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The Hilbert's Fifth Problem for Totally Intransitive Groupoids

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Abstract. We continue the study of the Hilbert's fifth problem for groupoids by giving results concerning the totally intransitive case. We start by constructing a counterexample to the problem in its most general form. We then continue by noting the key feature of this example to give a positive answer to the problem under the additional assumptions that among the Lie algebras of the automorphism groups there is at most a finite collection of pairwise non-isomorphic Lie algebras and the base is of dimension 1. On the way we reduce the problem (for arbitrary dimension of the base) to smoothing a continuous Lie algebra bundle derived from the groupoid.

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1. Introduction

The Hilbert's fifth problem states that a topological group which is a topological manifold is continuously isomorphic to a Lie group and was famously solved in [9]. In accordance with the trend of generalizing results about Lie groups to results about Lie groupoids and families of groups (most notable example of this is the long standing open problem of Lie's Third Theorem which was proven true for totally intransitive groupoids in [4] and was thoruoughly studied recently in [2]) we began the study of the Hilbert's fifth problem for groupoids by solving the transitive case [11, 12]. The key was providing the right setting for the problem by assuming that the base manifold is smooth as otherwise simple examples can be easily constructed. In this short paper we study the Hilbert's fifth problem for totally intransitive groupoids. We begin by giving an example that the problem even with the smoothness of base assumption in place does not have a possitive answer. However, the key feature of this example is that it has infinitely many isomorphism classes of Lie algebras among the Lie algebras of its automorphism groups. On the other hand, if there is only one isomorphism class then Theorem 3.3 gives a positive answer to the problem. This motivates us to eliminate the pathology found in the example by restricting our attention to groupoids which have only finitely many isomorphism classes of Lie algebras among its automorphism groups. Under this restriction we prove the main result of this paper:

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Theorem 1.1. *Let $\mathcal{G}$ be a topological totally intransitive groupoid such that the base space $\mathcal{G}_0$ is a manifold of dimension 1, the space of morphisms $\mathcal{G}_1$ and the automorphism group $\mathcal{G}_x^x$ of any point $x \in \mathcal{G}_0$ are topological manifolds and the source map is a topological submersion. Moreover, let us assume that among the Lie algebras of $\mathcal{G}_x^x$ one can find at most a finite collection of pairwise non-isomorphic Lie algebras. Then the groupoid $\mathcal{G}$ is continuously isomorphic to a Lie groupoid.*

Since the classical Hilbert's fifth problem can be considered as the special case of the Hilbert's fifth problem for totally intransitive groupoids (when the base is of dimension 0) this is a natural generalization of the classical problem. The first important step in the proof is reducing the problem to smoothing a certain topological Lie group bundle which is the content of Section 4. There we were able to conduct this reduction regardless of the dimension of the base which seems a necessary step in attempts to generalize this problem to higher dimensions. Section 5 is concerned with a technical result which will be used later on in Section 6 where we treat the 1-dimensional case. We also state our result in the languages of Lie algebra bundles and Lie group bundles for convenience and further use. Our findings here and in [11, 12] motivate the following conjecture:

Conjecture 1.2. Let $\mathcal{G}$ be a topological totally intransitive groupoid such that the base space $\mathcal{G}_0$ is a smooth manifold, the space of morphisms $\mathcal{G}_1$ and the automorphism group $\mathcal{G}_x^x$ of any point $x \in \mathcal{G}_0$ are topological manifolds and the source map is a topological submersion. Moreover, let us assume that among the Lie algebras of $\mathcal{G}_x^x$ one can find at most a finite collection of pairwise non-isomorphic Lie algebras. Then the groupoid $\mathcal{G}$ is continuously isomorphic to a Lie groupoid.

2. Preliminaries

2.1. Topological groupoids and Lie groupoids

Definition 2.1. A *groupoid* $\mathcal{G}$ is a small category in which all the morphisms are isomorphisms. Let us denote by $\mathcal{G}_0$ the set of objects of this category (also called the base of $\mathcal{G}$) and by $\mathcal{G}_1$ the set of morphisms of this category. This implies the existence of the following five structure maps:

1. the *source map* $s : \mathcal{G}_1 \to \mathcal{G}_0$ which associates to each morphism its source;
2. the *target map* $t : \mathcal{G}_1 \to \mathcal{G}_0$ which associates to each morphism its target;
3. the *identity map* $Id : \mathcal{G}_0 \to \mathcal{G}_1$ which associates to each object the identity over that object;
4. the *inverse map* $i : \mathcal{G}_1 \to \mathcal{G}_1$ which associates to each morphism its inverse;
5. the *multiplication (composition) map* $\circ : \mathcal{G}_2 \to \mathcal{G}_1$ which associates to each composable pair of morphisms its composition ($\mathcal{G}_2$ is the set of composable pairs).

A groupoid endowed with topologies on $\mathcal{G}_1$ and $\mathcal{G}_0$ which make all the structure maps continuous is called a *topological groupoid*. If additionally $\mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds, the source map is a surjective submersion and all the structure maps are smooth, then the topological groupoid is called a *Lie groupoid*.

