Homological Finiteness of Representations of Almost Linear Nash Groups

Yixin Bao and Yangyang Chen

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Abstract.   Let G be an almost linear Nash group, namely, a Nash group that admits a Nash homomorphism with finite kernel to some $\mathrm{GL}_k(\mathbb{R})$. A smooth Fréchet representation V with moderate growth of G is called homologically finite if the Schwartz homology $\mathrm{H}_i^{\mathcal{S}}(G;V)$ is finite dimensional for every $i \in \mathbb{Z}$. We show that the space of Schwartz sections $\Gamma^{\varsigma}(X, \mathsf{E})$ of a tempered G-vector bundle (X, E) is homologically finite as a representation of G, under some mild assumptions.

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Key Words: Schwartz homology, tempered vector bundle, Schwartz sections, homological finiteness.

1.   Introduction

Let G be an almost linear Nash group, namely, a Nash group that admits a Nash homomorphism $G \to \mathrm{GL}_k(\mathbb{R})$ with finite kernel, for some $k \geq 0$. For details about the structure and basic properties of these groups, we refer the reader to [12]. Denote by $\mathcal{S}\mathrm{mod}_G$ the category of smooth Fréchet representations of G of moderate growth. Recall that a representation V of G is said to be of moderate growth, if for every continuous seminorm $|\cdot|_\mu$ on V, there is a positive Nash function f on G and a continuous seminorm $|\cdot|_\nu$ on V such that

$$|g.v|_\mu \leq f(g)|v|_\nu, \quad \text{for all} \quad g \in G, v \in V.$$

Here and as usual, we do not distinguish a representation with its underlying space. For every representation V in the category $\mathcal{S}\mathrm{mod}_G$, the ith ($i \in \mathbb{Z}$) Schwartz homology $\mathrm{H}_i^{\mathcal{S}}(G;V)$ of V, which is naturally a locally convex topological vector space, was defined and studied by the second author and Sun in [6]. It turns out that the Schwartz homologies have many nice features. On the one hand, they coincide with the smooth homologies as defined by Blanc and Wigner [3], see [6, Theorem 1.8]. This implies that the Schwartz homologies share the properties of the usual smooth homologies. On the other hand, the powerful tool, Shapiro's lemma also holds in the setting of Schwartz homology, now for Schwartz induced representations, which can be taken as a generalization of the usual compactly supported smooth induction, see [9, Lemma 3.1].

Theorem 1.1. ([6], Theorem 1.11) *Let H be a Nash subgroup of an almost linear Nash group G, and let V_0 be a representation in $\mathcal{S}\mathrm{mod}_H$. Then there is an identification*

$$\mathrm{H}_i^{\mathcal{S}}(G; \mathrm{ind}_H^G V_0) = \mathrm{H}_i^{\mathcal{S}}(H; V_0 \otimes \delta_{G/H})$$

of topological vector spaces, for every $i \in \mathbb{Z}$.

Here ind_H^G denotes the Schwartz induction as defined by du Cloux, see [7, Section 2]. δ_G denotes the modular character of the group G and $\delta_{G/H} := (\delta_G)|_H \cdot \delta_H^{-1}$. Note that in [6, Theorem 1.11], the Schwartz produced representation was involved, which is related to the Schwartz induced representation via a twist by the relative modular character, see [6, Proposition 6.6]. It should be pointed out that the Schwartz homology $\mathrm{H}_i^{\mathcal{S}}(G; V)$ as a topological vector space, is not necessary Hausdorff. For applications to representation theory, it is important to show that $\mathrm{H}_i^{\mathcal{S}}(G; V)$ is Hausdorff, at least in some cases we are interested in. In this article, we study the Hausdorffness property of the Schwartz homologies of representations of almost linear Nash groups.

Definition 1.2. A representation V in the category $\mathcal{S}\mathrm{mod}_G$ is said to be *homologically separated* if the Schwartz homology $\mathrm{H}_i^{\mathcal{S}}(G; V)$ is separated (Hausdorff) for every $i \in \mathbb{Z}$. V is said to be *homologically finite* if the Schwartz homology $\mathrm{H}_i^{\mathcal{S}}(G; V)$ is finite dimensional for every $i \in \mathbb{Z}$.

