Left Invariant Spray Structure on a Lie Group

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Abstract. We use the technique of invariant frame to study the left invariant spray structure on a Lie group. We calculate its S-curvature and Riemann curvature, which generalizes L. Huang's formulae in homogeneous Finsler geometry. Using the canonical bi-invariant spray structure as the origin, any left invariant spray structure can be associated with a spray vector field on the Lie algebra. We find the correspondence between the geodesics for a left invariant spray structure and the inverse integral curves of its spray vector field. As an application for this correspondence, we provide an alternative proof of Landsberg Conjecture for homogeneous Finsler surfaces.

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Key Words: Finsler metric, Landsberg Conjecture, left invariant frame, Lie group, Riemann curvature, S-curvature, spray structure.

1. Introduction

A *Finsler metric* F on a smooth manifold M is a continuous function $F : TM \to [0, +\infty)$ such that $F|_{TM\setminus 0}$ is smooth and $F(x, \cdot) = F|_{T_xM}$ is a Minkowski norm for each $x \in M$ (see Definition 1.3.1 in [16]). This notion first appeared in G. F. B. Riemann's speech in 1854, and earned its name after P. Finsler's dissertation in 1918 [6]. Riemannian geometry is a special case of Finsler geometry, or as S. S. Chern claimed, Finsler geometry is just Riemannian geometry without the quadratic restriction [3].

The *geodesic spray* of a Finsler manifold (M, F), which is a smooth tangent vector field $\mathbf{G}_F$ on $TM\setminus 0$ with standard local coordinate presentation

$$\mathbf{G}_F = y^i \partial_{x^i} - 2\mathbf{G}_F^i \partial_{y^i}, \ \text{ in which } \ \mathbf{G}_F^i = \tfrac{1}{4}g^{il}([F^2]_{x^k y^l} y^k - [F^2]_{x^l}), \qquad (1)$$

is crucial for studying Finsler geometry [1, 15]. Many geometric notions, like geodesic, Riemann curvature, etc, may be defined from $\mathbf{G}_F$, without a direct appearance of the metric [2], so they can be studied in a more general context, i.e., the *spray geometry.*

In spray geometry, a manifold M is endowed with a *spray structure.* The spray structure is a smooth tangent vector field $\mathbf{G}$ on $TM\setminus 0$ satisfying certain mild requirement [15]. See Section 2.2 and Section 2.3 for its precise definition and some

other related notions. Besides those $\mathbf{G}_F$'s in (1), there exist many others which can not be induced by Finsler metrics [11, 23]. They exhibit for us interesting dynamic and geometric phenomena which are unseen in Riemannian or Finsler geometry.

Homogeneous spray geometry is a generalization of *homogeneous Finsler geometry* [5], where the Lie method plays the main role. A manifold M endowed with a spray structure $\mathbf{G}$, i.e., $(M, \mathbf{G})$, is called *homogeneous* (or *affinely homogeneous*), if it admits a smooth transitive Lie group action which preserves $\mathbf{G}$. Here the spray preserving property for a diffeomorphism ρ on M means that the lifting of ρ, i.e., its tangent map, $\widetilde{\rho} = \rho_* : TM \to TM$, satisfies $(\widetilde{\rho})_* \mathbf{G} = \mathbf{G}$, or equivalently, both ρ and ρ^{-1} maps geodesics to geodesics. The spray preserving property for a flow ρ_t of diffeomorphisms generated by a smooth tangent vector field V can be characterized by $[\widetilde{V}, \mathbf{G}] = 0$, where $\widetilde{V}$ is the (complete) lifting of V on $TM \backslash 0$ (See Section 2.1 below or [20] for more details on lifting vector fields).

Left invariant geometry on a (connected) Lie group is a typical model in homogeneous geometry which deserves to be singled out for study [4, 13, 22]. On a Lie group G, we have globally defined left invariant frame $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ and right invariant frame $\{\widetilde{V}_i, \partial_{v^i}, \forall i\}$ on TG. Take $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ for example. Here U_i is a left invariant tangent vector field on G and $\widetilde{U}_i$ is its lifting. The set $\{e_i = U_i(e), \forall i\}$ is a fixed basis for the Lie algebra $\mathfrak{g} = T_e G$. The smooth vector fields ∂_{u^i}'s are tangent to each $T_g G$, where they correspond to the linear coordinates (u^i) for $y = u^i U_i(g) \in T_g G$.

We show in this paper that, using these invariant frames, the geometry for a left invariant spray structure can be explicitly and globally described. Firstly, we prove the existence of a canonical bi-invariant spray structure on each Lie group (see Section 3.2), i.e.,

Theorem A. *The smooth tangent vector field $\mathbf{G}_0 = u^i \widetilde{U}_i = v^i \widetilde{V}_i$ on $TG \backslash 0$ is a bi-invariant affine spray structure on the Lie group G.*

Then we use $\mathbf{G}_0$ as the origin to present any other left invariant spray structure $\mathbf{G}$ as $\mathbf{G} = \mathbf{G}_0 - \mathbf{H}$, where $\mathbf{H} = \mathbf{H}^i \partial_{u^i}$ is a left invariant smooth vector field on $TG \backslash 0$. The left invariancy here means $(\widetilde{L}_g)_* \mathbf{H} = \mathbf{H}$, $\forall g \in G$, and implies $\widetilde{V}_j \mathbf{H}^i = 0$, $\forall i, j$. Using the data from $\mathbf{H}$, the left invariant frame on TG, and the Lie bracket coefficients c_{ij}^k for $\mathfrak{g} = \mathrm{Lie}(G)$, we can provide globally presented curvature formulae for $(G, \mathbf{G})$. For the S-curvature, we prove (see Section 3.3)

Theorem B. *For the left invariant spray structure $\mathbf{G} = \mathbf{G}_0 - \mathbf{H}^i \partial_{u^i}$ and any left invariant smooth measure on a Lie group G, the S-curvature satisfies*

$$\mathbf{S} = \tfrac{1}{2} \tfrac{\partial}{\partial u^i} \mathbf{H}^i + \tfrac{1}{2} c_{lj}^j u^l$$

at each $(g, y) \in TG \backslash 0$ with $y = u^i U_i(g)$.

For the Riemann curvature, we prove (see Section 3.4)

Theorem C. *For the left invariant spray structure $\mathbf{G} = \mathbf{G}_0 - \mathbf{H}^i \partial_{u^i}$, the Riemann curvature satisfies*

$$\begin{aligned}
\mathbf{R}_y(U_q(g)) &= \left(\tfrac{3}{4} c_{pq}^r u^p \tfrac{\partial}{\partial u^r} \mathbf{H}^i + \tfrac{1}{2} c_{qj}^i \mathbf{H}^j + \tfrac{1}{2} \mathbf{H}^p \tfrac{\partial^2}{\partial u^p \partial u^q} \mathbf{H}^i \right. \\
&\quad \left. - \tfrac{1}{4} \tfrac{\partial}{\partial u^q} \mathbf{H}^p \tfrac{\partial}{\partial u^p} \mathbf{H}^i + \tfrac{1}{4} c_{pr}^i u^r \tfrac{\partial}{\partial u^q} \mathbf{H}^p - \tfrac{1}{4} c_{qj}^p c_{pr}^i u^j u^r \right) U_i(g)
\end{aligned}$$

at each $(g, y) \in TG \backslash 0$ with $y = u^i U_i(g)$.

On the other hand, we may reduce a problem in homogeneous geometry to that in a tangent space. For the left invariant spray structure $\mathbf{G} = \mathbf{G}_0 - \mathbf{H}$, the *spray vector field* $\eta = \mathbf{H}|_{T_eG\backslash\{0\}}$ plays an important role [10]. Sometimes we view η as a smooth map $\eta : \mathfrak{g}\backslash\{0\} \to \mathfrak{g}$, i.e., $\eta(y) = \mathbf{H}^i(e, y)e_i$ for $y = u^ie_i \in \mathfrak{g}\backslash\{0\}$. Then we also have the *connection operator* $N(y, w) = \frac{1}{2}D\eta(y, w) - \frac{1}{2}[y, w] : (\mathfrak{g}\backslash\{0\} \times \mathfrak{g}) \to \mathfrak{g}$ [9]. Both notions were firstly defined by L. Huang in homogeneous Finsler geometry [7].

Using $\eta(\cdot)$ and $N(\cdot, \cdot)$ for a left invariant $\mathbf{G}$, Theorem B and Theorem C can be easily translated to Corollary 4.1, which generalizes the curvature formulae L. Huang found in homogeneous Finsler geometry [7, 8, 9].