Remark 2.2. Note that the identity map and the target map restricted to the image of the identity map are inverse to each other and so the identity map is an embedding. Hence, we can identify $\mathcal{G}_0$ with the image of the identity structure map.

Definition 2.3. A groupoid $\mathcal{G}$ is said to be *totally intransitive* if all the morphisms of $\mathcal{G}$ are automorphisms (i.e. $s(g) = t(g)$ for all $g \in \mathcal{G}$).

We denote by $\mathcal{G}_x^x$ the automorphism group of a point $x \in \mathcal{G}_1$. We also denote the image of $\mathcal{G}_0$ through the identity structure map by $Id_{\mathcal{G}}$.

2.2. Notions and results from the work of Douady and Lazard

We present the definitions and results from [4] which are used in the current work. Given a topological space Σ a pair (V, p) where V is a topological space and $p : V \to \Sigma$ is continuous is called a topological space over Σ.

Definition 2.4. Given a pair of integers $q \geq r \geq 0$ (or $q = \infty$ and $r \geq 0$, or $q = r = \infty$) and smooth manifolds Σ, X_1 and X_2 along with open sets $U_i \subset \Sigma \times X_i$ a map $f : U_1 \to U_2$ which commutes with the projections $p_i : \Sigma \times X_i \to \Sigma$ is said to be a *morphism of class* $\mathcal{C}^{r,q}$ if:

1. f is of class $\mathcal{C}^r$.
2. For all $s \in \Sigma$ the restriction $f_s : (\{s\} \times X_1) \cap U_1 \to (\{s\} \times X_2) \cap U_2$ is of class $\mathcal{C}^q$.
3. For all $k \leq q$ the k-th derivatives of f_s constitute a function of class $\mathcal{C}^r$ in (s, x).

We define a manifold of class $\mathcal{C}^{r,q}$ over Σ to be a topological space (V, p) over Σ together with an equivalence class of atlases of charts of class $\mathcal{C}^{r,q}$ (here by chart we understand a homeomorphism of an open set in V onto an open set of $\Sigma \times X$ which commutes with the projections). A product of manifolds over Σ is the fibered product of manifolds (and so in the remainder of this subsection we are going to denote it by $\times$).

Definition 2.5. A continuous (resp. smooth) Lie group bundle with base Σ is a *manifold* V over Σ of class $\mathcal{C}^0$ (resp. $\mathcal{C}^\infty$) together with group operations i.e. morphisms of class $\mathcal{C}^0$ (resp. $\mathcal{C}^\infty$) $(x, y) \to xy$ and $x \to x^{-1}$ which make each fiber into a Lie group.

Similarly:

Definition 2.6. A continuous (resp. smooth) Lie algebra bundle with base Σ is a *manifold* V over Σ of class $\mathcal{C}^0$ (resp. $\mathcal{C}^\infty$) which is a real vector bundle over Σ together with the Lie algebra operation i.e. a morphism of class $\mathcal{C}^0$ (resp. $\mathcal{C}^\infty$) $[\bullet, \bullet] : V \times V \to V$ which induces a Lie algebra structure on each fiber.

We now give a definition of a bouquet of Lie groups.

Definition 2.7. A *bouquet* of groups (over Σ) is given by the following:

- A family $\{G_s\}_{s\in\Sigma}$ of Lie groups.
- For each $s \in \Sigma$ a pair (T_s, D_s) where T_s is an open symmetric and generating neighbourhood of the identity in G_s and D_s is open in $T_s \times T_s$ such that for each $t \in T_s$ we have $(e,t); (t,e); (t,t^{-1}) \in D_s$ and the multiplication of two elements of D_s is in T_s.
- And a structure of a manifold over Σ on $T := \bigcup_{s\in\Sigma} T_s$.

Which satisfy the following additional conditions:

1. T_s is the fiber of T over s and the structures induced on T_s from T and G_s coincide.
2. $D := \bigcup_{s\in\Sigma} D_s$ is open in $T \times T$
3. The function $x \to x^{-1}$ is a morphism from T to T.
4. The function $(x,y) \to xy$ is a morphism from D to T.

Definition 2.8. Given a bouquet of groups, denote by L_x and M_x the free group T_x and the free monoid on two disjoint copies of T_x respectively. Both

$$L = \bigcup_{x\in\Sigma} L_x \text{ and } M = \bigcup_{x\in\Sigma} M_x$$

posses a canonical structure of a manifold over Σ (see [4]). Let $\phi : L \to G$ be the obvious map and let $\lambda : M \to L$ be the map constructed by treating one of the two copies of T_x generating M_x as the inverses of the other. We say that the bouquet of groups satisfies the *semi-continuity axiom* if for every $x \in \Sigma$ and every element $u \in Ker(\phi_x)$ there exists a neighbourhood U of x and a section A over U of the composition $\phi \circ \lambda$ such that $\lambda_x(A_x) = u$.

