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Structure and Representations for the Electrical Lie Algebra of Type D_4

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Abstract. We prove the dimension conjecture for electrical Lie algebra $\mathfrak{e}_{D_4}$ of type D_4. Moreover, we present a new method to construct 3-step nilpotent Lie algebras and show that $\mathfrak{e}_{D_4}$ is isomorphic to the semidirect product of $\mathfrak{sl}_2$ with a 3-step nilpotent Lie algebra constructed from the colored complete bipartible graph $K_{2,2}$. Also, we classify all simple highest weight modules for $\mathfrak{e}_{D_4}$.

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1. Introduction

We denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{C}$ and $\mathbb{C}^*$ the sets of all integers, non-negative integers, positive integers, complex numbers, and nonzero complex numbers, respectively. All vector spaces and algebras in this paper are over $\mathbb{C}$. We denote by $U(\mathfrak{a})$ the universal enveloping algebra of the Lie algebra $\mathfrak{a}$ over $\mathbb{C}$. Also, we denote by $\delta_{i,j}$ the Kronecker delta.

Electrical Lie algebras were introduced by T. Lam and P. Pylyavskyy to the study of electrical networks consisting only of nodes and resistors. All of the electrical properties of an electrical network N are captured by the response matrix $L(N)$, and $L(N)$ is preserved by a number of combinatorial moves including series and parallel reductions, and star-triangle $(Y-\Delta)$ relation. The space of circular planar electrical networks were studied in [6] and [7]. In [7], two operations on electrical networks were studied: *adding a boundary spike* and *adding a boundary edge* to planar electrical networks. In [12], Lam and Pylyavskyy introduced the electrical Lie algebra $\mathfrak{e}_{A_{2n}}$ of type A_{2n}, and the nonnegative part of the corresponding Lie group acts on the electrical networks exactly as the above operations. The star-triangle transformation translates to electrical Serre relations. They showed that $\mathfrak{e}_{A_{2n}}$ is isomorphic to symplectic Lie algebra $\mathfrak{sp}_{2n}$. In the end of [12], Lam and Pylyavskyy suggested a generalization of electrical Lie algebras of all finite Dynkin types. They conjectured that the dimension of electrical Lie algebra of type X is equal to $|\Phi(X)^+|$, here $\Phi(X)^+$ is the set of positive roots for the Dynkin diagram of type X.

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Y. Su studied electrical Lie algebras of classical Dynkin type in [16]. He proved the dimension conjecture for electrical Lie algebra of classical type, and determined the structure of $\mathfrak{e}_{A_{2n+1}}, \mathfrak{e}_{B_n}, \mathfrak{e}_{C_{2n}}$. Indeed, he proved that

$$\mathfrak{e}_{A_{2n+1}} \cong \mathfrak{sp}_{2n+1},\ \mathfrak{e}_{B_n} \cong \mathfrak{e}_{A_n} \oplus \mathfrak{e}_{A_{n-1}} \text{ and } \mathfrak{e}_{C_{2n}} \cong \mathfrak{e}_{A_{2n}} \ltimes (V_\lambda \oplus V_{\mathbf{0}}),$$

where V_λ and $V_{\mathbf{0}}$ are irreducible representations with highest weight $\lambda=(1,1,0,...,0)$ and $\mathbf{0} = (0,\dots,0)$ respectively. For the exceptional types, Lam and Pylyavskyy showed that $\dim \mathfrak{e}_{G_2} = 6$ in [12] and we will prove that $\mathfrak{e}_{G_2}$ is isomorphic to direct sum of the centerless Schrödinger algebra and a 1-dimensional Lie algebra.

As noted in [16], representation theory for electrical Lie algebra plays an important role in both the structure theory of electrical Lie algebra and the study of circular planar electrical networks. Representation theory for $\mathfrak{e}_{A_{2n}}$ is well studied. $\mathfrak{e}_{A_{2n+1}}$ is also known as the rank n symplectic oscillator Lie algebra, Jacobi Lie algebra, whose universal enveloping algebra is an infinitesimal Hecke algebra (see [3, 4, 10, 11]). The Jacobi groups were used to described the “squeezed coherent states” of quantum optics [2]. In Number Theory, the Jacobi forms are the automorphic forms on the Jacobi group, and have close relationship with the modular forms (see [4, 10]). Recently, Liu and Zhao classify the category of weight modules for $\mathfrak{e}_{A_{2n+1}}$ in [13]. The representation theory for $\mathfrak{e}_{G_2}$, known as the Schrödinger algebra, is well studied, see [1, 5, 9] and references therein.

In this paper, we will determine the structure for electrical Lie algebra of type D_4 and classify its irreducible highest weight modules. The paper is organized as follows. In section 2, we collect some basic definitions and results. The structure of electrical Lie algebra of type D_4 is determined in Section 3 (see Theorem 3.5). Finally, we classify all irreducible highest weight modules for $\mathfrak{e}_{D_4}$ in Section 4 (see Theorem 4.7).

2. Preliminaries

In this section, we collect some basic definitions and results for our study.

Electrical Lie algebras. Recall that the definition of electrical Lie algebra of finite Dynkin type is given as follows.

Definition 2.1. Let X be a Dynkin diagram of finite type, and $A = (a_{ij})$ be the corresponding Cartan matrix. Define the *electrical Lie algebra of type* X, $\mathfrak{e}_X$, to be the Lie algebra generated by the e_i's where i runs over the vertex indices in X, modulo the relations

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = \begin{cases} 0, & \text{if } i \neq j, a_{ij} \neq -1, \\ -2e_i, & \text{if } i \neq j, a_{ij} = -1. \end{cases}$$

A spanning set of $\mathfrak{e}_X$ was given in [12].