For the notion of homologically finite representations in the Lie algebra homology setting, see [2, Definition 3.1.2]. We study the Schwartz homologies $\mathrm{H}_\bullet^{\mathcal{S}}(G; V)$ of V at all degrees together instead of a single one. Moreover, the stronger finiteness property is much easier to deal with than the Hausdorffness property.

Proposition 1.3. *Every homologically finite representation is automatically homologically separated.*

This result is known to experts, see for example [4, Lemma 3.4] and [5, Proposition 6]. By [6, Theorem 7.7], every finite dimensional representation (with moderate growth) is homologically finite. In this article, we show that a special class of representations, which are constructed as Schwartz sections of tempered G-vector bundles, are homologically finite, thus also homologically separated.

Let X be a G-Nash manifold, namely, a Nash manifold that carries a Nash action $G \times X \to X$. Let E be a tempered G-vector bundle over X, namely, a tempered vector bundle over X together with a tempered bundle action $G \times \mathsf{E} \to \mathsf{E}$. Then the space of Schwartz sections $\Gamma^{\varsigma}(X, \mathsf{E})$ of the tempered vector bundle E over X carries a natural action of G, which as a representation of G, lies in the category $\mathcal{S}\mathrm{mod}_G$. We will recall the notions of tempered vector bundles and Schwartz sections in the next section.

For every $x \in X$, let G_x denote its stabilizer in G, and let E_x denote the fibre of E at x, which is a representation in $\mathcal{S}\mathrm{mod}_{G_x}$. Write

$$\mathrm{N}_x := \frac{\mathrm{T}_x(X)}{\mathrm{T}_x(G.x)} \otimes_{\mathbb{R}} \mathbb{C} \qquad (\mathrm{T}_x \text{ stands for the tangent space})$$

for the complexified normal space, and write

$$\mathrm{N}_x^* := \text{the dual space of } \mathrm{N}_x,$$

which is the complexified conormal space. They are both representations in $\mathcal{S}\mathrm{mod}_{G_x}$. Now we can state the main results of this article.

Theorem 1.4. *Let (X, E) be a tempered G-vector bundle such that G acts on X with finitely many orbits. Assume that for every $x \in X$, the following two conditions are satisfied:*

- $\mathsf{E}_x \otimes \mathrm{Sym}^k(\mathrm{N}_x^*) \otimes \delta_{G/G_x}$ *is homologically finite as a representation of G_x, for every integer $k \geq 0$,*

- *for every $i \in \mathbb{Z}$, $\mathrm{H}_i^{\mathcal{S}}(G_x; \mathsf{E}_x \otimes \mathrm{Sym}^k(\mathrm{N}_x^*) \otimes \delta_{G/G_x})$ vanishes for sufficiently large k,*

where Sym^k indicates the kth symmetric power. Then $\Gamma^\varsigma(X, \mathsf{E})$ is homologically finite as a representation of G.

Theorem 1.4 has the following consequence for finite rank vector bundles.

Theorem 1.5. *Let (X, E) be a tempered G-vector bundle such that G acts on X with finitely many orbits. Assume that all the fibres of E are finite dimensional and for every $x \in X$, the trivial representation of G_x does not occur as a subquotient of*

$$\mathsf{E}_x \otimes \mathrm{Sym}^k(\mathrm{N}_x^*) \otimes \delta_{G/G_x},$$

for sufficiently large k. Then $\Gamma^\varsigma(X, \mathsf{E})$ is homologically finite as a representation of G.

By Proposition 1.3 and Theorem 1.5 we know that the space of Schwartz sections $\Gamma^\varsigma(X, \mathsf{E})$ of a tempered G-vector bundle (X, E) is homologically separated, under some mild assumptions. Many important representations can be realized as Schwartz section space of a certain tempered G-vector bundle, see for example [6, Example 1.17]. Thus we have shown that many representations are homologically separated. This article is arranged as follows: In Section 2, we recall the notions of tempered vector bundles, tempered bundle maps and Schwartz sections of tempered vector bundles. Then in the last section, we prove the main results of this article. The main tools to prove Theorems 1.4 and 1.5 were already established by the second author and Sun in [6].