We are now ready to state two of the main results from [4] which provide the passage between the local Lie structure and global Lie structure:

Theorem 2.9. *A continuous (resp. smooth) Lie algebra bundles can be integrated to a continuous (resp. smooth) Lie group bundle with (possibly varying) simply connected fibers.*

Theorem 2.10. *A continuous (resp. smooth) bouquet of groups which satisfies the axiom of semicontinuity admits a unique structure of a continuous (resp. smooth) Lie group bundle. Moreover, this condition is necessary for a bouquet to admit a Lie group bundle structure.*

We are also going to state the following fact found at the beginning of Part 3 of [4].

Remark 2.11. Let M be a manifold over Σ of class $\mathcal{C}^{r,q}$ then the bundle tangent to the fibers of M is naturally a manifold of class $\mathcal{C}^{r,q-1}$. Moreover, given two sections X, Y of this bundle of class $\mathcal{C}^{r,q-1}$ their bracket $[X, Y]$ is also a section of this bundle of class $\mathcal{C}^{r,q-2}$. ■

2.3. Other preliminaries

There are two definitions of topological submersivity currently in use. We give both so that no confusion arises and distinguish them via the prefix "weak" (the implication justifying this prefix is obvious).

Definition 2.12. (cf. [6]) Let $f : X \to Y$ be a continuous map. Then f is a *topological submersion* if for every $x \in X$ there exists an open neighbourhood V of x and a subset $S_0 \subset f^{-1}(x)$ such that $f(V)$ is open and $V \cong f(V) \times S_0$ by a homeomorphism h over $f(V)$ (i.e. such that $f|_V = \pi_{f(V)} \circ h$).

Definition 2.13. (cf. [7]) Let $f : X \to Y$ be a continuous map. Then f is a *weak topological submersion* if for every $x \in X$ there exists an open neighbourhood V of $f(x)$ and a continuous function $s : V \to X$ such that $f \circ s = Id_V$ and $s(f(x)) = x$.

In this paper it is necessary to use topological submersions. We also wish to recall the following well known theorem (cf. [10]):

Theorem 2.14. *A principal bundle (P, B, G) with G a Lie group and B a smooth manifold is continuously isomorphic through a (base preserving isomorphism) to a unique (up to smooth isomorphism of smooth principal bundles) smooth principal bundle such that the initial smooth structure on B and the one induced from the smooth principal bundle coincide.*

We conclude this subsection by giving the definition of singular foliations along with a key property used in this paper (see. [1] for more information):

Definition 2.15. A *singular foliation* on M is a locally finitely generated submodule F of the $C^\infty(M)$-module of smooth vector fields on M which is closed under the Lie bracket. For such a module F there exists a partition of the manifold M into immersed disjoint submanifolds called leaves such that the tangent bundle to a leaf at x is equal to the vector spanned by the vector fields in the module F. We shall denote this partition by $\mathcal{F}$ and we shall identify it with the singular foliation F as defined above.

Theorem 2.16. *Let $(M, \mathcal{F})$, be a n-manifold with a foliation. Let $x \in M$, and set k to be the dimension of the leaf $\mathcal{F}_x$ passing through x. Moreover, let T be a submanifold of M passing through x, such that $\dim(T) + k = n$ and T is transverse to the foliation. Then there exists a neighbourhood U of x, such that:*

$$(U, \mathcal{F}|_U) \cong (T, \mathcal{F}|_T) \times ((0,1)^k, \mathcal{F}_1),$$

where $\mathcal{F}_1$ is the unique foliation on $(0,1)^k$ which has only one leaf, and the product foliation is given by the products of leaves in the product of manifolds. Henceforth, the leaves of $\mathcal{F}|_U$ will be called plaques.

3. Counterexample and motivation

The purpose of this section is to construct an example which shows the necessity of the assumption on the number of isomorphism classes of Lie algebras in Theorem 1.1.

Example 3.1. We start by taking a trivial real vector bundle of rank 3 over the real line. Let $\{x_1, x_2, x_3, t\}$ be its trivialization (the final coordinate t corresponds to the base real line). We endow it with the Lie bracket given by:

$$[x_3, x_1] = x_2 \quad [x_3, x_2] = f(t)x_1 + x_2 \quad [x_1, x_2] = 0,$$

where $f(t)$ is a composition of the Weierstrass nowhere differentiable function [5] with a homeomorphism of $\mathbb{R}$ onto an interval such that the Weierstrass function evaluated on this interval is strictly less than -1. Due to Theorem 2.9 we can integrate this to a continuous totally intransitive groupoid which is easily seen to satisfy the conditions of Theorem 1.1 except for the limitation on the number of different isomorphism classes of Lie algebras of automorphism groups (one can see due to the results from [3] that there are continuum different isomorphism classes of Lie algebras among the Lie algebras of automorphism groups of this groupoid). We will show that no choice of smooth atlas on the real line and no choice of smooth trivialization of this vector bundle make this into a smooth Lie algebra bundle. Let us assume to the contrary that this can be done. Then the subbundle spanned by x_1 and x_2 is also smooth as it is the image of the Lie brackets. This means that we can chose a homeomorphism A which restricted to any fiber gives a linear automorphism A_t of that fiber and passing from the trivialization given by $\{x_1, x_2, x_3\}$ to a smooth trivialization so that:

$$A_t x_1 = a_{11}(t)x_1 + a_{12}(t)x_2 \quad \text{and} \quad A_t x_2 = a_{21}(t)x_1 + a_{22}(t)x_2.$$

