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Transitive Lie Algebras of Nilpotent Vector Fields and their Tanaka Prolongations

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Abstract. Transitive nilpotent local Lie algebras of vector fields can be easily constructed from dilations h of $\mathbb{R}^n$ with positive weights (give me a sequence of n positive integers and I will give you a transitive nilpotent Lie algebra of vector fields on $\mathbb{R}^n$) as the Lie algebras $\mathfrak{g}_{<0}(h)$ of the polynomial vector fields of negative weights with respect to h.
We provide a condition for the dilation h such that the Lie algebras of polynomial vectors defined by h are exactly the Tanaka prolongations of the corresponding nilpotent Lie algebras $\mathfrak{g}_{<0}(h)$. However, in some cases of dilations h we can find some 'strange' elements of the Tanaka prolongations of $\mathfrak{g}_{<0}(h)$, which we describe in detail. In particular, we give a complete description of derivations of degree 0 for the Lie algebra $\mathfrak{g}_{<0}(h)$.

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1. Introduction

Origins of the theory of Lie groups go back to the 1870s when Sophus Lie studied objects called at that time continuous groups [15, 16]. To be precise, these objects were not groups in the modern sense of this term. They were rather locally defined families of diffeomorphisms, closed with respect to the operations of composition and inverse, whenever they were correctly defined. Nowadays such structures are called pseudogroups.

Special examples of pseudogroups studied by Lie were "finite continuous groups" i.e. representations of abstract finite dimensional Lie groups in diffeomorphisms on manifolds. The theory of Lie groups is now a standard and classical part of mathematical education, including also the theory of Lie algebras and their representations.

An important aspect of Lie theory is the problem of classification of Lie algebras. It started by the conjecture of Killing and Cartan that any finite-dimensional real Lie algebra $\mathfrak{g}$ is the semidirect product of a solvable ideal and a semisimple subalgebra. The conjecture was later proved by Eugenio Elia Levi in 1905 and is now known as *Levi decomposition.* The first component is the maximal solvable ideal i.e. the *radical* of the algebra. The semisimple subalgebra being the second component is called a *Levi subalgebra.* Since any finite dimensional Lie algebra is a a semidirect

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product of a solvable Lie algebra and a semisimple Lie algebra we can classify both classes of algebras separately.

Semisimple complex Lie algebras have been completely classified by E. Cartan [5], while the real case was solved by F. Gantmacher [7]. By definition every semisimple Lie algebra over an algebraically closed field of characteristic 0 is a direct sum of simple Lie algebras. Every finite-dimensional simple Lie algebra belongs to one of the four families: A_n, B_n, C_n, and D_n, or is isomorphic to one of the exceptional Lie algebras denoted usually by E_6, E_7, E_8, F_4, and G_2. Simple Lie algebras correspond to connected Dynkin diagrams. Semisimple Lie algebras can be classified by Dynkin diagrams that are not necessarily connected: each connected component of the diagram corresponds to a simple component in the decomposition of a semisimple Lie algebra into simple Lie algebras.

Classification of solvable components of the Levi decomposition has proven to be much more complicated – it is currently unsolved and moreover believed to be unsolvable in full generality, i.e. for arbitrarily large dimension. Classification of nilpotent Lie algebras is known up to dimension eight [17, 18, 22, 23] while the solvable case terminates in dimension six.

In this paper we discuss realizations of Lie algebras in vector fields. They are of special interest due to a wide range of applications in various aspects of differential geometry, e.g. the general theory of differential equations and their systems including classification problems for ODEs and PDEs, integration of differential equations, the theory of systems with superposition principles, geometric control theory, to name just a few. The literature on classification of local transitive realizations of Lie algebras in vector fields is already quite extensive. Theoretical results as well as powerful computational methods can be found e.g. in [1, 6, 13, 19]. A classification of Lie algebras of vector fields on the real plane is described in [8]. The solvable Lie algebras of vector fields are discussed thoroughly in the survey [19] and the references therein. Transitive solvable Lie algebras of vector fields have been also studied in the context of integrability by quadratures [3, 4]. Note that finite-dimensional solvable Lie algebras of vector fields are not far from the nilpotent ones, since they are characterized as those whose derived ideal is nilpotent.

We will be interested in the graded nilpotent Lie algebras of vector fields which have negative weights with respect to a dilation. This is not a very restrictive assumption since all transitive nilpotent Lie algebras of vector fields are subalgebras of algebras of this type [10]. Of course, we can consider the infinite dimensional algebra spanned by all homogeneous vector fields of the dilation. Another graded extension of the nilpotent algebra is the Tanaka prolongation [20, 21]. Our conjecture was that both prolongations coincide. However, unexpectedly it turned out that for some particular dilations there exist 'strange' elements of their Tanaka prolongations which do not come from vector fields. In this paper we analyze the situation in detail.

2. Nilpotent Lie algebras of vector fields

Probably the easiest way of constructing a nilpotent Lie algebra of vector fields is to consider vector fields with negative weight with respect to a positive dilation on $\mathbb{R}^n$. To be more precise, let us associate with the coordinates $(x^1, \dots, x^n)$ positive integer degrees (weights) $(r_1, r_2, \dots, r_n)$.