Remark 1.1. When the left invariant spray structure $\mathbf{G}$ is projective, i.e., its spray vector field $\eta(y) = P(y)y$ for some smooth function $P(\cdot)$ on $\mathfrak{g}\backslash\{0\}$, L. Huang and X. Mo calculated its Riemann curvature (see Lemma 5.1 in [10]) using a different approach.

The importance of spray vector field is further revealed by the correspondence between geodesics on $(G, \mathbf{G})$ and integral curves of $-\eta$ in $\mathfrak{g}\backslash\{0\}$. We prove the following theorem in Section 4.2.

Theorem D. *Let $\mathbf{G}$ be the left invariant spray structure on a Lie group G with the spray vector field η. Then for any open interval $(a, b) \subset \mathbb{R}$ containing 0, there is a one-to-one correspondence between the following two sets:*

(1) *the set of all $c(t)$ with $t \in (a, b)$ and $c(0) = e$, which are geodesics for $\mathbf{G}$;*

(2) *the set of all $y(t)$ with $t \in (a, b)$, which are integral curves of $-\eta$.*

The correspondence between these two sets is given by $y(t) = (L_{c(t)^{-1}})_*(\dot{c}(t))$.

The importance of spray vector field and connection operator will be further explored in the subsequent work [19].

There is an interesting application of Theorem D. Landsberg Conjecture is one of the most important open problems in Finsler geometry [12]. More references and some recent progress on this conjecture can be seen in [21]. Recently, A. Tayebi and B. Najafi proved Landsberg Conjecture for homogeneous Finsler surfaces [17]. We can use Theorem D to give a totally different proof for their theorem (see Theorem 4.3 and its proof in Section 4.4).

To summarize, we have two remarks. Firstly, the geometry of a left invariant spray structure has many similarities as its analog in Finsler geometry, but there are also many significant differences (see Section 4.3 for the details). This observation can provide many examples of left invariant spray structure which can not be induced by Finsler metrics [10]. Secondly, we believe that many results for left invariant spray structure may be generalized to homogeneous spray structure on a smooth coset space, where the proofs need a more delicate usage of special local frame.

This paper is organized as following. In Section 2, we recall the notion for lifting vector fields and basic knowledge on spray and Finsler geometries. In Section 3, we introduce the invariant spray structure on a Lie group and prove its S-curvature and Riemann curvature formulae. In Section 4, we discuss the spray vector field η for a left invariant spray structure, prove the correspondence between geodesics on the Lie group and integral curves of $-\eta$ in the Lie algebra, and provide an alternative proof for a theorem of A. Tayebi and B. Najafi.

2. Preliminaries

2.1. Complete lifting of a vector field

A smooth tangent vector field V on the smooth manifold M can be lifted to TM as following [20]. Locally around any $x \in M$, V generates a flow of local diffeomorphisms ρ_t. We first lift ρ_t to $\widetilde{\rho}_t = (\rho_t)_*$ on TM, and then define the smooth tangent vector field $\widetilde{V} = \frac{d}{dt}|_{t=0}\widetilde{\rho}_t$ on TM as the *complete lifting* (or *lifting* in short) of V.

We may use a *standard local coordinate* (x^i, y^i), i.e., $x = (x^i) \in M$ and $y = y^i \partial_{x^i}$, to present $\widetilde{V}$ as following (see Lemma 3.2 in [20])

Lemma 2.1.   *For a standard local coordinate* (x^i, y^i), *the complete lifting of a smooth tangent vector field* $V(x) = a^i(x)\partial_{x^i}$ *can be presented as*

$$\widetilde{V} = a^i(x)\partial_{x^i} + y^j \frac{\partial}{\partial x^j} a^i(x)\partial_{y^i}. \tag{2}$$

Notice that in the local frame $\{\partial_{x^i}, \partial_{y^i}, \forall i\}$ corresponding to the standard local coordinate (x^i, y^i), ∂_{x^i} on TM is the lifting of ∂_{x^i} on M, and ∂_{y^i} is tangent to each tangent space. So Lemma 2.1 implies the following transfer formulae for changing the standard local coordinate around a point,

$$\partial_{\bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\partial_{x^j} + y^j \frac{\partial}{\partial x^j}(\frac{\partial x^k}{\partial \bar{x}^i})\partial_{y^k} = \frac{\partial x^j}{\partial \bar{x}^i}\partial_{x^j} + \bar{y}^j \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j}\partial_{y^k},$$

$$\bar{y}^i = y^j \frac{\partial \bar{x}^i}{\partial x^j} \quad \text{and} \quad \partial_{\bar{y}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\partial_{y^j}.$$

Though geometric notions are sometimes introduced using a standard local coordinate, we can use above transfer formulae to verify they are globally well defined.

A *local frame* is referred to a set of smooth tangent vector fields $\{U_i, \forall i\}$ on some open subset $\mathcal{U}$ of M, such that at each $x \in \mathcal{U}$, $\{U_i(x), \forall i\}$ is a basis for $T_x M$. Associated with $\{U_i, \forall i\}$, there are functions u^i's on $T\mathcal{U}$ determined by $y = u^i U_i(x)$, $\forall x \in \mathcal{U}, y \in T_x M$. We denote ∂_{u^i}'s the smooth tangent vector fields which are tangent to and correspond to the linear coordinate (u^i) in each $T_x M$. Using the local frame $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ on $T\mathcal{U}$, Lemma 2.1 can be slightly generalized as following.

Lemma 2.2.   *For a local frame* $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ *on* $T\mathcal{U}$, *the complete lifting of the smooth tangent vector field* $V(x) = a^i(x)U_i$ *can be presented as*

$$\widetilde{V} = a^i \widetilde{U}_i + u^j U_j a^i \partial_{u^i}.$$

Proof.   Using a standard local coordinate (x^i, y^i), we have the presentations

$$U_i = A_i^j \partial_{x^j}, \quad u^i = y^j B_j^i, \quad \text{and} \quad \partial_{u^i} = A_i^j \partial_{y^j}, \tag{3}$$

where $(A_i^j) = (A_i^j(x))$ and $(B_i^j) = (B_i^j(x)) = (A_i^j(x))^{-1}$ (i.e. $A_i^j B_j^k = B_i^j A_j^k = \delta_i^k$) are matrix valued functions which only depend on $x \in M$. By Lemma 2.1, the lifting of $V = a^i U_i = a^i A_i^j \partial_{x^j}$ is

$$\widetilde{V} = a^i A_i^j \partial_{x^j} + y^j \frac{\partial}{\partial x^j}(a^i A_i^k)\partial_{y^k} = a^i(A_i^j \partial_{x^j} + y^j \frac{\partial}{\partial x^j} A_i^k \partial_{y^k}) + y^j A_i^k \frac{\partial}{\partial x^j} a^i \partial_{y^k}$$

$$= a^i \widetilde{U}_i + u^j U_j a^i \partial_{u^i},$$

which ends the proof.   ∎

By straight forward calculation using Lemma 2.1, we see

Lemma 2.3. (1) *For the diffeomorphism ϕ, and the smooth tangent vector fields V and ϕ_*V on M, their liftings satisfy $(\widetilde{\phi})_*\widetilde{V} = \widetilde{\phi_*V}$. So if ϕ preserves V, i.e., $\phi_*V = V$, then $\widetilde{\phi}$ preserves $\widetilde{V}$ as well.*
(2) *For the liftings $\widetilde{U}$, $\widetilde{V}$ and $\widetilde{[U,V]}$ of the smooth tangent vector fields U, V and $[U,V]$ on M, we have $\widetilde{[U,V]} = [\widetilde{U},\widetilde{V}]$. In particular, $\widetilde{U}$ and $\widetilde{V}$ commute when U and V do.*

Since lifting diffeomorphisms from M to TM might be viewed as a Lie group homomorphism, and lifting tangent vector fields the corresponding Lie algebra homomorphism, Lemma 2.3 can be naturally observed.

2.2. Spray structure, Finsler metric and geodesic

In the following three subsections, we summarize some basic knowledge on spray geometry and Finsler geometry.

