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Quadratic Forms on the 27-Dimensional Modules for E_6 in Characteristic Two

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Abstract. The purpose of this paper is to study the Chevalley group E of type $E_6(\mathbb{K})$ over fields $\mathbb{K}$ of characteristic two. We use the generalized quadrangle $(\mathbb{P},\mathcal{L})$ over $\mathbb{K}$ of type $O_6^-(2)$ to construct a trilinear form T on a 27-dimensional vector space A, this form preserves the action of E. We introduce an involution $g \to g^\alpha = g^* = (g^t)^{-1}$ on E, algebra structure on A and a quadratic map $\hat{Q} : A \to A$. Then we prove the following results:

(a) $\hat{Q}(x^g) = \hat{Q}(x)^{g^*}$ for all $x \in A$ and $g \in E$.

(b) For $x, y, z \in A$ and $g \in E$, the following holds true:
(1) $x^g\, y^g = (xy)^{g^*}$, and (2) $T(x^g, y^g, z^g) = T(x, y, z)$.

(c) The main results:
(1) The group G of isometries of T coincides with the group $G^* = \{g \in GL(A) \mid a^g b^g = (ab)^{g^*}\}$.
(2) The group $G_0 = \{g \in GL(A) \mid \hat{Q}(a^g) = \hat{Q}(a)^{g^*}\}$ is intermediate between E and G.
(3) The group $E = E^* = \{g^* = (g^t)^{-1} \mid g \in E\}$.

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1. Introduction

In [5] it has been shown that the Chevalley group $E_6(\mathbb{F})$ of type E_6 over a field $\mathbb{F}$ is the isometry group $G = O(V, f)$ of (V, f) where (V, f) is the 27-dimensional Dickson form over arbitrary field $\mathbb{F}$.

The construction of finite simple group $E_6(q)$ and their triple covers (which exist whenever $q \equiv 1 \mod 3$) goes back over 100 years to the work of Dickson [13, 14]. In [12], Chevalley gave a uniform construction of what are now called Chevalley groups which include five of the ten families of exceptional groups of Lie type, in particular $E_6(q)$.

The other major breakthrough since Dickson is the discovery of the exceptional Jordan algebra (Albert algebra). This 27-dimensional algebra consists of 3×3 Hermitian matrices over Cayley numbers with multiplication $X \circ Y = \frac{1}{2}(XY + YX)$. Freudental [15] showed that E_6 is the stabilizer of a certain cubic form defined on this space. Jacobson [16, 17, 18] studied the construction of F_4 and generalized the construction of E_6 to arbitrary fields of characteristic not 2 or 3. Fields of characteristic 2 or 3 are still problematic in the Jordan algebra context, although they were no

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obstacle to Dickson. The above observations are taken from [22]. Chevalley groups of type E_6 in the 27-dimensional representation, using computer, were studied in [21]. Simply connected Chevalley group $G(E_6, R)$ of type E_6 in a 27-dimensional representation is considered in [20] and it is shown that the following four groups coincide: the normalizer of the Chevalley group $G(E_6, R)$, the normalizer of the elementary subgroup $E(E_6, R)$, the transposer of $E(E_6, R)$ in $G(E_6, R)$ and the extended Chevalley group $\bar{G}(E_6, R)$ over an arbitrary commutative ring R, where all normalizers and transposers being taken in $GL(27, R)$.

To achieve this goal, an invariant cubic form and a system of quadrics is constructed, and a trilinear form on the 27-dimensional module $V = V(\bar{w})$, w is an element of the Weyl group $W(E_6)$, defined by $F(x, y, z) = \sum sgn(w)x_\lambda y_\mu z_\nu$, where the sum is taken over all triads $(\lambda, \mu, \nu) \in \theta$ the set of all triads, F takes the following values: $F(v^\lambda, v^\mu, v^n u) = \pm 1$ if $(\lambda, \mu, \nu) \in \theta$ and $F(v^\lambda, v^\mu, v^n u) = 0$ otherwise. The cubic form Q is defined in a similar manner but the coefficient 6 is avoided as it causes problem in characteristic 2 and 3. So the sum has been taken over the set θ_0 of unordered triads $\{\lambda, \mu, \nu\}$. The order of θ_0 is 45. The value of the form Q at a vector $x = \sum x_\lambda v^\lambda$ is defined by the formula $Q(x) = \sum sgn(w)x_\lambda x_\mu x_\nu$ where the sum is taken over $\{\lambda, \mu, \nu\} \in \theta_0$.

Our approach for defining the symmetric trilinear form and the quadratic map $\hat{Q}$ is completely different. Our construction is mainly based on incidence geometric notions, simply on the properties of the generalized quadrangle $(\mathbb{P}, \mathcal{L})$ of type $O_6^-(2)$. These properties were explained and applied in several papers, see [1, 2, 3, 4, 11]. A system composed of root-elements, root-bases, and root-generators was constructed, and it was shown that this system satisfies the properties of Lie algebras of type e_6 over fields of characteristic 2 [3]. Then the Chevalley group E of type E_6 corresponding to e_6 was also constructed [2].