Moreover, since $[x_1, x_2] = 0$ we can without loss of generality assume that $A_t x_3 = a_{33}(t)x_3$ (since other summands don't take part in the computation). For the sake of clarity we drop the variable t from the notation. Now we need to compute some brackets of these smooth fields (in particular the resulting vector fields are also smooth):

$$[Ax_3, Ax_1] = a_{33}((a_{11} + a_{12})x_2 + a_{12}fx_1)$$

$$[Ax_3, a_{33}((a_{11}+a_{12})x_2+a_{12}fx_1)] = a_{33}^2((a_{11}+a_{12})x_2+a_{12}fx_1)+a_{33}^2 f(a_{11}x_1+a_{12}x_2)$$

One can easily show that since $f < -1$ and $a_{33} \neq 0$ we have linear independence of Ax_1 and $a_{33}((a_{11} + a_{12})x_2 + a_{12}fx_1)$ (via computation of the appropriate determinant). Thus via the smoothness of the second of the above fields we get that both a_{33} and $a_{33}^2 f$ are smooth functions. However, due to the fact that the Weierstrass function is nowhere monotone there cannot exist a smooth nowhere vanishing function h such that hf is smooth.

Remark 3.2. The argument above shows in particular that the Lie algebra bundle on the interval defined as above is not continously isomorphic to a smooth Lie algebra bundle and hence not all continuous Lie algebra bundles admit a smooth Lie algebra bundle structure. ∎

Moreover, this example can be easily modified so that the space of objects is compact by taking a closed interval $[a, b]$ such that the Weierstrass function evaluated on both ends is equal and it is strictly smaller then -1 on this interval. Glue it to a circle while gluing the Lie algebras over a and b via the identity. Note that this counterexample is of importance to the Hilberts fifth problem for groupoids with arbitrary dimensional base as it shows that the smoothness of the space of objects is not sufficient on its own in the general case (as oppose to the transitive case; cf. [12]). On the other hand when there is only one isomorphism class among the Lie algebras of automorphism groups the question can be answered easily (even for arbitrary dimension).

Theorem 3.3. *Let $\mathcal{G}$ be a totally intransitive topological groupoid such that the object space $\mathcal{G}_0$ is a smooth manifold, the automorphism groups $\mathcal{G}_x^x$ and the space of morphisms $\mathcal{G}_1$ are topological manifolds, and the source map $s : \mathcal{G}_1 \to \mathcal{G}_0$ is a topological submersion. Moreover, assume that the Lie algebra $\mathfrak{g}_x$ of the Lie group $\mathcal{G}_x^x$ is isomorphic to the Lie algebra of $\mathcal{G}_y^y$ for any objects $x, y \in \mathcal{G}_0$. Then $\mathcal{G}$ admits a Lie groupoid structure.*

Proof. We pass from the groupoid $\mathcal{G}$ to its Lie algebra bundle. Note that the assumption that all the algebras are isomorphic comes down to the fact that the structure group of the underlying (continuous) vector bundle can be reduced to $Aut(\mathfrak{g}_x)$. Using the fibered product construction and the equivalence of smooth principal bundles of Lie groups over smooth manifolds with continuous principal bundles of Lie groups over smooth manifolds (Theorem 2.14) we get a smooth bundle of Lie algebras. Via the exponential map this gives a structure of a local Lie groupoid on a neighbourhood of the identity manifold $\mathcal{G}_0$. Now, either using local admissible sections, or Theorem 2.10, one gets a Lie groupoid structure on the initial groupoid. ∎

These opposing results justify the naturality of the question "what happens when the groupoid has only finitely many isomorphism classes among the Lie alebras of its automorphism groups?". We note that the topological submersivity assumption seems necessary, since it is assumed that the target and source maps are submersive and this assumption does not follow from the rest of the definition.

4. Reducing the problem to smoothing families of Lie algebras

In this section we start the proof of Theorem 1.1 by using the results from [4] to reduce it to the study of continuous Lie algebra bundles. Let us take a groupoid $\mathcal{G}$ as in Theorem 1.1. Throughout the remainder of this paper we shall assume without loss of generality that the base $\mathcal{G}_0$ is connected (since each connected component along with the restriction of the groupoid structure to it forms a totally intransitive groupoid to which we can apply the theorem proven with this assumption). We first show the following technical proposition which makes topological submersions more reminiscent of smooth submersions:

Proposition 4.1. *Let M and N be topological manifolds of dimensions m and n and let $f : M \to N$ be a topological submersion such that the fibers are manifolds. Then there exists an open neighbourhood V of x and a subset $S_0 \subset f^{-1}(x)$ such that $f(V)$ is open, $V \cong f(V) \times S_0$ by a homeomorphism h over $f(V)$ and the set S_0 is homeomorphic to $\mathbb{R}^{m-n}$.*

