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On Certain Classes of Algebras in which Centralizers are Ideals

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Abstract. This paper is primarily concerned with studying finite-dimensional anti-commutative nonassociative algebras in which every centralizer is an ideal. These are shown to be anti-associative and are classified over a field F of characteristic different from 2; in particular, they are nilpotent of class at most 3 and metabelian. These results are then applied to show that a Leibniz algebra over a field of charactersitic zero in which all centralizers are ideals is solvable.

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1. Introduction

Centralizers in algebras have been studied in many papers, including [1], [4], [5], [8], [10] (though this paper contains errors, as we will point out below) and [13]. In particular, in [4] the authors studied Leibniz algebras in which all of the centralizers are ideals. In this paper we will continue that study for other classes of algebras and answer one of the questions raised in that paper.

Throughout, A will denote a finite-dimensional nonassociative algebra over a field F. We will denote the product in A by juxtaposition unless A is a Lie or Leibniz algebra, in which case we will use the usual bracket notation, $[,]$. We will call A *anti-commutative* if $x^2 = 0$ for all $x \in A$; of course, in such an algebra $xy = -yx$ for all $x, y \in A$. Clearly, all Lie algebras are anti-commutative. In such an algebra, when specifying the non-zero products, we will only specify xy, leaving it assumed that $yx = -xy$

The *centralizer* of an element $x \in A$ is the set

$$C_A(x) = \{y \in A \mid xy = yx = 0 \text{ for all } y \in A\}.$$

Following [4] we call A a CL-algebra if every centralizer in A is an ideal of A. We will say that elements $x, y \in A$ have *commutative bonding (CB)* if $xy = 0$ implies that $(xz)y = 0$ for all $z \in A$. The algebra A is then defined to be a *CB-algebra* if every pair of elements of A have commutative bonding. For anti-commutative algebras we will see that these two conditions are equivalent. It is easy to see that not all nonassociative algebras have the CB-property. The smallest such example is the two-dimensional solvable Lie algebra L with basis e_1, e_2 and non-zero product $[e_1, e_2] = e_2$. Then $[e_1, e_1] = 0$ but $[[e_1, e_2], e_1] = [e_2, e_1] = -e_2$.

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The *Frattini subalgebra*, $F(A)$, of A is the intersection of the maximal subalgebras of A. The *Frattini ideal*, $\phi(A)$, of A is the biggest ideal contained in $F(A)$. If $\phi(A) = 0$ we say that A is *ϕ-free*. As pointed out above, there are errors in [10]. In particular, Proposition 3.4, which claims that a non-abelian Lie algebra L with $\phi(L) \neq 0$ has only one maximal abelian subalgebra, is false. For example, let L be the three-dimensional Heisenberg Lie algebra with basis e_1, e_2, e_3 and non-zero product $[e_1, e_2] = e_3$. Then $\phi(L) = L^2 = Fe_3 \neq 0$ and $Fe_1 + Fe_3$, $Fe_2 + Fe_3$ are maximal abelian subalgebras of L.

In the following four sections we will consider anti-commutative algebras. In Section 2 we introduce some terminology and notation that we use throughout. In Section 3 we introduce CB-elements and CB-algebras and show that anti-commutative algebras are CB-algebras if and only if they are anti-associative. It is also shown that anti-commutative algebras are CB-algebras if and only if they are CL-algebras and that the set of such CB-algebras is closed under subalgebras, factor algebras and direct sums.

In Section 4 we give a characterisation of all anti-commutative CB-algebras over a field F of characteristic different from 2. In particular, we show that they are all metabelian and nilpotent of class at most three. We also use our characterisation to give examples of non-Lie and non-Leibniz CB-algebras. In section 5 we determine which of the nilpotent Lie algebras of dimension at most six are CB-algebras.

In Section 6 we consider the consequences of our earlier results for Leibniz CL-algebras. In particular, we show that, over a field of characteristic zero, all such algebras are solvable, thereby answering a question raised in [4]. In the final section we consider group actions on algebras and show that CB-elements are preserved by such actions.

We will denote the subspace spanned by $e_1, \ldots, e_n$ by $Fe_1 + \ldots + Fe_n$. Algebra direct sums will be denoted by $\oplus$, whereas $\dot{+}$ will indicate a vector space direct sum.

