
$$

Using the assumptions (ii) and (iii) of the lemma and after a long but simple calculations, one can see that θ_1 and θ_2 preserve the relations $(A1)$–$(A6)$ and are thus Lie homomorphisms. Since $\theta_1(^b a) = {}^{\theta_2(b)}\theta_1(a)$ for all $a \in (M \oplus N) \otimes R$ and $b \in (P \oplus Q) \otimes K$, it follows from Proposition 2.1(iv) and its proof that the map $\theta = \langle \theta_1, \theta_2 \rangle : ((M \oplus N) \otimes R) \rtimes ((P \oplus Q) \otimes K) \longrightarrow \mathrm{coker}\alpha_1 \oplus \mathrm{coker}\alpha_2$ defined by $\theta(a,b) = \theta_1(a) + \theta_2(b)$ is a Lie homomorphism. As θ annihilates $\mathrm{Im}\alpha$, θ induces the Lie homomorphism ϕ_1. We now show that ϕ_1 is an isomorphism by giving an inverse for it. Consider the canonical homomorphisms $\psi_1 : \mathrm{coker}\alpha_1 \longrightarrow \mathrm{coker}\alpha$, $(m \otimes r, p \otimes k) + \mathrm{Im}\alpha_1 \longmapsto ((m,0) \otimes r, (p,0) \otimes k) + \mathrm{Im}\alpha$, and $\psi_2 : \mathrm{coker}\alpha_2 \longrightarrow \mathrm{coker}\alpha$, $(n \otimes r, q \otimes k) + \mathrm{Im}\alpha_2 \longmapsto ((0,n) \otimes r, (0,q) \otimes k) + \mathrm{Im}\alpha$. Then the homomorphism $\psi = \langle \psi_1, \psi_2 \rangle : \mathrm{coker}\alpha_1 \oplus \mathrm{coker}\alpha_2 \longrightarrow \mathrm{coker}\alpha$ in the coproduct of vector spaces is clearly an inverse for ϕ_1. So, (ϕ_1, ϕ_2) is a pair of isomorphisms such that the diagram (4) is commutative. It finally remains to show that $\phi_1(^y x) = {}^{\phi_2(y)}\phi_1(x)$ for all $x \in \mathrm{coker}\alpha$ and $y \in (P \oplus Q) \otimes R$. There is no loss of generality in assuming that $x = ((m,n) \otimes r, (p,q) \otimes k) + \mathrm{Im}\alpha$ and $y = (p', q') \otimes r'$. Then we have

$$
\begin{aligned}
\phi_1(^y x) = & (({}^r m \otimes ({}^{p'}r' + {}^{q'}r'), -{}^{r'}p' \otimes ({}^p k + {}^q k))) + \mathrm{Im}\alpha_1, \\
& ({}^r n \otimes ({}^{p'}r' + {}^{q'}r'), -{}^{r'}q' \otimes ({}^p k + {}^q k))) + \mathrm{Im}\alpha_2) \\
= & (({}^r m \otimes {}^{p'}r', -{}^{r'}p' \otimes {}^p k) + \mathrm{Im}\alpha_1, ({}^r n \otimes {}^{q'}r', -{}^{r'}q' \otimes {}^q k) + \mathrm{Im}\alpha_2) \\
= & ({}^{p' \otimes r'}(m \otimes r, p \otimes k) + \mathrm{Im}\alpha_1, {}^{q' \otimes r'}(n \otimes r, q \otimes k) + \mathrm{Im}\alpha_2) = {}^{\phi_2(y)}\phi_1(x).
\end{aligned}
$$

Note that the second equality follows from the assumptions (ii) and (iii) of the lemma. We therefore conclude that (ϕ_1, ϕ_2) is a morphism of Lie crossed modules and the proof is complete. ∎

Lemma 3.7. (i) *Let the Lie crossed modules* $\mathbf{M}$, $\mathbf{N}$ *and* $\mathbf{T}$ *be abelian. Then*

(a) *If* $\lambda = (\lambda_1, \lambda_2) : \mathbf{M} \longrightarrow \mathbf{T}$ *and* $\mu = (\mu_1, \mu_2) : \mathbf{N} \longrightarrow \mathbf{T}$ *are morphisms of Lie crossed modules, then* $\lambda * \mu = (\lambda_1 * \mu_1, \lambda_2 * \mu_2) : \mathbf{M} \oplus \mathbf{N} \longrightarrow \mathbf{T}$ *is a morphism of Lie crossed modules, where* $\lambda_1 * \mu_1(m, n) = \lambda_1(m) + \mu_1(n)$ *and* $\lambda_2 * \mu_2(p, q) = \lambda_2(p) + \mu_2(q)$.

(b) *There exists an isomorphism* $\nu = (\nu_1, \nu_2) : \mathbf{M} \otimes \mathbf{N} \longrightarrow \mathbf{N} \otimes \mathbf{M}$ *defined by*
$\nu_1((m \otimes q, p \otimes n) + Im\alpha) = (-n \otimes p, -q \otimes m) + Im\alpha'$ *and* $\nu_2(p \otimes q) = -q \otimes p$.

(ii) *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be ideal crossed submodules of* $\mathbf{T}$ *with* $[\mathbf{M}, \mathbf{N}] = 0$. *Then*
$\mathbf{M} \otimes \mathbf{N} \cong \mathbf{M}_{ab} \otimes \mathbf{N}_{ab}$.