Proof. Since V is open we have that $S_0 = V \cap f^{-1}(f(x))$ is open in $f^{-1}(f(x))$ and hence it is a manifold of the same dimension as the fiber of f over $f(x)$. This implies in particular that the dimension of this fiber is $m - n$ and all the fibers have the same dimension. We can reduce S_0 to S_0' so that $x \in f(V) \times S_0'$ and S_0' is homeomorphic to $\mathbb{R}^{m-n}$. By eliminating from V the fibers corresponding to the points excluded from S_0 we get the desired sets V' and S_0'. ■

We continue with the following lemma which constitutes a first step to reducing our problem:

Lemma 4.2. *Let $\mathcal{G}$ be as in Theorem* 1.1*. Then the Lie algebras of the automorphism groups form a continuous Lie algebra bundle $\mathcal{A}$ over $\mathcal{G}_0$.*

Proof. Since the source map is assumed to be a topological submersion we have the local product description needed in definition 2.4. Let us take a neighbourhood U of the identity submanifold in $\mathcal{G}_0$.

We show that U is a manifold of class $\mathcal{C}^{0,1}$. Due to the solution of Hilberts fifth problem each fiber is a local Lie group. Furthermore, the transition functions are on each fiber a continuous morphism of local Lie groups and hence are smooth (see e.g. [13]). This leaves us with proving that the transition functions satisfy the appropriate version of condition (3) in Definition 2.4. Note that it sufices to prove this property at identity and use the fact that Lie algebra homomorphisms preserve invariant vector fields. This is visible after treating the Lie algebra of the local group as 1-parameter subgroups and consequently identifying the derivative of the homomorphism as pushing the 1-parameter subgroups through the homomorphism. This shows (as indicated in Remark 2.11) that the Lie algebras form a continuous vector bundle $\mathcal{A}$.

By Remark 2.11 to prove that the bracket is continuous it suffices to prove that U is of class $\mathcal{C}^{0,2}$ (since the bracket in the Lie algebra comes from the Lie bracket of the local group). As before this comes down to proving that the second derivative is continuous. To see this note that the transition functions of the bundle tangent to the fibers are constant in the direction of fibers of U when written down with respect to a trivialization by invariant vector fields (again we use here that homorphisms of local Lie groups preserve invariant vector fields). Hence, the Lie algebras of automorphism groups form a continuous Lie algebra bundle $\mathcal{A}$ over $\mathcal{G}_0$. ■

Finally, we conclude this section with the result that allows us to reduce the Hilbert's fifth problem for totally intransitive groupoids to smoothing the bundle $\mathcal{A}$.

Theorem 4.3. *Let $\mathcal{G}$ be as in Conjecture* 1.2 *and let $\mathcal{A}$ be as above. Then the following conditions are equivalent:*

1. *The groupoid $\mathcal{G}$ is continuously isomorphic to a Lie groupoid.*
2. *The continuous Lie algebra bundle $\mathcal{A}$ can be given the structure of a smooth Lie algebra bundle.*

Proof. The implication from (1) to (2) is trivial since this is just taking the Lie algebroid of the totally intransitive Lie groupoid.

For the inverse implication after smoothing the Lie algebra bundle we use Theorem 2.9 and a neighbourhood of the identity in the resulting groupoid to get a smooth bundle of simply connected Lie groups. A neighbourhood of the identity submanifold gives a smooth family of local Lie groups which is continously isomorphic to a neighbourhood of the identity in $\mathcal{G}_1$ (this follows from the fact that Lie algebras and local Lie groups are in a one to one correspondence). Adding to this smooth family the family of Lie groups $\{\mathcal{G}_x^x\}_{x\in\mathcal{G}_0}$ we get a smooth bouquet of groups. This bouquet satisfies the axiom of semi-continuity since the corresponding topological bouquet satisfied it (which is evident since it comes from the groupoid $\mathcal{G}$). Hence, using Theorem 2.10 we assemble a new Lie groupoid. Due to the uniqueness in both the topological and smooth versions of Theorem 2.10 this groupoid is continuously isomorphic to the initial one. ■

Remark 4.4. We underline the fact that the results of this section are stated and proved for arbitrary dimension of the base space as we feel they are (possibly combined with the results of the subsequent section) of crucial importance to the Hilberts fifth problem for totally intransitive groupoids in general. ■

5. The space of Lie Algebras

This section is devoted to the study of certain structural properties of the subsets of the base of a continuous Lie algebra bundle $\mathcal{A} \to M$ over a manifold M corresponding to isomorphism classes of Lie algebras. We start with the following general remark:

Remark 5.1. Let G be a Lie group acting on a manifold M. Let Y be a subspace of M which is a sum of a finite number of orbits $Y_1, ..., Y_s$ of the action of G. The orbit of the point $x \in M$ is the image of an injective immersion from G/G_x where G_x is the isotropy group of x. Since $Y_1, ..., Y_s$ are immersed manifolds we can assume without loss of generality that if $i < j$ then $\dim(Y_i) \geq \dim(Y_j)$. Let l be the first index such that $\dim(Y_l) > \dim(Y_{l+1})$. Then $Y_1 \cup ... \cup Y_l$ is open in Y since this is precisely the part of Y on which the distribution given by the image of $\mathfrak{g}$ through the derivative of the Lie group action is of maximal rank. ■