In this paper we introduce the symmetric trilinear form T and the quadratic map $\hat{Q}$ on the 27-dimensional vector A and are defined by the elements of $\mathbb{P}$ and $\mathcal{L}$ with no need for Weyl group $W(E_6)$ or a cubic form, as it has been done in [20]. Then we use the important generalized quadrangle property, that is each point of $\mathbb{P}$ is contained in exactly 5 lines $L \in \mathcal{L}$ and if L is a line, p a point, $p \notin L$, then there is a unique point $q \in L$ such that p, q are collinear.

It is remarkable to mention that a ceratin class of cubic forms associated with trilinear forms over fields of characteristic $\neq 2$ or 3 are characterized in [19]. These forms are important in the study of algebraic groups of type E_6. This characterization is purely algebraic and it does not apply to our case. In this paper we prove the following main result:

$$E \leq G_0 \leq \hat{G} \;\text{ and }\; E = E^* = G, \;\text{ where}$$
$$\hat{G} = \{g \in GL(A) \,|\, \text{there exists } o \neq k_g \in \mathbb{K},\ T(a^g, b^g, c^g) = k_g T(a, b, c),\ \forall a, b, c \in A\}$$
$$G = \{g \in GL(A) \mid a^g b^g = (ab)^{g^\alpha} = (ab)^{g^*},\ \text{where } g^* = (g^t)^{-1},\ \forall a, b \in A\}$$
$$G_0 = \{g \in GL(A) \mid \hat{Q}(a^g) = \hat{Q}(a)^{g^*}\ ,\ \forall a \in A\} \;\text{ and }\; E^* = \{g^\alpha \mid g \in E\}.$$

The importance of this study is to provide descriptions which are sufficiently concrete to be used effectively in investigating subgroup structures of E_6. For more information about E_6 one may refer to [6, 7, 8, 9].

Remark 1.1. In Theorems 2 and 3 [20], the trilinear form F on the 27-dimensional module V is defined by a sum taken over all triads $(\lambda, \mu, \nu) \in \theta$ where $|\theta| = 270$, whilst the symmetric trilinear form in this paper is defined on the set of lines $\mathcal{L}$ where $|\mathcal{L}| = 45$. The investigation of the results in this paper is based on the quadratic map $\hat{Q}$ which is defined by the quadratic map Q_v , $v \in \mathbb{P}$ but the cubic form $Q(x)$ was used in the investigation in [20].

It remains to make a comparison between the triads in [20] and the lines in this article. As all quadratic spaces of "-" type are isomorphic, then one may consider the 6-dimensional vector space V over $\mathbb{F}_2$ endowed with quadratic form Q as $V = \{(x, y, z) \mid x, y, z \in \mathbb{F}_4\}$ with $Q(x, y, z) = x\overline{x} + y\overline{y} + z\overline{z}$, this leads to the quadric $\mathbb{P}$, the set of lines $\mathcal{L}$ and the set of exterior points of order 36. In contrast, the triads were built in a different manner. Finally, it remarkable to mention that this investigation and the investigations of Aschbacher, Freudental, Cartan and others in E_6 were in the spirit of Dickson's works [13, 14].

2. Preliminaries and general setup

To keep this paper self-contained the results below are taken from [1, 2, 3].

Let V be a 6-dimensional vector space over $\mathbb{F}_2$ and Q be a non-degenerate quadratic form on V of minimal Witt-index. The quadratic form Q is associated with a bilinear form on V defined by

$$(v|w) = Q(v + w) - Q(v) - Q(w).$$

Set $\mathbb{P} = \{0 \neq x \in V \mid Q(x) = 0\}$ and $\mathcal{L} = \{L \leq V \mid dim\, L = 2 \text{ and } Q(L) = 0\}$. Elements of $\mathbb{P}$ are called points and elements of $\mathcal{L}$ are called lines, and vectors $s \in V$ with $Q(s) = 1$ are called exterior points.

The incidence structure $(\mathbb{P}, \mathcal{L})$ is a generalized quadrangle with 27 points and 45 lines. Each line contains three points and each point is on five lines, [4].

Furthermore, let $W = Aut(V, Q) \cong O_6^-(2)$ be the corresponding orthogonal group. The group W is isomorphic to the Wely-group of type $\mathbb{E}_6$ and has order 51840. The group W is a 3-transposition group with respect to the 36-reflections σ_s at exterior points s, where $v^{\sigma_s} = v + (v|s)s$ for $v \in V$.

Definition 2.1. A subset B of $\mathbb{P}$ is a *root base* if:

(1) B is a F_2-base of V.

(2) $(x|y) = 1$ for distinct elements $x, y \in B$.