Let M be a smooth manifold and $\mathbf{G}$ be a smooth tangent vector field on $TM\backslash 0$. We call $\mathbf{G}$ a *spray structure* on M, if for every standard local coordinate (x^i, y^i), $\mathbf{G}$ can be presented as

$$\mathbf{G} = y^i\partial_{x^i} - 2\mathbf{G}^i\partial_{y^i},$$

where each $\mathbf{G}^i = \mathbf{G}^i(x,y)$ satisfies the positive 2-homogeneity for its y-entry, i.e., $\mathbf{G}^i(x,\lambda y) = \lambda^2\mathbf{G}^i(x,y)$ for every $x \in M$, $y \in T_xM\backslash\{0\}$ and $\lambda > 0$ (see Definition 4.1.1 in [15]). An important subclass of spray structures are those induced by Finsler metrics.

A *Finsler metric* on M is a continuous function $F : TM \to [0,+\infty)$ satisfying (see Section 1.1 in [1])

1. regularity, i.e., its restriction to $TM\backslash 0$ is a positive smooth function;

2. positive 1-homogeneity, i.e., $F(x,\lambda y) = \lambda F(x,y)$ for every $x \in M$, $y \in T_xM$ and $\lambda > 0$;

3. the strong convexity, i.e., for any standard local coordinate (x^i, y^i), the Hessian matrix $(g_{ij}) = (g_{ij}(x,y)) = (\frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^j}F^2(x,y))$ is positive definite when $y \neq 0$.

The restriction of a Finsler metric to each tangent space is a *Minkowski norm*. The Hessian matrices (g_{ij}) in (3) is called the *fundamental tensor* of F. We use (g_{ij}) and $(g^{ij}) = (g_{ij})^{-1}$ to move indices up and down. Denote $g_y(\cdot,\cdot)$, $\forall y \in T_xM\backslash\{0\}$, the inner product on T_xM determined by the fundamental tensor, i.e.,

$$g_y(u,v) = u^iv^jg_{ij}(x,y), \quad \forall u = u^i\partial_{x^i}, v = v^i\partial_{x^i} \in T_xM.$$

A Finsler metric is *Riemannian* if all fundamental tensors $(g_{ij}) = (g_{ij}(x))$ depend on $x \in M$ only. The *geodesic spray* $\mathbf{G}_F$, i.e., the spray structure induced by F is presented in (1), i.e., $\mathbf{G}_F = y^i\partial_{x^i} - 2\mathbf{G}_F^i\partial_{y^i}$, with $\mathbf{G}_F^i = \frac{1}{4}g^{il}([F^2]_{x^ky^l}y^k - [F^2]_{x^l})$, for any standard local coordinate (x^i, y^i) (see (5.7) and (5.8) in [16]).

The spray structure $\mathbf{G}$ one-to-one determines the set of all the geodesics (with specified parametrizations) on $(M, \mathbf{G})$. A smooth curve $c(t)$ on $(M, \mathbf{G})$ is called a *geodesic* if its tangent vector field $\dot{c}(t)$ is nonzero everywhere and its lifting $(c(t), \dot{c}(t))$

is an integral curve of $\mathbf{G}$ (see Definition 4.1.2 in [15]). For a standard local coordinate (x^i, y^i), a geodesic $c(t) = (c^i(t))$ is characterized as a solution of (see (4.6) in [15])

$$\ddot{c}^i(t) + 2\mathbf{G}^i(c(t), \dot{c}(t)) = 0, \quad \forall i. \tag{4}$$

We say $\mathbf{G}$ is *geodesically complete*, if every maximally extended geodesic $c(t)$ is defined for all $t \in (-\infty, \infty)$.

Remark 2.4.   When $\mathbf{G} = \mathbf{G}_F$ is the geodesic spray for the Finsler metric F, (4) characterizes geodesics with positive constant speeds. Using the Christoffel symbol $\Gamma^i_{jk} = \frac{\partial^2 \mathbf{G}^i}{\partial y^j \partial y^k}$, (4) can be reformulated as

$$\ddot{c}^i(t) + \Gamma^i_{jk}\dot{c}^j(t)\dot{c}^k(t) = 0, \quad \forall i,$$

which is the same geodesic equation as in Riemannian and Finsler geometries. When F is Riemannian, Γ^i_{jk} only depends on $x \in M$ and can be simplified as

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{il}\big(\tfrac{\partial g_{lj}}{\partial x^k} + \tfrac{\partial g_{lk}}{\partial x^j} - \tfrac{\partial g_{jk}}{\partial x^l}\big).$$

2.3. Riemann curvature

There are two ways to define the Riemann curvature of $(M, \mathbf{G})$.

The first definition uses variation of geodesics. Consider a smooth variation of geodesics, $c(s, t)$, such that each $c(s, \cdot)$ is a geodesic. Then the *Jacobi field* $J(t) = \frac{\partial}{\partial s}c(s, t)|_{s=0}$ along $c(t) = c(0, t)$ satisfies the following *Jacobi equation* (see Lemma 8.1.1 in [15]),

$$D_{\dot{c}(t)}D_{\dot{c}(t)}J(t) + \mathbf{R}_{\dot{c}(t)}(J(t)) = 0, \tag{5}$$

in which $D_{\dot{c}(t)}\cdot$ is the *covariant derivative* along $c(t)$ (see (7.27) in [15]), and the linear operator $\mathbf{R}_{\dot{c}(t)}(\cdot) : T_{c(t)}M \to T_{c(t)}M$ is the *Riemann curvature*. For any standard local coordinate (x^i, y^i), the Riemann curvature can be presented as

$$\mathbf{R}_y = R^i_k \mathrm{d}x^k \otimes \tfrac{\partial}{\partial x^i} : T_xM \to T_xM \text{ for any } y \in T_xM \backslash \{0\},$$

in which (see (8.14) and (8.15) in [15])

$$
\begin{aligned}
R^i_k &= 2\frac{\partial \mathbf{G}^i}{\partial x^k} - y^j \frac{\partial^2 \mathbf{G}^i}{\partial x^j \partial y^k} + 2\mathbf{G}^j \frac{\partial^2 \mathbf{G}^i}{\partial y^j \partial y^k} - \frac{\partial \mathbf{G}^i}{\partial y^j}\frac{\partial \mathbf{G}^j}{\partial y^k} \\
&= \big(\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls}\big)y^j y^l.
\end{aligned} \tag{6}
$$

The second definition uses the horizontal and vertical distributions. For any standard local coordinates (x^i, y^i), we denote $N^i_j = \frac{\partial \mathbf{G}^i}{\partial y^j}$ and $\delta_{x^i} = \partial_{x^i} - N^j_i \partial_{y^j}$. Then the tangent bundle $T(TM \backslash 0)$ is the direct sum of two linear subbundles. One is the *horizontal distribution* $\mathcal{H}$ linearly spanned by all δ_{x^i}'s at each point. The other is the *vertical distribution* $\mathcal{V}$ linearly spanned by all ∂_{y^i}'s at each point. For a smooth tangent vector field $V = a^i(x)\partial_{x^i}$, we call the smooth section $\widetilde{V}^{\mathcal{H}} = a^i(x)\delta_{x^i}$ of $\mathcal{H}$ the *horizontal lifting* of V.

In this context, the Riemannian curvature is defined as a smooth linear bundle map $\mathcal{R} : \mathcal{H} \to \mathcal{V}$. For each $w \in \mathcal{H} \cap T_{(x,y)}(TM \backslash 0)$, $\mathcal{R}(w)$ is defined as following. We first extend w to a smooth section W of $\mathcal{H}$, and then define $\mathcal{R}(w)$ as the $\mathcal{V}$-component of $[\mathbf{G}, W]$ at $(x, y) \in TM \backslash 0$. Obviously $\mathcal{R}(w)$ is irrelevant to the choice of W, so $\mathcal{R} : \mathcal{H} \to \mathcal{V}$ is well defined. Remark 8.1.3 in [15] or direct calculation indicates

$$[\mathbf{G}, \delta_{x^k}] \equiv R^i_k \partial_{y^i} \pmod{\mathcal{H}},$$

in which R^i_k is presented in (6). So $\mathcal{R}(\delta_{x^k}) = R^i_k \partial_{y^i}$, and more generally,

Lemma 2.5.   *For a standard local coordinate* (x^i, y^i) *and a smooth tangent vector field* $V = a^k(x)\partial_{x^k}$ *on* M, *we have* $\mathcal{R}(\widetilde{V}^{\mathcal{H}}) = a^k(x)R_k^i\partial_{y^i}$.