2. Tempered vector bundles and Schwartz sections

In this section, we recall briefly the notions of tempered vector bundles and Schwartz sections. For more details, we refer the reader to [6, Sections 2 and 6].

2.1. Tempered vector bundles

For basic knowledge on Nash manifolds and Nash maps, we refer the reader to [10, 11]. Let X be a Nash manifold and E_1 and E_2 be two Fréchet spaces. A map $\phi : X \times E_1 \to E_2$ is called a *linear family* if the map $\phi(x, \cdot) : E_1 \to E_2$ is linear for all $x \in X$.

Definition 2.1. Suppose that the Nash manifold X is affine. A linear family $\phi : X \times E_1 \to E_2$ is said to be of *moderate growth* if for every continuous seminorm $|\cdot|_2$ on E_2, there is a positive Nash function f on X and a continuous seminorm $|\cdot|_1$ on E_1 such that

$$|\phi(x, u)|_2 \leq f(x)|u|_1 \quad \text{for all } x \in X, u \in E_1.$$

Generally a linear family $\phi : X \times E_1 \to E_2$ is said to be of moderate growth if there is a finite covering $\{X_i\}_{i=1}^k$ $(k \geq 0)$ of X by affine open Nash submanifolds such that $\phi|_{X_i \times E_1}$ is of moderate growth for all $1 \leq i \leq k$.

Definition 2.2. Suppose that X is affine. A linear family $\phi : X \times E_1 \to E_2$ is said to be *tempered* if

- it is smooth as a map of infinite-dimensional manifolds; and

- for every Nash differential operator D on X, the linear family

$$D\phi : X \times E_1 \to E_2$$

  is of moderate growth.

For the notion of smooth maps between infinite-dimensional manifolds, we refer the reader to [8]. Generally a linear family $\phi : X \times E_1 \to E_2$ is said to be tempered if there is a finite covering $\{X_i\}_{i=1}^k$ $(k \geq 0)$ of X by affine open Nash submanifolds such that $\phi|_{X_i \times E_1}$ is tempered for all $1 \leq i \leq k$.

Definition 2.3. Let X_1, X_2 be Nash manifolds. A map $X_1 \times E_1 \to X_2 \times E_2$ is called a *tempered bundle map* if it has the form

$$(x, u) \mapsto (f(x), \phi(x, u)),$$

where $f : X_1 \to X_2$ is a Nash map, and $\phi : X_1 \times E_1 \to E_2$ is a tempered linear family.

Let X be a Nash manifold and let E be a Fréchet bundle over X, namely, a topological vector bundle over X such that all the fibres are Fréchet spaces. A *local chart* of E is defined to be a triple (U, E, ϕ), where U is an open Nash submanifold of X, E is a fibre of E, and $\phi : U \times E \to \mathsf{E}|_U$ is a topological isomorphism of vector bundles over U, where $\mathsf{E}|_U$ denotes the restriction of E to U.

Definition 2.4. A tempered structure on E is a subset $\mathcal{T}_\mathsf{E}$ of the set of all local charts of E with the following properties:

- every two elements (U_1, E_1, ϕ_1), (U_2, E_2, ϕ_2) in $\mathcal{T}_\mathsf{E}$ are compatible in the sense that the map $\phi_2^{-1} \circ \phi_1 : (U_1 \cap U_2) \times E_1 \to (U_1 \cap U_2) \times E_2$ and its inverse are both tempered bundle maps;

- for every local chart of E, if it is compatible with all elements of $\mathcal{T}_\mathsf{E}$, then it belongs to $\mathcal{T}_\mathsf{E}$;

- there exists a finite family $\{(U_i, E_i, \phi_i)\}_{i=1}^k$ $(k \geq 0)$ of elements of $\mathcal{T}_\mathsf{E}$ such that $\{U_i\}_{i=1}^k$ is a covering of X.