Proof. Straightforward. ∎

Theorem 3.8. *For Lie crossed modules* $\mathbf{M}, \mathbf{N}$, *there are isomorphisms*

$$(\mathbf{M} \oplus \mathbf{N}) \otimes (\mathbf{M} \oplus \mathbf{N}) \cong (\mathbf{M} \otimes \mathbf{M}) \oplus (\mathbf{N} \otimes \mathbf{N}) \oplus (\mathbf{M}_{ab} \otimes \mathbf{N}_{ab}) \oplus (\mathbf{N}_{ab} \otimes \mathbf{M}_{ab}), \quad (5)$$

$$(\mathbf{M} \oplus \mathbf{N}) \wedge (\mathbf{M} \oplus \mathbf{N}) \cong (\mathbf{M} \wedge \mathbf{M}) \oplus (\mathbf{N} \wedge \mathbf{N}) \oplus (\mathbf{M}_{ab} \wedge \mathbf{N}_{ab}). \quad (6)$$

Proof. Identifying $\mathbf{M}$ and $\mathbf{N}$ with their images in the Lie crossed module $\mathbf{M} \oplus \mathbf{N}$, it follows that $[\mathbf{M}, \mathbf{N}] = 0$ and so, according to Lemma 3.7(ii), $\mathbf{M} \otimes \mathbf{N} \cong \mathbf{M}_{ab} \otimes \mathbf{N}_{ab}$ and $\mathbf{N} \otimes \mathbf{M} \cong \mathbf{N}_{ab} \otimes \mathbf{M}_{ab}$. We therefore obtain the isomorphism (5) by applying Lemma 3.6 twice.

To prove the isomorphism (6), we consider the following diagram

where ψ, φ are isomorphisms obtained using (5), i is the inclusion morphism, and

$$id_{\mathbf{M}_{ab} \otimes \mathbf{N}_{ab}} * \nu : (\mathbf{M}_{ab} \otimes \mathbf{N}_{ab}) \oplus (\mathbf{N}_{ab} \otimes \mathbf{M}_{ab}) \longrightarrow \mathbf{M}_{ab} \otimes \mathbf{N}_{ab}$$

is the surjective morphism given in Lemma 3.7(i). An easy calculation shows that

$$\lambda\varphi((\mathbf{M} \oplus \mathbf{N})\square(\mathbf{M} \oplus \mathbf{N})) \subseteq (\mathbf{M}\square\mathbf{M}) \oplus (\mathbf{N}\square\mathbf{N}),$$

$$\psi i((\mathbf{M}\square\mathbf{M}) \oplus (\mathbf{N}\square\mathbf{N})) \subseteq (\mathbf{M} \oplus \mathbf{N})\square(\mathbf{M} \oplus \mathbf{N}),$$

from which we get the induced morphisms η and θ. One easily sees that η is surjective and θ is a left inverse of η, completing the proof. ∎

4. Relationship to the universal quadratic functor

In the category of Lie algebras, there are some connections between the Lie tensor product, the Lie exterior product and Whitehead's universal quadratic functor ([8]). In this section, we generalize these results to the category of Lie crossed modules, under some conditions. For this purpose, we first recall the following definition.

Definition 4.1. ([6]) Let (A, B, ∂) be an abelian crossed module of Lie algebras, and $B\underline{\otimes}A$ be the quotient of $B \otimes A$ by the subalgebra generated by the elements $\partial(a_1) \otimes a_2 - \partial(a_2) \otimes a_1$ with $a_1, a_2 \in A$. Then we define $\Gamma(A, B, \partial)$ to be the abelian crossed module $(\bar{\Gamma}(A, B, \partial), \Gamma(B), \partial_\Gamma)$, in which $\bar{\Gamma}(A, B, \partial)$ is the cokernel of the Lie homomorphism $f : A \otimes A \longrightarrow (B\underline{\otimes}A) \oplus \Gamma(A)$ given by

$$f(a_1 \otimes a_2) = (\overline{\partial(a_1) \otimes a_2}, -\Delta(a_1 \otimes a_2)),$$

and $$\partial_\Gamma(\overline{b \otimes a}, \gamma(a_1)) = \Delta(b \otimes \partial(a)) + \gamma(\partial(a_1)).$$

We consider the Lie crossed submodule $\mathbf{M} \times_{\mathbf{T}} \mathbf{N} := (M \times_T N, P \times_L Q, \partial_1 \oplus \partial_2)$ of the direct sum $\mathbf{M} \oplus \mathbf{N}$, where $M \times_T N = \{(m, n) \mid \lambda_1(m) = \mu_1(n)\}$ and $P \times_L Q = \{(p, q) \mid \lambda_2(p) = \mu_2(q)\}$. Remembering the Lie crossed modules

$$\mu_M : \mathrm{coker}\alpha \longrightarrow \mathrm{M}, \quad \mu_N : \mathrm{coker}\alpha \longrightarrow \mathrm{N}$$

from Proposition 2.6(iii) and $\lambda_P : P\otimes Q \longrightarrow P$, $\lambda_Q : P\otimes Q \longrightarrow Q$ from Proposition 2.1(i), we set

$$\langle M, N \rangle = \{(\mu_M(x), \mu_N(x)) \mid x \in \mathrm{coker}\alpha\}, \quad \langle P, Q \rangle = \{(\lambda_P(y), \lambda_Q(y)) \mid y \in P \otimes Q\}.$$

It is readily verified that $\langle \mathbf{M}, \mathbf{N} \rangle := (\langle M, N \rangle, \langle P, Q \rangle, \partial_1 \oplus \partial_2)$ is an ideal crossed submodule of $\mathbf{M} \times_{\mathbf{T}} \mathbf{N}$, and that the quotient crossed module $(\mathbf{M} \times_{\mathbf{T}} \mathbf{N})/\langle \mathbf{M}, \mathbf{N} \rangle$ is abelian.