Remark 2.6.   Besides Riemann curvature, there are other curvatures of Riemannian type in spray and Finsler geometries. Riemannian type means that you can find the analog in Riemannian geometry. For example, the Ricci curvature in spray and Finsler geometries can still be defined as the trace of the Riemann curvature $\mathbf{R}_y(\cdot) : T_xM \to T_xM$ (see Definition 8.1.7 in [15]). Flag curvature in Finsler geometry generalizes sectional curvature in Riemannian geometry (see Section 3.9 in [1]).

2.4. Curvatures of non-Riemannian type

There exist many geometric quantities in spray and Finsler geometries which vanish in Riemannian geometry. We call them curvatures of non-Riemannian type. For a spray structure $\mathbf{G} = y^i\partial_{x^i} - 2\mathbf{G}^i\partial_{y^i}$, where (x^i, y^i) is a standard local coordinate, we will meet the following.

The *Berwald curvature* for $\mathbf{G}$ is the tensor field

$$\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM, \ \forall x \in M, y \in T_xM\backslash\{0\},$$

satisfying $\qquad\qquad \mathbf{B}_y(\partial_{x^j}, \partial_{x^k}, \partial_{x^l}) = \frac{\partial^3\mathbf{G}^i}{\partial y^j\partial y^k\partial y^l}\partial_{x^i}.$

A spray structure $\mathbf{G}$ is called *affine* if its Berwald curvature vanishes everywhere, or equivalently, all $\mathbf{G}^i(x, \cdot)$'s are quadratic functions for their y-entries (see Definition 6.1.1 in [15]). A Finsler metric F is called a *Berwald metric* if its geodesic spray $\mathbf{G}_F$ is affine.

To introduce the S-curvature [14], we need to specify a smooth measure $d\mu$ on M. Suppose $d\mu$ is locally presented as $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$, where $\sigma(x)$ is nonvanishing everywhere. Then the *S-curvature* for $d\mu$ and $\mathbf{G}$ is a smooth function $\mathbf{S}(\cdot)$ on $TM\backslash 0$ satisfying (see (5.20) in [15])

$$\mathbf{S}(x, y) = \frac{\partial}{\partial y^i}\mathbf{G}^i(x, y) - \sigma(x)^{-1}y^m\frac{\partial}{\partial x^m}\sigma(x). \qquad (7)$$

In Finsler geometry, there are more curvatures of non-Riemannian type.

The *Cartan tensor* of (M, F) is the tensor field

$$\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}, \ \forall x \in M, y \in T_xM\backslash\{0\},$$

satisfying $\qquad\qquad \mathbf{C}_y(\partial_{x^i}, \partial_{x^j}, \partial_{x^k}) = \frac{1}{4}\frac{\partial^3(F^2)}{\partial y^i\partial y^j\partial y^k}.$

The Cartan tensor of (M, F) vanishes everywhere if and only if F is Riemannian.

The *Landsberg curvature* of (M, F) is the tensor field $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$, $\forall x \in M, y \in T_xM\backslash\{0\}$, which can be determined by the following lemma (see (7.16) in [16]).

Lemma 2.7.   *Let* $c(t)$ *be any geodesic on* (M, F) *with positive constant speed, and* $W_i(t)$ *with* $1 \le i \le 3$ *are linearly parallel vector fields along* $c(t)$. *Then we have*

$$\mathbf{L}_{\dot{c}(t)}(W_1(t), W_2(t), W_3(t)) = \frac{d}{dt}\mathbf{C}_{\dot{c}(t)}(W_1(t), W_2(t), W_3(t)).$$

A Finsler metric is called *Landsberg* if its Landsberg curvature vanishes. It is not hard to see Berwald metrics are Landsberg. The *Landsberg Conjecture* claims all smooth Landsberg metrics must be Berwald [12].

Remark 2.8.   The linearly parallelism along a smooth curve $c(t)$ is defined using the covariant derivative $D_{\dot{c}(t)}\cdot$ in the Jacobi equation (5), or (7.27) in [15]. For examples, the smooth vector field $X(t)$ along $c(t)$ is *linearly parallel* if $D_{\dot{c}(t)X(t)} \equiv 0$. See Section 5.3 in [16] for more details. When we prove Theorem 4.3 in Section 4.4, we will use the fact in Finsler geometry that linear parallel translation along a geodesic preserves the fundamental tensor (see Lemma 5.3.1 in [16]). It should be notified that there is a nonlinear parallelism in spray and Finsler geometries (see Section 7.3 in [15]). Comparing the two parallelisms is crucial for studying Landsberg Conjecture in Finsler geometry [21].

3.   Invariant spray structure on a Lie group

3.1. Notations for invariant frame

Let G be a Lie group. We denote $L_g(g') = gg'$ and $R_g(g') = g'g$ its left and right translations. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G, for which we fix a basis $\{e_1, \cdots, e_n\}$, and denote c_{ij}^k the corresponding bracket coefficients in $[e_i, e_j] = c_{ij}^k e_k$. Each e_i determines a left invariant vector field $U_i(g) = (L_g)_*(e_i)$. Any tangent vector $y \in T_gG$ can be uniquely written as $y = u^i U_i(g)$. Let ∂_{u^i}'s be the sections of $\mathcal{V}$ which correspond to the linear coordinate (u^i) in each T_gG. We will simply call $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ the *left invariant frame on TG*.

Similarly, we also have the right invariant vector fields $V_i(g) = (R_g)_*(e_i)$, the functions v^i on TG determined by $y = v^i V_i(g)$ for every $y \in T_gG$, and the *right invariant frame* $\{\widetilde{V}_i, \partial_{v^i}, \forall i\}$ *on TG*. By their invariancies and Lemma 2.3, we have the following obvious facts for every i and j,

$$[U_i, U_j] = c_{ij}^k U_k, \quad [V_i, V_j] = -c_{ij}^k V_k, \quad [U_i, V_j] = 0,$$

$$[\widetilde{U}_i, \widetilde{U}_j] = c_{ij}^k \widetilde{U}_k, \quad [\widetilde{V}_i, \widetilde{V}_j] = -c_{ij}^k \widetilde{V}_k, \quad [\widetilde{U}_i, \widetilde{V}_j] = 0,$$

$$\widetilde{U}_i v^j = 0, \quad [\widetilde{U}_i, \partial_{v^j}] = 0, \quad \widetilde{V}_i u^j = 0, \quad [\widetilde{V}_i, \partial_{u^j}] = 0. \tag{8}$$

Denote by ϕ_i^j and ψ_i^j the functions on G with $\mathrm{Ad}(g)e_i = \phi_i^j e_j$, $\mathrm{Ad}(g^{-1})e_i = \psi_i^j e_j$ (so we have $(\psi_i^j) = (\phi_i^j)^{-1}$, i.e., $\psi_i^j \phi_j^k = \phi_i^j \psi_j^k = \delta_i^k$). At each $g \in G$,

$$U_i(g) = (L_g)_*(e_i) = (R_g)_*(R_{g^{-1}})_*(L_g)_*(e_i) = (R_g)_*(\mathrm{Ad}(g)e_i)$$

$$= (R_g)_*(\phi_i^j e_j) = \phi_i^j V_j(g).$$

So we have $U_i = \phi_i^j V_j, \quad u^i = \psi_j^i v^j, \quad \partial_{u^i} = \phi_i^j \partial_{v^j}.$ \hfill (9)

In the later discussion we need the following results.

Lemma 3.1.   *Keeping the above notations, then at each point of G the following conditions are fulfilled:*

(1) $\phi_l^j V_j \phi_i^k = c_{li}^j \phi_j^k,$ (2) $\widetilde{U}_i = \phi_i^j \widetilde{V}_j + c_{pi}^q u^p \partial_{u^q},$

(3) $\widetilde{U}_i u^j = c_{li}^j u^l,$ (4) $[\widetilde{U}_i, \partial_{u^l}] = c_{il}^p \partial_{u^p}.$

Proof. (1) At each $g \in G$, we have $\mathrm{Ad}(g)e_i = \phi_i^k e_k$ and then

$$
\begin{aligned}
V_j \phi_i^k e_k &= V_j(\phi_i^k e_k) = \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}(\exp t e_j \cdot g)e_i) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}(\exp t e_j)(\mathrm{Ad}(g)e_i) = [e_j, \mathrm{Ad}(g)e_i].
\end{aligned}
$$

So we have

$$
\phi_l^j V_j \phi_i^k e_k = [\mathrm{Ad}(g)e_l, \mathrm{Ad}(g)e_i] = \mathrm{Ad}(g)[e_l, e_i] = c_{li}^j \mathrm{Ad}(g)e_j = c_{li}^j \phi_j^k e_k. \qquad (10)
$$

Comparing the coefficients of e_k in both sides of (10), then (1) is proved.