Lemma 2.2. *$\mathfrak{e}_X$ has a spanning set indexed by positive roots of the type X root system.*

Indeed, if $\{\tilde{e}_\alpha = [\tilde{e}_{i_1}, [\tilde{e}_{i_2}, [\cdots, [\tilde{e}_{i_{l-1}}, \tilde{e}_{i_l}]]]] \mid \alpha \in \Phi(X)^+\}$ is a basis for the positive nilpotent Lie subalgebra of the simple Lie algebra of type X, then

$$\left\{ e_\alpha = [e_{i_1}, [e_{i_2}, [\cdots, [e_{i_{l-1}}, e_{i_l}]]]] \right\}$$

spans $\mathfrak{e}_X$. Lam and Pylyavskyy also conjectured that this spanning set is a basis for $\mathfrak{e}_X$. And the conjecture is true for type A_n, B_n, C_n and G_2, see [12, 16].
The following lemma gives a description of $\mathfrak{e}_{G_2}$.

Lemma 2.3. *We have $\mathfrak{e}_{G_2} \cong \mathfrak{sch} \oplus \mathbb{C}z$, where $\mathfrak{sch}$ is the centerless Schrödinger algebra with basis $\{h, e, f, p, q\}$ and nontrivial brackets*

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h,$$
$$[h,p] = p, \quad [h,q] = -q, \quad [e,q] = p, [f,p] = q.$$

Proof. The Cartan matrix of type G_2 is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. Hence, from the definition, $\mathfrak{e}_{G_2}$ is generated by e_1, e_2 with relations

$$[e_1, [e_1, e_2]] = -2e_1, \quad [e_2, [e_2, [e_2, [e_2, e_1]]]] = 0.$$

From [12], a basis for $\mathfrak{e}_{G_2}$ is $\{X_1 = e_1,\ X_2 = e_2,\ X_3 = [e_2, e_1],\ X_4 = [e_2, [e_2, e_1]],\ X_5 = [e_2, [e_2, [e_2, e_1]]],\ X_6 = [e_1, [e_2, [e_2, [e_2, e_1]]]]\}$.
From direct computations, we get the brackets between basis elements in $\mathfrak{e}_{G_2}$.

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$-X_3$	$2X_1$	$2X_3$	X_6	0
X_2	X_3	0	X_4	X_5	0	X_5
X_3	$-2X_1$	$-X_4$	0	$-X_6 + 2X_4$	X_5	$-X_6$
X_4	$-2X_3$	$-X_5$	$X_6 - 2X_4$	0	0	$-2X_5$
X_5	$-X_6$	0	$-X_5$	0	0	0
X_6	0	$-X_5$	X_6	$2X_5$	0	0

Now it is easy to check that the linear map from $\mathfrak{sch} \oplus \mathbb{C}z$ to $\mathfrak{e}_{G_2}$ defined by

$$h \mapsto X_3,\ e \mapsto \frac{1}{3}X_6 - X_4,\ f \mapsto \frac{1}{2}X_1,\ p \mapsto 2X_5,\ q \mapsto X_6,\ z \mapsto 2X_2 + X_4 - X_6$$

is an isomorphism of Lie algebras. ∎

Derivations and semidirect product of Lie algebras. To determine the structure of $\mathfrak{e}_{D_4}$, we need some results on derivations and semidirect product for Lie algebras.

Definition 2.4. A linear map σ is called a *derivation* on a Lie algebra $\mathfrak{g}$ if

$$\sigma([a,b]) = [\sigma(a), b] + [a, \sigma(b)],\ \forall a, b \in \mathfrak{g}.$$

It is well known that the set of all derivations $\mathrm{Der}\mathfrak{g}$ on a Lie algebra $\mathfrak{g}$ forms a Lie algebra under $[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1$. We have the following proposition which can be checked directly.

Proposition 2.5. *Let* $\mathfrak{g}_1$ *be a Lie algebra and* $\mathfrak{g}_2$ *is a Lie subalgebra of* $\mathrm{Der}\mathfrak{g}_1$*, then the vector space* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *is a Lie algebra, denoted by* $\mathfrak{g}_2 \ltimes \mathfrak{g}_1$*, under the following brackets*

$$[x, y]_{\ltimes} = \begin{cases} [x, y]_{\mathfrak{g}_i}, \textit{ if } x, y \in \mathfrak{g}_i, i = 1, 2, \\ x(y), \textit{ if } x \in \mathfrak{g}_2, y \in \mathfrak{g}_1. \end{cases}$$

Nilpotent Lie algebras and colored diagrams. We also need some results for nilpotent Lie algebras and colored diagrams. Studying nilpotent Lie algebras with simple graphs was started by S. G. Dani and M. G. Mainkar in 2004 [8]. They presented a method for constructing two-step nilpotent Lie algebras. Later on, many mathematicians generalized their work (see [14, 15] and references therein).

For a Lie algebra $\mathfrak{g}$ and $n > 0$, let $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}]$. A Lie algebra is called n-step nilpotent if $\mathfrak{g}^n = 0$. Recall a 2-step nilpotent Lie algebra constructed form a colored diagram is as follows.

Definition 2.6. Let $\Gamma = (V, E)$ be a simple graph with edge coloring $c : E \to S$. Let $V = \{v_i\}_{i=1}^q$ and $S = \{z_k\}_{k=1}^p$, and $\mathfrak{v}$ and $\mathfrak{z}$ be the vector spaces of $\mathbb{C}$-linear combinations of V and S, respectively. We define

$$[v_i, v_j] = \sum_{k=1}^{p} \alpha_{ij}^k z_k, \quad [z_i, z_j] = [z_i, v_k] = 0,$$

where
$$\alpha_{ij}^k = \begin{cases} 1, & e_{ij} = (v_i, v_j) \in E \text{ and } c(e_{ij}) = z_k, \\ -1, & e_{ji} = (v_j, v_i) \in E \text{ and } c(e_{ji}) = z_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathfrak{v} \oplus \mathfrak{z}$ is a 2-step nilpotent Lie algebra, denoted by $\mathfrak{g}(\Gamma, c)$.