The root bases contained in $\mathbb{P}$ are denoted by Φ.

Remark 2.2. Equivalently, any set of six pairwise non-orthogonal points is a root base. If B is a root base, then $s_B = \sum_{x \in B} x$ is an exterior point, and $B^* = s_B + B$ is also a root base. We call B and B^* corresponding root bases. Furthermore, we denote by B_0 the set of all points which are orthogonal to s_B, so that $\mathbb{P} = B \cup B^* \cup B_0$. A line is either contained in B_0 or intersects each B, B^* and B_0, forming a generalized quadrangle of type $S_{P_4}(2)$. A subset E of $\mathbb{P}$ is called a *coclique* if for distinct vectors x, y in E, $(x|y) = 1$.

Proposition 2.3. ([3])

(1) $\mathbb{P}$ *contains exactly* 72 *root bases,* $|\Phi| = 72$.

(2) W *acts transitively on* Φ *with stabilizer* S_6.

(3) W *acts imprimitively on* Φ *with* 36 *blocks* $\{B, B^*\}$, $B \in \Phi$.

Remark 2.4. Root bases can be constructed as follows. For points x and y with $(x|y) = 1$, set $B_{x,y} = \{x\} \cup \{z \in \mathbb{P} \mid (x|z) = 1\ ,\ (y|z) = 0\}$. Then $B_{x,y}$ is a root base of this form. For a root base B and $x \in B$ we find $B = B_{x,x+s_B}$. For a point x there exists 16 points y with $(x|y) = 1$. Hence again

$$|\Phi| = \frac{27 \cdot 16}{6} = 72.$$

Remark 2.5. The root base B corresponds to the exterior point $s_B = \sum_{b \in B} b$, every exterior s or reflection σ_s corresponds to two root bases B_1 and B_2, such that $s = s_B$. Moreover, $B_2 = B_1^* = B_1^{\sigma_s} = B_1 + s$ and $B_1 \cup B_2 = \{z \in \mathbb{P} \mid (s|z) = 1\}$. We abbreviate σ_{s_B} as σ_B

Proposition 2.6. ([2]) *For a root base* B *and* C *with* $s_B \neq s_C$ *and* $(s_B|s_C) = 0$ *we have:*

$$|B \cap C| = |B \cap C^*| = 1 \text{ and } B^{\sigma_C} = B$$

Proposition 2.7. ([2]) *For root bases* B *and* C *with* $(s_B|s_C) = 1$ *we have:*

(1) $|B \cap C| = |B^* \cap C^*| = 3$ *and* $B \cap C^* = B^* \cap C = \emptyset$ *or*

(2) $|B \cap C^*| = |B^* \cap C| = 3$ *and* $B \cap C = B^* \cap C^* = \emptyset$.

(3) *If* $B \cap C = \emptyset$, *then* $(B^* \cap C)^{\sigma_B} \cup (C^* \cap B)^{\sigma_C} = B^{\sigma_C} = C^{\sigma_B}$ *is a root base corresponding to* $s_B + s_C$.

3. A Lie algebra of type E_6 and of characteristic 2

Let $\mathbb{K}$ be a field of characteristic 2 and $(\mathbb{P}, \mathcal{L})$ a generalized quadrangle of type $O_6^-(2)$, defined as above. Further, let A be a vector space over $\mathbb{K}$ with a base $\{e_x \mid x \in \mathbb{P}\}$.

Let $GL(A)$ denote the corresponding Lie algebra End $_{\mathbb{K}}(A)$ with Lie bracket $[X, Y] = XY - YX$. For $v \in V$ define a linear transformation H_v of A by

$$e_x^{H_v} = (x|v)e_x \ \ x \in \mathbb{P}.$$

For a root bases B, define linear transformations R_B of A by

$$e_x^{R_B} = \begin{cases} e_x^{\sigma_B} = e_{x+s_B} & , \ x \in B \\ 0 & , \ \text{otherwise.} \end{cases}$$

The linear transformation R_B has rank six, R_B and R_{B^*} are transposed to each other with respect to the base $\{e_x \mid x \in \mathbb{P}\}$ and $R_B^2 = 0$.

Proposition 3.1. ([3]) *For root bases B and C and $v, w \in V$ we have:*

(1) $[H_v, H_w] = 0$ *and* $[H_v, R_B] = (s_B|v)R_B$.

(2) $[R_B, R_{B^*}] = H_{s_B}$.

(3) *If s_B and s_C are distinct and orthogonal, then* $[R_B, R_C] = 0$.

(4) *If $(s_B|s_C) = 1$ and $B \cap C \neq \emptyset$, then* $R_B R_C = R_C R_B = [R_B, R_C] = 0$.

(5) *If $(s_B|s_C) = 1$ and $B \cap C = \emptyset$, then* $[R_B, R_C] = R_{B^{\sigma_C}} = R_{C^{\sigma_B}}$.