Definition 2.5. A tempered vector bundle is a triple $(X, \mathsf{E}, \mathcal{T}_\mathsf{E})$, where X is a Nash manifold, E is a Fréchet bundle over X and $\mathcal{T}_\mathsf{E}$ is a tempered structure on E.

When $\mathcal{T}_\mathsf{E}$ is understood, we simply call E a tempered vector bundle over X. For every tempered vector bundle $(X, \mathsf{E}, \mathcal{T}_\mathsf{E})$ and every Nash submanifold Z of X, $\mathsf{E}|_Z$ is obviously a tempered vector bundle over Z.

Definition 2.6. Let $(X_1, \mathsf{E}_1, \mathcal{T}_{\mathsf{E}_1})$ and $(X_2, \mathsf{E}_2, \mathcal{T}_{\mathsf{E}_2})$ be two tempered vector bundles. A map $f : \mathsf{E}_1 \to \mathsf{E}_2$ is called a *tempered bundle map* if there is a Nash map $f_0 : X_1 \to X_2$ such that the diagram

$$
\begin{array}{ccc}
\mathsf{E}_1 & \xrightarrow{\;f\;} & \mathsf{E}_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\;f_0\;} & X_2
\end{array}
$$

commutes, and for every $(U_1, E_1, \phi_1) \in \mathcal{T}_{\mathsf{E}_1}$ and $(U_2, E_2, \phi_2) \in \mathcal{T}_{\mathsf{E}_2}$ with $f_0(U_1) \subset U_2$, the map

$$\phi_2^{-1} \circ f \circ \phi_1 : U_1 \times E_1 \to U_2 \times E_2$$

is a tempered bundle map in the sense of Definition 2.3.

2.2. Schwartz sections

For the definition of Schwartz functions on Nash manifolds, we refer the reader to [1].

Let $(X, \mathsf{E}, \mathcal{T}_\mathsf{E})$ be a tempered vector bundle. Suppose that $\{(U_i, E_i, \phi_i)\}_{i=1}^k$ ($k \geq 0$) are elements of $\mathcal{T}_\mathsf{E}$ such that $\{U_i\}_{i=1}^k$ is a covering of X. Write $\Gamma^\varsigma(U_i, U_i \times E_i)$ for the space of the sections which correspond to Schwartz functions in $\mathcal{S}(U_i, E_i)$. This is obviously a Fréchet space. Define

$$\Gamma^\varsigma(U_i, \mathsf{E}|_{U_i}) := \{\phi_i \circ s \; : \; s \in \Gamma^\varsigma(U_i, U_i \times E_i)\},$$

which is also obviously a Fréchet space. Denote by $\Gamma(X, \mathsf{E})$ the space of continuous sections of the bundle E over X. Then extension by zero gives a continuous linear map

$$\bigoplus_{i=1}^k \Gamma^\varsigma(U_i, \mathsf{E}|_{U_i}) \to \Gamma(X, \mathsf{E}).$$

Definition 2.7. With the notation as above, the Schwartz section $\Gamma^\varsigma(X, \mathsf{E})$ of the tempered vector bundle E over the Nash manifold X is defined to be the image of the above map, equipped with the quotient topology of the domain.

The definition of $\Gamma^\varsigma(X, \mathsf{E})$ (as a topological vector space) does not depend on the choice of the local charts $\{(U_i, E_i, \phi_i)\}_{i=1}^k$, see [6, Proposition 6.2].

Let X be a G-Nash manifold, namely, a Nash manifold carrying a Nash action $G \times X \to X$. By a tempered G-vector bundle over X, we mean a tempered vector bundle E over X, together with an action $G \times \mathsf{E} \to \mathsf{E}$ which is a tempered bundle map. Here $G \times \mathsf{E}$ is obviously viewed as a tempered vector bundle over $G \times X$.

