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Ten-Dimensional Lie Algebras with $\mathfrak{so}(3)$ Semi-Simple Factor. Erratum

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Abstract. In our recently published paper with the same title [J. Lie Theory 31/1 (2021) 93–118] we have classified Lie algebras of dimension ten that have a non-trivial Levi decomposition and semi-simple factor $\mathfrak{so}(3)$. In this note we make some corrections concerning the small class of algebras for which the semi-simple factor is six dimensional.

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1. Ten-dimensional Lie algebras with six-dimensional semi-simple factor

We recently published a paper in this journal [1] concerned primarily with the classification of non-trivial Levi decomposition algebras in dimension ten, and for which the semi-simple factor is $\mathfrak{so}(3)$. In addition to this class of Lie algebras, we also included non-trivial Levi decomposition algebras in dimension ten for which, the semi-simple factor is of dimension six. Unfortunately it is necessary to correct some errors relating to this latter class.

In the first place there appears in [1] the following Theorem:

Theorem 1.1. *Let a semi-simple Lie algebra S have a faithful representation in $End(N)$ for some vector space N. Then there is a Lie algebra $S \rtimes N$ that has a Levi decomposition with N being an abelian radical. Conversely, every Lie algebra that has a Levi decomposition with abelian radical arises in this way. Such a Lie algebra is decomposable if and only if the representation of S, being completely reducible, contains a trivial subrepresentation.*

Unfortunately the "only if" part of the Theorem is incorrect and we need to assume that S is simple rather than semi-simple; a counterexample is provided by $L_{10.3}$ in the listing given at the end of [1]. Clearly, on closer examination, this algebra is a direct sum of two copies of $A_{5.40}$, to use the numbering in [3]. Accordingly, this algebra has to be removed from the list.

We will explore the issue of the semi-simple factor N being isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ further. Amazingly, we find it difficult to provide a standard reference, so we shall give our own account. It is important to appreciate from the outset that the

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classification of simple Lie algebras is performed, in the first instance, for *complex* simple Lie algebras. Some of the basic Theorems such as Schur's Lemma and Weyl's complete reducibility Theorem need to modified when dealing with real Lie algebras.

1.1. Subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$

Now we shall find all the subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ that are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, that is, all representations of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{sl}(4,\mathbb{R})$ up to conjugacy. Looking at the classification of $\mathfrak{sl}(2,\mathbb{C})$ in [2], we take the brackets in the form

$$[e_1,e_2]=2e_2,\quad [e_1,e_3]=-2e_3,\quad [e_2,e_3]=e_1. \tag{1}$$

From the classification of $\mathfrak{sl}(2,\mathbb{C})$ modules in [2], we may assume that e_1 is represented by a semi-simple matrix. Over $\mathbb{R}$ every 4×4 semi-simple matrix is equivalent under change of basis to one of the following three:

$$\begin{bmatrix} a & b & 0 & 0\\ -b & a & 0 & 0\\ 0 & 0 & d & c\\ 0 & 0 & -c & d\end{bmatrix}\qquad \begin{bmatrix} a & b & 0 & 0\\ -b & a & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & d\end{bmatrix}\qquad \begin{bmatrix} a & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & d\end{bmatrix}. \tag{2}$$

We may further assume that the matrix has trace zero since it is a commutator; see also the Lemma on page 25 in [2]. We take the first matrix in (2) with $d=-a$ and where $bc\neq 0$. For e_2 and e_3 it is convenient to write the three matrices concerned in the form of 2×2 blocks. Then we impose the three brackets in eqs.(1), considered as giving linear conditions on the components of e_2 and e_3. It is easy to see, first of all, that the diagonal blocks of e_2 and e_3 are zero. Furthermore, in order to have non-zero e_2 and e_3, we must have $a=1$ and $b=\pm c$, conditions, that come from the determinant of a matrix of coefficients being zero. Necessarily, e_2 and e_3 are strictly upper and lower block triangular, respectively, and if $b=c$ such blocks are "complex", that is commute with $J=\left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]$ whereas if $b=-c$, such blocks are "anti-complex", that is anti-commute with J. However, in both of these cases it is impossible to satisfy the third bracket in eq. (1). The conclusion is that it is impossible for the matrix spanning the Cartan subalgebra to be of the first type in (2).