2. Preliminaries for anti-commutative algebras

Definition 2.1. An *ideal* of an algebra A is a subspace I with the property that $IA \subseteq A$.

Note that, as A is anti-commutative, all ideals are two-sided.

Example 2.2. The center $Z(A) = \{x \in A \mid xy = 0, \text{ for all } y \in A\}$ of A is an ideal of A.

Example 2.3. We define the subalgebras A^k inductively in the following way: $A^2 = \text{span}\{xy \mid x, y \in A\}$, $A^k = A^{k-1}A$ for all $k \geq 3$. Then a straightforward induction argument shows that $A^k \subseteq A^{k-1}$ for all $k \geq 2$, and A^k is an ideal of A for all $k \geq 1$.

Definition 2.4. A is said to be *nilpotent of class n* if $A^{n+1} = 0$ but $A^n \neq 0$.

Example 2.5. Similarly, we define the subalgebras $A^{(k)}$ inductively by $A^{(0)} = A$, $A^{(k)} = A^{(k-1)}A^{(k-1)}$ for all $k \geq 2$. Then a straightforward induction argument shows that $A^{(k)} \subseteq A^{(k-1)}$ for all $k \geq 1$, and $A^{(k)}$ is an subalgebra of A for all $k \geq 1$, but may not be an ideal of A.

Definition 2.6. A is said to be *solvable* if $A^{(n)} = 0$ for some $n \geq 1$. If $A^{(2)} = 0$ we say that A is metabelian.

Definition 2.7. We define the *centralizer* of x in A as

$$C_A(x) = \{y \in A : xy = 0\}.$$

Definition 2.8. We say that A is a *CL-algebra* if $C_A(x)$ is an ideal of A for all $x \in A$.

3. Anti-commutative CB-algebras

Definition 3.1. Two elements $x, y \in A$ are said to have *commutative bonding* if $xy = 0$ imples $(xz)y = 0$ for all $z \in A$.

Definition 3.2. The anti-commutative algebra A is called a *CB-algebra* if it satisfies the following property: whenever $x, y \in A$ are such that $xy = 0$ then $(xz)y = 0$ for all $z \in A$.

Example 3.3. An algebra A in which $A^2 = 0$ is automatically a CB-algebra.

Example 3.4. Any nilpotent algebra A of class 2 is a CB-algebra.

Definition 3.5. We define the linear transformation $R_x : A \to A : a \mapsto ax$. An element $x \in L$ such that $R_x^2 = 0$ is called an *absolute zero divisor*.

Remark 3.6. In the early 1960's Kostrikin showed that absolute zero divisors play a special and very important role in the theory of Lie algebras over fields of prime characteristic. Since Lie algebras containing absolute zero divisors have a degenerate Killing form, Kostrikin called them algebras with strong degeneration.

Theorem 3.7. *The following are equivalent:*

(i) *A is a CB-algebra;*

(ii) *every element of A is an absolute zero divisor;*

(iii) *A is anti-associative.*

Proof. (i)$\Rightarrow$(ii): Let $x, y \in A$. Then $x^2 = 0$, so $0 = -(xy)x = (yx)x$. As this is true for all $x, y \in A$ we have $R_x^2 = 0$ for all $x \in A$, so every element of A is an absolute zero divisor, giving (ii).

(ii)$\Rightarrow$(iii): Let x, y, z be arbitrary elements of A. Then $(xy)x = 0$. Hence

$$0 = ((x+z)y)(x+z) = (xy)x + (xy)z + (zy)x + (zy)z = (xy)z + (zy)x.$$

It follows that $(xy)z = -x(yz)$ and A is anti-associative.

(iii)$\Rightarrow$(i): Suppose that $xy = 0$. Then

$$(xz)y = -y(xz) = (yx)z = -(xy)z = 0.$$

Thus, A is a CB-algebra. ∎

Remark 3.8. Algebras which are anti-commutative and anti-associative have also been called *dual mock-Lie algebras* (see [3]). A *mock-Lie* algebra is a commutative algebra satisfying the Jacobi identity. These algebras have generated considerable interest (see [14] and the references therein). The dual mock-Lie algebras are the algebras over the operad Koszul dual to the mock-Lie operad.