(6) *If $B \neq C$, then* $R_B R_C R_B = R_B[R_B, R_C] = 0$.

Definition 3.2. The subalgebra $\mathbb{E}$ of $\overline{GL}(A)$ is generated by the elements H_v, $v \in V$, and R_B, B a root base.

The group W acts on A by $e_x^g = e_{x^g}$. So W permutes the roots R_B and induces a Wely-group on the Lie algebra $\mathbb{E}$, normalizing the Cartan subalgebra generated by the elements H_v. The algebra $\mathbb{E}$ contains naturally embedded subalgebras.

The above results were proved in [3].

4. The Chevalley group E of type E_6

Definition 4.1. Let B be a root base. For $k \in \mathbb{K}$ set $r_B(k) = I + kR_B$, and $U_B = \{r_B(k) \mid k \in \mathbb{K}\}$. The root elements $r_B(k)$, $k \neq 0$, are called *Chevalley involutions*, the groups U_B are called *Chevalley root groups*.

Obviously, the following holds for a root base B and elements $r_B(k)$.

Proposition 4.2. ([2])

(1) $U_B \cong \mathbb{K}$

(2) *If $k \neq 0$, then $C_A(r_B(k)) = \langle e_x \mid x \in B^* \cup B_0 \rangle$ has dimension 21, and $[A, r_B(k)] = \langle a + a^{r_B(k)}, a \in A \rangle = \langle e_x \mid x \in B^* \rangle$ has dimension 6.*

For root bases B and $C \neq B^*$ and k, $m \in \mathbb{K}$ we find

$$\begin{aligned}(I + kR_B)(I + mR_C)(I + kR_B) &= (I + kR_B + mR_C + kmR_B R_C)(I + kR_B) \\ &= I + mR_C + km[R_B, R_C]\end{aligned}$$

as $R_B^2 = R_B R_C R_B = 0$ by Proposition 3.1. Hence, $[r_B(k), r_C(m)] = I + km[R_B, R_C]$ as $R_C[R_B, R_C] = 0$.

Proposition 4.3. ([2]) *Let B and $C \neq B$ be root bases and $k, m \in \mathbb{K}$, then:*

(1) *If $B \cap C \neq \emptyset$, then* $[r_B(k), r_C(m)] = I$.

(2) *If $B \cap C = \emptyset$, then $[r_B(k), r_C(m)] = r_D(km)$ for the root base $D = B^{\sigma_C} = C^{\sigma_B}$.*

(3) *U_B and U_{B^*} generate a group isomorphic $SL_2(\mathbb{K})$, where $r_B(1)$ and $r_{B^*}(1)$ are standard Chevalley generators in the group.*

Definition 4.4. Let $E = E(\mathbb{K})$ denote the group generated by the Chevalley root groups U_B. Conjugates in E of Chevalley root groups are called root groups. This group is the Chevalley group of type E_6.

For $g \in E$ let g^* denote the transposed inverse of g with respect to the base $\{e_x \mid x \in \mathbb{P}\}$ and g^* is also contained in E, as $r_B(k)^* = r_{B^*}(k)$.
The above results were proved in [2].

5. Invariant forms

Let A be a 27-dimensional vector space over $\mathbb{K}$. Then

Definition 5.1. For $x, y, z \in \mathbb{P}$, $T(e_x, e_y, e_z) := \begin{cases} 1 & , \quad \langle x, y, z \rangle \in \mathcal{L} \\ 0 & , \quad \text{otherwise.} \end{cases}$
So T defines a symmetric trilinear form on A.

Remark 5.2. The 27-dimensional vector space A can be turned into a (non-associative) algebra by defining a multiplication on the base vectors

$$e_x e_y = \begin{cases} e_{x+y} & , \quad x \neq y \text{ and } (x|y) = 0 \ , \ x, y \in \mathbb{P} \\ 0 & , \quad \text{otherwise} \end{cases}$$

This multiplication can be linearly extended to A.

Definition 5.3. We define a symmetric trilinear form T on A by $T : A \times A \times A \to \mathbb{K}$, such that

(1) $T(a_1, a_2, a_3) = T(a_{1\pi}, a_{2\pi}, a_{3\pi})$ for all $\pi \in S_3$.
(2) For any given $a, b \in A$, the maps $x \mapsto T(x, a, b)$, $x \mapsto T(a, x, b)$ and $x \mapsto T(a, b, x)$ from A into $\mathbb{K}$ are linear forms.

Remark 5.4. If $x = \sum_i x_i e_i$, $y = \sum_j y_j e_j$ and $z = \sum_k z_k e_k$ for X, Y, Z in A, then by the above definition

$$T(x, y, z) = \sum_{i,j,k} x_i y_j z_k T(e_i, e_j, e_k),$$

and T is uniquely determined by the values on triples of base elements.