In the second type of matrix in (2), we may assume that $b\neq 0$. Then, introducing an arbitrary 4×4 matrix for e_2, we impose the condition $[e_1,e_2]-2e_2=0$. If we solve for all the linear conditions on e_2, including $2a+c+d=0$, we find that it can only be of the form $\left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\\0&0&0&f\\0&0&0&0\end{smallmatrix}\right]$. Similarly, if we use the condition $[e_1,e_3]+2e_3=0$, we find that e_3 must be of the form $\left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&g&0\end{smallmatrix}\right]$. However, now the condition $[e_2,e_3]-e_1=0$ implies that $b=0$, contrary to the hypothesis.

Concerning the third type of matrix in (2), now that we have reduced to the case where e_1 has real eigenvalues, we can follow the standard procedure for classifying the irreducible modules of $\mathfrak{sl}(2,\mathbb{C})$ modules in [2]. Hence there are, up to conjugacy, three inequivalent representations of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{sl}(4,\mathbb{R})$:

$$\begin{bmatrix} s_1 & s_2 & 0 & 0\\ s_3 & -s_1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\qquad \begin{bmatrix} 2s_1 & 2s_2 & 0 & 0\\ s_3 & 0 & s_2 & 0\\ 0 & 2s_3 & -2s_1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\qquad \begin{bmatrix} s_1 & s_2 & 0 & 0\\ s_3 & -s_1 & 0 & 0\\ 0 & 0 & s_1 & s_2\\ 0 & 0 & s_3 & -s_1\end{bmatrix}. \tag{3}$$

1.2. Subalgebras of $\mathfrak{sl}(4,\mathbb{R})$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$

Now our goal is to obtain inequivalent representations of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{sl}(4,\mathbb{R})$. In the first type of matrix in (3), it is easy to see that the centralizer consists of matrices $\left[\begin{smallmatrix} kI & 0\\ 0 & B\end{smallmatrix}\right]$ where the matrix is written in 2×2 blocks, I being the identity and B arbitrary. This case easily leads to the block diagonal representation of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ and algebra $L_{10.3}$ in [1], which, as we have pointed out above, is decomposable and therefore should be removed from the list.

In the second type of matrix in (3),the centralizer is two-dimensional and so there can be no representation of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ in this case. Finally, in the third type of matrix in (3), the centralizer is of the form, again written in 2×2 blocks, $\left[\begin{smallmatrix} aI & bI\\ cI & dI\end{smallmatrix}\right]$. Now we do obtain a representation of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ as

$$\begin{bmatrix} s_1+s_4 & s_2 & s_5 & 0\\ s_3 & s_4-s_1 & 0 & s_5\\ s_6 & 0 & s_1-s_4 & s_2\\ 0 & s_6 & s_3 & -s_1-s_4\end{bmatrix}. \tag{4}$$

The representation (4) is more familiar in the guise of $\mathfrak{so}(2,2)$, as we shall now explain. We take $\mathfrak{so}(2,2)$ in the form

$$S=\begin{bmatrix} 0 & s_3+s_6 & s_1+s_4 & s_2+s_5\\ -s_3-s_6 & 0 & -s_2+s_5 & s_1-s_4\\ s_1+s_4 & -s_2+s_5 & 0 & -s_3+s_6\\ s_2+s_5 & s_1-s_4 & s_3-s_6 & 0\end{bmatrix}. \tag{5}$$

We have written $\mathfrak{so}(2,2)$ in the form (5) so as to make evident the direct sum decomposition. In fact s_1,s_2,s_3 correspond to the first factor of $\mathfrak{sl}(2,\mathbb{R})$ and s_4,s_5,s_6 to the second. If we conjugate S in eq. (5) by

$$U=\begin{bmatrix} 0 & 1 & 2 & 0\\ -2 & 0 & 0 & 1\\ 0 & 1 & -2 & 0\\ 2 & 0 & 0 & 1\end{bmatrix} \tag{6}$$

we find that

$$U^{-1}SU=\begin{bmatrix} -s_1+s_4 & \frac{s_2+s_3}{2} & s_6+s_5 & 0\\ -2\,s_3+2\,s_2 & s_1+s_4 & 0 & s_6+s_5\\ -s_6+s_5 & 0 & -s_1-s_4 & \frac{s_2+s_3}{2}\\ 0 & -s_6+s_5 & -2\,s_3+2\,s_2 & s_1-s_4\end{bmatrix}. \tag{7}$$