Consider now for a given vector space $\mathfrak{g}$ of dimension n the space $(\wedge^2\mathfrak{g}^*) \otimes \mathfrak{g}$ of antisymmetric bilinear forms on $\mathfrak{g}$ with values in $\mathfrak{g}$. The group $Gl(n)$ acts on this space in the natural way (i.e. $A(\alpha \wedge \beta \otimes v) = \alpha(A^{-1}) \wedge \beta(A^{-1}) \otimes Av$) which can be thought of as changing the basis of $\mathfrak{g}$ in which we write the elements of $(\wedge^2\mathfrak{g}^*) \otimes \mathfrak{g}$.

Remark 5.2. It is crucial to note that two Lie algebra structures on the space $\mathfrak{g}$ are isomorphic if and only if the orbits (with respect to the above action) of their brackets coincide (to see this note that each isomorphism of Lie algebra structures is an element of $Gl(n)$ which transforms one of the above structures into the other). ■

These considerations allow us to prove the following theorem:

Theorem 5.3. *Let $\mathcal{A} \to M$ be a continuous bundle of Lie algebras over the manifold M such that among the fibers of this bundle one cannot choose an infinite collection of non-isomorphic Lie algebras. Let us denote the isomorphism classes of the Lie algebras occuring among the fibers of $\mathcal{A}$ by $\mathfrak{g}_1, ..., \mathfrak{g}_k$ and by X_i the set of all points in M such that the fibers over them are isomorphic to $\mathfrak{g}_i$. Then there exists $i \in \{1, ..., k\}$ such that X_i is open in M.*

Proof. It suffices to prove that the set X_i is open when intersected with some open neighbourhood U_x of any point $x \in M$ since then X_i is equal to the sum over $x \in M$, of the sets $X_i \cap U_x$. Choose a point $x \in M$ and let U_x be a neighbourhood homeomorphic to an open disc with a choosen trivialization of $\mathcal{A}$ as a vector bundle. This trivialization allows us to view the Lie algebra structure of the fibers as a continuous map $f : U_x \to (\wedge^2 \mathfrak{g}^*) \otimes \mathfrak{g}$. Using Remark 5.1 along with the continuity of f we see that $f^{-1}(Y_1 \cup ... \cup Y_l)$ is open (where Y is the collection of orbits corresponding to $\mathfrak{g}_1, ..., \mathfrak{g}_k$). Moreover, we can rearrange the collection $X_1, .., X_k$ using Remark 5.2 so that $X_i = f^{-1}(Y_i)$. Note that $f(U_x)$ is path-connected since it is an image of a path connected space.

We prove that there is no path connecting X_{i_0} with X_{i_1} for $i_0, i_1 \in \{1, ..., l\}$ and $i_0 \neq i_1$ such that it does not pass through any X_i with $i > l$. Let us assume to the contrary that such a path γ exists. Then $f \circ \gamma$ connects Y_{i_0} and Y_{i_1} without passing through any other orbit of the action. Denote $f(\gamma(0)) = y_0$ and let Δ be a closed disc in $Gl(n)$ which intersects the isotropy group G_{y_0} of y_0 transversely in a single point (the identity). By taking Δ small enough we can assume that $\Delta \cdot \gamma(t)$ is compact for any $t \in [0, \delta]$ (for δ small enough) and so the intersection of $f \circ \gamma$ and $\Delta \cdot f(\gamma(t))$ is compact. This implies that if for some open interval $(t' - \epsilon, t' + \epsilon) \subset [0, \delta]$ the set $f(\gamma((t' - \epsilon, t' + \epsilon)))$ is contained in $\gamma(t') \cdot \Delta$ then so is the image of the closure of this interval.

Hence, there exists a point (which we assume without loss of generality to be 0) such that arbitrarily close to it there exists a point $t_0 \in [0, 1]$ such that $f(\gamma(t_0)) \notin y_0 \cdot \Delta$ (as otherwise the image of the path $f \circ \gamma$ would be contained in a single orbit). We now note that smooth sections of the distribution tangent to the orbits form a singular foliation (since the image of the Lie algebra through the derivative of the group action is obviously closed under the Lie bracket). Using Theorem 2.16 we get a foliated neighbourhood $(U, \mathcal{F}|_U)$ of y_0 which is isomorphic to a product foliation and there is a point y' lying on the image of the path $f \circ \gamma$ through this diffeomorphism and not lying on the plaque of y_0. Let us restrict this path to $[0, \delta']$ so that it ends at y' without leaving U. Taking the projection of this restricted path onto the transversal T we get a path which passes only through plaques consisting of single points (treated as plaques of the foliation of T) and joining two distinct plaques.