(2) Using Lemma 2.2 for $U_i = \phi_i^j V_j$, (1) of Lemma 3.1 and (9), we get

$$
\begin{aligned}
\widetilde{U}_i &= \phi_i^j \widetilde{V}_j + v^j V_j \phi_i^k \partial_{v^k} = \phi_i^j \widetilde{V}_j + u^p \phi_p^j V_j \phi_i^k \partial_{v^k} \\
&= \phi_i^j \widetilde{V}_j + u^p c_{pi}^q \phi_q^k \partial_{v^k} = \phi_i^j \widetilde{V}_j + c_{pi}^q u^p \partial_{u^q},
\end{aligned}
$$

which proves (2).

(3) By (2) of Lemma 3.1, we have

$$
\widetilde{U}_i u^j = \phi_i^k \widetilde{V}_k u^j + c_{pi}^q u^p \tfrac{\partial}{\partial u^q} u^j = c_{li}^j u^l,
$$

which proves (3).

(4) Using (2) of Lemma 3.1 again, we get

$$
[\widetilde{U}_i, \partial_{u^l}] = [\phi_i^j \widetilde{V}_j + c_{qi}^p u^q \partial_{u^p}, \partial_{u^l}] = [c_{qi}^p u^q \partial_{u^p}, \partial_{u^l}] = c_{il}^p \partial_{u^p},
$$

which ends the proof of Lemma 3.1. ∎

We will keep above notations in the discussion below.

3.2. Invariant spray on a Lie group

A spray structure $\mathbf{G}$ on the Lie group G is called *left invariant* (or *right invariant*), if $(\widetilde{L}_g)_* \mathbf{G} = \mathbf{G}$ (or $(\widetilde{R}_g)_* \mathbf{G} = \mathbf{G}$) for all $g \in G$. In the Lie algebra level, it means $[\widetilde{V}_i, \mathbf{G}] = 0$ (or $[\widetilde{U}_i, \mathbf{G}] = 0$ respectively) for each i. It is *bi-invariant* if it is both left and right invariant.

For example, when G is endowed with a left invariant Riemannian or Finsler metric F, all left translations are isometries for F, which preserve the set of geodesics. So the geodesic spray $\mathbf{G}_F$ for a left invariant F is also left invariant.

Theorem A claims the existence of a canonical bi-invariant spray structure $\mathbf{G}_0$, which will be set as the origin for the space of all left invariant spray structures on G.

Proof of Theorem A. We prove first that $\mathbf{G}_0 = u^i \widetilde{U}_i$ is an affine spray structure. Obviously, it is a globally defined smooth tangent vector field on $TG \backslash 0$.

Let (x^i, y^i) be any standard local coordinate on TM. We apply the notations in (3), i.e., $U_i = A_i^j \partial_{x^j}$, and $(B_i^j) = (A_i^j)^{-1}$. By Lemma 2.1, $\widetilde{U}_i = A_i^j \partial_{x^j} + y^j \tfrac{\partial}{\partial x^j} A_i^k \, \partial_{y^k}$. So we have

$$
\mathbf{G}_0 = u^i \widetilde{U}_i = u^i A_i^j \partial_{x^j} + u^i y^j \tfrac{\partial}{\partial x^j} A_i^k \, \partial_{y^k} = y^j \partial_{x^j} + y^l y^j (B_l^i \tfrac{\partial}{\partial x^j} A_i^k) \, \partial_{y^k}, \qquad (11)
$$

Notice that the functions A^i_j and B^i_j only depend on the local coordinate (x^i) on G. So we see from (11) that $\mathbf{G}_0$ is an affine spray structure.

Secondly, we prove $u^i \widetilde{U}_i = v^i \widetilde{V}_i$. By (2) in Lemma 3.1 and (9),

$$u^i \widetilde{U}_i = u^i \phi^j_i \widetilde{V}_j + u^i u^p c^q_{pi} \partial_{u^q} = v^j \widetilde{V}_j + u^i u^p c^q_{pi} \partial_{u^q} = v^j \widetilde{V}_j,$$

where $u^i u^p c^q_{pi}$ vanishes because $c^q_{pi} = -c^q_{ip}$.

Finally, we prove the bi-invariance of $\mathbf{G}_0$. By (8), we see for each i,

$$[\widetilde{V}_i, \mathbf{G}_0] = [\widetilde{V}_i, u^j \widetilde{U}_j] = \widetilde{V}_i u^j \widetilde{U}_j + u^j [\widetilde{V}_i, \widetilde{U}_j] = 0,$$

i.e., $\mathbf{G}_0$ is left invariant. Using $\mathbf{G}_0 = v^j \widetilde{V}_j$, we can similarly argue $[\widetilde{U}_i, \mathbf{G}_0] = 0$ for each i, i.e., $\mathbf{G}_0$ is right invariant.   ∎

Then we consider any left invariant spray structure $\mathbf{G}$ on G. Comparing $\mathbf{G}$ with $\mathbf{G}_0 = u^i \widetilde{U}_i = v^i \widetilde{V}_i$, we see $\mathbf{H} = \mathbf{G}_0 - \mathbf{G} = \mathbf{H}^i \partial_{u^i}$ is a left invariant section of $\mathcal{V}$. The left invariance of $\mathbf{H}$ implies each coefficient function $\mathbf{H}^i$ is preserved by the action of $\widetilde{L}_g$, $\forall g \in G$. Since the right invariant vector fields generate the left translations, we have $\widetilde{V}_j \mathbf{H}^i = 0$, $\forall i, j$. Further more, every $\mathbf{H}^i$ is positive 2-homogeneous when restricted to each $T_g G \backslash \{0\}$.

The technique of invariant frame enable us to assemble the Lie bracket coefficients, the invariant u^i-coordinates and $\mathbf{H}^i$'s to global clean curvature formulae for $\mathbf{G}$. The calculation for S-curvature and Riemann curvature are carried out below as examples.

3.3. Left invariant S-curvature formula

Now we use the left invariant frame $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ to calculate the S-curvature for the left invariant spray structure $\mathbf{G} = \mathbf{G}_0 - \mathbf{H}^i \partial_{u^i}$ and any left invariant smooth measure.

To apply the formula (7), we specify a standard local coordinate (x^i, y^i) on TG, with the corresponding notations in (3), i.e., $U_i = A^j_i \partial_{x^j}$ and $u^i = B^i_j y^j$, where (A^j_i) and $(B^j_i) = (A^j_i)^{-1}$ are matrix valued local functions on G.

By Lemma 2.1 and Theorem A,

$$\mathbf{G} = u^i \widetilde{U}_i - \mathbf{H}^i \partial_{u^i} = u^i (A^j_i \partial_{x^j} + y^j \frac{\partial}{\partial x^j} A^k_i \partial_{y^k}) - A^k_j \mathbf{H}^j \partial_{y^k}$$

$$= y^j \partial_{x^j} - (A^k_i \mathbf{H}^i - u^i y^j \frac{\partial}{\partial x^j} A^k_i) \partial_{y^k},$$

so we have
$$\mathbf{G}^k = \tfrac{1}{2} A^k_i \mathbf{H}^i - \tfrac{1}{2} u^i y^l \frac{\partial}{\partial x^l} A^k_i. \tag{12}$$

Further calculation shows

$$N^k_j = \frac{\partial}{\partial y^j} \mathbf{G}^k = \tfrac{1}{2} A^k_i \frac{\partial}{\partial y^j} \mathbf{H}^i - \tfrac{1}{2} u^i \frac{\partial}{\partial x^j} A^k_i - \tfrac{1}{2} B^i_j y^l \frac{\partial}{\partial x^l} A^k_i, \tag{13}$$

and its trace is
$$N^j_j = \tfrac{1}{2} A^j_i \frac{\partial}{\partial y^j} \mathbf{H}^i - \tfrac{1}{2} u^i \frac{\partial}{\partial x^j} A^j_i - \tfrac{1}{2} B^i_j y^l \frac{\partial}{\partial x^l} A^j_i. \tag{14}$$

We denote by $\{U^*_i, \forall i\}$ the dual frame for $\{U_i, \forall i\}$. Then the left invariant volume form can be presented as

$$d\mu = c\, U^*_1 \wedge \cdots \wedge U^*_n = c \det(B^j_i) dx^1 \wedge \cdots \wedge dx^n,$$

in which c is some nonzero constant.