Finally, if we make the substitution

$$s_1=-t_1,\quad s_2=\frac{2t_2+t_3}{4},\quad s_3=\frac{2t_2-t_3}{4},\quad s_4=t_4,\quad s_5=\frac{t_5+t_6}{2},\quad s_6=\frac{t_5-t_6}{2}, \tag{8}$$

we obtain

$$\begin{bmatrix} t_1+t_4 & t_2 & t_5 & 0\\ t_3 & t_4-t_1 & 0 & t_5\\ t_6 & 0 & t_1-t_4 & t_2\\ 0 & t_6 & t_3 & -t_1-t_4\end{bmatrix} \tag{9}$$

making obvious the relationship between $\mathfrak{so}(2,2)$ and the representation of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ in (4). If we use (4) as the semi-simple factor of a Levi decomposition algebras of dimension ten with $\mathbb{R}^4$ as radical we obtain:

$$\begin{aligned}
&[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_7]=e_7,\ [e_1,e_8]=-e_8,\ [e_1,e_9]=e_9,\\
&[e_1,e_{10}]=-e_{10},[e_2,e_3]=e_1,[e_2,e_8]=e_7,[e_2,e_{10}]=e_9,[e_3,e_7]=e_8,[e_3,e_9]=e_{10},\\
&[e_4,e_5]=2e_5,\ [e_4,e_6]=-2e_6,\ [e_4,e_7]=e_7,\ [e_4,e_8]=e_8,\ [e_4,e_9]=-e_9,\\
&[e_4,e_{10}]=-e_{10},[e_5,e_6]=e_4,[e_5,e_9]=e_7,[e_5,e_{10}]=e_8,[e_6,e_7]=e_9,[e_6,e_8]=e_{10}.
\end{aligned} \tag{10}$$

We propose to use (10) as $L_{10.3}$ in our revised list in [1].

2. $\mathfrak{so}(3,1)$

By definition, we take the algebra $\mathfrak{so}(3,1)$ to be the following space of matrices:

$$\begin{bmatrix} 0 & -s_6 & -s_5 & s_1 \\ s_6 & 0 & s_4 & s_2 \\ s_5 & -s_4 & 0 & s_3 \\ s_1 & s_2 & s_3 & 0 \end{bmatrix}. \tag{11}$$

From eq.(11) the Lie brackets of $\mathfrak{so}(3,1)$ are

$$\begin{aligned}
&[e_1,e_2]=-e_6,\ [e_1,e_3]=-e_5,\ [e_1,e_5]=-e_3,\ [e_1,e_6]=-e_2,\\
&[e_2,e_3]=e_4,\ [e_2,e_4]=e_3,\ [e_2,e_6]=e_1,\ [e_3,e_4]=-e_2,\\
&[e_3,e_5]=e_1,\ [e_4,e_5]=e_6,\ [e_4,e_6]=-e_5,\ [e_5,e_6]=e_4.
\end{aligned} \tag{12}$$

We will find all faithful representations of $\mathfrak{so}(3,1)$ in $\mathfrak{gl}(4,\mathbb{R})$ up to conjugacy. Now $\mathfrak{so}(3,1)$ contains a subalgebra isomorphic to $\mathfrak{so}(3)$, spanned by e_4,e_5,e_6 in eq. (12). As such, there are two possibilities for $\mathfrak{so}(3)$. Either the representation of $\mathfrak{so}(3)$ is by the 3×3 or 4×4 irreducible representation. These representations were denoted by $\operatorname{ad}\mathfrak{so}(3)$ and R_4, respectively, in [1] and [6].