Definition 5.5. Set $x = \sum_{a \in \mathbb{P}} x_v e_v \in A$. Then we define a quadratic map $\hat{Q} : A \to A$ by

$$\hat{Q}(x) = \sum_{v \in \mathbb{P}} Q_v(x) e_v \ , \text{ where } \ Q_v(x) = \sum_{v \in L \in \mathcal{L}} \Big(\prod_{w \in L \setminus \{v\}} x_w \Big).$$

In particular, $Q_v(x)$ is a quadratic form on A, and can therefore be written as

$$Q_v(x) = \sum_{L = \{v, a, b\} \in \mathcal{L}} x_a x_b .$$

Proposition 5.6. (1) *$\hat{Q}(kx) = k^2 \hat{Q}(x)$ for $x \in A$ and $k \in \mathbb{K}$.*

(2) *$\hat{Q}(x + y) = \hat{Q}(x) + \hat{Q}(y) + xy$ where $x = \sum_{a \in \mathbb{P}} x_a e_a$ and $y = \sum_{b \in \mathbb{P}} y_b e_b \in A$.*

Proof. (1) Obvious.

(2) $\hat{Q}(x+y) = \sum_{v\in\mathbb{P}} Q_v(x+y)e_v$ and

$$\begin{aligned} Q_v(x+y) &= \sum_{\{v,a,b\}\in\mathcal{L}} (x_a+y_a)(x_b+y_b) \\ &= \sum_{\{v,a,b\}\in\mathcal{L}} x_a\, x_b + \sum_{\{v,a,b\}\in\mathcal{L}} y_a\, y_b + \sum_{\{v,a,b\}\in\mathcal{L}} (x_a\, y_b + y_a\, x_b) \\ &= Q_v(x) + Q_v(y) + \sum_{\{v,a,b\}\in\mathcal{L}} (x_a y_b + y_a x_b). \end{aligned}$$

On the other hand

$$\begin{aligned} xy &= \left(\sum_{a\in\mathbb{P}} x_a\, e_e\right)\left(\sum_{b\in\mathbb{P}} y_b\, e_b\right) = \sum_{a,b\in\mathbb{P}} x_a\, y_b\, e_a\, e_b \\ &= \sum_{\substack{(a,b)\in\mathbb{P}\times\mathbb{P} \\ a\neq b,(a,b)=0}} x_a\, y_b\, e_a\, e_b = \sum_{\substack{(a,b)\in\mathbb{P}\times\mathbb{P} \\ a\neq b,(a,b)=0}} x_a\, y_b\, e_{a+b} \\ &= \sum_{v\in\mathbb{P}} \Bigg(\sum_{\substack{(a,b)\in\mathbb{P}\times\mathbb{P} \\ a\neq b,(a,b)=0}} x_a\, y_b \Bigg) e_v. \end{aligned}$$

Hence, $\hat{Q}(x+y) = \hat{Q}(x) + \hat{Q}(y) + xy$. ■

Remark 5.7. This result can be generalized to $X = \sum_{i=1}^{n} X_i$, $X_i \in A$, so one has

$$\hat{Q}(X) = \sum_i \hat{Q}(X_i) + \sum_{i<j} X_i X_j\,.$$

Definition 5.8. Let $x = \sum_{a\in\mathbb{P}} x_a\, e_a \in A$.
Then the *support* (x) is defined as the set $\{a \in \mathbb{P} \mid x_a \neq 0\}$.

Proposition 5.9. *If* $\hat{Q}(x) \neq 0$*, then the support* (x) *is not a coclique.*

Proof. This means that there exists $v \in \mathbb{P}$ such that $Q_v(x) \neq 0$, that is $\sum_{\{v,a,b\}\in\mathcal{L}} x_a\, x_b \neq 0$. In particular there exist a, b such that $a \neq b$, $(a|b) = 0$ and $x_a\, x_b \neq 0$. Hence the claim. ■

Corollary 5.10. *Let* B *be a root base and* $\sigma = \sigma_{s_{\mathbb{P}}}$. *Then* $\hat{Q}(x^{R_B}) = 0$.

Proof. As $x^{R_B} = \sum_a x_a\, e_a^{R_B} = \sum_{a\in B} x_a\, e_{a^\sigma} = \sum_{w\in B^\sigma} x_w\, e_w$. Hence, a support $(x^{R_B}) \subseteq B^\sigma$ which is a coclique. The claim follows by Proposition 5.9. ■

Remark 5.11. By abuse of notation we denote the corresponding matrix of R_B with respect to the base e_x , $x \in \mathbb{P}$, also by R_B so that $(R_B)_{xy} = 1$ if and only if $x \in B$ and $y = x^\sigma$.

Remark 5.12. Let B be a root base with corresponding vector $s = s_B$ and reflection $\sigma = \sigma_{s_B}$, then $(R_B)^t = R_{B^\sigma}$.