Proposition 2.8. ([6], Proposition 6.3) *Suppose that the Nash manifold X carries a Nash G-action. Let E be a tempered G-vector bundle over X. Then for every $g \in G$ and $\phi \in \Gamma^{\varsigma}(X, \mathsf{E})$,*

$$g.\phi : X \to \mathsf{E}, \quad x \mapsto g.(\phi(g^{-1}.x))$$

is a section in $\Gamma^{\varsigma}(X, \mathsf{E})$. Moreover, the Fréchet space $\Gamma^{\varsigma}(X, \mathsf{E})$ is a representation in $\mathcal{S}\mathrm{mod}_G$ under the action

$$(g, \phi) \mapsto g.\phi.$$

3.   Proof of the main results

We prove Theorem 1.4 by induction on the number of G-orbits in X. Firstly, assume that G acts on X transitively. By [6, Proposition 6.7],

$$\Gamma^{\varsigma}(X, \mathsf{E}) \cong \mathrm{ind}_{G_x}^G \mathsf{E}_x$$

as representations of G, for any $x \in X$. For every $i \in \mathbb{Z}$, by Theorem 1.1,

$$\mathrm{H}_i^{\mathcal{S}}(G; \Gamma^{\varsigma}(X, \mathsf{E})) \cong \mathrm{H}_i^{\mathcal{S}}(G; \mathrm{ind}_{G_x}^G \mathsf{E}_x) \cong \mathrm{H}_i^{\mathcal{S}}(G_x; \mathsf{E}_x \otimes \delta_{G/G_x}),$$

which is finite dimensional by the first assumption in Theorem 1.4. Thus $\Gamma^{\varsigma}(X, \mathsf{E})$ is homologically finite in this case.

Now assume that the number n of G-orbits in X is larger than 1 and Theorem 1.4 holds for any tempered G-vector bundle (X', E') (satisfying the conditions in Theorem 1.4) with the number of G-orbits in X' less than n. Let Z be a G-orbit in X with minimal dimension among all the orbits. Then by [12, Proposition 3.6], Z must be closed in X. Denote by $U := X \setminus Z$, which is a G-invariant open Nash submanifold of X. Clearly, $(U, \mathsf{E}|_U)$ is a tempered G-vector bundle satisfying the conditions in Theorem 1.4 and the number of G-orbits in U is less than n. By the induction assumption, $\Gamma^{\varsigma}(U, \mathsf{E}|_U)$ is homologically finite. By [1, Theorem 5.4.3], the extension by zero yields a closed linear embedding

$$\Gamma^{\varsigma}(U, \mathsf{E}|_U) \hookrightarrow \Gamma^{\varsigma}(X, \mathsf{E}),$$

and we identify $\Gamma^{\varsigma}(U, \mathsf{E}|_U)$ with its image in $\Gamma^{\varsigma}(X, \mathsf{E})$. Note that in [1, Theorem 5.4.3], the authors considered Nash vector bundles, but their result generalizes naturally to the case of tempered vector bundles. See also the remarks before [6, Proposition 6.4]. Define

$$\Gamma_Z^{\varsigma}(X, \mathsf{E}) := \Gamma^{\varsigma}(X, \mathsf{E}) / \Gamma^{\varsigma}(U, \mathsf{E}|_U).$$

Lemma 3.1.   *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence in the category $\mathcal{S}\mathrm{mod}_G$. If any two of V_1, V_2 and V_3 are homologically finite, then so is the third one.*

Proof.   By [6, Corollary 7.8], the short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ yields a long exact sequence

$$\cdots \to \mathrm{H}_{i+1}^{\mathcal{S}}(G; V_3) \to \mathrm{H}_i^{\mathcal{S}}(G; V_1) \to \mathrm{H}_i^{\mathcal{S}}(G; V_2) \to \mathrm{H}_i^{\mathcal{S}}(G; V_3) \to \cdots$$

of Schwartz homologies. Now the lemma follows directly from the definition of homological finiteness.   ∎

From the short exact sequence

$$0 \to \Gamma^\varsigma(U, \mathsf{E}|_U) \to \Gamma^\varsigma(X, \mathsf{E}) \to \Gamma^\varsigma_Z(X, \mathsf{E}) \to 0$$

and Lemma 3.1, to complete the proof it is enough to show that $\Gamma^\varsigma_Z(X, \mathsf{E})$ is homologically finite. For this quotient space, we have the following characterization.