This is equivalent to picking up the *weight vector field*

$$\nabla^h = \sum_i r_i x^i \partial_{x^i}\,,$$

or a one-parametric family of positive *dilations* (called in [2, 12], a *homogeneity structure*)

$$h_t(x^1,\dots,x^n) = \left(t^{r_1}x^1,\dots,t^{r_n}x^n\right)\,. \tag{1}$$

The biggest r_i is called the *weight* of the homogeneity structure and is denoted by $\mathsf{w}(h)$. By permuting variables, any dilation is determined by its *signature* $(r_1 \le r_2 \le r_3 \le \dots \le r_n)$. We will always use the above ordered signature, so that $\mathsf{w}(h) = r_n$. The weight vector field defines *homogeneous functions of weight* k as those $f \in C^\infty(\mathbb{R}^n)$ for which $\nabla^h(f) = kf$. One can prove that only non-negative homogeneity degrees are allowed and each homogeneous function is polynomial in $(x^1,\dots,x^n)$ ([11, 12]). Thus we have simultaneously two degrees: the degree of a homogeneous function as the polynomial in variables x^i, which we will denote by deg, and the weight with respect to the dilation, denoted $\mathsf{w}(x^i)$ or just r_i. For instance x^n is a first degree polynomial with weight r_n. This can be extended to tensors, e.g. a vector field X is homogeneous of degree k (which we denote by $X \in \mathfrak{g}_k(h)$) if $[\nabla^h, X] = kX$. Here k may be negative, for instance $\mathsf{w}(\partial_{x^i}) = -r_i$, while $\mathsf{w}(x^j\partial_{x^i}) = r_j - r_i$. For homogeneous vector fields X, Y,

$$\mathsf{w}([X,Y]) = \mathsf{w}(X) + \mathsf{w}(Y)\,,$$

which trivially implies that the family of vector fields of negative degrees

$$\mathfrak{g}_{<0}(h) = \bigoplus_{i=-\mathsf{w}(h)}^{-1} \mathfrak{g}_i(h)$$

is a (finite-dimensional) nilpotent Lie algebra (cf. [10, 14]). This (except for finite-dimensionality) remains valid for general graded supermanifolds [24], in particular graded bundles in the sense of [12]. Of course, for a given dilation the whole graded Lie algebra

$$\mathfrak{g}_\infty(h) = \mathfrak{g}_{<0}(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_1(h) \oplus \mathfrak{g}_2(h) \oplus \dots\,,$$

where

$$\mathfrak{g}_{<0}(h) = \mathfrak{g}_{-\mathsf{w}(h)}(h) \oplus \cdots \oplus \mathfrak{g}_{-1}(h)\,,$$

is defined. Interestingly enough, the total graded algebra is determined (with some exceptions) by means of the so called *Tanaka prolongation* [20, 21].

On the other hand, a strong result [10] states that any transitive nilpotent Lie algebra L of vector fields defined on a neighborhood of $0 \in \mathbb{R}^n$ is locally a subalgebra of $\mathfrak{g}_{<0}(h)$ for some dilation h on $\mathbb{R}^n$ (cf. also [14], where the assumptions are a little bit stronger). The nilpotent algebra L is therefore automatically finite-dimensional and polynomial in appropriate homogeneous coordinates $(x^1,\dots,x^n)$. A similar result is valid for solvable Lie algebras of vector fields [9]. The homogeneity structure is uniquely defined by the structure of L, so that $\mathfrak{g}_{<0}(h)$ can be viewed as nilpotent 'enveloping' of L. We also note that every finite-dimensional (real) Lie algebra is a transitive Lie algebra of vector fields once identified with the Lie algebra of left-invariant vector fields on the corresponding 1-connected Lie group.

The homogeneity structures defined in [12] are a little bit more general: they admit coordinates of degree 0, so that the manifold is a fibration over the quotient manifold corresponding to zero-degree coordinates (graded bundle). In particular, graded bundles of degree 1 are just vector bundles. The family of vector fields $\mathfrak{g}_{<0}(h)$ is a nilpotent subalgebra in the family of vector fields with non-positive degrees,

$$\mathfrak{g}_{\leq 0}(h) = \bigoplus_{i=-\mathsf{w}(h)}^{0} \mathfrak{g}_i(h)\,.$$

The latter is generally not solvable, but any finite-dimensional Lie algebra L in $\mathfrak{g}_{\leq 0}(h)$ such that $[L,L] \subset \mathfrak{g}_{<0}(h)$ is solvable [9]. This is because, for finite-dimensional Lie algebras, L is solvable if and only if $[L,L]$ is nilpotent (this is not true in infinite dimensions). In other words, the *nilradical* $\mathfrak{nr}(L)$ of L contains $[L,L]$. We will call solvable Lie algebras of vector fields in $\mathfrak{g}_{\leq 0}(h)$ *dilational* (with respect to h) (cf. [9]).

The method of defining dilational transitive solvable Lie algebras of vector fields is illustrated by the following example.

