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Applications of Lie Theory to Daws' Conjecture on Ultrapowers of Locally Compact Group Algebras

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Abstract. Focusing on the fact that a locally compact group G may be approximated by Lie groups, we show that for a given locally compact group G, $L^1(G)$ is ultra-amenable if and only if G is finite. Thus we answer a question raised by M. Daws in 2009.

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1. Introduction

Let A be a Banach algebra, and let X be a Banach A-bimodule, that is, a Banach space which is algebraically an A-bimodule, and for which there is a constant C such that for $a \in A, x \in X$ we have

$$\|a \cdot x\| \leq \|a\|\|x\|.$$

A *derivation* $D : A \to X$ is a linear map satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

For $x \in X$, $ad_x : A \to X$, with $ad_x(a) = a \cdot x - x \cdot a$ is a derivation. Such derivations are called *inner derivations.* A is called *amenable* if for any A-bimodule X, any continuous derivation $D : A \to X'$ is inner, where X' is the dual of X, and it is also a Banach A-bimodule with the operations defined by

$$< x, a \cdot \lambda >=< x \cdot a, \lambda >, \quad < x, \lambda \cdot a >=< a \cdot x, \lambda > \quad (a \in A, x \in X, \lambda \in X');$$

see [1] for more details. The concept of amenability for Banach algebras which has been proved to be of enormous importance in Banach algebra theory, is introduced by Johnson [6]. Several modifications of this notion have been introduced and studied over various classes of Banach algebras so far, which the concept of ultra-amenability is one of the newest ones. M. Daws [2] called A to be ultra-amenable if every ultrapower of A is amenable.

For an ultrafilter $\mathcal{U}$ on an index set I and a Banach algebra A, if

$$\ell^\infty(A) = \{(x_i)_i : \sup_i \|(x_i)\| < \infty\}$$

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is the space of bounded sequences, then

$$N_{\mathcal{U}} = \{(x_i)_i \in \ell^\infty(A) : \lim_{\mathcal{U}} \|x_i\| = 0\}$$

is a closed ideal in the Banach algebra $\ell^\infty(A)$, and the ultrapower of A with respect to the ultrafilter $\mathcal{U}$, is the quotient space $\ell^\infty(A)/N_{\mathcal{U}}$, which equipped with the canonical quotient norm and the pointwise product is also a Banach algebra, and it is denoted by $(A)_{\mathcal{U}}$. Ultra-amenability of C*-algebras, group algebras and semigroup algebras have been investigated so far [2, 11]. Recently in [4], the operator ultra-amenability of $A(G)$, for a locally compact group G, has been investigated.

In the case when G is a discrete, compact or abelian group, M. Daws shows that $L^1(G)$ is ultra-amenable if and only if G is finite; see [2, Theorem 5.11 and Theorem 5.9]. Also, for a locally compact group G, he introduced a family of locally compact groups, which he called relatively *[IN]*-groups. A non-compact locally compact group G is called relatively *[IN]* if there exists a compact, symmetric, non-null (with respect to Haar measure) set K in G and a non relatively compact subset $A \subseteq G$ with $aK = Ka$ for each $a \in A$, so that (K, A) is called a witness.

Example 1.1. (i) Let G be a locally compact, non-compact abelian group. Then G is relatively *[IN]*: for every compact symmetric neighborhood K of e in G, the pair (K, G) is a witness.

(ii) If H is an open subgroup of G and H is relatively *[IN]*, then so is G.

Daws shows that in the case G is a relatively *[IN]*-group, ultra-amenability of $L^1(G)$ implies that G is finite; [2, Theorem 5.16]. Then he presented the following result:

Theorem 1.2. ([2, Theorem 5.17]) *Let G be a locally compact group such that $L^1(G)$ is ultra-amenable. Then G is finite; or G satisfies all of the following properties:*

(i) *G is amenable;*

(ii) *G is neither compact nor a relatively-[IN] group (so that G is not abelian or discrete);*

(iii) *The space $AP(G)$ of almost periodic functions, is finite dimensional;*

(iv) *If H is a closed normal subgroup of G, then either G/H is finite or G/H satisfies properties* (i), (ii), (iii) *above.*

He strongly suspected that for an arbitrary locally compact group G, $L^1(G)$ is ultra-amenable if and only if G is finite. In this paper, we confirm Daws' conjecture by proving the following result:

Theorem 1.3. *Let G be a locally compact group. Then the following statements are equivalent:*

(i) *$M(G)$ is ultra-amenable;*

(ii) *$L^1(G)$ is ultra-amenable;*

(iii) *G is finite.*

Indeed, we use the fact that amenable semisimple connected Lie groups are compact. Then we concentrate on the relation of locally compact groups with Lie groups. To reach our goal, we consider a locally compact group G in the cases G is connected, locally connected and totally disconnected.