Based on this construction, we can construct a class of 3-step nilpotent Lie algebras. Let Γ and c be as above and $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}(\Gamma, c)$ be the vector space $\mathfrak{g}(\Gamma, c) \oplus \mathbb{C}v^+ \oplus \mathbb{C}v^-$. Define $[v^+, \widehat{\mathfrak{g}}] = [v^-, \widehat{\mathfrak{g}}] = 0$ and

$$[v_i, z_j] = \begin{cases} v^+, & \text{if there is only one edge } e \text{ colored with } z_j, \text{ and } s(e) = v_i, \\ v^-, & \text{if there is only one edge } e \text{ colored with } z_j, \text{ and } t(e) = v_i, \\ 0, & \text{otherwise,} \end{cases}$$

where $s(e)$ and $t(e)$ means the source point and target point of e respectively. Then we have

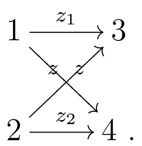
Proposition 2.7. $\widehat{\mathfrak{g}}(\Gamma, c)$ *is a* 3*-step nilpotent Lie algebra.*

Proof. The proposition follows from direct computations. ∎

3. Electrical Lie algebra of type D_4

In this section, we will determine the structure of electrical Lie algebra of type D_4.

The Lie algebras $\widehat{\mathfrak{g}}(K_{2,2}, c)$ **and** $\mathfrak{sl}_2 \ltimes \widehat{\mathfrak{g}}(K_{2,2}, c)$. First, we consider the 3-step nilpotent Lie algebra constructed from the colored complete bipartite graph $K_{2,2}$ with coloring map c as follows:



Set $v_1 = v_1, v_{-1} = v_3, w_1 = v_2, w_{-1} = w_4$, then $\widehat{\mathfrak{g}}(K_{2,2}, c)$ is a 9-dimensional 3-step nilpotent Lie algebra with basis $\{v_1, v_{-1}, w_1, w_{-1}, z_1, z_2, z, v^+, v^-\}$, the brackets are listed in Table 1.

Table 1: brackets in $\widehat{\mathfrak{g}}(K_{2,2}, c)$

$[x,y]$	v_1	w_1	v^+	v_{-1}	w_{-1}	v^-	z_1	z_2	z
v_1	0	0	0	z_1	z	0	v^+	0	0
w_1	0	0	0	z	z_2	0	0	v^+	0
v^+	0	0	0	0	0	0	0	0	0
v_{-1}	$-z_1$	$-z$	0	0	0	0	v^-	0	0
w_{-1}	$-z$	$-z_2$	0	0	0	0	0	v^-	0
v^-	0	0	0	0	0	0	0	0	0
z_1	$-v^+$	0	0	$-v^-$	0	0	0	0	0
z_2	0	$-v^+$	0	0	$-v^-$	0	0	0	0
z	0	0	0	0	0	0	0	0	0

Proposition 3.1 determines the derivations for $\widehat{\mathfrak{g}}(K_{2,2}, c)$.

Proposition 3.1. *Set $x_1 = v_1$, $x_2 = w_1$, $x_3 = v_{-1}$, $x_4 = w_{-1}$, $x_5 = z_1$, $x_6 = z_2$, $x_7 = z$, $x_8 = v^+$, $x_9 = v^-$, then with respect to this basis, the derivation Lie algebra of $\widehat{\mathfrak{g}}(K_{2,2}, c)$ is*

$$\mathrm{Der}\,\widehat{\mathfrak{g}}(K_{2,2}, c)) = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{13} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{31} & 0 & a_{33} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{61} & a_{53} & a_{63} & a_{11}+a_{33} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & 0 & a_{11}+a_{33} & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & 0 & 0 & a_{11}+a_{33} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{53} & a_{64} & a_{63} & 2a_{11}+a_{33} & a_{13} \\ a_{91} & a_{92} & a_{93} & a_{94} & -a_{51} & -a_{62} & -a_{61} & a_{31} & a_{11}+2a_{33} \end{pmatrix} \right\}.$$

In particular, $\mathfrak{sl}_2 \subseteq \mathrm{Der}\widehat{\mathfrak{g}}(K_{2,2}, c)$.

Proof. It is easy to check that any linear transformation defined by a matrix on the right hand side is a derivation on $\widehat{\mathfrak{g}}(K_{2,2}, c)$ and

$$\{E_{11} + E_{22} - E_{33} - E_{44} + E_{88} - E_{99},\ E_{13} + E_{24} + E_{89},\ E_{31} + E_{42} + E_{98}\}$$

span a subalgebra isomorphic to $\mathfrak{sl}_2$, where E_{ij} represents the matrix with 0 everywhere except a 1 on position (i, j). For any $\sigma \in \mathrm{Der}\widehat{\mathfrak{g}}(K_{2,2}, c)$, denote by σ the matrix with respect to the fixed basis. Since $\sigma(\widehat{\mathfrak{g}}(K_{2,2}, c)^i) \subseteq \widehat{\mathfrak{g}}(K_{2,2}, c)^i$ and

$$\widehat{\mathfrak{g}}(K_{2,2}, c)^1 = \mathrm{span}\{z_1, z_2, z, v^+, v^-\},\ \widehat{\mathfrak{g}}(K_{2,2}, c)^2 = \mathrm{span}\{v^+, v^-\},\ \widehat{\mathfrak{g}}(K_{2,2}, c)^3 = 0,$$

we may assume that

$$\sigma = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} \end{pmatrix}.$$

Denote by X_i the matrix of ad_{x_i}, then from $\sigma([x,y]) = [\sigma(x), y] + [x, \sigma(y)]$, we have

$$(I_9 \otimes \sigma)\begin{pmatrix} X_1 \\ \vdots \\ X_9 \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_9 \end{pmatrix}\sigma + (\sigma^T \otimes I_9)\begin{pmatrix} X_1 \\ \vdots \\ X_9 \end{pmatrix}.$$

Here I_9 is the 9×9 identity matrix and $\otimes$ is the Kronecker product of matrices. Solving this matrix equation, we get the required results. ■

Following from this proposition, we know $\mathfrak{sl}_2 \ltimes \widehat{\mathfrak{g}}(K_{2,2}, c)$ is a well defined Lie algebra with basis $\{e, h, f, v_1, v_{-1}, w_1, w_{-1}, z_1, z_2, z, v^+, v^-\}$ and brackets