Corollary 3.9. *The set of finite-dimensional CB-algebras form a pseudo-variety; that is, they are closed under subalgebras, factor algebras and direct sums.*

Proof. It is clear that anti-associativity is preserved under the taking of subalgebras, factor algebras and direct sums. ∎

Theorem 3.10. *The anti-commutative algebra A is a CL-algebra if and only if it is a CB-algebra.*

Proof. Suppose $C_A(x)$ is an ideal of A for all $x \in A$. Let $xy = 0$ for some $x, y \in A$. This implies that $x \in C_A(y)$. Now for any $z \in A$, $xz \in C_A(y)$ as $C_A(y)$ is an ideal of A. Thus, $(xz)y = 0$. Therefore, A is a CB-algebra.

Conversely, supoose A is a CB-algebra. We need to show $C_A(x)$ is an ideal of A for all $x \in A$. Let $y \in C_A(x)$ and $z \in A$. We need to verify that $yz \in C_A(x)$. This clearly follows from the definition of a CB-algebra. ∎

Definition 3.11. An element $z \in A$ is said to have the *CB-property* if $(xz)y = 0$ for all $x \in A$ and $y \in C_A(x)$. We will call such an element a *CB-element*.

Remark 3.12. Note that CB-elements are those elements of the algebra which do not break the commutativity between any two elements. For example, 0 is a CB-element. All the elements in a CB-algebra are CB-elements.

Lemma 3.13. *If $z \in A$ is a CB-element then $x(zy) = -(xz)y$ for all $x, y \in A$.*

Proof. If $z \in A$ is a CB-element then $(xz)x = 0$ for all $x \in A$. Observe that

$$0 = ((x+y)z)(x+y) = (xz)x + (yz)x + (xz)y + (yz)y = (yz)x + (xz)y.$$

Thus, we have $x(zy) = -(xz)y$. ∎

Lemma 3.14. *Let $x \in A$, $y \in C_A(x)$. Then $z \in A$ is a CB-element if and only if $zy \in C_A(x)$.*

Proof. Let $z \in A$ be a CB-element. For all $x \in A$ and $y \in C_A(x)$, we have

$$(zy)x = -x(zy) = (xz)y = 0,$$

using anti-commutativity and Lemma 3.13. Thus, $zy \in C_A(x)$.

Conversely, suppose $zy \in C_A(x)$ for some $z \in A$. Interchanging the role of x and y, we have $zx \in C_A(y)$. Observe that $(xz)y = 0$. Therefore, z is a CB-element. ∎

Proposition 3.15. *The collection K of all CB-elements is a subalgebra of A. Thus, K is a CB-algebra.*

Proof. As $0 \in K$, the set K is non-empty; it is clearly a subspace of A. Let $z_1, z_2 \in K$ and $y \in C_A(x)$. We need to show that $z = z_1 z_2 \in K$.

As z_1, z_2 are CB-elements, using Lemma 3.13 and 3.14 for z_1 and z_2, we have

$$zy = (z_1z_2)y = -z_1(z_2y) \in C_A(x).$$

Again using the Lemma 3.14 for z, we get $z = z_1z_2$ is a CB-element. Thus, the collection K of CB-elements is a subalgebra, and K is a CB-algebra. ∎

Proposition 3.16. *Let A_1 and A_2 be non-associative algebras with $x^2 = 0$ for all $x \in A_1, A_2$ and $\phi : A_1 \to A_2$ be a homomorphism. If z is a CB-element of A_1 then $\phi(z)$ is a CB-element of $\phi(A_1)$.*

Proof. Let $z_1 \in A_1$ be a CB-element and $z_2 = \phi(z_1)$. Suppose $x_2 \in \phi(A_1)$ and $y_2 \in C_{\phi(A_1)}(x_2)$, so $x_2 = \phi(x_1)$ and $y_2 = \phi(y_1)$ for some $x_1, y_1 \in A_1$. Then

$$(x_2z_2)y_2 = (\phi(x_1)\phi(z_1))\phi(y_1) = \phi(x_1z_1)\phi(y_1) = \phi((x_1z_1)y_1) = \phi(0) = 0.$$

Therefore, $z_2 = \phi(z_1)$ is a CB-element in $\phi(A_1)$. ∎

Corollary 3.17. *Let A_1 and A_2 be non-associative algebras with $x^2 = 0$ for all $x \in A_1, A_2$ and $\phi : A_1 \to A_2$ be an isomorphism. If A_1 is a CB-algebra then so is A_2.*

4. Classification of anti-commutative CB-algebras

Here we assume that F has characteristic different from 2. First we show that all anti-commutative CB-algebras are metabelian and nilpotent of index at most three.