Proof. The entry $(R_B)_{xy} \neq 0$ if and only $y = x^\sigma$, $x \in B$.
The entry $(R_{B^\sigma})_{yx} \neq 0$ if and only if $y \in B^\sigma$ and $x = y^\sigma$, which is equivalent to $y^\sigma \in B$ and $x = y^\sigma$ and this is equivalent to $(R_B)_{xy} \neq 0$. ■

Remark 5.13. Let $E = \langle r_B(k) \mid k \in \mathbb{K}\ ,\ B \text{ is a root base}\rangle$ be the Chevalley group of type E_6. For $g \in E$, set $g^* = (g^{-1})^t$, where the transpose is taken with respect to the base $e_v\ ,\ v \in \mathbb{P}$.

Theorem 5.14. *$\hat{Q}(x^g) = \hat{Q}(x)^{g^*}$ for all $x \in A$ and $g \in E$.*

Proof. It suffices to prove the claim for root elements $r_B(k)$ because if we have $\hat{Q}(x^g) = \hat{Q}(x)^{g^*}$ and $\hat{Q}(x^h) = \hat{Q}(x)^{h^*}$, then

$$\hat{Q}(x^{gh}) = \hat{Q}(x^g)^{h^*} = \hat{Q}(x)^{g^* h^*} = \hat{Q}(x)^{(gh)^*},$$

as $(gh)^* = ((gh)^{-1})^t = (h^{-1}g^{-1})^t = (g^{-1})^t(h^{-1})^t = g^*h^*$. Hence

$$\begin{aligned} r_B(k)^* &= ((I + kR_B)^{-1})^t = (I + kR_B)^t = I + kR_B^t = I + kR_B^\sigma \\ &= r_{B^\sigma}(k) \ \text{ by the above observation.} \end{aligned}$$

So it suffices to show $\hat{Q}\left(x^{r_B(k)}\right) = \left(\hat{Q}(x)\right)^{r_{B^\sigma}(k)}$ or $\hat{Q}(x+kx^{R_B}) = \hat{Q}(x) + k(\hat{Q}(x))^{R_{B^\sigma}}$.

Moreover
$$\begin{aligned} \hat{Q}(x + kx^{R_B}) &= \hat{Q}(x) + \hat{Q}(kx^{R_B}) + kx\, x^{R_B} \\ &= \hat{Q}(x) + k^2\hat{Q}(x^{R_B}) + kx\, x^{R_B} = \hat{Q}(x) + kx\, x^{R_B} \end{aligned}$$

by Proposition 5.6 and Corollary 5.10.

Our claim is then equivalent to $x\, x^{R_B} = \hat{Q}(x)^{R_{B^\sigma}}$. We prove the induction on the support of $x = \sum x_v\, e_v\ ,\ v \in \mathbb{P}$. Without loss of generality, assume that a support $(x) = \{v\}$ i.e. $x = e_v$, hence $\hat{Q}(e_v) = 0$ and

$$e_v\, e_v^{R_B} = \begin{cases} e_v e_{v^\sigma} & ,\ v \in B \\ 0 & ,\ \text{otherwise} \end{cases}$$

As $v \in B$, this implies $(v|s) = 1$ and $v^\sigma = v+s$, hence $(v|v^\sigma) = (v|v+s) = 1$, which implies $e_v\, e_{v^\sigma} = 0$. Assume that $y\, y^{R_B} = \left(\hat{Q}(y)\right)^{R_{B^\sigma}}$ and assume that $x = e_v + y$ where $|\text{support}(y)| < |\text{support}(x)|$. Then

$$\begin{aligned} x\, x^{R_B} &= (e_v + y)(e_v^{R_B} + y^{R_B}) = e_v\, e_v^{R_B} + e_v\, y^{R_B} + y\, e_v^{R_B} = y\, y^{R_B} \\ &= e_v\, y^{R_B} + y\, e_v^{R_B} + y\, y^{R_B}, \quad \text{and} \end{aligned}$$
$$\hat{Q}(e_v + y)^{R_{B^\sigma}} = (\hat{Q}(e_v) + \hat{Q}(y) + e_v\, y)^{R_{B^\sigma}} = (\hat{Q}(y) + e_v\, y)^{R_{B^\sigma}} = \hat{Q}(y)^{R_{B^\sigma}} + (e_v\, y)^{R_{B^\sigma}}.$$

As $\hat{Q}(y)^{R_{B^\sigma}} = yy^{R_B}$ the induction proof can be completed if we can show that $e_v y^{R_B} + y e_v^{R_B} = (e_v y)^{R_{B^\sigma}}$. This follows if we can show that:

$$e_v e_w^{R_B} + e_v^{R_B} e_w = (e_v e_w)^{R_{B^\sigma}} \ \text{ for } v, w \in \mathbb{P}.$$

Hence we prove the following proposition. ■

Proposition 5.15. *For $x, y \in \mathbb{P}$, and B is a root base with corresponding vector $s = \sigma_B$ and reflection $\sigma = \sigma_{s_B}$, the following relations hold:*

(1) $e_x e_y^{R_B} + e_x^{R_B} y = (e_x e_y)^{R_{B^\sigma}}$.