Table 2: brackets in $sl_2 \ltimes \widehat{\mathfrak{g}}(K_{2,2}, c)$

$[x,y]$	h	e	f	v_1	w_1	v^+	v_{-1}	w_{-1}	v^-	z_1	z_2	z
h	0	$2e$	$-2f$	v_1	w_1	v^+	$-v_{-1}$	$-w_{-1}$	$-v^-$	0	0	0
e	$-2e$	0	h	0	0	0	v_1	w_1	v^+	0	0	0
f	$2f$	$-h$	0	v_{-1}	w_{-1}	v^-	0	0	0	0	0	0
v_1	$-v_1$	0	$-v_{-1}$	0	0	0	z_1	z	0	v^+	0	0
w_1	$-w_1$	0	$-w_{-1}$	0	0	0	z	z_2	0	0	v^+	0
v^+	$-v^+$	0	$-v^-$	0	0	0	0	0	0	0	0	0
v_{-1}	v_{-1}	$-v_1$	0	$-z_1$	$-z$	0	0	0	0	v^-	0	0
w_{-1}	w_{-1}	$-w_1$	0	$-z$	$-z_2$	0	0	0	0	0	v^-	0
v^-	$-v^-$	$-v^+$	0	0	0	0	0	0	0	0	0	0
z_1	0	0	0	$-v^+$	0	0	$-v^-$	0	0	0	0	0
z_2	0	0	0	0	$-v^+$	0	0	$-v^-$	0	0	0	0
z	0	0	0	0	0	0	0	0	0	0	0	0

Electrical Lie algebra of type D_4. Now let us determine the electrical Lie algebra of type D_4. The Cartan matrix of type D_4 is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},$$

hence $\mathfrak{e}_{D_4}$ is generated by e_1, e_2, e_3, e_4 modulo

$$[e_1, e_3] = [e_1, e_4] = [e_3, e_4] = 0,$$
$$[e_2, [e_2, e_i]] = -2e_2, \quad [e_i, [e_i, e_2]] = -2e_i, \ i = 1, 3, 4.$$

Following [12], $\mathfrak{e}_{D_4}$ is spanned by $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = [e_2, e_1]$, $x_6 = [e_2, e_3]$, $x_7 = [e_2, e_4]$, $x_8 = [e_3, [e_2, e_1]]$, $x_9 = [e_4, [e_2, e_1]]$, $x_{10} = [e_4, [e_2, e_3]]$, $x_{11} = [e_4, [e_3, [e_2, e_1]]]$, $x_{12} = [e_2, [e_4, [e_3, [e_2, e_1]]]]$, the brackets are

Table 3: brackets in $\mathfrak{e}_{D_4}$ I

$[x_i, x_j]$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	$-x_5$	0	0	$2x_1$	x_8
x_2	x_5	0	x_6	x_7	$-2x_2$	$-2x_2$
x_3	0	$-x_6$	0	0	x_8	$2x_3$
x_4	0	$-x_7$	0	0	x_9	x_{10}
x_5	$-2x_1$	$2x_2$	$-x_8$	$-x_9$	0	$-x_5 + x_6$
x_6	$-x_8$	$2x_2$	$-2x_3$	$-x_{10}$	$x_5 - x_6$	0
x_7	$-x_9$	$2x_2$	$-x_{10}$	$-2x_4$	$x_5 - x_7$	$x_6 - x_7$
x_8	0	$-x_5 - x_6$	0	$-x_{11}$	$2x_1 + x_8$	$2x_3 + x_8$
x_9	0	$-x_5 - x_7$	$-x_{11}$	0	$2x_1 + x_9$	$x_8 + x_{10} - x_{12}$
x_{10}	$-x_{11}$	$-x_6 - x_7$	0	0	$x_8 + x_9 - x_{12}$	$2x_3 + x_{10}$
x_{11}	0	$-x_{12}$	0	0	x_{11}	x_{11}
x_{12}	$-x_{11}$	0	$-x_{11}$	$-x_{11}$	$-x_{12}$	-x_{12}

Table 4: brackets in $\mathfrak{e}_{D_4}$ II

$[x_i, x_j]$	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
x_1	x_9	0	0	x_{11}	0	x_{11}
x_2	$-2x_2$	$x_5 + x_6$	$x_5 + x_7$	$x_6 + x_7$	x_{12}	0
x_3	x_{10}	0	x_{11}	0	0	x_{11}
x_4	$2x_4$	x_{11}	0	0	0	x_{11}
x_5	$-x_5 + x_7$	$-2x_1 - x_8$	$-2x_1 - x_9$	$-x_8 - x_9 + x_{12}$	$-x_{11}$	x_{12}
x_6	$-x_6 + x_7$	$-2x_3 - x_8$	$-x_8 - x_{10} + x_{12}$	$-2x_3 - x_{10}$	$-x_{11}$	x_{12}
x_7	0	$-x_9 - x_{10} + x_{12}$	$-2x_4 - x_9$	$-2x_4 - x_{10}$	$-x_{11}$	x_{12}
x_8	$x_9 + x_{10} - x_{12}$	0	0	0	0	$2x_{11}$
x_9	$2x_4 + x_9$	0	0	0	0	$2x_{11}$
x_{10}	$2x_4 + x_{10}$	0	0	0	0	$2x_{11}$
x_{11}	x_{11}	0	0	0	0	0
x_{12}	$-x_{12}$	$-2x_{11}$	$-2x_{11}$	$-2x_{11}$	0	0

Indeed, we claim that this is a basis for $\mathfrak{e}_{D_4}$, that is the dimension conjecture is true for $\mathfrak{e}_{D_4}$. To prove this, we first have

Lemma 3.2. *The subalgebra* $\mathfrak{b}$ *generated by* x_2 *and* $\frac{1}{3}(x_1+x_3+x_4)$ *is isomorphic to* $\mathfrak{e}_{G_2}$*. In particular,* $2x_1+2x_3+2x_4-x_8-x_9-x_{10}+x_{12}\neq 0$.