Theorem 4.1. *Let A be a CB-algebra over any field F. If F has characteristic different from two then $A^4 = 0$. Moreover, A is metabelian (that is, $A^{(2)} = 0$). If A is a Lie algebra and F has characteristic different from three then $A^3 = 0$. If A is an associative algebra and F has characteristic different from two then $A^3 = 0$.*

Proof. Let x, y, z, w be arbitrary elements of A. Then A is anti-associative, by Proposition 3.7 and so $(xy)z = -x(yz)$. Now

$$((xy)z)w = -(x(yz))w = x((yz)w) = -x(y(zw)).$$

But also
$$((xy)z)w = -(xy)(zw) = x(y(zw)).$$

Hence, if F has characteristic different from two, we have that $x(y(zw)) = 0$ and $(xy)(zw) = 0$. Now suppose that A satisfies the Jacobi identity. Then

$$\begin{aligned} 0 &= x(yz) + y(zx) + z(xy) = x(yz) - (zx)y - (xy)z \\ &= x(yz) + (xz)y + x(yz) = 2x(yz) - x(zy) = 3x(yz). \end{aligned}$$

If F has characteristic different from three this implies that $x(yz) = 0$, whence $A^3 = 0$. If A is associative, then $x(yz) = -(xy)z = (xy)z$ which implies that $x(yz) = 0$ if F has characteristic different from two. ∎

Proposition 4.2. *Let A be a CB-algebra with* $\dim(A/A^2) = 2$. *Then $A^3 = 0$.*

Proof. Suppose that $A^3 \neq 0$. Then there exist $x, y \in A$ such that $xy \in A^2 \setminus Z(A)$. If either x or y is in A^2 we have $xy \in A^3 \subseteq Z(A)$, since $A^4 = 0$, by Theorem 4.1.

Hence $A = Fx + Fy + A^2$. Now $x(xy) = y(xy) = 0$ implies that $A(xy) = 0$, since A is metabelian, by Theorem 4.1 again. It follows that either $x(xy) \neq 0$ or $y(xy) \neq 0$, both of which imply that A is not a CB-algebra, a contradiction. The result follows. ■

Definition 4.3. A nilpotent Lie algebra L of dimension n is called *filiform* if $\dim L^i = n - i$ for each $i \geq 2$.

Filiform Lie algebras are nilpotent Lie algebras with maximal nilindex: a filiform Lie algebra L of dimension n has $L^n = 0$. They were introduced by Vergne in [12] and have attracted much attention since then as they have important properties; in particular, every filiform Lie algebra can be obtained by a deformation of the model filiform algebra. However, very few are Lie CB-algebras as the following corollary shows.

Corollary 4.4. *If L is a filiform Lie CB-algebra then L is two- or three-dimensional abelian or the three-dimensional Heisenberg algebra.*

Proof. If L is filiform then $\dim(L/L^2) = 2$. It follows from Proposition 4.2 that $\dim L \leq 3$. Hence L is two- or three-dimensional abelian or the three-dimensional Heisenberg algebra. ■

In order to classify CB-algebras we follow the ideas in [9, Theorem 2.4(a)]. Let A be a CB-algebra, let B be a subspace of A^2 which is complementary to $Z(A)$ and let C be a subspace of A which is complementary to A^2, so that $A = (Z(A) \dot{+} B) \dot{+} C$. Choose a basis $\{e_i, \ldots, e_r\}$ for C and put

(1) $e_i e_j = e_{ij} + z_{ij}$ and $e_i e_{jk} = z_{ijk}$, where $e_{ij} \in B$, $z_{ij}, z_{ijk} \in Z(A)$ for $1 \leq i, j, k \leq r$.

(2) $B^2 = 0$ (since A is metabelian) and $AZ(A) = 0$.