Theorem 4.2. *Let the Lie crossed modules* $\mathbf{M}$ *and* $\mathbf{N}$ *be simply connected. Then there is a natural exact sequence of Lie crossed modules*

$$\Gamma(\frac{\mathbf{M} \times_{\mathbf{T}} \mathbf{N}}{\langle \mathbf{M}, \mathbf{N} \rangle}) \xrightarrow{\psi} \mathbf{M} \otimes \mathbf{N} \twoheadrightarrow \mathbf{M} \wedge \mathbf{N}.$$

Proof. It suffices to define a surjective morphism $\psi = (\psi_1, \psi_2)$

$$\begin{array}{ccc}
\bar{\Gamma}(\frac{M \times_T N}{\langle M, N \rangle}, \frac{P \times_L Q}{\langle P, Q \rangle}, \overline{\partial_1 \oplus \partial_2}) & \xrightarrow{\psi_1} & I \\
\partial_\Gamma \downarrow & & \downarrow \delta \\
\Gamma(\frac{P \times_L Q}{\langle P, Q \rangle}) & \xrightarrow{\psi_2} & P \Box Q.
\end{array}$$

We take the second component ψ_2 equal to the Lie epimorphism introduced in Proposition 2.2(i). Setting

$$\overline{M \times_T N} = (M \times_T N)/\langle M, N \rangle \quad \text{and} \quad \overline{P \times_L Q} = (P \times_L Q)/\langle P, Q \rangle,$$

we construct ψ_1 in the following three steps.

Step 1. Here we prove the existence of a Lie homomorphism

$$\widetilde{\theta}_1 : (\overline{P \times_L Q})\underline{\otimes}(\overline{M \times_T N}) \longrightarrow I.$$

Consider the map

$$\theta_1 : (\overline{P \times_L Q}) \times (\overline{M \times_T N}) \longrightarrow I, \quad (\overline{(p,q)}, \overline{(m,n)}) \longmapsto (m \otimes q, p \otimes n) + \mathrm{Im}\,\alpha.$$

We first prove that θ_1 is well-defined. Suppose $(p_1, q_1) = (p_2, q_2) + (\lambda_P(y), \lambda_Q(y))$

and
$$m_1 = m_2 + \mu_M((x_1, x_2) + \mathrm{Im}\,\alpha) = m_2 + \lambda'_M(x_1) + \mu'_M(x_2),$$
$$n_1 = n_2 + \mu_N((x_1, x_2) + \mathrm{Im}\,\alpha) = n_2 + \mu'_N(x_1) + \lambda'_N(x_2),$$

for some $y \in P \otimes Q$ and $(x_1, x_2) + \mathrm{Im}\,\alpha \in \mathrm{coker}\,\alpha$. Then we have

$$\theta_1(\overline{(p_1,q_1)}, \overline{(m_1,n_1)}) = (m_2 \otimes q_2, p_2 \otimes n_2) + (\lambda'_M(x_1) \otimes q_2, p_2 \otimes \mu'_N(x_1))$$
$$+ (\mu'_M(x_2) \otimes q_2, p_2 \otimes \lambda'_N(x_2)) + (m_2 \otimes \lambda_Q(y), \lambda_P(y) \otimes n_2)$$
$$+ (\lambda'_M(x_1) \otimes \lambda_Q(y), \lambda_P(y) \otimes \mu'_N(x_1))$$
$$+ (\mu'_M(x_2) \otimes \lambda_Q(y), \lambda_P(y) \otimes \lambda'_N(x_2)) + \mathrm{Im}\,\alpha.$$

The subjectivity of the maps ∂_1 and ∂_2 yields that $p_2 = \partial_1(m')$, $q_2 = \partial_2(n')$, $x_1 = id_M \otimes \partial_2(z_1)$, $x_2 = \partial_1 \otimes id_N(z_2)$ and $y = \partial_1 \otimes \partial_2(z)$ for some $m' \in M$, $n' \in N$, $z, z_1, z_2 \in M \otimes N$. By using the following commutative diagrams
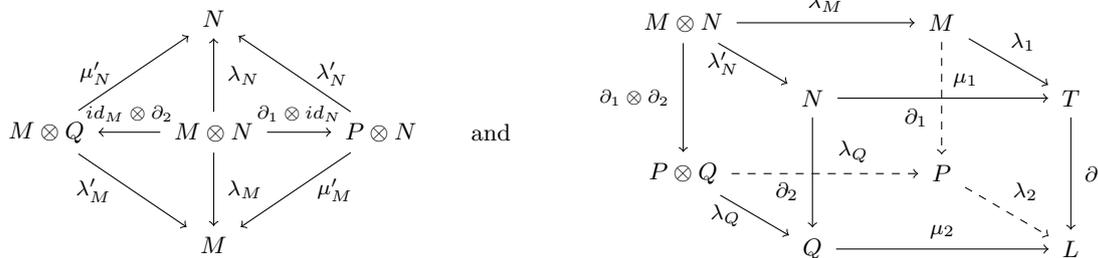
one observes that $\lambda'_M(x_1) = \lambda_M(z_1)$, $\mu'_N(x_1) = \lambda_N(z_1)$, $\mu'_M(x_2) = \lambda_M(z_2)$, $\lambda'_N(x_2) = \lambda_N(z_2)$, $\lambda_P(y) = \partial_1\lambda_M(z)$, $\lambda_Q(y) = \partial_2\lambda_N(z)$, and $\lambda_2\partial_1(m') = \mu_2\partial_2(n')$. These equalities together with Proposition 2.1(vi) and Lemma 2.4(iii), yield