Lemma 3.2. (Borel's Lemma) $\Gamma^\varsigma_Z(X, \mathsf{E})$ *has a natural countable decreasing filtration* $\{\Gamma^\varsigma_Z(X, \mathsf{E})_k\}_{k \geq 0}$ *by closed subspaces such that*

(a) *The natural map* $\Gamma^\varsigma_Z(X, \mathsf{E}) \to \varprojlim_k \Gamma^\varsigma_Z(X, \mathsf{E})/\Gamma^\varsigma_Z(X, \mathsf{E})_k$ *is a topological linear isomorphism.*

(b) *For all integers $k \geq 0$, there is naturally a topological linear isomorphism*

$$\Gamma^\varsigma_Z(X, \mathsf{E})_k/\Gamma^\varsigma_Z(X, \mathsf{E})_{k+1} \cong \Gamma^\varsigma(Z, \mathrm{Sym}^k(\mathrm{N}^*_Z(X)) \otimes \mathsf{E}|_Z),$$

where $\Gamma^\varsigma_Z(X, \mathsf{E})_0 = \Gamma^\varsigma_Z(X, \mathsf{E})$.

Here
$$\mathrm{N}_Z(X) := \bigsqcup_{x \in Z} \frac{\mathrm{T}_x(X)}{\mathrm{T}_x(Z)} \otimes_{\mathbb{R}} \mathbb{C}$$

denotes the complexified normal bundle of Z in X and $\mathrm{N}^*_Z(X)$ its dual bundle, which is called the complexified conormal bundle. In our situation, the closed subspaces $\Gamma^\varsigma_Z(X, \mathsf{E})_k$ are naturally G-invariant and the isomorphisms in Lemma 3.2 are G-equivariant. For a proof of Lemma 3.2, see for example [6, Propositions 8.2 and 8.3].

Lemma 3.3. $\Gamma^\varsigma_Z(X, \mathsf{E})/\Gamma^\varsigma_Z(X, \mathsf{E})_k$ *is homologically finite for every integer $k \geq 0$.*

Proof. By [6, Proposition 6.7] and Theorem 1.1,

$$\begin{aligned}
\mathrm{H}^{\mathcal{S}}_i(G; \Gamma^\varsigma(Z, \mathrm{Sym}^j(\mathrm{N}^*_Z(X)) \otimes \mathsf{E}|_Z)) &\cong \mathrm{H}^{\mathcal{S}}_i(G; \mathrm{ind}^G_{G_z}(\mathrm{Sym}^j(\mathrm{N}^*_z) \otimes \mathsf{E}_z)) \\
&\cong \mathrm{H}^{\mathcal{S}}_i(G_z; \mathrm{Sym}^j(\mathrm{N}^*_z) \otimes \mathsf{E}_z \otimes \delta_{G/G_z}),
\end{aligned}$$

which by the first assumption of Theorem 1.4, is finite dimensional for every $i \in \mathbb{Z}$ and $j \geq 0$, here $z \in Z$. Thus $\Gamma^\varsigma(Z, \mathrm{Sym}^j(\mathrm{N}^*_Z(X)) \otimes \mathsf{E}|_Z)$ is homologically finite, for every $j \geq 0$. By Lemma 3.2, we conclude that $\Gamma^\varsigma_Z(X, \mathsf{E})_j/\Gamma^\varsigma_Z(X, \mathsf{E})_{j+1}$ is homologically finite, for every $j \geq 0$. Now the lemma follows from Lemma 3.1. ∎

Lemma 3.4. *The natural map*

$$\mathrm{H}^{\mathcal{S}}_i(G; \Gamma^\varsigma_Z(X, \mathsf{E})) \to \varprojlim_k \mathrm{H}^{\mathcal{S}}_i(G; \Gamma^\varsigma_Z(X, \mathsf{E})/\Gamma^\varsigma_Z(X, \mathsf{E})_k)$$

is a linear isomorphism, for every $i \in \mathbb{Z}$.