(3) $e_i^2 = 0$, $e_{ij} = -e_{ji}$, $z_{ij} = -z_{ji}$ (by anticommutativity) and, for all permutations $\sigma \in S_3$,
$$z_{\sigma(i)\sigma(j)\sigma(k)} = sign(\sigma) z_{ijk}$$
(by anti-associativity).

(4) The set $\{e_{ij} \mid 1 \leq i, j \leq r\}$ span B, since $e_1, \ldots, e_r$ are the generators of A and $AB \subseteq Z(A)$.

(5) $\sum \lambda_{jk} e_{ij} = 0 \Rightarrow \sum \lambda_{jk} z_{ijk} = 0$ for all $1 \leq i \leq r$ $(\lambda_{jk} \in F)$.

(6) If $x \in B + C$ then there is $y \in C$ such that $xy \neq 0$.

Conversely, if we have three subspaces, Z, A, B such that $A = (Z \dot{+} B) \dot{+} C$ with $\{e_i, \ldots, e_r\}$ a basis for C and satisfying (1)–(6) then A is a well-defined algebra, $x^2 = 0$ for all $x \in A$ and A is anti-associative, by (1), (2), (3). It follows from Theorem 3.7 that A is a CB-algebra.

We have proved the following result.

Theorem 4.5. *An algebra A is a CB-algebra if and only if it has three subspaces Z, B, C such that $A = (Z \dot{+} B) \dot{+} C$ and satisfying* (1)–(6) *above.*

Clearly, any algebra A in which $A^3 = 0$ satisfies the Jacobi identity, and so every CB-algebra for which this is the case is a Lie algebra. Theorem 4.5 can be used to construct examples which are non-Lie and non-Leibniz as follows.

Example 4.6. The smallest example of a CB-algebra such that $A^3 \neq 0$ will be seven-dimensional in which C is spanned by e_1, e_2, e_3, B is spanned by e_1e_2, e_1e_3, e_2e_3 and Z is spanned by $e_1(e_2e_3)$. If we denote e_1e_2 by e_4, e_1e_3 by e_5, e_2e_3 by e_6 and $e_1(e_2e_3)$ by e_7, this has multiplication

$$e_1e_2 = e_4 \qquad e_1e_3 = e_5 \qquad e_2e_3 = e_6$$
$$e_1e_6 = e_7 \qquad e_2e_5 = -e_7 \qquad e_3e_4 = e_7$$

Notice that this is Lie algebra if and only if F has characteristic three, and is an associative algebra if and only if F has characteristic two.

Example 4.7. Let C be spanned by e_1, e_2, e_3, e_4, B be spanned by e_1e_2, e_1e_3, e_2e_3 and Z be spanned by $e_1(e_2e_3) = e_3e_4$. If we denote e_1e_2 by e_5, e_1e_3 by e_6, e_2e_3 by e_7 and $e_1(e_2e_3)$ by e_8, this has multiplication

$$e_1e_2 = e_5 \qquad e_1e_3 = e_6 \qquad e_2e_3 = e_7$$
$$e_1e_7 = e_8 \qquad e_2e_6 = -e_8 \qquad e_3e_5 = e_8 \qquad e_3e_4 = e_8$$

Then this is a CB-algebra.

Example 4.8. Let C be spanned by e_1, e_2, e_3, e_4, e_5, B be spanned by e_1e_2, e_1e_3, e_2e_3 and Z be spanned by $e_1(e_2e_3) = e_4e_5$. If we denote e_1e_2 by e_6, e_1e_3 by e_7, e_2e_3 by e_8 and $e_1(e_2e_3)$ by e_9, this has multiplication

$$e_1e_2 = e_6 \qquad e_1e_3 = e_7 \qquad e_2e_3 = e_8$$
$$e_1e_7 = e_9 \qquad e_2e_6 = -e_9 \qquad e_3e_5 = e_9 \qquad e_4e_5 = e_9$$

This, again, is a CB-algebra.

5. Low dimensional Lie CB-algebras

Here we look at the nilpotent Lie algebras of dimension less than or equal to six to see which of them are CB-algebras. This will give all CB-algebras of dimension less than seven. We use the classification over a field of characteristic different from two given in [6], and will employ the same notation as there. For the reader's convenience we list the algebras here. Throughout, I will denote a one-dimensional ideal of L.