(2) $e_x^{R_B}\, e_y^{R_B} = 0$.

Proof. (1) **Case 1:** $(x|s)=(y|s)=0$, this implies $e_x^{R_B}=e_y^{R_B}=0$ and $e_x\,e_y=0$ or $e_x\,e_y=e_{x+y}$. In the later case $(x+y|s)=0$ and hence $e_{x+y}^{R_{B^\sigma}}=0$.

Case 2: $(x|s)=0$, $(y|s)=1$ implies $e_x^{R_B}=0$, $e_xe_y^{R_B}\neq 0$ if and only if $y\in B$. In this case, one has

$$e_xe_y^{R_B}=\begin{cases} e_{x+y^\sigma} & ,\ y\in B \text{ and } (x|y^\sigma)=(x|s)\\ 0 & ,\ \text{otherwise}\end{cases}$$

$$(e_xe_y)^{R_{B^\sigma}}=\begin{cases} e_{(x+y)^\sigma} & ,\ (x|y)=0 \text{ and } x+y\in B^\sigma\\ 0 & ,\ \text{otherwise}\end{cases}$$

As $(x|s)=0$, it follows that $x^\sigma=x$, hence we have $(x|y^\sigma)=(x^\sigma|y)=(x|y)$ and $(x+y)^\sigma=x^\sigma+y^\sigma=x+y^\sigma$. $x+y\in B^\sigma$ if and only if $(x+y|s)=1$, which implies $x+y\in B\cup B^\sigma$ and $y\in B\cup B^\sigma$ and $(x|y)=0$. If $y\in B$ then $x+y\in B^\sigma$ as B is a root base. If $y\in B^\sigma$ then $x+y\in B$. So $y\in B$ if and only if $x+y\in B^\sigma$. Hence the claim.

Case 3: $(x|s)=(y|s)=1$ implies $x,y\in B\cup B^\sigma$

(a) If $x,y\in B$, it follows that $(x|y)=1$, $(x|y^\sigma)=(x^\sigma|y)=0$. Hence

$$\begin{aligned} e_xe_y^{R_B}+e_x^{R_B}e_y &= e_xe_{y^\sigma}+e_{x^\sigma}e_y=e_{x+y^\sigma}+e_x^{R_B}e_y\\ &= e_{x+y+s}+e_{x+s+y}=0=(e_xe_y)^{R_{B^\sigma}}\end{aligned}$$

(b) If $x,y\in B^\sigma$, then $e_x^{R_B}=e_y^{R_B}=(e_xe_y)^{R_{B^\sigma}}=0$.

(c) If $x\in B$, $y\in B^\sigma$, then $e_xe_y^{R_B}+e_x^{R_B}e_y=e_x\cdot 0+e_{x^\sigma}e_y=0$ as $x^\sigma,y\in B^\sigma$ and $(e_xe_y)^{R_{B^\sigma}}=0$, because either $e_xe_y=0$ or $e_xe_y=e_{x+y}$. As $(x+y|s)=0$, it follows that $(e_{x+y})^{R_{B^\sigma}}=0$.

(d) If $x\in B^\sigma$ and $y\in B$, then $e_x^{R_B}=0$ and $e_xe_y^{R_B}=e_xe_{y^\sigma}=0$ and $(e_xe_y)^{R_{B^\sigma}}=0$ as either $e_xe_y=0$ or $e_xe_y=e_{x+y}$ and $(x+y|s)=0$.

The proof of (2) is easy. Hence the claim and this completes the proof of Theorem 5.14. ∎

Corollary 5.16. *The above result can be expanded to elements of A, i.e. for $a,b\in A$*

(1) $ab^{R_B}+a^{R_B}b=(ab)^{R_{B^\sigma}}$.

(2) $a^{R_B}b^{R_B}=0$.

Corollary 5.17. *Let E,A be as defined above. Then, for $g\in E$ and $a,b\in A$, we have $a^gb^g=(ab)^{g^*}$.*

Proof. For $r=r_B(k)\in E$ and $a\in A$, $a^{r_B(k)}=a+ka^{R_B}$ and $r^*=r_{B^\sigma}(k)=(r^{-1})^t$. This implies

$$\begin{aligned} a^rb^r &= (a+ka^{R_B})(b+kb^{R_B})\\ &= ab+k(ab^{R_B}+a^{R_B}b)+k^2a^{R_B}b^{R_B}\\ &= ab+k(ab)^{R_{B^\sigma}}+0\ ,\ \text{by Corollary 5.2 (2).}\\ &= (ab)^{r^*}.\end{aligned}$$