$$(\lambda'_M(x_1) \otimes q_2, p_2 \otimes \mu'_N(x_1)) = (\lambda_M(z_1) \otimes \partial_2(n'), -\partial_1\lambda_M(z_1) \otimes n'),$$
$$(\mu'_M(x_2) \otimes q_2, p_2 \otimes \lambda'_N(x_2)) = (\lambda_M(z_2) \otimes \partial_2(n'), -\partial_1\lambda_M(z_2) \otimes n'),$$
$$(m_2 \otimes \lambda_Q(y), \lambda_P(y) \otimes n_2) = (m_2 \otimes \partial_2\lambda_N(z), -\partial_1(m_2) \otimes \lambda_N(z)),$$
$$(\lambda'_M(x_1) \otimes \lambda_Q(y), \lambda_P(y) \otimes \mu'_N(x_1)) = (\lambda_M(z_1) \otimes \partial_2\lambda_N(z), -\partial_1\lambda_M(z_1) \otimes \lambda_N(z)),$$
$$(\mu'_M(x_2) \otimes \lambda_Q(y), \lambda_P(y) \otimes \lambda'_N(x_2)) = (\lambda_M(z_2) \otimes \partial_2\lambda_N(z), -\partial_1\lambda_M(z_2) \otimes \lambda_N(z)).$$

and so, the left-hand side of the above equalities belong to $\mathrm{Im}\,\alpha$. It consequently follows that $\theta_1(\overline{(p_1,q_1)}, \overline{(m_1,n_1)}) = (m_2 \otimes q_2, p_2 \otimes n_2) + \mathrm{Im}\,\alpha = \theta_1(\overline{(p_2,q_2)}, \overline{(m_2,n_2)})$. We now show that θ_1 satisfies the relations (B1)–(B5). It is clear for relations (B1)–(B4). For any $\overline{(p_1,q_1)}, \overline{(p_2,q_2)} \in \overline{P \times_L Q}$ and $\overline{(m_1,n_1)}, \overline{(m_2,n_2)} \in \overline{M \times_T N}$, using Proposition 2.1(v) and Lemma 2.4(iv), we have

$$[\theta_1(\overline{(p_1,q_1)}, \overline{(m_1,n_1)}), \theta_1(\overline{(p_2,q_2)}, \overline{(m_2,n_2)})]$$
$$= [(m_1 \otimes q_1, p_1 \otimes n_1), (m_2 \otimes q_2, p_2 \otimes n_2)] + \mathrm{Im}\,\alpha$$
$$= (-{}^{q_1}m_1 \otimes {}^{m_2}q_2 + {}^{q_2}m_2 \otimes \partial_2({}^{p_1}n_1) - {}^{q_1}m_1 \otimes \partial_2({}^{p_2}n_2), {}^{n_2}p_2 \otimes {}^{p_1}n_1) + \mathrm{Im}\,\alpha$$
$$= (-{}^{q_1}m_1 \otimes {}^{m_2}q_2 + {}^{q_2}m_2 \otimes \partial_2({}^{p_1}n_1) + {}^{q_1}m_1 \otimes {}^{m_2}q_2, -\partial_1({}^{q_2}m_2) \otimes {}^{p_1}n_1) + \mathrm{Im}\,\alpha$$
$$= \mathrm{Im}\,\alpha.$$

On the other hand, the trivially of the action of $\overline{P \times_L Q}$ on $\overline{M \times_T N}$ forces

$$\theta_1(\overline{{}^{(m_1,n_1)}(p_1,q_1)}, \overline{{}^{(p_2,q_2)}(m_2,n_2)}) = 0.$$

So, θ_1 is a Lie pairing and the universal property of the tensor product then gives rise to the Lie homomorphism $\overline{\theta}_1 : (\overline{P \times_L Q}) \otimes (\overline{M \times_T N}) \longrightarrow I$. It is routine to indicate that $\overline{\theta}_1$ annihilates the ideal generated by the elements of the form $\overline{\partial_1 \oplus \partial_2(m_1, n_1)} \otimes \overline{(m_2, n_2)} - \overline{\partial_1 \oplus \partial_2(m_2, n_2)} \otimes \overline{(m_1, n_1)}$, $m_i \in M$, $n_i \in N$ $(i = 1, 2)$, so $\overline{\theta}_1$ induces a Lie homomorphism $\widetilde{\theta}_1 : (\overline{P \times_L Q}) \underline{\otimes} (\overline{M \times_T N}) \longrightarrow I$.

Step 2. Here we show that the existence of a Lie homomorphism

$$\widetilde{\theta}_2 : \Gamma(\overline{M \times_T N}) \longrightarrow I.$$

Let the map $\theta_2 : \overline{M \times_T N} \longrightarrow I$ be defined by $\theta_2(\overline{(m,n)}) = (0, \partial_1(m) \otimes n) + \mathrm{Im}\,\alpha$. We claim that θ_2 is well-defined. Suppose

$$m_1 = m_2 + \mu_M((x_1, x_2) + \mathrm{Im}\,\alpha) = m_2 + \lambda'_M(x_1) + \mu'_M(x_2),$$
$$n_1 = n_2 + \mu_N((x_1, x_2) + \mathrm{Im}\,\alpha) = n_2 + \mu'_N(x_1) + \lambda'_N(x_2),$$

for some $(x_1, x_2) + \mathrm{Im}\,\alpha \in \mathrm{coker}\,\alpha$. An easy verification shows that

$$
\begin{aligned}
\theta_2(\overline{(m_1,n_1)}) &= (0, \partial_1(m_2) \otimes n_2) + (0, \partial_1(m_2) \otimes \lambda'_N(x_2) + \partial_1\mu'_M(x_2) \otimes n_2) \\
&\quad + (0, \partial_1\lambda'_M(x_1) \otimes n_2 + \partial_1(m_2) \otimes \mu'_N(x_1)) \\
&\quad + (0, \partial_1\mu'_M(x_2) \otimes \mu'_N(x_1) + \partial_1\lambda'_M(x_1) \otimes \lambda'_N(x_2)) \\
&\quad + (0, \partial_1\lambda'_M(x_1) \otimes \mu'_N(x_1) + \partial_1\mu'_M(x_2) \otimes \lambda'_N(x_2)) + \mathrm{Im}\,\alpha \qquad (7)
\end{aligned}
$$