Example 2.1. On $\mathbb{R}^3$ with coordinates (x,y,z) we define the dilation h declaring that x is of weight 1, y is of weight 2, z is of weight 3. The nilpotent part $\mathfrak{g}_{<0}(h)$ is a vector space over $\mathbb{R}$ generated by polynomial vector fields

$$\mathfrak{g}_{-3}(h) = \langle\, \partial_z \,\rangle, \quad \mathfrak{g}_{-2}(h) = \langle\, x\partial_z,\, \partial_y \,\rangle, \quad \mathfrak{g}_{-1}(h) = \langle\, \partial_x,\, y\partial_z \rangle\,,$$

while
$$\mathfrak{g}_0(h) = \langle\, x^3\partial_z\,, x^2\partial_y\,, xy\partial_z\,, x\partial_x\,, y\partial_y\,, z\partial_z \rangle\,.$$

As polynomial vector fields have an additional grading related to the degree of polynomial (linear, quadratic, etc), we will use both gradings to read the necessary information about the Lie algebras of transitive vector fields.

2.1. Tanaka prolongation of nilpotent algebra associated with a dilation.

The following is well known. Let L be a Lie algebra. We define the *lower central series of* L inductively:

$$L = L^1\,,\ L^{i+1} = [L, L^i] \text{ for } i \geq 1\,.$$

Definition 2.2. If the lower central series *stops at zero*, i.e. $L^{k+1} = \{0\}$ for some k, then we say that L is *nilpotent*. The smallest such k we call the *height* of the nilpotent Lie algebra.

Let us recall from section 2 that on $\mathbb{R}^n$ we consider a positive dilation, i.e. an action h of the multiplicative monoid $(\mathbb{R}, \cdot)$ of the form

$$h_t(x^1, \dots, x^n) = \left(t^{r_1}x^1, \dots, t^{r_n}x^n\right)\,,\ t \in \mathbb{R}\,,$$

where $r_i \in \mathbb{N}$, $r_i > 0$, $\mathsf{w}(h) = r_n = \max\{r_1 \leq \ldots \leq r_n\}$. The associated *weight vector field* reads $\nabla^h = \sum_i r_i x^i \partial_{x^i}$. A smooth function f defined in a neighborhood of $0 \in \mathbb{R}^n$ is homogeneous of weight a if $\nabla^h(f) = af$. Note that only non-negative integer homogeneity degrees are possible, and any such function admits a global polynomial form. Similarly, a vector field X is homogeneous of weight a if $[\nabla^h, X] = aX$, however now negative weights not less than $-\mathsf{w}(h)$ are allowed. The family of homogeneous vector fields (of weight a) is denoted by $\mathfrak{g}_a(h)$.

We observe that the family of vector fields of negative weights

$$\mathfrak{m}=\mathfrak{g}_{<0}(h)=\bigoplus_{i=-\mathsf{w}(h)}^{-1}\mathfrak{g}_i(h)$$

is a transitive (finite-dimensional) graded nilpotent Lie algebra of vector fields (cf. [10, 14]). The transitivity means that the vector fields of negative weights span $\mathsf{T}\mathbb{R}^n$. Of course, any transitive subalgebra of $\mathfrak{g}_{<0}(h)$ is again a transitive nilpotent Lie algebra of vector fields.

It can be easily checked that the product of two homogeneous functions of weight v_1 and v_2 is homogeneous of weight v_1+v_2. Consequently, for any multi-index $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{N}^n$, the function $x^\alpha:=x_1^{\alpha_1}\dots x_n^{\alpha_n}$ is homogeneous of weight $r_1\alpha_1+...+r_n\alpha_n$.

Let us fix a homogeneity structure h on $\mathbb{R}^n$ and let us denote the space of vector fields generated by homogeneous vector fields of negative degree by $\mathfrak{g}_{<0}(h)$. The algebra generated by these vector fields is a nilpotent algebra

$$\mathfrak{m}=\mathfrak{g}_{-\mathsf{w}(h)}(h)\oplus\dots\oplus\mathfrak{g}_{-1}(h)\,.$$

The l-th *Tanaka prolongation* is given by

$$\mathfrak{g}^T_{(l)}(\mathfrak{m})=\mathfrak{m}\oplus\mathfrak{g}^T_0(\mathfrak{m})\oplus\mathfrak{g}^T_1(\mathfrak{m})\oplus\dots\oplus\mathfrak{g}^T_l(\mathfrak{m})\,,$$

where the spaces $\mathfrak{g}^T_k(\mathfrak{m})$, for $k\geq 0$, are defined inductively by

$$\mathfrak{g}^T_k(\mathfrak{m})=\left\{v\in\bigoplus_{p<0}\mathfrak{g}^T_{p+k}(\mathfrak{m})\otimes\mathfrak{g}_p(h)^*\mid v[X,Y]=[v(X),Y]+[x,v(Y)]\right\}.$$

Here we put $\mathfrak{g}^T_i(\mathfrak{m})=\mathfrak{g}_i(h)$ for $i<0$. The full Tanaka prolongation is

$$\mathfrak{g}^T(\mathfrak{m})=\mathfrak{m}\oplus\bigoplus_{k=0}^{\infty}\mathfrak{g}^T_k(\mathfrak{m})\,.$$

Note that $\mathfrak{g}^T_i(\mathfrak{m})$ for $i\geq 0$ may be in principle bigger than $\mathfrak{g}_i(h)$. On the other hand, $\mathfrak{g}_i(h)\subset\mathfrak{g}^T_i(\mathfrak{m})$, as vector fields of weight i define derivations of weight i in an obvious way.