Then we have $\sigma(x) = c \det(B_i^j)$ and

$$-\sigma(x)^{-1} y^m \tfrac{\partial}{\partial x^m} \sigma(x) = -y^m A_j^i \tfrac{\partial}{\partial x^m} B_i^j = y^m B_i^j \tfrac{\partial}{\partial x^m} A_j^i. \tag{15}$$

Adding (14) and (15), we get

$$
\begin{aligned}
\mathbf{S} &= \tfrac{1}{2} A_i^j \tfrac{\partial}{\partial y^j} \mathbf{H}^i + \tfrac{1}{2} y^m B_i^j \tfrac{\partial}{\partial x^m} A_j^i - \tfrac{1}{2} u^i \tfrac{\partial}{\partial x^j} A_i^j \\
&= \tfrac{1}{2} A_i^j \tfrac{\partial}{\partial y^j} \mathbf{H}^i + \tfrac{1}{2} B_i^j u^l A_l^m \tfrac{\partial}{\partial x^m} A_j^i - \tfrac{1}{2} u^i \tfrac{\partial}{\partial x^j} A_i^j \\
&= \tfrac{1}{2} A_i^j \tfrac{\partial}{\partial y^j} \mathbf{H}^i + \tfrac{1}{2} B_i^j u^l (A_l^m \tfrac{\partial}{\partial x^m} A_j^i - A_j^m \tfrac{\partial}{\partial x^m} A_l^i) + (\tfrac{1}{2} B_i^j u^l A_j^m \tfrac{\partial}{\partial x^m} A_l^i - \tfrac{1}{2} u^i \tfrac{\partial}{\partial x^j} A_i^j) \\
&= \tfrac{1}{2} A_i^j \tfrac{\partial}{\partial y^j} \mathbf{H}^i + \tfrac{1}{2} B_i^j u^l (A_l^m \tfrac{\partial}{\partial x^m} A_j^i - A_j^m \tfrac{\partial}{\partial x^m} A_l^i) + (\tfrac{1}{2} u^l \tfrac{\partial}{\partial x^i} A_l^i - \tfrac{1}{2} u^i \tfrac{\partial}{\partial x^j} A_i^j) \\
&= \tfrac{1}{2} \tfrac{\partial}{\partial u^i} \mathbf{H}^i + \tfrac{1}{2} B_i^j u^l (A_l^m \tfrac{\partial}{\partial x^m} A_j^i - A_j^m \tfrac{\partial}{\partial x^m} A_l^i). \tag{16}
\end{aligned}
$$

We compare the coefficients of ∂_{x^k} in

$$c_{qi}^p A_p^k \partial_{x^k} = c_{qi}^p U_p = [U_q, U_i] = (A_q^j \tfrac{\partial}{\partial x^j} A_i^k - A_i^j \tfrac{\partial}{\partial x^j} A_q^k) \partial_{x^k},$$

$$\text{and get} \qquad A_q^j \tfrac{\partial}{\partial x^j} A_i^k - A_i^j \tfrac{\partial}{\partial x^j} A_q^k = c_{qi}^p A_p^k. \tag{17}$$

With the help of (17), the calculation in (16) can be continued as

$$
\begin{aligned}
\mathbf{S} &= \tfrac{1}{2} \tfrac{\partial}{\partial u^i} \mathbf{H}^i + \tfrac{1}{2} B_i^j u^l (A_l^m \tfrac{\partial}{\partial x^m} A_j^i - A_j^m \tfrac{\partial}{\partial x^m} A_l^i) \\
&= \tfrac{1}{2} \tfrac{\partial}{\partial u^i} \mathbf{H}^i + \tfrac{1}{2} B_i^j u^l c_{lj}^p A_p^i = \tfrac{1}{2} \tfrac{\partial}{\partial u^i} \mathbf{H}^i + \tfrac{1}{2} c_{lj}^j u^l,
\end{aligned}
$$

which ends the proof of Theorem B.

3.4. Left invariant Riemann curvature formula

By Lemma 2.5 and (3), the Riemann curvature formula in Theorem C is equivalent to the following

$$
\begin{aligned}
\mathcal{R}(\widetilde{U}_q^{\mathcal{H}}) &= \left(\tfrac{3}{4} c_{pq}^r u^p \tfrac{\partial}{\partial u^r} \mathbf{H}^i + \tfrac{1}{2} c_{qj}^i \mathbf{H}^j + \tfrac{1}{2} \mathbf{H}^p \tfrac{\partial^2}{\partial u^p \partial u^q} \mathbf{H}^i \right. \\
&\qquad \left. - \tfrac{1}{4} \tfrac{\partial}{\partial u^q} \mathbf{H}^p \tfrac{\partial}{\partial u^p} \mathbf{H}^i + \tfrac{1}{4} c_{pr}^i u^r \tfrac{\partial}{\partial u^q} \mathbf{H}^p - \tfrac{1}{4} c_{qj}^p c_{pr}^i u^j u^r \right) \partial_{u^i}. \tag{18}
\end{aligned}
$$

Now we use the left invariant frame $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ to calculate $\mathcal{R}(\widetilde{U}_q^{\mathcal{H}})$, i.e., the $\mathcal{V}$-component in $[\mathbf{G}, \widetilde{U}_q^{\mathcal{H}}]$, for the left invariant spray structure $\mathbf{G} = \mathbf{G}_0 - \mathbf{H}^i \partial_{u^i}$. Firstly, we have

Lemma 3.2. *The horizontal lifting of U_q is $\widetilde{U}_q^{\mathcal{H}} = \widetilde{U}_q - (\tfrac{1}{2} \tfrac{\partial}{\partial u^q} \mathbf{H}^i - \tfrac{1}{2} u^j c_{qj}^i) \partial_{u^i}$.*

Proof. Let (x^i, y^i) be the standard local coordinate on TM with $U_q = A_q^j \partial_{x^j}$ and $u^i = y^j B_j^i$. By (13) and (17), we have

$$
\begin{aligned}
\widetilde{U}_q^{\mathcal{H}} &= A_q^j \delta_{x^j} = A_q^j \partial_{x^j} - A_q^j N_j^k \partial_{y^k} \\
&= \widetilde{U}_q - y^j \tfrac{\partial}{\partial x^j} A_q^l \partial_{y^l} - \tfrac{1}{2} A_q^j A_i^k \tfrac{\partial}{\partial y^j} \mathbf{H}^i \partial_{y^k} + \tfrac{1}{2} u^i A_q^j \tfrac{\partial}{\partial x^j} A_i^k \partial_{y^k} + \tfrac{1}{2} y^l \tfrac{\partial}{\partial x^l} A_q^k \partial_{y^k} \\
&= \widetilde{U}_q - \tfrac{1}{2} \tfrac{\partial}{\partial u^q} \mathbf{H}^i \partial_{u^i} + \tfrac{1}{2} u^i (A_q^j \tfrac{\partial}{\partial x^j} A_i^k - A_i^j \tfrac{\partial}{\partial x^j} A_q^k) \partial_{y^k} \\
&= \widetilde{U}_q - \tfrac{1}{2} \tfrac{\partial}{\partial u^q} \mathbf{H}^i \partial_{u^i} + \tfrac{1}{2} c_{qi}^l A_l^k u^i \partial_{y^k} = \widetilde{U}_q - (\tfrac{1}{2} \tfrac{\partial}{\partial u^q} \mathbf{H}^i - \tfrac{1}{2} c_{qj}^i u^j) \partial_{u^i},
\end{aligned}
$$

which ends the proof of Lemma 3.2. ∎

Next, we use Lemma 3.1 and Lemma 3.2 to get

$$[-\mathbf{H}^i\partial_{u^i}, \widetilde{U}_q^{\mathcal{H}}]$$

$$= [-\mathbf{H}^i\partial_{u^i}, \widetilde{U}_q - \tfrac{1}{2}\tfrac{\partial}{\partial u^q}\mathbf{H}^i\partial_{u^i} + \tfrac{1}{2}c_{qj}^i u^j\partial_{u^i}]$$

$$= \widetilde{U}_q\mathbf{H}^i\partial_{u^i} + \mathbf{H}^i[\widetilde{U}_q, \partial_{u^i}] + \tfrac{1}{2}[\mathbf{H}^i\partial_{u^i}, \tfrac{\partial}{\partial u^q}\mathbf{H}^i\partial_{u^i}] - \tfrac{1}{2}c_{qj}^i[\mathbf{H}^p\partial_{u^p}, u^j\partial_{u^i}]$$