By arguments similar to those used in Step 1, we find an element $z_2 \in M \otimes N$ such that $\lambda'_N(x_2) = \lambda_N(z_2)$, $\mu'_M(x_2) = \lambda_M(z_2)$, and then, owing to Lemma 2.4(iii), we have

$$
\begin{aligned}
\partial_1(m_2) \otimes \lambda'_N(x_2) + \partial_1\mu'_M(x_2) \otimes n_2 &= \partial_1(m_2) \otimes \lambda_N(z_2) + \partial_1\lambda_M(z_2) \otimes n_2 \\
&= \partial_1(m_2) \otimes \lambda_N(z_2) - \partial_1(m_2) \otimes \lambda_N(z_2) = 0
\end{aligned}
$$

Analogously, one can check that in the right-hand side of the equality (7), except the first term, other terms are vanished. Consequently, $\theta_2(\overline{(m_1, n_1)}) = \theta_2(\overline{(m_2, n_2)})$ and hence θ_2 induces a well-defined map $\widetilde{\theta}_2 : \Gamma(\overline{M \times_T N}) \longrightarrow I$. As $\widetilde{\theta}_2$ preserves the defining relations of $\Gamma(-)$ and I is an abelian Lie algebra, it follows that $\widetilde{\theta}_2$ is a Lie homomorphism.

Step 3. Here we define the Lie homomorphism

$$\psi_2 : \overline{\Gamma(M \times_T N, P \times_L Q, \partial_1 \oplus \partial_2)} \longrightarrow I.$$

Since the Lie algebras $(\overline{P \times_L Q}) \underline{\otimes} (\overline{M \times_T N})$, $\Gamma(\overline{M \times_T N})$ and I are abelian, we obtain the Lie homomorphism $\widetilde{\theta} = \langle \widetilde{\theta}_1, \widetilde{\theta}_2 \rangle$ in the coproduct of vector spaces

$$\widetilde{\theta} : (\overline{P \times_L Q}) \underline{\otimes} (\overline{M \times_T N}) \oplus \Gamma(\overline{M \times_T N}) \longrightarrow I,$$

which is evidently surjective. It is easy to see that $\widetilde{\theta}(\mathrm{Im}\, f) = 0$, where f is the Lie homomorphism introduced in Definition 4.1. We thus get the Lie epimorphism ψ_1 induced by $\widetilde{\theta}$.

Finally, it is readily checked that the pair (ψ_1, ψ_2) is a Lie crossed module morphism. The proof of theorem is complete. ∎

As an immediate consequence of the above theorem, we have

Corollary 4.3. *In Theorem* 4.2*, if* **M** *and* **N** *are ideals of a crossed module, then there is an exact sequence*

$$\Gamma(\frac{\mathbf{M} \cap \mathbf{N}}{[\mathbf{M}, \mathbf{N}]}) \xrightarrow{\psi} \mathbf{M} \otimes \mathbf{N} \twoheadrightarrow \mathbf{M} \wedge \mathbf{N}. \tag{8}$$

In next theorem, we show that in the case of $\mathbf{M} = \mathbf{N}$, the morphism ψ in (8) is injective.

Theorem 4.4. *For any simply connected crossed module* $\mathbf{T} = (T, L, \partial)$*, there is an exact sequence*

$$\Gamma(\mathbf{T}_{ab}) \xrightarrow{\psi} \mathbf{T} \otimes \mathbf{T} \twoheadrightarrow \mathbf{T} \wedge \mathbf{T}.$$

Proof. In virtue of Corollary 4.3 and [8, Proposition 17], we have only to prove that

$$\psi_1 : \bar{\Gamma}(\mathbf{T}_{ab}) = \frac{(\bar{L} \underline{\otimes} \bar{T}) \oplus \Gamma(\bar{T})}{\mathrm{Im} f} \longrightarrow I,$$

given by $\psi_1(\overline{(\bar{l} \otimes t, \gamma(\bar{t}'))}) = (t \otimes l, l \otimes t) + (0, \partial(t') \otimes t') + \mathrm{Im}\alpha)$ is injective, where $\bar{T} = T/[L, T]$ and $\bar{L} = L/L^2$. Suppose $\pi : L \otimes L \longrightarrow \bar{L} \otimes \bar{L}$ is the Lie homomorphism induced by the projection $L \longrightarrow \bar{L}$. Let J be a well-ordered set, $\{l_j \mid j \in J\}$ a basis in $\bar{L}$, and for any $j \in J$, $\bar{\partial}(e_j) = l_j$ where $e_j \in \bar{T}$ is any element in the pre-image of l_j via the epimorphism $\bar{\partial} : \bar{T} \longrightarrow \bar{L}$. Evidently, the set $\{e_j \mid j \in J\}$ is linearly independent in $\bar{T}$ and so, it can be extended to a basis for $\bar{T}$ by adding a set of vectors, say $\{e_k \mid k \in K\}$, where we assume that K is a well-ordered set such that for all $j \in J$ and $k \in K$, $j < k$. For the sake of clarity, we divide the rest of the proof into three steps.