Hence $a^rb^r=(ab)^{r^*}$. As $(gh)^*=g^*h^*$ we get $g^*=(g^{-1})^t$ and $a^gb^g=(ab)^{g^*}$ for all $g\in E$. Hence the claim. ∎

Definition 5.18. We define the inner product $\langle | \rangle$ on A by:

$$\langle e_i | e_j \rangle = \begin{cases} 1 & , \quad i = j \\ 0 & , \quad \text{otherwise} \end{cases}$$

Lemma 5.19. *Let $x = \sum_{i \in \mathbb{P}} x_i \, e_i$, $r = r_B(k) \in E$ for some root base B. Then*

$$\hat{Q}(x^r) = \hat{Q}(x)^{r^*} \text{ where } r^* = (r^{-1})^t.$$

Proof. By Proposition 5.6, it follows that

$$\begin{aligned} \hat{Q}(x^r) = \hat{Q}\left(\sum x_i \, e_i^r\right) &= \sum x_i^2 \hat{Q}(r_i^r) + \sum_{i<j} x_i \, x_j \, e_i^r \, e_j^r \\ &= \sum x_i^2 \hat{Q}(e_i)^{r^*} + \sum x_i \, x_j (e_i \, e_j)^{r^*} = \hat{Q}(x)^{r^*} \end{aligned}$$

■

Remark 5.20. The inner product $\langle x^g | x^{g^*} \rangle = \langle x | y \rangle$, $\forall x, y \in A, g \in E$.

Lemma 5.21. *The trilinear form $T(x^g, y^g, z^g) = T(x, y, z)$ for all $x, y, z \in A$ and for all $g \in E$.*

Proof. As the trilinear form $T(x, y, z) = \langle xy | z \rangle$. Hence

$$T(x^g, y^g, z^g) = \langle x^g \, y^g | z^g \rangle = \langle (xy)^{g^*} | z^g \rangle = \langle xy | z \rangle$$

using Remark 5.4. So $T(x^g, y^g, z^g) = T(x, y, z)$. ■

The above results can be summarized in the following

Theorem 5.22. *For $x, y, z \in A$ and $g \in E$, the following holds true:*

(1) $x^g \, y^g = (xy)^{g^*}$.

(2) $T(x^g, y^g, z^g) = T(x, y, z)$.

Definition 5.23. Define the groups

$$\begin{aligned} \hat{G} &= \{g \in GL(A) \mid \text{there exists } 0 \neq k_g \in \mathbb{K},\ T(a^g, b^g, c^g) = k_g T(a, b, c)\ ,\ \forall a, b, c \in A\}, \\ G &= \{g \in GL(A) \mid T(a^g, b^g, c^g) = T(a, b, c)\ ,\ \forall a, b, c \in A\}, \\ G^* &= \{g \in GL(A) \mid a^g b^g = (ab)^{g^*}\ ,\ \text{ where } g^* = (g^t)^{-1}\ ,\ \forall a, b \in A\}, \\ G_0 &= \{g \in GL(A) \mid \hat{Q}(a^g) = \hat{Q}(a)^{g^*}\ ,\ \forall a \in A\}, \\ E^* &= \left\{g^* = (g^t)^{-1} \mid g \in E\right\}. \end{aligned}$$

Then we prove the following

Theorem 5.24. *Let $\hat{G}, G, G^*, G_0, E$ and E^* be the groups defined above. Then*

(i) $E = E^*$,

(ii) $G = G^*$,

(iii) $E \leq G_0 \leq G \leq \hat{G}$.

Proof. (i) Clear.

(ii) It suffices to show that $g \in GL(A)$ preserves T if and only if $a^g b^g = (ab)^{g^*}$ for all $a, b \in A$. Hence

$$T\left(a^g, b^g, c^g\right) = \langle a^g b^g \mid c^g \rangle = \left\langle \left(a^g b^g\right)^{(g^*)^{-1}} \mid c \right\rangle = T(a, b, c),$$

which is equivalent to $\left(a^g b^g\right)^{(g^*)^{-1}} = ab$. This implies that $a^g \cdot b^g = (ab)^{g^*}$, where $c \in A$. Hence (ii) follows.

(iii) Theorem 5.14, Corollary 5.17, Lemma 5.19 and Theorem 5.22, imply $E \leq G_0$. To show that $G_0 \leq G$, let $g \in G_0$ and $a, b \in A$, then by Proposition 5.6 it follows $a^g b^g = \hat{Q}(a^g + b^g) + \hat{Q}(a^g) + \hat{Q}(b^g) = \hat{Q}(a+b)^{g^*} + \hat{Q}(a)^{g^*} + \hat{Q}(b)^{g^*} = (ab)^{g^*}$, using Theorem 5.14 and Proposition 5.6 once again, this implies $G_0 \leq G$. As the simply-connected Chevalley group over a field coincides with its elementary subgroups, so $E = G$ and this completes the proof of the main result. ■

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