2.1. $\mathfrak{so}(3)$ factor 3×3 irreducible

Suppose the representation of $\mathfrak{so}(3)$ is by $\operatorname{ad}\mathfrak{so}(3)$. We take the $\mathfrak{so}(3)$ representation corresponding to s_4,s_5,s_6 in eq. (11). In other words it will be specified by the parameters s_4,s_5,s_6 in matrix (11). Now we proceed as follows. We take e_1 as an arbitrary 4×4 matrix. Then we define e_2 by $-[e_1,e_6]$ and e_3 by $-[e_1,e_5]$, respectively. Now we impose all the remaining Lie brackets in eqs.(12). In the end we find that all the conditions are satisfied by the following matrix:

$$S=\begin{bmatrix} 0 & -s_6 & -s_5 & as_1 \\ s_6 & 0 & s_4 & as_2 \\ s_5 & -s_4 & 0 & as_3 \\ \frac{1}{a}s_1 & \frac{1}{a}s_2 & \frac{1}{a}s_3 & 0 \end{bmatrix}. \tag{13}$$

Finally we conjugate the matrix S in eq.(13) by the matrix $P=\left[\begin{smallmatrix} aI_3 & 0 \\ 0 & 1\end{smallmatrix}\right]$ so as to obtain $P^{-1}SP$, which, in effect, reduces a to 1 in eq. (13). In conclusion, if the $\mathfrak{so}(3)$ factor is given by the 3×3 irreducible representation, then the associated representation of $\mathfrak{so}(3,1)$ in $\mathfrak{gl}(4,\mathbb{R})$ is just the standard one given by matrix (11).

The corresponding Levi decomposition algebras of dimension ten with $\mathbb{R}^4$ as radical was denoted by $L_{10.1}$ in [1].

2.2. $\mathfrak{so}(3)$ factor 4×4 irreducible

Now suppose the representation of $\mathfrak{so}(3)$ is by R_4. This time, instead of starting from the matrix corresponding to s_4, s_5, s_6 in eq.(11), we begin with

$$\begin{bmatrix} 0 & -s_5 & -s_6 & -s_4 \\ s_5 & 0 & -s_4 & s_6 \\ s_6 & s_4 & 0 & -s_5 \\ s_4 & -s_6 & s_5 & 0 \end{bmatrix}. \tag{14}$$

Actually, it is necessary to multiply the matrix (14) by an overall factor of $\frac{1}{2}$ so as to obtain precisely the Lie brackets in eq.(12). We proceed similarly as in subsection 2.1.1, taking e_1 as an arbitrary 4×4 matrix and then satisfying all the Lie brackets in eq.(12), so as to obtain:

$$\begin{bmatrix} s_1 & s_3 - s_5 & s_2 - s_6 & -s_4 \\ s_3 + s_5 & -s_1 & -s_4 & s_2 + s_6 \\ s_2 + s_6 & s_4 & -s_1 & -s_3 - s_5 \\ s_4 & s_2 - s_6 & -s_3 + s_5 & s_1 \end{bmatrix}. \tag{15}$$

If we put $s_1 = t_1$, $s_2 = \frac{t_3+t_6}{2}$, $s_3 = \frac{t_2+t_5}{2}$, $s_4 = -t_4$, $s_5 = \frac{t_5-t_2}{2}$, $s_6 = \frac{t_6-t_3}{2}$ we obtain the following matrix, which displays $\mathfrak{so}(3,1)$ as a subalgebra of $\mathfrak{sp}(4)$:

$$\begin{bmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & -t_1 & t_4 & t_6 \\ t_6 & -t_4 & -t_1 & -t_5 \\ -t_4 & t_3 & -t_2 & t_1 \end{bmatrix}. \tag{16}$$

The corresponding Levi decomposition algebras of dimension ten with $\mathbb{R}^4$ as radical is given by

$$\begin{aligned} &[e_1,e_2] = 2e_2,\ [e_1,e_3] = 2e_3,\ [e_1,e_5] = -2e_5,\ [e_1,e_6] = -2e_6,\ [e_1,e_7] = e_7, \\ &[e_1,e_8] = -e_8,\ [e_1,e_9] = -e_9,\ [e_1,e_{10}] = e_{10}, [e_2,e_4] = 2e_3,\ [e_2,e_5] = e_1, \\ &[e_2,e_6] = e_4,\ [e_2,e_8] = e_7,\ [e_2,e_9] = -e_{10},\ [e_3,e_4] = -2e_2,\ [e_3,e_5] = -e_4, \\ &[e_3,e_6] = e_1,\ [e_3,e_8] = e_{10},\ [e_3,e_9] = e_7,\ [e_4,e_5] = -2e_6,\ [e_4,e_6] = 2e_5, \\ &[e_4,e_7] = -e_{10},\ [e_4,e_8] = -e_9,\ [e_4,e_9] = e_8,\ [e_4,e_{10}] = e_7,\ [e_5,e_7] = e_8, \\ &[e_5,e_{10}] = -e_9,\ [e_6,e_7] = e_9,\ [e_6,e_{10}] = e_8. \end{aligned} \tag{17}$$