Step 1. We show that the set $A := \{\overline{\bar{\partial}(e_j) \otimes e_r} \mid j, r \in J, \ j \leq r\}$ is a basis in $\bar{L} \underline{\otimes} \bar{T}$. It is obvious that the set $B := \{\bar{\partial}(e_j) \otimes e_r \mid j \in J, \ r \in J \cup K\}$ is a basis for the a belian Lie algebra $\bar{L} \otimes \bar{T}$. Remembering that $\bar{L} \underline{\otimes} \bar{T}$ is the quotient of $\bar{L} \otimes \bar{T}$ by the subalgebra X generated by the elements $\bar{\partial}(\bar{t}_1) \otimes \bar{t}_2 - \bar{\partial}(\bar{t}_2) \otimes \bar{t}_1$ $(\bar{t}_1, \bar{t}_2 \in \bar{T})$, one regards that the set

$$\{\bar{\partial}(e_j) \otimes e_r - \bar{\partial}(e_r) \otimes e_j \mid j \in J, \ r \in J \cup K, \ j < r\}$$

is generating X. Hence, in the vector space $\bar{L} \underline{\otimes} \bar{T}$, we have $\overline{\bar{\partial}(e_j) \otimes e_k} = \overline{\bar{\partial}(e_k) \otimes e_j}$ for all $j \in J$, $k \in K$. But the elements $\overline{\bar{\partial}(e_k) \otimes e_j}$ are finite linear combinations of the elements of the form $\overline{\bar{\partial}(e_r) \otimes e_j}$ $(j, r \in J, \ r \leq j)$. It consequently follows that $\bar{L} \underline{\otimes} \bar{T} = span(A)$. Now, assume that

$$\sum_{\substack{j,r \in J \\ j \leq r}} c_{jr}(\overline{\bar{\partial}(e_j) \otimes e_r}) = 0,$$

where the c_{jr}'s belong to Λ.

Then we have

$$
\sum_{\substack{j,r\in J \\ j\le r}} c_{jr}(\bar\partial(e_j)\otimes e_r) =
$$

$$
= \sum_{\substack{j,r\in J \\ j<r}} d_{jr}(\bar\partial(e_j)\otimes e_r) - \bar\partial(e_r)\otimes e_j) + \sum_{\substack{j\in J \\ k\in K}} d'_{jk}(\bar\partial(e_j)\otimes e_k - \bar\partial(e_k)\otimes e_j), \qquad (9)
$$

where the d_{jr}'s and the d'_{jk}'s are in Λ. From the above equality, we get

$$
\sum_{j,r\in J} c'_{jr}(\bar\partial(e_j)\otimes e_r) + \sum_{\substack{j\in J \\ k\in K}} d'_{jk}(\bar\partial(e_j)\otimes e_k) = 0,
$$

for some scalars $c'_{jr}\in\Lambda$. So, d_{jr}'s must be zero. We deduce from (9) that the d_{jr}'s and then the c_{jr}'s are zero.

Step 2. Here we show that the set $C := \{((\overline{\bar\partial(e_j)\otimes e_r}),0)+\mathrm{Im}f \mid j,r\in J,\ j\le r\}$ if char $\Lambda \ne 2$, and the set

$$
D := \{((\overline{\bar\partial(e_j)\otimes e_r}),0)+\mathrm{Im}f, (0,\gamma(e_s))+\mathrm{Im}f \mid j,r\in J,\ s\in J\cup K,\ j<r\}
$$

if char $\Lambda = 2$, are bases in $\bar\Gamma(\mathbf{T}_{ab})$.

We first assume that char $\Lambda \ne 2$. By virtue of Proposition 2.2(iii) and Step 1, the elements $((\overline{\bar\partial(e_j)\otimes e_r}),0)+\mathrm{Im}f$ $(j,r\in J, j\le r)$, $(0,\gamma(e_j))+\mathrm{Im}f$ $(j\in J\cup K)$, and $(0,z_{rs})+\mathrm{Im}f$ $(r,s\in J\cup K, r<s)$ form a generating set for $\bar\Gamma(\mathbf{T}_{ab})$. An easy verification shows that

$$
\mathrm{Im}f = \langle(\overline{\bar\partial(e_j)\otimes e_j}, -2\gamma(e_j)), ((\overline{\bar\partial(e_r)\otimes e_s}), -z_{rs}) \mid j,r,s\in J\cup K,\ r<s\rangle.
$$

Hence, we have

$$
(0,\gamma(e_j))+\mathrm{Im}f = -\frac{1}{2}\{(-\overline{\bar\partial(e_j)\otimes e_j},0) + (\overline{\bar\partial(e_j)\otimes e_j}, -2\gamma(e_j))\}+\mathrm{Im}f
$$

$$
= \frac{1}{2}(\overline{\bar\partial(e_j)\otimes e_j},0)+\mathrm{Im}f.
$$

Since the similar result holds for the element $(0,z_{rs})+\mathrm{Im}f$, one deduces that $\bar\Gamma(\mathbf{T}_{ab}) = span(C)$. Also, applying again Step 1 and using the generating set introduced for X, it is seen that the set C is linearly independent. The second case is proved similarly.