Hence, it has to pass through a continuum of different plaques. By taking the preimage of these plaques in the manifold $\bigcup G/G_{p_i}$ (where $p_i \in Y_i$) we get a continuum of disjoint open submanifolds in this manifold. Which would give a submanifold not satysfying the Lindelöf property which gives the desired contradiction.

This shows that each X_i for $i \in \{1, ..., l\}$ is a sum of path-connected components of the manifold $X_1 \cup ... \cup X_l = f^{-1}(Y_1 \cup ... \cup Y_l)$. Since for manifolds path-connected components coincide with connected components this proves that each X_i for $i \in \{1, ..., l\}$ is in fact open. ■

6. Solution of the problem for 1 dimensional base

In this section we prove the following theorem which constitutes the final step in the proof of Theorem 1.1.

Theorem 6.1. *Every continuous Lie algebra bundle $\mathcal{A}$ over a 1-dimensional base $\mathcal{G}_0$ such that there are only finitely many isomorphism classes of Lie algebras among the fibers can be given a smooth structure so that it becomes a smooth Lie algebra bundle.*

Proof. We split the manifold $\mathcal{G}_0$ into subsets $X_1, ..., X_n$ corresponding to different isomorphism classes of the fibers of $\mathcal{A}$. Using Theorem 5.3 we establish that among these sets there is an open set (without loss of generality we assume that it is X_1). After this we apply Theorem 5.3 to the interior of the complement of X_1. In doing so we split this complement into 3-parts: the boundary of the complement which is closed with empty interior, the interior of (without loss of generality) X_2 and the complement of the sum of the previous sets.

We apply this procedure again to the interior of the third set and continue in this manner until the interior turns out to be empty. At the end we are left with a family of sets $V_j \subset X_j$ which are open in $\mathcal{G}_0$ and its complement Y which is a closed set which due to the construction has empty interior. This in particular implies that the sum of all V_j is dense in $\mathcal{G}_0$. Let us denote the connected components of $\bigcup V_j$ by U_i and let $\phi_i : U_i \to (0,1)$ be the coordinate charts on each U_i and let $\overline{\phi}_i : \overline{U}_i \to [0,1]$ denote the extension of ϕ_i to the boundary of U_i.

Applying now Theorem 3.3 to each of the U_i we find local trivialization which make $\mathcal{A}|_{U_i}$ into a smooth Lie algebra bundle. Since any topological vector bundle is isomorphic to a smooth one we can choose trivializations around the points of Y along with smooth local trivializations of each U_i so that the transition functions are smooth. Hence, the Lie algebra bundle $\mathcal{A}$ is smooth and the bracket is smooth in the sum of U_i.

Remark 6.2. At this point we assume without loss of generality that the bundle $\mathcal{A}$ is trivial as a vector bundle. We can do this via taking a direct sum of this bundle with an appropriate vector bundle Q endowed with the 0 bracket. If the bracket on $\mathcal{A} \oplus Q$ is smooth then so will be its restriction to $\mathcal{A}$. Let us fix the trivialization $(v_1, ...v_k)$ of $\mathcal{A}$ and define the functions f_{ijm} by the formula:

$$[v_i, v_j] = \sum_{m=1}^{k} f_{ijm} v_m.$$

These functions are smooth on each U_i. ∎

We are now going to change the smooth structure of $\mathcal{G}_0$ (without changing the diffeomorphism type of $\mathcal{G}_0$) so that $\mathcal{A}$ becomes a smooth Lie algebra bundle which is equivalent to the functions f_{ijm} being smooth and $\mathcal{A}$ remaining a smooth vector bundle. Let us fix an open subset U_{i_0} from the collection of U_i.

Let us focus on one side of the boundary of U_{i_0} treating its closed neighbourhood as the interval $[0, \epsilon]$. In this interval we can find a strictly decreasing function f_- and a strictly increasing function f_+ which are smooth on $(0, \epsilon]$ and satisfy for all $x_0 \in [0, \epsilon]$ and $i, j, m \in \{1, ..., k\}$ the conditions:

$$f_-(x_0) \leq f_{ijm}(x_0) - f_{ijm}(0) \leq f_+(x_0), \quad \lim_{x \to 0} f_+(x) = \lim_{x \to 0} f_-(x) = 0.$$

To construct f_+ (f_- is constructed analogously) we put $\lambda_s := \max(f_{ijm}(x) - f_{ijm}(0))$ where the maximum is taken over all $i, j, m \in \{1, ..., k\}$ and $x \in [0, \frac{\epsilon}{s}]$. We define the function f_+ to be $\lambda_1 + 1$ at ϵ and $f_+(\frac{\epsilon}{s+1}) = \lambda_s + \frac{1}{s+1}$ for natural numbers s. We now extend f_+ to the rest of $[0.\epsilon]$ so that it is monotone and its derivative is everywhere non-zero (this guaranties that its inverse is smooth). Since both λ_s and $\frac{1}{s+1}$ tend to 0 as s tends to infinity it is apparent that the limit of this function at 0 is 0.