Before we present the proof, let us fix some notations and recall some notions and results, which play essential roles in our argument.

Throughout, for a locally compact group G, we denote by e the identity of G. Furthermore, the connected component of G containing e is denoted by G_e. For a subgroup H of G, the subgroup $N_G(H) = \{g \in G : gH = Hg\}$ is called the *normalizer* of H in G; we note that H is normal in $N_G(H)$. Also, for a subset U of G, we may define $N_G(U) = \{g \in G : gU = Ug\}$. Also, for a solvable group G, we denote by $\ell(G)$ the length of derived series of G, and for each $1 \leq i \leq \ell(G)$, the i-derived subgroup of G is denoted by G^i. We note that $G^{\ell(G)}$ is the trivial group. Furthermore, by [7, Section 5], each connected locally compact group G has a maximal (closed) connected solvable normal subgroup, which is called the radical of G, and it is denoted by $rad(G)$. If G is any locally compact group, we define the radical of G by $rad(G) = rad(G_e)$.

Now, we present some easy but useful lemmas.

Lemma 1.4. *The property "$L^1(G)$ is ultra-amenable" is inherited by quotients of G by closed normal subroups.*

Proof. By [2, Proposition 5.2], the quotient of an ultra-amenable Banach algebra by a closed ideal, is ultra-amenable. ■

Lemma 1.5. *Let G be a non-compact locally compact group. If G has a compact open subgroup N (resp. a compact open symmetric non-empty subset U) such that $N_G(N)$ (resp. $N_G(U)$) is not relatively compact, then G is a relatively [IN]-group.*

Proof. With our hypothesis, $(N, N_G(N))$ (resp. $(U, N_G(U))$) is a witness to the fact that G is a relatively *[IN]*-group. ■

Lemma 1.6. *Let G be a locally compact group with a finite number of connected components. If $L^1(G)$ is ultra-amenable, and G_e is solvable, then G is finite.*

Proof. As the number of connected components of G is finite, therefore G/G_e is finite. If G_e is compact, then the finiteness of G/G_e implies that G is compact, and applying Theorem 1.2 implies that G is finite, as required. Now, we show by contradiction that the case "G_e is non-compact" can not happen. Indeed, as G_e is solvable, we use induction on the length $\ell(G_e)$ to show that non-compactness of G_e implies that G is finite. This contradiction completes the proof.

If $\ell(G_e) = 1$, then the group G_e is commutative. Again, we use the fact that G/G_e is finite, and we see that G_e is open in G. By Example 1.1(i), we see that G_e is a relatively *[IN]*-group, and by Example 1.1(ii), we obtain that G is a relatively *[IN]*-group. Now, Theorem 1.2 shows that G is finite.

Now, suppose that $\ell(G_e) = n,\ n \geq 2$, and consider the following derived series with abelian factors

$$\{e\} = G_e^n < G_e^{n-1} < \cdots < G_e^1 = G_e,$$

and observe that for each $1 \le k \le n$, $G_e^k \lhd G$. Set $L = G/\overline{G_e^{n-1}}$. We note that $L_e = G_e/\overline{G_e^{n-1}}$, and by [10, Corollary 14.17],

$$rad(L) = rad(G)/\overline{G_e^{n-1}} = G_e/\overline{G_e^{n-1}} = L_e.$$

Furthermore, by Lemma 1.4, $L^1(L)$ is ultra-amenable. We note that $L/L_e \simeq G/G_e$ is finite. By [8, Theorem 15.18(iv)], $\ell(L_e) = n-1$, and so by our hypothesis we see that L is finite. Since $L_e = \{e\}$, this gives $G_e = \overline{G_e^{n-1}}$, so G_e is abelian. Applying Example 1.1 implies that G is a relatively *[IN]*-group. Now, Theorem 1.2 shows that G is finite. ■

2. Proof of Theorem 1.3

(i)⇒ (ii). Suppose that $M(G)$ is ultra-amenable. As $L^1(G)$ has a bounded approximate identity, by [2, Proposition 5.2], $L^1(G)$ being a closed ideal of $M(G)$ is ultra-amenable.

(iii)⇒ (i). Suppose that G is finite. Then $L^1(G)$ is finite-dimensional, and for each ultrafilter $\mathcal{U}$, $(L^1(G))_{\mathcal{U}} \simeq L^1(G)$. Furthermore, $M(G) = L^1(G) = \ell^1(G)$ is amenable whenever G is finite. Therefore, $M(G)$ is ultra-amenable.