The space $\mathfrak{g}^T_k(\mathfrak{m})$ for $k\geq 0$ will be called, with some abuse of terminology, the *space of graded derivations of* $\mathfrak{m}$ *of weight* k. It is clear that every vector field $X\in\mathbb{R}^n$ of weight p defines a derivation D_X such that $D_X(Y)=[X,Y]$.

So, if Y is a vector field of weight $i<0$ then $D_X(Y)$ is a vector field of weight $p+i$. Therefore, $[\mathfrak{g}^T_p(\mathfrak{m}),\mathfrak{g}_{-1}(h)]\subset\mathfrak{g}^T_{p-1}$. However, we will not assume that $\mathfrak{m}$ is fundamental, i.e. $[\mathfrak{g}_p(h),\mathfrak{g}_{-1}(h)]=\mathfrak{g}_{p-1}(h)$. In particular, if the nonzero weights are only even (the signature consists of even numbers), then $\mathfrak{g}_{-1}(h)=\{0\}$.

Thus we will work with the graded nilpotent Lie algebra $\mathfrak{m}=\mathfrak{g}_{-w}(h)\oplus\cdots\oplus\mathfrak{g}_{-1}(h)$, trying to find its Tanaka prolongations. We want to find the condition for the dilation h assuring that $\mathfrak{g}^T_s=\mathfrak{g}_s(h)$ for all s, i.e. the derivations of weight s come from vector fields of weight s. The main result is the following.

Theorem 2.3. *For the dilation h in* (1) *denote $\ell = r_n - 2r_{n-1}$. Then, the Tanaka prolongation consists of only vector fields,*

$$\mathfrak{g}^T(\mathfrak{m}) = \mathfrak{m} \oplus \bigoplus_{k=0}^{\infty} \mathfrak{g}_k(h), \qquad \textit{if and only if } \ell < 0.$$

The 'only if' part follows directly from the following Lemma.

Lemma 2.4. *Let h be a dilation on $\mathbb{R}^n$ such that $\ell = r_n - 2r_{n-1} \geq 0$. Let $A = (a_i^j)_{i,j=1}^{n-1}$ be a symmetric matrix of real numbers such that $a_i^j \neq 0$ only for $r_i = r_j = r_{n-1}$. Then there is a unique derivation of weight ℓ,*

$$D_\ell^A \in \bigoplus_{p<0} \mathfrak{g}_{p+\ell}(h) \otimes \mathfrak{g}_p^*(h),$$

which vanishes on constant vector fields and on linear vector fields and reads

$$D_\ell^A(x_i\partial_j) = \delta_j^n a_i^k \partial_k. \tag{2}$$

Since it vanishes on constant vector fields and has weight $\ell \geq 0$, this derivation does not come from a vector field of weight ℓ.

Proof. It is clear that there is exactly one extension of D_ℓ^A defined as above on constant and linear vector fields to

$$D_\ell^A \in \bigoplus_{p<0} \mathfrak{g}_{p+\ell}(h) \otimes \mathfrak{g}_p^*(h),$$

such that for polynomials f, g we have another derivation property:

$$D_\ell^A(fg\partial_j) = fD_\ell^A(g\partial_j) + gD_\ell^A(f\partial_j). \tag{3}$$

We will show that D_ℓ^A is a derivation of the Lie algebra $\mathfrak{m} = \mathfrak{g}_{<0}(h)$ of weight ℓ.
First, let us consider the weight of D_ℓ^A. Note that D_ℓ^A is of weight ℓ on constant and linear vector fields. Indeed, from (2) we read that the weight of $D_\ell^A(x_i\partial_j)$ is $-r_{n-1}$. Since $D_\ell^A(x_i\partial_j)$ is nonzero only for $j = n$ and $r_i = r_{n-1}$, we calculate the weight of D_ℓ^A as $-r_{n-1} - (r_{n-1} - r_n) = r_n - 2r_{n-1} = \ell$. Then inductively, with the use of (3), we prove that D_ℓ^A is of weight ℓ on higher polynomial degree vector fields. The weight of $x_i f \partial_j$ is $r_i + \mathsf{w}(f) - r_j$. Acting with D_ℓ^A, we get

$$D_\ell^A(x_i f\partial_j) = fD_\ell^A(x_i\partial_j) + x_i D_\ell^A(f\partial_j).$$

The weight of the first summand is $\mathsf{w}(f) + r_i - r_j + \ell$ by the definition of D_ℓ^A, and the weight of the second summand is also $r_i + \mathsf{w}(f) - r_j + \ell$ by the inductive assumption. Then, we conclude that D_ℓ^A is indeed of weight ℓ.
Now we show that D_ℓ^A is a Lie algebra derivation, i.e. all polynomials f and g satisfy

$$D_\ell^A([f\partial_j, g\partial_k]) = [D_\ell^A(f\partial_j), g\partial_k] + [f\partial_j, D_\ell^A(g\partial_k)], \tag{4}$$

which means the Leibniz rule holds. Note that if $\deg(f) = 0$ or $\deg(g) = 0$ and $\deg(f) + \deg(g) = 1$, the Leibniz rule (4) is satisfied trivially, because D_ℓ^A vanishes on all vector fields of the form ∂_i.