Proof. From

$$\begin{aligned}
&\left[x_2, \frac{1}{3}(x_1+x_3+x_4)\right] = \frac{1}{3}(x_5+x_6+x_7),\\
&\left[x_2, \left[x_2, \frac{1}{3}(x_1+x_3+x_4)\right]\right] = -2x_2,\\
&\left[\frac{1}{3}(x_1+x_3+x_4), \left[\frac{1}{3}(x_1+x_3+x_4), x_2\right]\right] = -\frac{2}{9}(x_1+x_3+x_4+x_8+x_9+x_{10}),\\
&\left[\frac{1}{3}(x_1+x_3+x_4), \left[\frac{1}{3}(x_1+x_3+x_4), \left[\frac{1}{3}(x_1+x_3+x_4), x_2\right]\right]\right] = -\frac{2}{9}x_{11},\\
&\left[\frac{1}{3}(x_1+x_3+x_4), \left[\frac{1}{3}(x_1+x_3+x_4), \left[\frac{1}{3}(x_1+x_3+x_4), \left[\frac{1}{3}(x_1+x_3+x_4), x_2\right]\right]\right]\right] = 0,
\end{aligned}$$

we know that x_2 and $\frac{1}{3}(x_1+x_3+x_4)$ satisfy the electrical Serre relations of type G_2. It is easy to check that

$$\frac{1}{3}(x_1+x_3+x_4)\mapsto \begin{pmatrix}1&1&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, x_2\mapsto \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

gives a faithful representation for $\mathfrak{b}$. Hence, $\dim\mathfrak{b}=6$ and $\mathfrak{g}\cong\mathfrak{e}_{G_2}$ via the map $x_2\mapsto X_1, \frac{1}{3}(x_1+x_3+x_4)\mapsto X_2$. Following from Lemma 2.3, $2X_2+X_4-X_6$ is a nonzero central element in $\mathfrak{e}_{G_2}$, so that

$$\begin{aligned}
&\frac{2}{3}(x_1+x_3+x_4)-\frac{2}{9}(x_1+x_3+x_4+x_8+x_9+x_{10})+\frac{2}{9}x_{12}\\
&=\frac{2}{9}(2x_1+2x_3+2x_4-x_8-x_9-x_{10}+x_{12})\neq 0.
\end{aligned}$$ ■

Now we can prove the dimension conjecture for $\mathfrak{e}_{D_4}$.

Theorem 3.3. $\dim\mathfrak{e}_{D_4}=12$.

Proof. It suffices to show that $\{x_i\}_{i=1}^{12}$ is linearly independent.
Suppose $\sum_{i=1}^{12}k_ix_i=0$. From $0=[x_1,[x_1,\sum_{i=1}^{12}k_ix_i]]=-2k_2x_1$, we have $k_2=0$.

Therefore,
$$0=[x_3,[x_1,\sum_{i=1}^{12}k_ix_i]]=k_7x_{11},$$

which implies that $k_7=0$. Similarly,

$$0=[x_4,[x_1,\sum_{i=1}^{12}k_ix_i]]=k_6x_{11}$$

implies that $k_6=0$. From

$$0=[x_1,\sum_{i=1}^{12}k_ix_i]=2k_5x_1+(k_{10}+k_{12})x_{11},$$

$$0=[x_5,[x_1,\sum_{i=1}^{12}k_ix_i]]=[x_5,2k_5x_1+(k_{10}+k_{12})x_{11}]=-4k_5x_1-(k_{10}+k_{12})x_{11},$$

we obtain $k_5=0$ and $k_{10}+k_{12}=0$. Thus,

$$k_1x_1+k_3x_3+k_4x_4+k_8x_8+k_9x_9+k_{10}x_{10}+k_{11}x_{11}-k_{10}x_{12}=0.$$

And hence

$$\begin{aligned}0&=[x_3,k_1x_1+k_3x_3+k_4x_4+k_8x_8+k_9x_9+k_{10}x_{10}+k_{11}x_{11}-k_{10}x_{12}]\\&=(k_9-k_{10})x_{11},\\0&=[x_4,k_1x_1+k_3x_3+k_4x_4+k_8x_8+k_9x_9+k_{10}x_{10}+k_{11}x_{11}-k_{10}x_{12}]\\&=(k_8-k_{10})x_{11},\end{aligned}$$

So, $k_8=k_9=k_{10}$, that is

$$k_1x_1+k_3x_3+k_4x_4+k_8(x_8+x_9+x_{10}-x_{12})+k_{11}x_{11}=0.$$

From

$$\begin{aligned}&[x_4,[x_1,[x_2,k_1x_1+k_3x_3+k_4x_4+k_8(x_8+x_9+x_{10}-x_{12})+k_{11}x_{11}]]]\\&=[x_4,[x_1,(k_1+2k_8)x_5+(k_3+2k_8)x_6+(k_4+2k_8)x_7+k_{11}x_{12}]]\\&=[x_4,-2(k_1+2k_8)x_1+(k_3+2k_8)x_8+(k_4+2k_8)x_9+k_{11}x_{11}]\\&=(k_3+2k_8)x_{11},\end{aligned}$$

we get: $k_3=-2k_8$. Similarly, $k_1=k_4=-2k_8$. Therefore,

$$-k_8(2x_1+2x_3+2x_4-x_8-x_9-x_{10}+x_{12})+k_{11}x_{11}=0.$$

And

$$0=[x_2,-k_8(2x_1+2x_3+2x_4-x_8-x_9-x_{10}+x_{12})+k_{11}x_{11}]=k_{11}x_{12},$$

implies $k_{11}=0$. With Lemma 3.2, we deduce that $k_8=0$. So, $k_i=0, i=1,\ldots,12$, that is $\{x_i\}_{i=1}^{12}$ is linearly independent. ∎

Remark 3.4. When revising the paper, suggested by the referee, we learned about the unpublished preprint [17] by Y. Su, where the dimension of $\mathfrak{e}_{D_n}$ was given.

Moreover, we can show that $\mathfrak{e}_{D_4}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2\ltimes\widehat{\mathfrak{g}}(K_{2,2},c)$ constructed above.

Theorem 3.5. $\mathfrak{e}_{D_4}\cong\mathfrak{sl}_2\ltimes\widehat{\mathfrak{g}}(K_{2,2},c)$.