Proof. By Lemma 3.3, $\mathrm{H}^{\mathcal{S}}_{i+1}(G; \Gamma^\varsigma_Z(X, \mathsf{E})/\Gamma^\varsigma_Z(X, \mathsf{E})_k)$ is finite dimensional, for every $k \geq 0$. Now the lemma follows directly from [6, Lemma 8.4]. ∎

We need to show that $\mathrm{H}^{\mathcal{S}}_i(G; \Gamma^\varsigma_Z(X, \mathsf{E}))$ is finite dimensional for every $i \in \mathbb{Z}$.

Lemma 3.5. *Let $i \in \mathbb{Z}$, the natural map*

$$\mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1}) \to \mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_k)$$

induced from the surjection

$$\Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1} \to \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_k$$

is a linear isomorphism for sufficiently large k.

Proof.   By [6, Corollary 7.8], the short exact sequence

$$0 \to \Gamma_Z^{\varsigma}(X, \mathsf{E})_k/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1} \to \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1} \to \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_k \to 0$$

in the category $\mathcal{S}\mathrm{mod}_G$ yields a long exact sequence

$$
\begin{aligned}
\to \quad & \mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})_k/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1}) \to \mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1}) \\
\to \quad & \mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_k) \to \mathrm{H}_{i-1}^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})_k/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1}) \to
\end{aligned}
$$

of Schwartz homologies. By Lemma 3.2 and Theorem 1.1,

$$
\begin{aligned}
\mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})_k/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1}) &\cong \mathrm{H}_i^{\mathcal{S}}(G; \Gamma^{\varsigma}(Z, \mathrm{Sym}^k(\mathrm{N}_Z^*(X)) \otimes \mathsf{E}|_Z)) \\
&\cong \mathrm{H}_i^{\mathcal{S}}(G; \mathrm{ind}_{G_z}^{G}(\mathrm{Sym}^k(\mathrm{N}_z^*) \otimes \mathsf{E}_z)) \\
&\cong \mathrm{H}_i^{\mathcal{S}}(G_z; \mathrm{Sym}^k(\mathrm{N}_z^*) \otimes \mathsf{E}_z \otimes \delta_{G/G_z}),
\end{aligned}
$$

which vanishes for sufficiently large k, by the second assumption in Theorem 1.4. Similarly, $\mathrm{H}_{i-1}^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})_k/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1})$ vanishes for sufficiently large k. Thus the natural map

$$\mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_{k+1}) \to \mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E})/\Gamma_Z^{\varsigma}(X, \mathsf{E})_k)$$

is a linear isomorphism for sufficiently large k.   ∎

By Lemmas 3.3, 3.4 and 3.5, we conclude that $\mathrm{H}_i^{\mathcal{S}}(G; \Gamma_Z^{\varsigma}(X, \mathsf{E}))$ is finite dimensional for every $i \in \mathbb{Z}$, i.e., $\Gamma_Z^{\varsigma}(X, \mathsf{E})$ is homologically finite, which completes the proof of Theroem 1.4.

Now we prove Theorem 1.5. Let $x \in X$. Since all the fibres of E are finite dimensional, the representation $\mathsf{E}_x \otimes \mathrm{Sym}^k(\mathrm{N}_x^*) \otimes \delta_{G/G_x}$ of G_x is finite dimensional, for every $k \geq 0$. By [6, Theorem 7.7], it must be homologically finite. Since the trivial representation of G_x does not occur as a subquotient of $\mathsf{E}_x \otimes \mathrm{Sym}^k(\mathrm{N}_x^*) \otimes \delta_{G/G_x}$ for sufficiently large k, it follows from [6, Proposition 7.11 and Corollary 7.8] that $\mathrm{H}_i^{\mathcal{S}}(G_x; \mathrm{Sym}^k(\mathrm{N}_x^*) \otimes \mathsf{E}_x \otimes \delta_{G/G_x})$ vanishes for every $i \in \mathbb{Z}$ and sufficiently large k. Now Theorem 1.5 follows from Theorem 1.4.

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Yixin Bao, School of Sciences, Harbin Institute of Technology, Shenzhen, China; mabaoyixin1984@163.com.

Yangyang Chen, School of Sciences, Jiangnan University, Wuxi, China; 8202007345@jiangnan.edu.cn.