$$= c_{pq}^r u^p\tfrac{\partial}{\partial u^r}\mathbf{H}^i\partial_{u^i} + c_{qp}^i\mathbf{H}^p\partial_{u^i} + \tfrac{1}{2}\mathbf{H}^p\tfrac{\partial^2}{\partial u^p\partial u^q}\mathbf{H}^i\partial_{u^i} - \tfrac{1}{2}\tfrac{\partial}{\partial u^p}\mathbf{H}^p\tfrac{\partial}{\partial u^p}\mathbf{H}^i\partial_{u^i}$$

$$\qquad -\tfrac{1}{2}c_{qj}^i\mathbf{H}^j\partial_{u^i} + \tfrac{1}{2}c_{qj}^p u^j\tfrac{\partial}{\partial u^p}\mathbf{H}^i\partial_{u^i}$$

$$= \tfrac{1}{2}c_{pq}^r u^p\tfrac{\partial}{\partial u^r}\mathbf{H}^i\partial_{u^i} + \tfrac{1}{2}c_{qj}^i\mathbf{H}^j\partial_{u^i} + \tfrac{1}{2}\mathbf{H}^p\tfrac{\partial^2}{\partial u^p\partial u^q}\mathbf{H}^i\partial_{u^i} - \tfrac{1}{2}\tfrac{\partial}{\partial u^q}\mathbf{H}^p\tfrac{\partial}{\partial u^p}\mathbf{H}^i\partial_{u^i}. \quad (19)$$

Then we compute the $\mathcal{V}$-component in $[\mathbf{G}_0, \widetilde{U}_q^{\mathcal{H}}]$ and get

$$[\mathbf{G}_0, \widetilde{U}_q^{\mathcal{H}}]$$

$$= [\mathbf{G}_0, \widetilde{U}_q - \tfrac{1}{2}(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)\partial_{u^i}]$$

$$= [\mathbf{G}_0, \widetilde{U}_q] + [u^p\widetilde{U}_p, -\tfrac{1}{2}(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)\partial_{u^i}]$$

$$= [u^p\widetilde{U}_p, -\tfrac{1}{2}(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)\partial_{u^i}]$$

$$= \tfrac{1}{2}(\tfrac{\partial}{\partial u^q}\mathbf{H}^p - c_{qj}^p u^j)\widetilde{U}_p - \tfrac{1}{2}u^p(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)[\widetilde{U}_p, \partial_{u^i}] - \tfrac{1}{2}u^p\widetilde{U}_p(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)\partial_{u^i}$$

$$= \tfrac{1}{2}(\tfrac{\partial}{\partial u^q}\mathbf{H}^p - c_{qj}^p u^j)\widetilde{U}_p - \tfrac{1}{2}u^p(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)[\widetilde{U}_p, \partial_{u^i}] - \tfrac{1}{2}v^p\widetilde{V}_p(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)\partial_{u^i}$$

$$= \tfrac{1}{2}(\tfrac{\partial}{\partial u^q}\mathbf{H}^p - c_{qj}^p u^j)\widetilde{U}_p - \tfrac{1}{2}u^p(\tfrac{\partial}{\partial u^q}\mathbf{H}^i - c_{qj}^i u^j)[\widetilde{U}_p, \partial_{u^i}]$$

$$= \tfrac{1}{2}(\tfrac{\partial}{\partial u^q}\mathbf{H}^p - c_{qj}^p u^j)(\widetilde{U}_p^{\mathcal{H}} + \tfrac{1}{2}(\tfrac{\partial}{\partial u^p}\mathbf{H}^i - c_{pr}^i u^r)\partial_{u^i}) - \tfrac{1}{2}u^p(\tfrac{\partial}{\partial u^q}\mathbf{H}^r - c_{qj}^r u^j)c_{pr}^i\partial_{u^i}$$

$$= \tfrac{1}{4}(\tfrac{\partial}{\partial u^q}\mathbf{H}^p - c_{qj}^p u^j)(\tfrac{\partial}{\partial u^p}\mathbf{H}^i - c_{pr}^i u^r)\partial_{u^i} - \tfrac{1}{2}u^p(\tfrac{\partial}{\partial u^q}\mathbf{H}^r - c_{qj}^r u^j)c_{pr}^i\partial_{u^i} \quad (\text{mod } \mathcal{H})$$

$$= \tfrac{1}{4}\tfrac{\partial}{\partial u^q}\mathbf{H}^p\tfrac{\partial}{\partial u^p}\mathbf{H}^i\partial_{u^i} - \tfrac{1}{4}c_{qj}^p u^j\tfrac{\partial}{\partial u^p}\mathbf{H}^i\partial_{u^i} + \tfrac{1}{4}c_{pr}^i u^r\tfrac{\partial}{\partial u^q}\mathbf{H}^p\partial_{u^i} - \tfrac{1}{4}c_{qj}^p c_{pr}^i u^j u^r\partial_{u^i}, \quad (20)$$

in which the first summand in the third line of (20) vanishes because $\mathbf{G}_0$ is right invariant, and the third summand in the sixth line vanishes because $\mathbf{H}^i$'s, u^i's and ∂_{u^i}'s are left invariant, i.e., $\widetilde{V}_j\mathbf{H}^i = 0$, $\widetilde{V}_j u_i = 0$ and $[\widetilde{V}_j, \partial_{u^i}] = 0$, $\forall i, j$.

Finally, we add (19) and (20), then we get (18) and end the proof of Theorem C.

4. Spray vector field for a left invariant spray structure

4.1. Spray vector field and left invariant curvature formulae

For a left invariant spray structure $\mathbf{G} = \mathbf{G}_0 - \mathbf{H} = \mathbf{G}_0 - \mathbf{H}^i\partial_{u^i}$ on the Lie group G, the restriction $\eta = \mathbf{H}|_{T_eG\backslash\{0\}}$ is a smooth tangent vector field on $T_eG\backslash\{0\} = \mathfrak{g}\backslash\{0\}$. Following the convention of L. Huang [7], we sometimes view it as a smooth map $\eta : \mathfrak{g}\backslash\{0\} \to \mathfrak{g}$, i.e., $\eta(y) = \mathbf{H}^i(e, y)e_i$, $\forall y = u^i e_i \in \mathfrak{g}\backslash\{0\}$, and call it the *spray vector field* of $\mathbf{G}$.

We denote by $D\eta(y, v)$ the derivative of $\eta : \mathfrak{g}\backslash\{0\} \to \mathfrak{g}$ at $y \in \mathfrak{g}\backslash\{0\}$ in the direction of v, and we call $N(\cdot, \cdot) : (\mathfrak{g}\backslash\{0\}) \times \mathfrak{g} \to \mathfrak{g}$, $N(y, v) = \tfrac{1}{2}D\eta(y, v) - \tfrac{1}{2}[y, v]$, the *connection operator* of $\mathbf{G}$. Notice that $N(\cdot, \cdot)$ is linear for the second entry.

When $\mathbf{G} = \mathbf{G}_F$ is induced by a left invariant Finsler metric F, Theorem 3.1 in [20] implies that its spray vector field η satisfies

$$g_y(\eta(y), u) = g_y(y, [u, y]), \quad \forall y \in \mathfrak{g} \backslash \{0\}, u \in \mathfrak{g}, \tag{21}$$

in which $g_y(\cdot, \cdot)$ is the inner product defined by the fundamental tensor of the Minkowski norm $F(e, \cdot) : \mathfrak{g} \to [0, +\infty)$. L. Huang found (21) independently (see Definition 3 in [8]), and he also found the following formula for the connection operator of a left invariant Finsler metric (see Definition 4 in [8] and (4) in [9])

$$2g_y(N(y, v), u) = g_y([u, v], y) + g_y([u, y], v) + g_y([v, y], u) - 2\mathbf{C}_y(u, v, \eta(y)).$$

Using the above notations, the curvature formulae in Theorem B and Theorem C can be reformulated as following.