Proof. Direct computation shows that $\rho : sl_2 \ltimes \widehat{\mathfrak{g}}(K_{2,2},c) \to \mathfrak{e}_{D_4}$ defined by:

$$\rho\begin{pmatrix} h \\ e \\ f \\ v_1 \\ w_1 \\ v_{-1} \\ w_{-1} \\ z_1 \\ z_2 \\ z \\ v^+ \\ v^- \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & 0 & -\omega^2 & -1 & -\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega & -1 & -\omega^2 & 0 & 0 & 0 & 0 & 0 \\ -2\omega^2 & 0 & -2 & -2\omega & 0 & 0 & 0 & \omega & 1 & \omega^2 & 0 & 0 \\ -2\omega & 0 & -2 & -2\omega^2 & 0 & 0 & 0 & \omega^2 & 1 & \omega & 0 & 0 \\ -6\omega & 0 & -6 & -6\omega^2 & 0 & 0 & 0 & -6\omega^2 & -6 & -6\omega & 0 & 0 \\ -6\omega^2 & 0 & -6 & -6\omega & 0 & 0 & 0 & -6\omega & -6 & -6\omega^2 & 0 & 0 \\ -6 & 0 & -6 & -6 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 54 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{pmatrix},$$

with $\omega = e^{\frac{2\pi i}{3}}$, is a Lie algebra homomorphism. It is a bijective map since

$$\det \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & 0 & -\omega^2 & -1 & -\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega & -1 & -\omega^2 & 0 & 0 & 0 & 0 & 0 \\ -2\omega^2 & 0 & -2 & -2\omega & 0 & 0 & 0 & \omega & 1 & \omega^2 & 0 & 0 \\ -2\omega & 0 & -2 & -2\omega^2 & 0 & 0 & 0 & \omega^2 & 1 & \omega & 0 & 0 \\ -6\omega & 0 & -6 & -6\omega^2 & 0 & 0 & 0 & -6\omega^2 & -6 & -6\omega & 0 & 0 \\ -6\omega^2 & 0 & -6 & -6\omega & 0 & 0 & 0 & -6\omega & -6 & -6\omega^2 & 0 & 0 \\ -6 & 0 & -6 & -6 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 54 & 0 \end{pmatrix}$$

$$= \begin{vmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -2\omega^2 & 0 & -2 & -2\omega & \omega & 1 & \omega^2 \\ -2\omega & 0 & -2 & -2\omega^2 & \omega^2 & 1 & \omega \\ -6\omega & 0 & -6 & -6\omega^2 & -6\omega^2 & -6 & -6\omega \\ -6\omega^2 & 0 & -6 & -6\omega & -6\omega & -6 & -6\omega^2 \\ -6 & 0 & -6 & -6 & 3 & 3 & 3 \end{vmatrix} \begin{vmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\omega^2 & -1 & -\omega \\ -\omega & -1 & -\omega^2 \end{vmatrix} \begin{vmatrix} 0 & 18 \\ 54 & 0 \end{vmatrix}$$

$$= 54^4(\omega - \omega^2) \neq 0.$$

■

4. Highest weight modules for $\mathfrak{e}_{D_4}$

In this section, we will classify all simple highest weight modules for $\mathfrak{e}_{D_4}$. First, $\mathfrak{e} = \mathfrak{e}_{D_4}$ has a triangular decomposition as follows:

$$\mathfrak{e}_{D_4} = \mathfrak{e}^+ \oplus \mathfrak{H} \oplus \mathfrak{e}^-,$$

where $\mathfrak{e}^+ = \operatorname{span}\{e, v_1, w_1, v^+\}$, $\mathfrak{H} = \operatorname{span}\{h, z_1, z_2, z\}$, $\mathfrak{e}^- = \operatorname{span}\{f, v_{-1}, w_{-1}, v^-\}$. Consider the quotient Lie algebra $\overline{\mathfrak{e}} = \mathfrak{e}_{D_4}/\operatorname{span}\{v^+, v^-\}$ with triangular decomposition $\overline{\mathfrak{e}} = \overline{\mathfrak{e}}^+ \oplus \overline{\mathfrak{H}} \oplus \overline{\mathfrak{e}}^-$, where

$$\overline{\mathfrak{e}}^+ = \operatorname{span}\{e, v_1, w_1\}, \quad \overline{\mathfrak{H}} = \operatorname{span}\{h, z_1, z_2, z\}, \quad \overline{\mathfrak{e}}^- = \operatorname{span}\{f, v_{-1}, w_{-1}\}.$$

Let $\mathfrak{g}$ be $\mathfrak{e}$ or $\overline{\mathfrak{e}}$. A $\mathfrak{g}$-module M is called a *weight module* if h acts diagonally. It is called a highest weight module if there is a vector $\mathbb{1}_\lambda$ such that $M = U(\mathfrak{g})\mathbb{1}_\lambda$ and $\mathfrak{g}^+\mathbb{1}_\lambda = 0, h\mathbb{1}_\lambda = \lambda\mathbb{1}_\lambda$. Clearly, as vector spaces, $M = U(\mathfrak{g}^{\leq 0})\mathbb{1}_\lambda$.

Let M be a simple highest weight $\mathfrak{e}$-module with highest weight vector $\mathbb{1}_\lambda$, then $v^-\mathbb{1}_\lambda = 0$, for otherwise $U(\mathfrak{e})v^-\mathbb{1}_\lambda$ is a nonzero proper submodule of M. Therefore,

$$v^+ \sum a_{ijk}(z_1, z_2, z) v_{-1}^i w_{-1}^j f^k \mathbb{1}_\lambda = -\sum k a_{ijk}(z_1, z_2, z) v_{-1}^i w_{-1}^j f^{k-1} v^- \mathbb{1}_\lambda = 0,$$

where $a_{ijk}(z_1, z_2, z) \in \mathbb{C}[z_1, z_2, z]$. So, v^+, v^- act trivially on simple highest weight $\mathfrak{e}$-modules. Thus, we have

Theorem 4.1. *The category of simple highest weight $\mathfrak{e}$-modules is equivalent to the category of simple highest weight $\bar{\mathfrak{e}}$-modules.*

So, we only need to consider simple highest weight $\bar{\mathfrak{e}}$-modules. For later discussion, we need the following relations in $U(\bar{\mathfrak{e}})$.