The first nontrivial case is then $\deg(f) = 1$ and $\deg(g) = 1$. First, let us assume that $j < n$ and $l < n$ and consider $[x_i\partial_j, x_k\partial_l]$. Since both vector fields should be of negative weight, we also assume that $r_i < r_j$ and $r_k < r_l$. We have

$$D^A_\ell([x_i\partial_j, x_k\partial_l]) = \delta_{jk}D^A_\ell(x_i\partial_l) - \delta_{li}D^A_\ell(x_k\partial_j) = 0\,,$$

because D^A_ℓ vanishes on all linear vector fields $x_i\partial_j$ with $j < n$. For the same reason

$$[D^A_\ell(x_i\partial_j), x_k\partial_l] + [x_i\partial_j, D^A_\ell(x_k\partial_l)] = 0,$$

Leibniz' rule is then satisfied. Next, we consider $[x_i\partial_n, x_j\partial_k]$ for $k < n$, so

$$D^A_\ell([x_i\partial_n, x_j\partial_k]) = \delta_{ik}D^A_\ell(-x_j\partial_n) = -\delta_{ik}a^s_j\partial_s = 0\,,$$

because if $k < n$ then $r_j < r_{n-1}$ and $a^s_j = 0$. On the other hand,

$$[D^A_\ell(x_i\partial_n), x_j\partial_k] + [x_i\partial_n, D^A_\ell(x_j\partial_k)] = [a^s_i\partial_s, x_j\partial_k] = a^j_i\partial_k = 0,$$

and again Leibniz' rule holds. At the end we are left with the brackets $[x_i\partial_n, x_j\partial_n]$ which are zero. Since $i < n$ and $j < n$, then of course $D^A_\ell([x_i\partial_n, x_j\partial_n]) = 0$ and also

$$[D^A_\ell(x_i\partial_n), x_j\partial_n] + [x_i\partial_n, D^A_\ell(x_j\partial_n)] = [a^s_i\partial_s, x_j\partial_n] + [x_i\partial_n, a^s_j\partial_s)] = (a^j_i - a^i_j)\partial_n = 0\,,$$

because a^i_j is symmetric.
For the inductive step we have to show that

$$D^A_\ell([x_if\partial_j, g\partial_k]) = [D^A_\ell(x_if\partial_j), g\partial_k] + [x_if\partial_j, D^A_\ell(g\partial_k)]\,, \tag{5}$$

for $\deg(f) \leq \alpha$ and $\deg(g) \leq \beta$, $\alpha, \beta \geq 1$, and

$$D^A_\ell([f\partial_j, g\partial_k]) = [D^A_\ell(f\partial_j), g\partial_k] + [f\partial_j, D^A_\ell(g\partial_k)],$$

for $\deg(f) \leq \alpha$ and $\deg(g) \leq \beta$, $\alpha, \beta \geq 1$. We start from the left hand side. For the simplicity of notation we shall write $f_k = \partial_k f$, so we get

$$\begin{aligned} &D^A_\ell([x_if\partial_j, g\partial_k]) = D^A_\ell(x_ifg_j\partial_k - \delta_{ik}gf\partial_j - gx_if_k\partial_j) \\ &= x_ifD^A_\ell(g_j\partial_k) + x_ig_jD^A_\ell(f\partial_k) + fg_jD^A_\ell(x_i\partial_k) - \delta_{ik}gD^A_\ell(f\partial_j) - \delta_{ik}fD^A_\ell(g\partial_j) \\ &\quad - x_igD^A_\ell(f_k\partial_j) - x_if_kD^A_\ell(g\partial_j) - gf_kD^A_\ell(x_i\partial_j)\,. \end{aligned}$$

The sum of all the terms containing x_i equals $x_iD^A_\ell([f\partial_j, g\partial_k])$. Thus, for the left hand side of (5) we get

$$\begin{aligned} &D^A_\ell([x_if\partial_j, g\partial_k]) \\ &= x_iD^A_\ell([f\partial_j, g\partial_k]) + fg_jD^A_\ell(x_i\partial_k) - \delta_{ik}gD^A_\ell(f\partial_j) - \delta_{ik}fD^A_\ell(g\partial_j) - gf_kD^A_\ell(x_i\partial_j)\,. \end{aligned} \tag{6}$$