Let us now take the function $f := f_+ - f_-$. Note that f is a strictly increasing continuous function on $[0, \epsilon]$ which is smooth on $(0, \epsilon]$ and satsfies for all $x_0 \in [0, \epsilon]$ and $i, j, m \in \{1, ..., k\}$ the conditions:

$$-f(x_0) \leq f_{ijm}(x_0) - f_{ijm}(0) \leq f(x_0), \quad \lim_{x \to 0} f(x) = 0.$$

It suffices to change the smooth structure so that f is flat at 0 (due to the above inequalities this would imply that all the f_{ijm} are flat at 0 in this structure). Due to monotonicity of f and the fact that its differential is non-vanishing on $(0, \epsilon]$ it has a well defined inverse f^{-1} which is smooth on $(0, \epsilon]$. Let us now take the homeomorphism $g_{i_0}^-$ defined by the formula

$$(g_{i_0}^-)^{-1} := e^{-\frac{1}{f^{-1}(x)}} : [0, \delta] \to [0, \epsilon].$$

This homeomorphism is in fact a diffeomorphism when restricted to $(0, \epsilon]$ onto $(0, \delta]$. It is apparent that by changing coordinates from ϕ_{i_0} to $g_{i_0}^-$ the function f becomes flat. A similar construction on the other end of U_{i_0} will give a similar function:

$$g_{i_0}^+ : [1 - \epsilon, 1] \to [1 - \delta, 1].$$

By taking small enough neighbourhoods of the end points of $\overline{U}_{i_0}$ we can smoothly extend these homeomorphisms to a homeomorphism:

$$g_{i_0} : \overline{U}_{i_0} \to [0, 1].$$

smooth in the interior. This in turn allows us to define the homeomorphism:

$$h_{i_0} := \overline{\phi}_{i_0}^{-1} \circ g_{i_0} : \overline{U}_{i_0} \to \overline{U}_{i_0}.$$

which changes the coordinates appropriately. Moreover, since it keeps the boundary of $\overline{U}_{i_0}$ unchanged we can extend it to a homeomorphism $\tilde{h}_{i_0}$ of $\mathcal{G}_0$ by specifying that $\tilde{h}_{i_0}|_{\mathcal{G}_0 \setminus U_{i_0}} = Id_{\mathcal{G}_0 \setminus U_{i_0}}$. We use this homeomorphism to change the smooth structure of $\mathcal{G}_0$. These homeomorphisms have a well defined composition since the areas they change intersect only on their boundaries which are preserved by all of them.

Remark 6.3. Note that the vector bundle $\mathcal{A}$ remains smooth throughout these operations since all these maps are smooth diffeomorphisms on the overlaps of the local trivilaizations. ■

Finally, we need to make all the derivatives of f_{ijm} continuous at points which do not belong to any $\overline{U}_i$ or points that are boundary points of some $\overline{U}_i$ only from one side. To do that we need to modify f_{ijm} on any U_i which is contained in some chosen small neighbourhood of such a point t_0. Note that since f_{ijm} are continuous the smaller neighbourhood of t_0 we choose the closer the values of the functions have to be to $f_{ijm}(t_0)$. In particular, if it is chosen small enough we can take the middle part (corresponding to $[\delta, 1-\delta]$) of each U_i contained in it and replace the functions f_{ijm} over that part by a nicer function joining the ends of the middle part (corresponding to δ and $1-\delta$). To be precise we consider the action of $GL(n)$ on $(\wedge^2\mathfrak{g}^*)\otimes\mathfrak{g}$, take the element $g\in GL(n)$ such that $g\{f_{ijm}(\delta)\}_{i,j,m\in\{1,\dots,n\}}=\{f_{ijm}(1-\delta)\}_{i,j,m\in\{1,\dots,n\}}$ and replace the middle part by functions $\tilde{f}_{ijm}$ satysfying $\{\tilde{f}_{ijm}(t)\}_{i,j,m\in\{1,\dots,n\}}=g_t\{f_{ijm}(\delta)\}_{i,j,m\in\{1,\dots,n\}}$, where g_t is the interval in $GL(n)$ connecting identity to g parametrized over $[\delta,1-\delta]$. After smoothing the corners these new functions will have the desired property (since as we tend to t_0 (and consequently the values of f_{ijm} tend to $f_{ijm}(t_0)$) the derivatives of these functions will tend to zero). ■

To prove Theorem 1.1 it suffices to apply this Theorem to the continuous Lie algebra bundle $\mathcal{A}$ arising from the groupoid $\mathcal{G}$ as described in Section 4. Note that for Lie group bundles the Theorem 1.1 can be stated as follows:

Corollary 6.4. *Let $\mathcal{G}$ be a continuous bundle of Lie groups such that the base $\mathcal{G}_0$ is a manifold of dimension 1, its total space is a topological manifold and for which the projection is a topological submersion. Moreover, let us assume that among the Lie algebras of the fibers one can find at most a finite collection of pairwise non-isomorphic Lie algebras. Then the groupoid $\mathcal{G}$ is continuously isomorphic to a Lie groupoid.*

Moreover, just like for groupoids, Example 3.1 justifies the assumption on the number of isomorphism classes among the Lie algebras of the fibers.

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