Lemma 4.2. *For $k \in \mathbb{Z}_+$, we have*

$$[v_1, v_{-1}^k] = k v_{-1}^{k-1} z_1,\ [v_1, f^k] = -k v_{-1} f^{k-1},\ [w_1, w_{-1}^k] = k w_{-1}^{k-1} z_2,$$
$$[w_1, f^k] = -k w_{-1} f^{k-1},\ [h, v_{-1}^k] = -k v_{-1}^k,\ [e, v_{-1}^k] = \frac{k(k-1)}{2} v_{-1}^{k-2} z_1 + k v_{-1}^{k-1} v_1,$$
$$[h, w_{-1}^k] = -k w_{-1}^k,\ [e, w_{-1}^k] = \frac{k(k-1)}{2} v_{-1}^{k-2} z_2 + k w_{-1}^{k-1} w_1.$$

Proof. The lemma follows from direct computations. ■

For $\lambda \in \mathbb{C}$, $\mathbb{c} = (c_1, c_2, c) \in \mathbb{C}^3$, define the Verma module over $\bar{\mathfrak{e}}$ as follows: Let $\mathbb{C}_{\lambda,\mathbb{c}} = \mathbb{C}\mathbb{1}_\lambda$ be the 1-dimensional $\bar{\mathfrak{e}}^+ \oplus \overline{\mathfrak{H}}$-module defined by

$$\bar{\mathfrak{e}}^+ \mathbb{1}_\lambda = 0,\ h\mathbb{1}_\lambda = \lambda \mathbb{1}_\lambda,\ z_1 \mathbb{1}_\lambda = c_1 \mathbb{1}_\lambda,\ z_2 \mathbb{1}_\lambda = c_2 \mathbb{1}_\lambda,\ z\mathbb{1}_\lambda = c\mathbb{1}_\lambda.$$

The Verma module $M(\lambda, \mathbb{c})$ is the induced module $\mathrm{Ind}_{\bar{\mathfrak{e}}^+ \oplus \overline{\mathfrak{H}}}^{\bar{\mathfrak{e}}} \mathbb{C}_{\lambda,\mathbb{c}}$. Since z_1, z_2, z are central elements in $\bar{\mathfrak{e}}$, we know by Schur's Lemma that they act on a simple $\bar{\mathfrak{e}}$-module by scalars c_1, c_2, c, we call such a module at level $\mathbb{c} = (c_1, c_2, c)$. Also, the Verma module $M(\lambda, \mathbb{c})$ has the following universal property: any simple highest weight module with highest weight λ at level $\mathbb{c}$ is a simple quotient of $M(\lambda, \mathbb{c})$. Thus, to classify simple highest weight modules for $\bar{\mathfrak{e}}$, it suffices to classify all simple quotients for $M(\lambda, \mathbb{c})$.

If M is a simple highest weight module with $c \neq 0$, consider the automorphisms of $\bar{\mathfrak{e}}$ as follows, we denote them by θ, $\theta|_{\mathfrak{sl}_2} = \mathrm{Id}$,

$$\theta(v_1) = \begin{cases} v_1 + w_1, & \text{if } c_1 = c_2 = 0, \\ c_2 v_1 - c w_1, & \text{if } c_2 \neq 0, \\ c_1 w_1 - c v_1, & \text{if } c_1 \neq 0, \end{cases} \qquad \theta(v_{-1}) = \begin{cases} v_{-1} + w_{-1}, & \text{if } c_1 = c_2 = 0, \\ c_2 v_{-1} - c w_{-1}, & \text{if } c_2 \neq 0, \\ c_1 w_{-1} - c v_{-1}, & \text{if } c_1 \neq 0, \end{cases}$$

$$\theta(w_1) = \begin{cases} v_1 - w_1, & \text{if } c_1 = c_2 = 0, \\ w_1, & \text{if } c_2 \neq 0, \\ v_1, & \text{if } c_1 \neq 0, \end{cases} \qquad \theta(w_{-1}) = \begin{cases} v_{-1} - w_{-1}, & \text{if } c_1 = c_2 = 0, \\ w_{-1}, & \text{if } c_2 \neq 0, \\ v_{-1}, & \text{if } c_1 \neq 0, \end{cases}$$

and

$$(\theta(z_1), \theta(z_2), \theta(z)) = \begin{cases} (z_1 + z_2 + 2z, z_1 + z_2 - 2z, z_1 - z_2), & \text{if } c_1 = c_2 = 0, \\ (c_2^2 z_1 + c^2 z_2 - 2c_2 c z, z_2, c_2 z - c z_2), & \text{if } c_2 \neq 0, \\ (c_1^2 z_2 + c^2 z_1 - 2c_1 c z, z_1, c_1 z - c z_1), & \text{if } c_1 \neq 0. \end{cases}$$

Define the module $M^\theta = M$ (as vector spaces) by $x \cdot v := \theta(x)v$, then on M^θ, we have $c = 0$. Denote by $\theta(\mathbb{c})$ the level of M^θ. Let $L(\lambda, \mathbb{c})$ be the simple highest weight module over $\bar{\mathfrak{e}}$ with highest weight λ at level $\mathbb{c} = (c_1, c_2, 0)$. Then we have the following theorem.