Now, on the right hand side we use first (3) and then perform the necessary calculation:

$$\begin{aligned} &[D^A_\ell(x_if\partial_j), g\partial_k] + [x_if\partial_j, D^A_\ell(g\partial_k)] \\ &= [x_iD^A_\ell(f\partial_j) + fD^A_\ell(x_i\partial_j), g\partial_k] + x_i[f\partial_j, D^A_\ell(g\partial_k)] - fD^A_\ell(g\partial_k)(x_i)\partial_j \\ &= x_i[D^A_\ell(f\partial_j), g\partial_k] - \delta_{ik}gD^A_\ell(f\partial_j) + f[D^A_\ell(x_i\partial_j), g\partial_k] - gf_kD^A_\ell(x_i\partial_j) \\ &\quad + x_i[f\partial_j, D^A_\ell(g\partial_k)] - fD^A_\ell(g\partial_k)(x_i)\partial_j\,. \end{aligned}$$

Again, the terms containing x_i sum up to $x_i D^A_\ell([f\partial_j, g\partial_k])$. Finally, for the right hand side we get

$$[D^A_\ell(x_i f\partial_j), g\partial_k] + [x_i f\partial_j, D^A_\ell(g\partial_k)] \tag{7}$$
$$= x_i D^A_\ell([f\partial_j, g\partial_k]) - \delta_{ik} g D^A_\ell(f\partial_j) + f[D^A_\ell(x_i\partial_j), g\partial_k] - g f_k D^A_\ell(x_i\partial_j) - f D^A_\ell(g\partial_k)(x_i)\partial_j .$$

The difference between (6) and (7) then reads

$$f\left([D^A_\ell(x_i\partial_j), g\partial_k] - D^A_\ell(g\partial_k)(x_i)\partial_j - g_j D^A_\ell(x_i\partial_k) + \delta_{ik} D^A_\ell(g\partial_j)\right) .$$

The first summand can be replaced with $D^A_\ell([x_i\partial_j, g\partial_k]) - [x_i\partial_j, D^A_\ell(g\partial_k)]$ and the last two summands can be replaced with $-D^A_\ell([x_i\partial_j, g\partial_k]) + x_i D^A_\ell(g_j\partial_k)$. To see this, we use the explicit calculation of $D^A_\ell([x_i\partial_j, g\partial_k])$. The difference between the left hand side and the right hand side of (5) reads now

$$f\left(-[x_i\partial_j, D^A_\ell(g\partial_k)] - D^A_\ell(g\partial_k)(x_i)\partial_j + x_i D^A_\ell(g_j\partial_k)\right) . \tag{8}$$

Note that $[x_i\partial_j, D^A_\ell(g\partial_k)] = x_i[\partial_j, D^A_\ell(g\partial_k)] - D^A_\ell(g\partial_k)(x_i)\partial_j$ and

$$D^A_\ell(g_j\partial_k) = D^A_\ell([\partial_j, g\partial_k]) = [\partial_j, D^A_\ell(g\partial_k)] .$$

We can therefore transform (8) in the following way

$$f\left(-[x_i\partial_j, D^A_\ell(g\partial_k)] - D^A_\ell(g\partial_k)(x_i)\partial_j + x_i D^A_\ell(g_j\partial_k)\right)$$
$$= f\left(-x_i[\partial_j, D^A_\ell(g\partial_k)] + D^A_\ell(g\partial_k)(x_i)\partial_j - D^A_\ell(g\partial_k)(x_i)\partial_j + x_i[\partial_j, D^A_\ell(g\partial_k)]\right) = 0 ,$$

which concludes the inductive proof. ∎

Example 2.5. Take $\mathbb{R}^3$ with coordinates (x, y, z) of weights 1, 2, and 4 respectively. Here $\mathfrak{g}_{<0}(h)$ is spanned by $\mathfrak{g}_{-4}(h) = \langle\partial_z\rangle$ of weight -4, $\mathfrak{g}_{-3}(h) = \langle x\partial_z\rangle$ of weight -3, $\mathfrak{g}_{-2}(h) = \langle\partial_y, x^2\partial_z, y\partial_z\rangle$ and $\mathfrak{g}_{-1}(h) = \langle\partial_x, x\partial_y, yx\partial_z, x^3\partial_z\rangle$ of weight -1. Consider the linear map $D : \mathfrak{g}_{<0}(h) \to \mathfrak{g}_{<0}(h)$ defined by

$$D(y\partial_z) = \partial_y, \ \ D(yx\partial_z) = x\partial_y$$

and vanishes on the rest of the basis. One can verify that D is a derivation. This is a derivation of weight 0 which vanishes on coordinate vector fields, so it cannot come from a vector field.

Example 2.6. Another example of a dilation h is the one on $\mathbb{R}^3 = \{(x_1, x_2, y)\}$ with x_i of degree 1 and y of degree 3. The derivation D of $\mathfrak{g}_{<0}(h)$ such that

$$D(x_1\partial_y) = \partial_{x_2} , \ D(x_2\partial_y) = \partial_{x_1} ,$$
$$D(x_1^2\partial_y) = 2x_1\partial_{x_2} , \ D(x_2^2\partial_y) = 2x_2\partial_{x_1} , \ D(x_1x_2\partial_y) = x_1\partial_{x_1} + x_2\partial_{x_2} ,$$

and it maps the rest of the homogeneous basis to 0, is of degree 1 and does not come from a vector field.