Step 3. Here we complete the proof by showing that ψ_1 is injective.
Consider the composite homomorphism

$$
\theta:\ \bar\Gamma(\mathbf{T}_{ab}) \xrightarrow{\psi_1} I \xrightarrow{\subseteq} \mathrm{coker}\alpha \xrightarrow{\delta} L\otimes L \xrightarrow{\pi} \bar L\otimes\bar L.
$$

A trivial verification shows that this homomorphism maps the basis C of $\bar\Gamma(\mathbf{T}_{ab})$ (obtained in Step 2) onto a linearly independent set of vectors in $\bar L\otimes\bar L$. It therefore follows that θ and then ψ_1 are injective, as required. ∎

Corollary 4.5. *Let $\mathbf{M}$ and $\mathbf{N}$ be simply connected and abelian. Then there is an isomorphism*

$$
\Gamma(\mathbf{M}\oplus\mathbf{N}) \cong \Gamma(\mathbf{M})\oplus\Gamma(\mathbf{N})\oplus(\mathbf{M}\otimes\mathbf{N}).
$$

Proof. This follows directly from the above theorem and Theorem 3.8. ∎

5. Applications to the homology of Lie crossed modules

In this section, using the results obtained so far, we generalize some known basic facts concerning the second homology of Lie algebras to the second homology of Lie crossed modules. First of all, let us remind the following useful results.

• Casas, Inassaridze, and Ladra [2] proved that the category of Lie crossed modules is an algebraic category, that is, there is a tripleable forgetful functor from the category $\mathfrak{XLie}$ to the category of sets $\mathfrak{Set}$, and deduced that every Lie crossed module admits a projective presentation. Also, it is showed in [15] that if (Y, F, μ) is a projective crossed module, then μ can be assumed to be the inclusion map, and the Lie algebras F, F/Y are free objects in the category of Lie algebras.

• In [2], using the theory of cotriple homology of Barr and Beck, the nth homology crossed modules $H_n(\mathbf{T})$ of a Lie crossed module $\mathbf{T} = (T, L, \partial)$ are defined as simplicial derived functors of the abelianization functor from the category $\mathfrak{XLie}$ to the category $\mathfrak{XVect}$, which are generalizations of the Eilenberg-Maclane homologies of Lie algebras. Furthermore, it is proved that $H_1(\mathbf{T}) \cong \mathbf{T}_{ab}$ and

$$H_2(\mathbf{T}) \cong (\frac{V \cap [F, Y]}{[R, Y] + [F, V]}, \frac{R \cap F'}{[F, R]}, \overline{\mu}),$$

where $(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, L, \partial)$ is a projective presentation of (T, L, ∂). Certainly, if L is considered as a Lie crossed module in the two usual ways, then we have $H_2(L, L, id) = (H_2(L), H_2(L), id)$ or $H_2(0, L, i) = (0, H_2(L), i)$, which gives the classic formula of Hopf [10]. Also, for any projective Lie crossed module (Y, F, μ), we have $H_2(Y, F, \mu) = 0$.

• For any Lie crossed module $\mathbf{T}$, if we denote by $J_2(\mathbf{T})$ the kernel of the commutator morphism $\mathbf{T} \otimes \mathbf{T} \longrightarrow \mathbf{T}$, then Proposition 3.2(ii) implies that $J_2(\mathbf{T})$ is abelian.

The following proposition from [14] gives a description of the second homology of Lie crossed modules in terms of their exterior products.

Proposition 5.1. ([14, Theorem 2.5]) *For any Lie crossed module* $\mathbf{T}$ *we have* $H_2(\mathbf{T}) = \ker(\mathbf{T} \wedge \mathbf{T} \longrightarrow \mathbf{T})$.

The following theorem is a generalization of [8, Proposition 21] and [16, Corollary 2.1].

Theorem 5.2. *For the Lie crossed modules* $\mathbf{M}$ *and* $\mathbf{N}$, *there are isomorphisms*

$$J_2(\mathbf{M} \oplus \mathbf{N}) \cong J_2(\mathbf{M}) \oplus J_2(\mathbf{N}) \oplus (\mathbf{M}_{ab} \otimes \mathbf{N}_{ab}) \oplus (\mathbf{N}_{ab} \otimes \mathbf{M}_{ab}),$$

$$H_2(\mathbf{M} \oplus \mathbf{N}) \cong H_2(\mathbf{M}) \oplus H_2(\mathbf{N}) \oplus (\mathbf{M}_{ab} \otimes \mathbf{N}_{ab}).$$

Proof. It follows a direct consequence of the about results and Theorem 3.8. ∎

Combining Proposition 5.1 with Theorem 4.4, we obtain the following result.

Theorem 5.3. *For any simply connected* $\mathbf{T}$, *there is an exact sequence*

$$\Gamma(\mathbf{T}^{ab}) \rightarrowtail J_2(\mathbf{T}) \twoheadrightarrow H_2(\mathbf{T}).$$

If we consider a Lie algebra as a Lie crossed module on any of the two usual way, then we get the following result from Theorem 5.3, which was already proved in [13, Corollary 2.6] by another technique.

Corollary 5.4.   *For any Lie algebra L, there is an isomorphism*

$$J_2(L) \cong H_2(L) \oplus \Gamma(L^{ab}).$$

Ellis [8], using topology techniques, proved that for any Lie algebra L with an ideal P, there is an eight-term exact sequence in homology of Lie algebras

$$H_3(L) \longrightarrow H_3(\frac{L}{P}) \longrightarrow \ker(L \wedge P \longrightarrow L) \longrightarrow H_2(L) \longrightarrow H_2(\frac{L}{P})$$
$$\longrightarrow P/[L, P] \longrightarrow H_1(L) \twoheadrightarrow H_1(\frac{L}{P}).$$

In particular, if F/R is a free presentation of L and S/R is the induced presentation of P for some ideal S of F, it is deduced from this sequence that

$$H_3(L) \cong \ker(F \wedge R \longrightarrow F) \quad \text{and} \quad H_3(L/P) \cong \ker(F \wedge S \longrightarrow F).$$