Theorem 4.3. *Let M be a simple highest weight module over $\bar{\mathfrak{e}}$ with highest weight λ at level $\mathbb{c}$. Then $M \cong L(\lambda, \mathbb{c})$ or $M^\theta \cong L(\lambda, \theta(\mathbb{c}))$.*

Thus, we only need to determine all simple highest weight modules $L(\lambda, \mathbb{c})$ over $\bar{\mathfrak{e}}$ with $\mathbb{c} = (c_1, c_2, 0)$, which are simple quotients of $M(\lambda, \mathbb{c})$. Moreover, the category of simple highest weight modules over $\bar{\mathfrak{e}}$ at level $(c_1, c_2, 0)$ is equivalent to the category of simple highest weight modules over $\bar{\mathfrak{e}}_1 := \bar{\mathfrak{e}}/\mathbb{C}z$ at level (c_1, c_2). Similar to the proof of Theorem 4.1, we know that when $c_1 = 0$ ($c_2 = 0$, respectively), v_1, v_{-1} (w_1, w_{-1}, respectively) act trivially on any simple highest weight $\bar{\mathfrak{e}}_1$-module. That is we have

Theorem 4.4. (1) *If $c_1 = c_2 = 0$, then the category of simple highest weight modules over $\bar{\mathfrak{e}}$ at level $(0, 0, 0)$ is equivalent to the category of simple highest weight modules over $\mathfrak{sl}_2$.*

(2) *If $c_1 = 0, c_2 = \mu \neq 0$ or $c_1 = \mu \neq 0, c_2 = 0$, then the category of simple highest weight modules over $\bar{\mathfrak{e}}$ at level $(c_1, c_2, 0)$ is equivalent to the category of simple highest weight modules over the Schrödinger algebra at level μ.*

Remark 4.5. (1) Theorem 4.4 follows from the fact that

$$\bar{\mathfrak{e}}_1/(\mathbb{C}v_1 + \mathbb{C}v_{-1} + \mathbb{C}z_1) \cong \bar{\mathfrak{e}}_1/(\mathbb{C}w_1 + \mathbb{C}w_{-1} + \mathbb{C}z_2)$$

is isomorphic to the Schrödinger algebra.

(2) All simple highest weight modules over the Schrödinger algebra were classified in [9].

It remains to classify all $L(\lambda, \mathbb{c})$ with $\mathbb{c} = (c_1, c_2, 0)$ and $c_1 c_2 \neq 0$, that is all simple quotients of $M(\lambda, \mathbb{c})$. Recall that a vector u in a module M is called a singular vector if $\bar{\mathfrak{e}}^+ u = 0$. Proposition 4.6 determines all singular vectors in $M(\lambda, \mathbb{c})$.

Proposition 4.6. *Let $\mathbb{c} = (c_1, c_2, 0), c_1 c_2 \neq 0$, and $\lambda \in \mathbb{C}$. Then the set of singular vectors in $M(\lambda, \mathbb{c})$ is*

$$S(\lambda, \mathbb{c}) = \begin{cases} \mathbb{C}\mathbb{1}_\lambda, & \text{if } \lambda \notin \mathbb{N} - 2, \\ \mathbb{C}(f + \frac{1}{2c_1}v_{-1}^2 + \frac{1}{2c_2}w_{-1}^2)^{\lambda+2}\mathbb{1}_\lambda + \mathbb{C}\mathbb{1}_\lambda, & \text{if } \lambda \in \mathbb{N} - 2. \end{cases}$$

Proof. Let $F = f + \frac{1}{2c_1}v_{-1}^2 + \frac{1}{2c_2}w_{-1}^2$. Then any vector u in $M(\lambda, \mathbb{c})$ can be written as

$$\sum_{i,j} a_{ij}(F) v_{-1}^i w_{-1}^j \mathbb{1}_\lambda = \sum_{i=0}^{n} b_i w_{-1}^i \mathbb{1}_\lambda = \sum_{j=0}^{m} b'_j v_{-1}^j \mathbb{1}_\lambda,$$

where $a_{ij}(F) \in \mathbb{C}[F]$, $b_i \in \mathbb{C}[F, v_{-1}]$, $b'_j \in \mathbb{C}[F, w_{-1}]$ and $b_n b'_m \neq 0$.

If u is a singular vector, then $v_1 u = 0$, that is $\sum_{j=0}^{m} j b'_j v_{-1}^j \mathbb{1}_\lambda = 0$. Hence, $m = 0$.

Similarly, we have $n=0$. So, $u=a(F)\mathbb{1}_\lambda$ with $a(F)=\sum_{k=0}^{r} a_kF^k\in\mathbb{C}[F]$.

Following from $0=eu=\sum_{k=0}^{r} ka_k(\lambda-k+2)F^{k-1}\mathbb{1}_\lambda$, we know that $a_k=0$ unless $k=\lambda+2$. And the proposition follows. ∎

Let $\mathbb{c}=(c_1,c_2,0)$ with $c_1c_2\neq 0$. Following from this proposition, we see that $M(\lambda,\mathbb{c})$ is simple if and only if $\lambda\notin\mathbb{N}-2$. And when $\lambda\in\mathbb{N}-2$, $M(\lambda,\mathbb{c})$ has a unique maximal submodule generated by $F^{\lambda+2}\mathbb{1}_\lambda$. Denote the unique simple quotient by $M_1(\lambda,\mathbb{c})$.

To summarize, we classify all simple highest weight modules for electrical Lie algebra of type D_4 as follows.

Theorem 4.7. *Let $\mathbb{c}=(c_1,c_2,c)\in\mathbb{C}^3$, $\lambda\in\mathbb{C}$ and M be a simple highest weight module over $\bar{\mathfrak{e}}$ with highest weight λ at level $\mathbb{c}$. Then M (M^θ) when $c=0$ $(c\neq 0)$ is isomorphic to*

(1) *a simple highest weight $\mathfrak{sl}_2$ module, if $c_1=c_2=0$ $(\theta(c_1)=\theta(c_2)=0)$;*

(2) *a simple highest weight module over the Schrödinger algebra, if $c_1c_2=0$, $c_1+c_2\neq 0$ $(\theta(c_1)\,\theta(c_2)=0,\ \theta(c_1)+\theta(c_2)\neq 0)$;*

(3) $M(\lambda,\mathbb{c})$ $\big(M(\lambda,\theta(\mathbb{c}))\big)$*, if $c_1c_2\neq 0,\ \lambda\notin\mathbb{N}-2$ $(\theta(c_1)\theta(c_2)\neq 0,\ \lambda\notin\mathbb{N}-2)$;*

(4) $M_1(\lambda,\mathbb{c})$ $\big(M_1(\lambda,\theta(\mathbb{c}))\big)$ *if $c_1c_2\neq 0,\ \lambda\in\mathbb{N}-2$ $(\theta(c_1)\theta(c_2)\neq 0,\ \lambda\in\mathbb{N}-2)$.*

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