(ii)⇒ (iii). The proof consists several steps:

Step 1. Suppose that G is an almost connected group. Then G/G_e is a compact group. By Lemma 1.4, $L^1(G/G_e)$ is ultra-amenable, and by Theorem 1.2, G/G_e is finite. It is easy to see that G_e is open in G. On the other hand, by [9, page 175], there is a compact normal subgroup K of G such that G/K is a Lie group. We set $H = G/K$, so that H is a Lie group with finitely many connected components. Let S be the solvable radical of H_e. As G is amenable (by Theorem 1.2), so is H_e, hence H/S is compact. By Theorem 1.2, H/S is finite, so $H_e = S$. By lemma 1.6, H is finite, and therefore G is compact. Now, applying Theorem 1.2 implies that G is finite.

Step 2. Suppose that G is a totally disconnected group. We show that ultra-amenability of $L^1(G)$ implies that G is finite. Assume by contradiction that G is infinite. By [12], G has a base of neighborhoods consisting of compact open subgroups. We may choose a compact open subgroup M such that M is an element of the base for identity, and fix it. Then M as a compact totally disconnected group, has a base $\mathcal{S}$ of neighborhoods consisting of compact open normal subgroups; see [5]. Suppose that $\mathcal{L}$ is the family of all compact open symmetric subsets of M. Then we see that $\mathcal{S} \subseteq \mathcal{L}$. Now, consider the following set

$$\Sigma = \{U \in \mathcal{L} : U \neq M, N_G(U) \text{ is relatively compact}\}.$$

If M is the only element of $\mathcal{L}$, it is also the only element of $\mathcal{S}$. On the other hand, $\cap_{N\in\mathcal{S}} N = \{e\}$, which implies that $M = \{e\}$. Since M is an open subgroup of G, it follows that G is discrete and Theorem 1.2 implies that G is finite. Hence we may assume that there exists $U \in \mathcal{L}$ with $U \neq M$. If $\Sigma = \emptyset$, then $N_G(U)$ is not relatively compact. By combining Lemma 1.5 and Theorem 1.2, it follows that G is finite. Thus we may suppose that Σ is not empty. Furthermore, Σ, equipped with the relation inclusion, is a partially ordered set.

We show that every chain in Σ has an upper bound. Consider a totally ordered family $\{U_i\}_{i\in I}$. Then we claim that $U_1 = \cup_{i\in I} U_i \in \Sigma$. Suppose that $M = U_1$. Since M is compact and $\{U_i\}_{i\in I}$ is an open cover for M, a finite number of U_i covers M. Since $\{U_i\}_{i\in I}$ is totally ordered, $M = U_i$ for some i, which is in contradiction with $U_i \in \Sigma$ and hence $M \neq U_1$. If $N_G(U_1)$ is not relatively compact, then Lemma 1.5 implies that G is a relatively $[IN]$-group, and so by Theorem 1.2, it follows that G is finite. Therefore, we may suppose that $U_1 \in \Sigma$. Now, Zorn's Lemma implies that Σ has a maximal element, say U_0. We may choose $x \in M\backslash U_0$, and since U_0 is symmetric, we also have $x^{-1} \in M\backslash U_0$. We may also choose $y \in M\backslash U_0$ with $y \neq x, x^{-1}$. If not, $M\backslash U_0 = \{x, x^{-1}\}$ is an open set in M, which implies that M is discrete. Since M is an open set in G, we conclude that G is discrete, and by applying Theorem 1.2 it follows that G is finite. Hence we may choose $y \in M\backslash U_0$ with $y \neq x, x^{-1}$. Now, $\{x, x^{-1}\}$ and $U_0 \cup \{y, y^{-1}\}$ are two closed subsets of M with $\{x, x^{-1}\} \cap (U_0 \cup \{y, y^{-1}\}) = \emptyset$. Since M, as a locally compact group (indeed, as a compact group) is a normal topological space, there exists an open set W in M with $\{x, x^{-1}\} \in W$, and $W \cap (U_0 \cup \{y, y^{-1}\}) = \emptyset$. Then, there exists an open neighborhood V_0 of identity such that $xV_0 \cup V_0x^{-1} \subseteq W$. Hence there is $N \in \mathcal{S}$ such that $N \subseteq V_0$, and so we obtain

$$xN \cup Nx^{-1} \subseteq xV_0 \cup V_0x^{-1} \subseteq W.$$

If $V = U_0 \cup xN \cup Nx^{-1}$ then V is a compact open symmetric subset of M with $U_0 \subseteq V$. Since $y \notin V$, then $V \neq M$ and since U_0 is a maximal element of Σ, we have $V \notin \Sigma$, which implies that $N_G(V)$ is not relatively compact. Therefore, Lemma 1.5 implies that G is a relatively *[IN]*-group. Finally, by Theorem 1.2, we conclude that G is finite, as required.

Step 3. Suppose that G is an arbitrary locally compact group. Then G/G_e is a totally disconnected group. Therefore, by Step 2, G/G_e is finite. So, G is an almost connected group, and by Step 1, we see that G is finite. The proof of the theorem is now complete.

Final remark. The author of [2] published an erratum on that paper, see [3]. The results from [2] that we use are not concerned by this erratum.

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