Proposition 5.1. *Nilpotent Lie algebras of dimensions one or two are abelian; in dimension three there are two non-isomorphic algebras, $L_{3,1}$, which is abelian, and the Heisenberg algebra $L_{3,2}$ with $[e_1, e_2] = e_3$. All of these are CB-algebras.*

Proof. These all have $L^3 = 0$. ■

Proposition 5.2. *In dimension 4 there are three non-isomorphic nilpotent Lie algebras:*

- $L_{4,1} = L_{3,1} \oplus I$;
- $L_{4,2} = L_{3,2} \oplus I$; *and*
- $L_{4,3}$ *with non-zero products* $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$.

Of these, $L_{4,1}$ and $L_{4,2}$ are CB-Lie algebras, but $L_{4,3}$ is not.

Proof. The first two have $L^3 = 0$. The third has $[e_1, [e_1, e_2]] \neq 0$, so this is not a CB-algebra. ∎

Proposition 5.3. *In dimension 5 there are nine non-isomorphic nilpotent Lie algebras:*

- $L_{5,k} = L_{4,k} \oplus I$ *for* $k = 1, 2, 3$;
- $L_{5,4}: \ [e_1, e_2] = e_5, \ [e_3, e_4] = e_5$;
- $L_{5,5}: \ [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$;
- $L_{5,6}: \ [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$;
- $L_{5,7}: \ [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$;
- $L_{5,8}: \ [e_1, e_2] = e_4, \ [e_1, e_3] = e_5$; *and*
- $L_{5,9}: \ [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$.

Of these, $L_{5,1}, L_{5,2}, L_{5,4}$ *and* $L_{5,8}$ *are CB-Lie algebras, but the others are not.*

Proof. $L_{5,1}, L_{5,2}, L_{5,4}$ and $L_{5,8}$ all have $L^3 = 0$.

$L_{5,3}$ is not a CB-Lie algebra because $L_{4,3}$ isn't one.

$L_{5,5}$, $L_{5,6}$, $L_{5,7}$ and $L_{5,9}$ all have $[e_1, [e_1, e_2]] \neq 0$, and so they are not CB-Lie algebras. ∎

Proposition 5.4. *In dimension 6 we get the following nilpotent Lie algebras:*

- $L_{6,k} = L_{5,k} \oplus I$ *for* $k = 1, \ldots, 9$;
- $L_{6,10}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_6, \ [e_4, e_5] = e_6$;
- $L_{6,11}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_6, \ [e_2, e_3] = e_6, \ [e_2, e_5] = e_6$;
- $L_{6,12}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_6, \ [e_2, e_5] = e_6$;
- $L_{6,13}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_5, \ [e_2, e_4] = e_5, \ [e_1, e_5] = e_6, \ [e_3, e_4] = e_6$;
- $L_{6,14}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5, \ [e_2, e_3] = e_5, \ [e_2, e_5] = e_6, \ [e_3, e_4] = -e_6$;
- $L_{6,15}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5, \ [e_2, e_3] = e_5, \ [e_1, e_5] = e_6, \ [e_2, e_4] = e_6$;
- $L_{6,16}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5, \ [e_2, e_5] = e_6, \ [e_3, e_4] = -e_6$;
- $L_{6,17}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5, \ [e_1, e_5] = e_6, \ [e_2, e_3] = e_6$;
- $L_{6,18}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5, \ [e_1, e_5] = e_6$;
- $L_{6,19}(\epsilon): \ [e_1, e_2] = e_4, \ [e_1, e_3] = e_5, \ [e_2, e_4] = e_6, \ [e_3, e_5] = \epsilon e_6$;
- $L_{6,20}: \ [e_1, e_2] = e_4, \ [e_1, e_3] = e_5, \ [e_1, e_5] = e_6, \ [e_2, e_4] = e_6$;
- $L_{6,21}(\epsilon): \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_2, e_3] = e_5, \ [e_1, e_4] = e_6, \ [e_2, e_5] = \epsilon e_6$;
- $L_{6,22}(\epsilon): \ [e_1, e_2] = e_5, \ [e_1, e_3] = e_6, \ [e_2, e_4] = \epsilon e_6, \ [e_3, e_4] = e_5$;
- $L_{6,23}: \ [e_1, e_2] = e_3, \ [e_1, e_3] = e_5, \ [e_1, e_4] = e_6, \ [e_2, e_4] = e_5$;