Example 2.7. Consider the dilation h as the one on $\mathbb{R}^3 = \{(x_1, x_2, y)\}$ with x_i of degree 1 and y of degree 2. The Lie algebra $\mathfrak{g}_{<0}(h)$ is the Heisenberg algebra and its derivation defined by

$$D(x_1\partial_y) = \partial_{x_1}\,,\ D(x_2\partial_y) = \partial_{x_2},$$

maps the rest of the homogeneous basis to 0, is of degree 0 and does not come from a vector field.

Proof of the Theorem. Now we will prove the 'if' part of Theorem 2.3.

Suppose that $\ell < 0$. We shall prove inductively with respect to the weight of the derivation that $\mathfrak{g}^T_k(\mathfrak{m})$ consists of vector fields only. Since the proof of the case of derivation of weight 0 is essentially the same as the proof of inductive step, we shall proceed with the derivation D_ι of weight ι and the assumption that all the derivations of weight less than ι are given by vector fields. Acting by D_ι on a vector field of negative weight, we obtain a derivation of weight less than ι, i.e. a vector field. We put then $D_\iota(\partial_u) = F^v_u\partial_v$. Since $[\partial_u, \partial_s] = 0$ and $D([\partial_u, \partial_s]) = [D(\partial_u), \partial_s] + [\partial_u, D(\partial_s)]$, we have

$$[F^v_u\partial_v, \partial_s] + [\partial_u, F^w_s\partial_w] = \left(\frac{\partial F^w_s}{\partial x_u} - \frac{\partial F^w_u}{\partial x_s}\right)\partial_w = 0\,.$$

This shows that the 1-forms $\omega^w = F^w_u \mathsf{d}x^u$ are closed on $\mathbb{R}^n$. Hence, there exist functions F^w such that $F^w_u = \partial F^w/\partial x_u$.

Now, if we put

$$Z = F^w\partial_w\,, \tag{9}$$

we see that $D_\iota(\partial_u) = [Z, \partial_u]$. Then, we consider $D'_\iota = D_\iota - \mathrm{ad}_Z$, which is a derivation of weight ι vanishing on coordinate vector fields. Now, it remains to prove that such D'_ι has to be 0. We shall proceed inductively with respect to the polynomial degree of vector field.

First, consider the linear vector fields $x_a\partial_b$. Since $[\partial_c, x_a\partial_b] = \delta_{ac}\partial_b$, we have

$$0 = D'_\iota([\partial_c, x_a\partial_b]) = [\partial_c, D'_\iota(x_a\partial_b)]\,.$$

By the inductive assumption, $D'_\iota(x_a\partial_b)$ is a vector field annihilated by any coordinate vector field, therefore it is also a coordinate vector field

$$D'_\iota(x_a\partial_b) = d^c_{ab}\partial_c\,.$$

Here, clearly $r_a < r_b$ and $r_c = r_b - r_a - \iota$.

Let us fix $x_a\partial_b$ such that $D'_\iota(x_a\partial_b) \neq 0$, e.g. $d^s_{ab} \neq 0$ for some s. We will show that $b = n$. As $r_s = r_b - r_a - \iota < r_b$ and $r_a < r_n$, we have $[x_a\partial_b, x_s\partial_n] = 0$ and

$$D'_\iota([x_a\partial_b,\, x_s\partial_n]) = [D'_\iota(x_a\partial_b),\, x_s\partial_n] + [x_a\partial_b,\, D'_\iota(x_s\partial_n)] = d^s_{ab}\partial_n - D'_\iota(x_s\partial_n)(x_a)\partial_b\,,$$

which is definitely not zero if $b \neq n$, therefore $D'_\iota(x_a\partial_b) \neq 0$ only for $b = n$. In particular this also means that there is only one coordinate of highest weight, i.e. $r_{n-1} < r_n$.

Let $i < n$ and put $D'_\iota(x_i\partial_n) = a^k_i\partial_k$. We can think of numbers a^k_i as matrix elements of a real $n \times n$ matrix A. As the weight of $D'_\iota(x_i\partial_n)$ is $r_i - r_n + \iota$ and greater than

$-r_n$, we have $a_i^n = 0$. It is also clear that $a_n^i = 0$, because we act only on vector fields of negative weight. Then, the matrix A has the form

$$A = \left[\begin{array}{ccc|c} * & \dots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & * & 0 \\ \hline 0 & \dots & 0 & 0 \end{array}\right].$$

By applying the derivative on the bracket $[x_i\partial_n, x_j\partial_n] = 0$, we get

$$a_i^j = a_j^i \quad \text{for all} \quad i, j < n\,, \tag{10}$$

which means that the matrix A is symmetric. Next, we will show that $D'_\iota(x_i\partial_n) = 0$ for $r_i < r_{n-1}$. For, suppose $r_i < r_{n-1}$ and consider the bracket $[x_i\partial_{n-1}, x_{n-1}\partial_n] = x_i\partial_n$. Applying the derivation, we get

$$[D'_\iota(x_i\partial_{n-1}), x_{n-1}\partial_n] + [x_i\partial_{n-1}, a_{n-1}^k\partial_k] = a_i^j\partial_j\,.$$