We generalize the above sequence for Lie crossed modules as follows:

Theorem 5.5.   *Let $\mathbf{T}$ be a Lie crossed module with an ideal crossed submodule $\mathbf{M}$. Let $\mathbf{V} \rightarrowtail \mathbf{Y} \twoheadrightarrow \mathbf{T}$ be a projective presentation of $\mathbf{T}$ and $\mathbf{M} \cong \mathbf{X}/\mathbf{V}$ for some ideal crossed submodule $\mathbf{X}$ of $\mathbf{Y}$. Then there is a natural exact sequence of Lie crossed modules*

$$\ker(\mathbf{Y} \wedge \mathbf{V} \longrightarrow \mathbf{Y}) \longrightarrow \ker(\mathbf{Y} \wedge \mathbf{X} \longrightarrow \mathbf{Y}) \longrightarrow \ker(\mathbf{T} \wedge \mathbf{M} \longrightarrow \mathbf{T}) \longrightarrow H_2(\mathbf{T})$$
$$\longrightarrow H_2(\frac{\mathbf{T}}{\mathbf{M}}) \longrightarrow \frac{\mathbf{M}}{[\mathbf{T}, \mathbf{M}]} \longrightarrow H_1(\mathbf{T}) \twoheadrightarrow H_1(\frac{\mathbf{T}}{\mathbf{M}}).$$

In order to prove this theorem, we need the following lemma. We omit the proof which is quite similar to the proof of [14, Proposition 3.4].

Lemma 5.6.   *Let $\mathbf{M}$ and $\mathbf{N}$ be ideal crossed submodules of a Lie crossed module $\mathbf{T}$ with $\mathbf{M} \subseteq \mathbf{N}$. Then there is an exact sequence of Lie crossed modules*

$$\mathbf{T} \wedge \mathbf{M} \longrightarrow \mathbf{T} \wedge \mathbf{N} \twoheadrightarrow \frac{\mathbf{T}}{\mathbf{M}} \wedge \frac{\mathbf{N}}{\mathbf{M}}.$$

Proof of Theorem 5.5.   Consider the following commutative diagrams:

$$
\begin{array}{ccccc}
\mathbf{T} \wedge \mathbf{M} & \longrightarrow & \mathbf{T} \wedge \mathbf{T} & \twoheadrightarrow & \frac{\mathbf{T}}{\mathbf{M}} \wedge \frac{\mathbf{T}}{\mathbf{M}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{M} & \rightarrowtail & \mathbf{T} & \twoheadrightarrow & \frac{\mathbf{T}}{\mathbf{M}}
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
\mathbf{Y} \wedge \mathbf{V} & \longrightarrow & \mathbf{Y} \wedge \mathbf{X} & \twoheadrightarrow & \mathbf{T} \wedge \mathbf{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{V} & \rightarrowtail & \mathbf{X} & \twoheadrightarrow & \mathbf{M}
\end{array}
$$

in which, the top rows are exact by Lemma 5.6.

Applying the snake lemma to these diagrams, we get the exact sequences

$$\ker(\mathbf{T} \wedge \mathbf{M} \longrightarrow \mathbf{T}) \xrightarrow{\alpha} \ker(\mathbf{T} \wedge \mathbf{T} \longrightarrow \mathbf{T}) \longrightarrow \ker(\frac{\mathbf{T}}{\mathbf{M}} \wedge \frac{\mathbf{T}}{\mathbf{M}} \longrightarrow \frac{\mathbf{T}}{\mathbf{M}}) \xrightarrow{\Delta_1} \frac{\mathbf{M}}{[\mathbf{T}, \mathbf{M}]}$$

$$\longrightarrow H_1(\mathbf{T}) \twoheadrightarrow H_1(\frac{\mathbf{T}}{\mathbf{M}}),$$

$$\ker(\mathbf{Y} \wedge \mathbf{V} \longrightarrow \mathbf{Y}) \longrightarrow \ker(\mathbf{Y} \wedge \mathbf{X} \longrightarrow \mathbf{Y}) \longrightarrow \ker(\mathbf{T} \wedge \mathbf{M} \longrightarrow \mathbf{T}) \xrightarrow{\Delta_2} \frac{\mathbf{V}}{[\mathbf{Y}, \mathbf{V}]},$$

where Δ_i, $i = 1, 2$, is the connecting homomorphism. It now remains to show that $\ker \Delta_2 = \ker \alpha$. But this immediately follows from the commutative diagram

$$
\begin{array}{ccc}
\ker(\mathbf{T} \wedge \mathbf{M} \longrightarrow \mathbf{T}) & \xrightarrow{\Delta_2} & \dfrac{\mathbf{V}}{[\mathbf{Y}, \mathbf{V}]} \\
\downarrow{\alpha} & & \uparrow{\subseteq} \\
\ker(\mathbf{T} \wedge \mathbf{T} \longrightarrow \mathbf{T}) & \xrightarrow{\beta} & H_2(\mathbf{T}),
\end{array}
$$

where β is the isomorphism obtained in [14, Theorem 4.2]. The proof is complete. ∎

Theorem 5.5 improves on [14, Theorem 4.4] (which in turn is a generalization of a result of Edalatzadeh [6]). Also, using this theorem and the explanations before it, we end this paper with the following conjecture.

Conjecture 5.7. With the assumptions of Theorem 5.5,
$$H_3(\mathbf{T}) \cong \ker(\mathbf{Y} \wedge \mathbf{V} \longrightarrow \mathbf{Y}).$$

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