- $L_{6,24}(\epsilon)$: $[e_1, e_2] = e_3$, $[e_1, e_3] = e_5$, $[e_1, e_4] = \epsilon e_6$, $[e_2, e_3] = e_6$, $[e_2, e_4] = e_5$;
- $L_{6,25}$: $[e_1, e_2] = e_3$, $[e_1, e_3] = e_5$, $[e_1, e_4] = e_6$; *and*
- $L_{6,26}$: $[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_2, e_3] = e_6$,

where $\epsilon \in F$. Of these, $L_{6,1}, L_{6,2}, L_{6,4}, L_{6,8}, L_{6,22}(\epsilon)$ and $L_{6,26}$ are CB-Lie algebras, but the others are not.

Proof. $L_{6,1}, L_{6,2}, L_{6,4}, L_{6,8}, L_{6,22}(\epsilon)$ and $L_{6,26}$ all have $L^3 = 0$.

$L_{6,3}, L_{6,5}, L_{6,6}, L_{6,7}$ and $L_{6,9}$ are not CB-Lie algebras because the corresponding $L_{5,k}$ isn't one.

$L_{6,10}, L_{6,11}, L_{6,12}, L_{6,13}, L_{6,14}, L_{6,15}, L_{6,16}, L_{6,17}, L_{6,18}, L_{6,21}(\epsilon), L_{6,23}, L_{6,24}(\epsilon)$ and $L_{6,25}$ all have $[e_1, [e_1, e_2]] \neq 0$ and so are not CB-Lie algebras.

$L_{6,19}$ and $L_{6,20}$ have $[[e_1, e_2], e_2] \neq 0$ and so are not CB-Lie algebras. ∎

Note that the above results confirm that the seven-dimensional Lie algebra given in the Example 4.6 is the smallest nilpotent CB-Lie algebra L with $L^3 \neq 0$.

6. Leibniz algebras

An algebra L over a field F is called a *Leibniz algebra* if we have

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad \text{for every} \quad x, y, z \in L,$$

In other words the right multiplication operator $R_x : L \to L : y \mapsto [y, x]$ is a derivation of L. As a result such algebras are sometimes called *right* Leibniz algebras, and there is a corresponding notion of *left* Leibniz algebras, which satisfy

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

Clearly the opposite of a right (left) Leibniz algebra is a left (right) Leibniz algebra, so, in most situations, it doesn't matter which definition we use.

Every Lie algebra is a Leibniz algebra and every Leibniz algebra satisfying $[x, x] = 0$ for every element is a Lie algebra. The usual definitions for subalgebra, right (left) ideal, ideal, homomorphism apply for Leibniz algebras. Put $I = \langle \{x^2 : x \in L\} \rangle$. Then

$$\begin{aligned} [y, x^2] &= [[y, x], x] - [[y, x], x] = 0 \text{ and} \\ [x^2, y] &= [x, [x, y]] + [[x, y], x] = (x + [x, y])^2 - x^2 - [x, y]^2 \in I, \end{aligned}$$

so I is an ideal; in fact, I is the smallest ideal of L such that L/I is a Lie algebra; L/I is sometimes called the *liesation* of L.

In [4] the authors asked whether Leibniz CL-algebras are solvable. For Leibniz algebras over a field F of characteristic zero, using our previous results, we can answer this in the affirmative.

Theorem 6.1. *Let L be a Leibniz CL-algebra over a field of characteristic zero. Then L is solvable.*

Proof. By Levi's Theorem (see [2]) we have $L = R\dot{+}S$ where R is the radical and S is a semisimple Lie subalgebra of L. Now S is a Lie CB-algebra and so is nilpotent, by Theorem 4.1. It follows that $S = 0$, whence the result. ∎

Over more general fields we have the following results.

Theorem 6.2. *If L is a ϕ-free Leibniz CL-algebra over any field then $L^{(3)} = 0$.*

Proof. Since L is ϕ-free we have $L = I\dot{+}B$ where B is a subalgebra of L, by [11, Lemma 7.2]. Now B is a Lie CB-algebra and so $B^{(2)} = 0$, by Theorem 4.1. It follows that $L^{(3)} \subseteq I^2 = 0$, as claimed. ∎

The following is proved in [7, Lemma 1].