But $D'_\iota(x_i\partial_{n-1}) = 0$ and thus the left hand side equals

$$[x_i\partial_{n-1}, a_{n-1}^k\partial_k] = -a_{n-1}^i\partial_{n-1}\,.$$

Therefore, what we get is $-a_{n-1}^i\partial_{n-1} = a_i^j\partial_j$, and then combining this with (10) we get $-a_i^{n-1}\partial_{n-1} = a_i^j\partial_j$. This equation means that, first, $a_i^j = 0$ for $j < n-1$ and i such that $r_i < r_{n-1}$, and second, that for $j = n-1$ the coefficients $a_i^{n-1} = 0$ for i with weight lower than r_{n-1}. As a result $D'_\iota(x_i\partial_n) = 0$ for $r_i < r_{n-1}$. Moreover, by the symmetry condition (10) we have $a_i^j = a_j^i$, so $a_{n-1}^i = 0$ for $r_i < r_{n-1}$. The matrix A has now the form

$$A = \left[\begin{array}{ccc|c|c} 0 & \dots & 0 & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & * & 0 \\ \hline * & \dots & * & * & 0 \\ \hline 0 & \dots & 0 & 0 & 0 \end{array}\right].$$

The only nonzero column (and row) represents coefficients corresponding to possibly more than one coordinate with weights equal to r_{n-1}. This implies that the only possibility to have $D'_\iota(x_i\partial_n) = a_i^j\partial_j \neq 0$ is when $r_i = r_j = r_{n-1}$. Calculating the weight of both sides, we get

$$r_{n-1} - r_n + \iota = -r_{n-1}\,,$$

and so $r_n = 2r_{n-1} + \iota$. This contradicts our assumption that $\iota < \ell$. We conclude that the derivative D'_ι of weight $\iota < \ell$ vanishes on constant vector fields, and vanishes also on linear vector fields.

What we will show now is that, for $\iota < \ell$, if the derivative D'_ι of weight ι of $\mathfrak{m}$ vanishes on coordinate and linear vector fields, then it vanishes totally. First, we prove inductively that if D'_ι vanishes on vector fields of degree q, then it takes constant values for vector fields of degree $q+1$. Indeed, if g is a polynomial of degree $q+1$, then $[g\partial_i, \partial_j]$ is a vector field of degree q, so that

$$0 = D'_\iota([g\partial_i, \partial_j]) = [D'_\iota(g\partial_i), \partial_j]\,.$$

In consequence, $D'_\iota(g\partial_i)$ commutes with all coordinate vector fields, hence it is a constant vector field by the inductive assumption.

Now we take $f\partial_j$ with f a homogeneous polynomial which is at least quadratic and such that $D'_\iota(f\partial_j) \neq 0$. We have $D'_\iota(f\partial_j) = \partial_c$. Consider now $[f\partial_j, x_c\partial_j]$. Note that, since $r_c = -r_f + r_j - \iota$, we have $\mathsf{w}(x_c\partial_j) = r_c - r_j = -r_f - \iota$, so that $x_c\partial_j \in \mathfrak{m}$. We get $[f\partial_j, x_c\partial_j] = 0$, since $r_c < r_j$ and $r_j > r_f$. Applying D'_ι, we get

$$0 = D'_\iota([f\partial_j, x_c\partial_j]) = [D'_\iota(f\partial_j), x_c\partial_j] = [\partial_c, x_c\partial_j] = \partial_j\,,$$

which is a contradiction. Hence, any derivation of weight ι vanishing on constant vector fields vanishes globally, and the vector field (9) is the 'inner' derivation we are looking for. This finishes the proof. ■

Combining the arguments of the above proof and the proof of Lemma 2.4 we can easily derive the following.

Corollary 2.8. *The derivations of degree* 0 *of the nilpotent Lie algebra* $\mathfrak{g}_{<0}(h)$ *are just polynomial vector fields of weight* 0 *if* $\ell = r_n - 2r_{n-1} \neq 0$ *and* $\mathfrak{g}_0(h) \oplus \langle D_0^A \rangle$, *where the derivations* D_0^A *are described by* (2) *for the appropriate symmetric matrices* A, *if* $\ell = r_n - 2r_{n-1} = 0$.

3. Concluding remarks

We have found a necessary and sufficient condition for the signature $(r_1, \ldots, r_n)$ of a dilation h on $\mathbb{R}^n$ assuring that the Tanaka prolongation of the nilpotent Lie algebra $\mathfrak{g}_{<0}(h)$ of negative vector fields with respect to h is the Lie algebra

$$\bigoplus_{k=-\mathsf{w}(h)}^{\infty} \mathfrak{g}_k(h)$$

of polynomial vector fields. Surprisingly enough, for dilations with $\ell = r_n - 2r_{n-1} \geq 0$ there are some 'strange' derivations of degree ℓ which we described in detail. In particular, we gave a complete description of derivations of the Lie algebra $\mathfrak{g}_{<0}(h)$ of degree 0.

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