and we propose to name it $L_{10.1'}$. The semi-simple factor is $\mathfrak{so}(3,1)$, although its brackets do not appear in the standard basis.

3. Relationship to another paper of Turkowksi

We conclude this Erratum by making a few comments about our work [1] and another paper by Turkowski [4] at the behest of the referee. Turkowski's article appeared at the advent of string theory and was motivated by the desire to extend four-dimensional spacetime by adding some extra “compact” dimensions.

At the Lie algebra level, one is considering a Levi decomposition Lie algebra, which may or may not be a direct sum of its radical and semi-simple factor. Although the extra dimensions added are "compact", they are not necessarily "compact semi-simple"; however, these dimensions must be a direct sum of compact semi-simple and abelian Lie algebras. At the Lie group level, these abelian dimensions may be integrated to give tori. Turkowski is primarily concerned with spaces whose overall dimension is ten and to a lesser extent nine. Because of the relatively small overall dimension of the spaces involved, the extra compact dimensions may be constructed by using either a compact semi-simple factor and borrowing an abelian part of the radical, or else having a non-compact semi-simple factor and using an abelian radical of suitable dimension. Turkowski uses a compact semi-simple factor which consists of direct sums of copies of $\mathfrak{so}(3)$ and for non-compact semi-simple factor, a single copy of $\mathfrak{sl}(2,\mathbb{R})$.

In [4] Turkowski compiles three lists of Lie algebras. The first list (Table I) consists of ten-dimensional Levi decomposition Lie algebras which, however, are all decomposable and are direct sums of their semi-simple factors and radicals. In (Table II) Turkowski gives a list of ten-dimensional Levi decomposition Lie algebras that contain a compact subalgebra of dimension at least seven. In this case the Lie algebras concerned are *not* direct sums of their semi-simple factors and radicals; nonetheless, the majority of them are still decomposable Lie algebras. In fact, of the 30 Lie algebras listed in Table II, all but six are decomposable. Each of these six has either $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$ as semi-simple factor and seven-dimensional abelian radical. In the case of $\mathfrak{so}(3)$ there are two possible inequivalent R-representations and for $\mathfrak{sl}(2,\mathbb{R})$ there are four of them. We shall not explain the meaning of the R-representation here, since it is amply discussed in each of [1], [4] and [6]. Unfortunately, it is true that Turkowski's numbering for the two $\mathfrak{so}(3)$ Lie algebras does conflict with the list given in [1]. However, in [5] and [6], Turkowski used his "L" notation to document Levi decomposition Lie algebras that are indecomposable up to and including dimension nine. Our notation in [1] sought to extend his classification to dimension ten. Finally we will remark that the first three of our algebras in the revised list, $L_{10.1}$, $L_{10.1'}$, $L_{10.2}$ should be added to Table II in [4]. We thank the referee for suggesting that we look at [4] and allowing us to comment on its relationship to our work.

References

[1] N. M. P. S. K. Bandara, G. Thompson: *Ten-dimensional Lie algebras with $\mathfrak{so}(3)$ semi-simple factor*, J. Lie Theory 31/1 (2021) 93–118.

[2] J. Humphreys: *Lie Algebras and their Representations*, Springer, Berlin (1997).

[3] L. Snobl, P. Winternitz: *Classification and Identification of Lie algebras*, CRM Monograph Series 33, American Mathematical Society, Providence (2014).

[4] P. Turkowski: *Classification of multidimensional spacetimes*, J. Geometry Physics 4/2 (1987) 119–132.

[5] P. Turkowski: *Low-dimensional real Lie algebras*, J. Math. Phys. 29 (1988) 2139–2144.

[6] P. Turkowski: *Structure of real Lie algebras*, Linear Algebra Appl. 171 (1992) 197–212.

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