Lemma 6.3. *If a right Leibniz algebra L is also a left Leibniz algebra, then, for all $x, y, z \in L$,*

(i) $[[x, y], z] + [z, [x, y]] = 0$*; and*

(ii) $2[[x, y], [x, y]] = 0$.

Definition 6.4. We call L a symmetric Leibniz algebra if it is both a right and left Leibniz algebra and $[[x, y], [x, y]] = 0$. Note that if F has characteristic different from two the added identity is not needed because of Lemma 6.3(ii).

Proposition 6.5. *If L is a symmetric Leibniz algebra over any field then L^2 is a Lie algebra.*

Proof. This follows easily from Lemma 6.3 and remarks made at the beginning of this section. ∎

Corollary 6.6. *If L is a symmetric Leibniz CL-algebra over any field then $L^{(3)} = 0$.*

Proof. It follows from Proposition 6.5 that L^2 is a Lie CB-algebra, so $L^{(3)} = (L^{(1)})^{(2)} = 0$, by Theorem 4.1. ∎

7. Group actions on CB-algebras

In this section, we study a finite group action on algebras and show that CB-elements is preserved under the group action.

Definition 7.1. Let A be an algebra and G be a finite group. We say the group G is acting on A if there exists a function

$$\phi : G \times A \to A, \quad (g, x) \mapsto \phi(g, x) = gx$$

satisfying the following conditions.

1. For each $g \in G$ the map $x \mapsto gx$, denoted by ψ_g is linear.
2. $ex = x$ for all $x \in A$, where $e \in G$ is the group identity.
3. $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$ and $x \in A$.
4. $g(xy) = (gx)(gy)$ for all $g \in G$ and $x, y \in A$.

The above definition of group action can be stated equivalently as follows:

Proposition 7.2. *A finite group G acts on A if and only if there exists a group homomorphism*

$$\psi : G \to Aut(A), \quad g \mapsto \psi(g) = \psi_g,$$

from the group G to the automorphism group of A, where $\psi_g(x) = gx$ is the left translation by g.

Remark 7.3. Let G be a finite group and $F[G]$ be the associated group algebra. If G acts on a algebra A then A may be viewed as a $F[G]$-module.

Theorem 7.4. *Let $x \in A$ be a CB-element. Then x is preserved under the action of G, that is, gx is also a CB-element for all $g \in G$.*

Proof. Let A be equipped with an action of a group G. Let $z \in A$ be a CB-element. For all $x \in A$, $y \in C_A(x)$, $g \in G$, we have

$$(x(gz))y = g(((g^{-1}x)z)g^{-1}y) = 0. \tag{1}$$

Observe that in the above computation we have used the fact that $g^{-1}y \in C_A(g^{-1}x)$ as $(g^{-1}x)(g^{-1}y) = g^{-1}(xy) = g^{-1}0 = 0$. Equation (1) shows that gz are also CB-elements for all $g \in G$. Therefore, CB-elements are preserved under the group action. ∎

Suppose the given algebra A is equipped with an action of a finite group G. Let $x \in A$, we define an orbit of x under the action of G as follows:

$$G(x) = \{gx \mid g \in G\} \subseteq A.$$

It is easy to check that the orbits $G(x)$ and $G(y)$ of any two points $x, y \in A$ are either equal or disjoint. Note that if $x \in A$ is a CB-element then all the elements in the corresponding orbit $G(x)$ are also CB-elements. Let S be the collection of all CB-elements of A.

Proposition 7.5. *The set $B = \bigcup_{x \in S} G(x)$ is a CB-subalgebra of A.*

Proof. Note that B is non-empty as $0 \in B$. It is clear from the construction of B that every element of B is a CB-element. We only need to show that B is a subalgebra of A. Let $b_1, b_2 \in B$. Then $b_1 = g_1x$ and $b_2 = g_2y$ for some $g_1, g_2 \in G$ and $x, y \in A$. Observe that

$$ab = (g_1x)(g_2y) = g_1(x(g_1^{-1}g_2y)) \in B.$$

As the group action is linear, this proves that B is a subalgebra of A. Therefore, B is a CB-algebra. ∎

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