Corollary 4.1. *Let* $\mathbf{G}$ *be a left invariant spray structure on the Lie group* G *with the spray vector field* $\eta : \mathfrak{g} \backslash \{0\} \to \mathfrak{g}$, *then its S-curvature and Riemann curvature satisfy*

$$\mathbf{S}(e, y) = \mathrm{tr}_{\mathbb{R}}(N(y, \cdot) + \mathrm{ad}(y)), \quad and \tag{22}$$

$$\mathbf{R}_y(v) = DN(y, v, \eta(y)) - N(y, N(y, v)) + N(y, [y, v]) - [y, N(y, v)], \tag{23}$$

respectively, for any $y \in \mathfrak{g} \backslash \{0\}$ *and* $v \in \mathfrak{g}$. *Here* $DN(y, v, \eta(y))$ *is the derivative of* $N(\cdot, v) : \mathfrak{g} \backslash \{0\} \to \mathfrak{g}$ *at* y *in the direction of* $\eta(y)$.

The formulae (22) and (23) coincide with those L. Huang found in homogeneous Finsler geometry [7]. By the left invariance, they contain all information for the S-curvature and Riemann curvature of $\mathbf{G}$ respectively.

4.2. Proof of Theorem D

Let $\mathbf{G} = \mathbf{G}_0 - \mathbf{H}^i \partial_{u^i}$ be a left invariant spray structure on the Lie group G, and $\eta : \mathfrak{g} \backslash \{0\} \to \mathfrak{g}$ its spray vector field. Now we prove Theorem D, i.e., the correspondence between geodesics for $\mathbf{G}$ and integral curves of $-\eta$.

Let $c(t)$ be any geodesic for $\mathbf{G}$. Using a standard local coordinate (x^i, y^i), it can be presented as $c(t) = (c^i(t))$ which satisfies

$$\ddot{c}^i(t) = -2\mathbf{G}^i(c(t), \dot{c}(t)), \quad \forall i, \tag{24}$$

where $-2\mathbf{G}^i$ is coefficient in the presentation $\mathbf{G} = y^i \partial_{x^i} - 2\mathbf{G}^i \partial_{x^i}$.

Now we switch to left invariant frame $\{\widetilde{U}_i, \partial_{u^i}, \forall i\}$ with notations in (3), i.e., $U_i = A_i^j \partial_{x^j}$ and $u^i = B_j^i y^j$, in which (A_i^j) and $(B_i^j) = (A_i^j)^{-1}$ are local functions on G with matrix values. We denote $\dot{c}(t) = u^i(t) U_i(c(t))$, then $\dot{c}^i(t) = A_j^i(c(t)) u^j(t)$ satisfies

$$\ddot{c}^i(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}(A_j^i(c(t)) u^j(t)) = A_j^i(c(t)) \dot{u}^j(t) + u^j(t) \dot{c}^k(t) \tfrac{\partial}{\partial x^k} A_j^i(c(t)). \tag{25}$$

Meanwhile, using (12), we can change (24) to

$$\ddot{c}^i(t) = -A_j^i(c(t)) \mathbf{H}^j(c(t), \dot{c}(t)) + u^j(t) \dot{c}^k(t) \tfrac{\partial}{\partial x^k} A_j^i(c(t)). \tag{26}$$

Comparing (25) and (26), we get

$$\dot{u}^j(t) + \mathbf{H}^j(c(t), \dot{c}(t)) = 0, \quad \forall j. \tag{27}$$

By the left invariance we obtain $\mathbf{H}^j(c(t), \dot{c}(t)) = \mathbf{H}^j(e, y(t))$, where

$$y(t) = (L_{c(t)^{-1}})_*(\dot{c}(t)) = u^j(t)e_j,$$

so that (27) implies that $y(t) = (L_{c(t)^{-1}})_* \dot{c}(t) = u^j(t)e_j$ is an integral curve of $-\eta = -\mathbf{H}|_{T_eG\backslash\{0\}}$.

The above argument provides the correspondence from (1) to (2) in Theorem D. Now we discuss its inverse.

For any smooth map $y(t) : (a, b) \to \mathfrak{g}\backslash\{0\} = T_eG\backslash\{0\}$, $a < 0 < b$, we claim the ordinary differential equation (ODE in short)

$$\dot{c}(t) = (L_{c(t)})_* y(t) \tag{28}$$

has a unique solution for $t \in (a, b)$ satisfying $c(0) = e$. To prove our claim, we only need to show the existence of $c(t)$ for $t \in (a + \epsilon, b - \epsilon)$, where $\epsilon > 0$ is arbitrarily small.

Using the existence theory for ODE and by the compactness of $[a + \epsilon, b - \epsilon]$, we can find a sufficiently small positive $\delta > 0$, such that for each $t_0 \in [a + \epsilon, b - \epsilon]$, the solution $c_{t_0}(t)$ of

$$\dot{c}_{t_0}(t) = (L_{c_{t_0}(t)})_*(y(t - t_0)), \quad \text{with} \quad c_{t_0}(0) = e \tag{29}$$

uniquely exists for $t \in (-\delta, \delta)$.

Let (a', b') be the maximal open sub-interval in (a, b) on which the solution $c(t)$ of (28) with $c(0) = e$ exists. Obviously a' and b' exist and $a' \leq -\delta$ and $b' \geq \delta$.

Now we prove $b' > b - \epsilon$ and $a' < a + \epsilon$. Assume conversely $b' \leq b - \epsilon$. Then we take $t_0 = b' - \frac{\delta}{2}$ and denote $c_{t_0}(t)$ with $t \in (-\delta, \delta)$ the corresponding solution of (29). By the left invariancy of the equation (28), $c_1(t) = c(t_0)c_{t_0}(t - t_0)$ with $t \in (t_0 - \delta, t_0 + \delta)$ is a solution of (28) satisfying $c_1(t_0) = c(t_0)$. By the uniqueness theory for ODE, the solution $c(t)$ for (28) can be extended to $(a', b' + \frac{\delta}{2})$. This is a contradiction with our assumption for b'. Similarly, we can also prove $a' < a + \epsilon$.

To summarize, the solution $c(t)$ of (28) with $c(0) = e$ exists uniquely on $(a+\epsilon, b-\epsilon)$ for arbitrarily small $\epsilon > 0$. So the solution $c(t)$ exists uniquely on (a, b), which proves our claim.

Finally, we can apply similar argument as for the correspondence from (1) to (2) to prove $c(t)$ is a geodesic on $(G, \mathbf{G})$ when $y(t)$ is an integral curve of $-\eta$. This ends the proof of Theorem D.

4.3. Remarks on bi-invariant spray structure

Applying Theorem D and Corollary 4.1 to the canonical bi-invariant spray structure $\mathbf{G}_0$, we get the following immediate consequence.

Corollary 4.2. *For the canonical bi-invariant spray structure $\mathbf{G}_0$ on a Lie group G, the set of all maximally extended geodesics consists of $c(t) = g \cdot \exp tX$ for all $g \in G$ and $X \in \mathfrak{g}$, its S-curvature is $\mathbf{S}(e, y) = \frac{1}{2}\mathrm{tr}_{\mathbb{R}}\mathrm{ad}(y)$, $\forall y \in \mathfrak{g}\backslash\{0\}$, and its Riemann curvature satisfies $\mathbf{R}_y(v) = -\frac{1}{4}[y, [y, v]]$, $\forall y \in \mathfrak{g}\backslash\{0\}, v \in \mathfrak{g}$.*

Here we can see many differences between spray geometry and Finsler geometry. Some observations below were independently discovered by L. Huang and X. Mo [10].

Firstly, a bi-invariant Finsler metric F exists on a Lie group G iff its Lie algebra $\mathfrak{g}$ is compact. By (21), the spray vector field for the bi-invariant F vanishes, so the geodesic spray $\mathbf{G}_F$ must be $\mathbf{G}_0$. Furthermore, bi-invariant Finsler metrics always have vanishing S-curvature and non-negative flag curvature, which implies that the linear map $\mathbf{R}_y(\cdot)$ for any $y \in \mathfrak{g}\backslash\{0\}$ is diagonalizable and has no negative eigenvalues.

But spray geometry tells us different stories. Theorem A claims at least one bi-invariant spray structure $\mathbf{G}_0$ on every Lie group. A Lie group may have many different bi-invariant spray structures (see Example 4.1.3 in [15] for another bi-invariant spray structure on $\mathbb{R}^2$). Corollary 4.2 indicates the S-curvature for $\mathbf{G}_0$ does not vanish when G is not unimodular (a 2-dimensional non-Abelian Lie group for example), and $\mathbf{R}_y(v) = -\frac{1}{4}[y, [y, v]]$ may have negative eigenvalues and may be non-diagonalizable.

Secondly, for a left invariant Riemannian or Finsler metric on a non-Abelian nilpotent Lie group, its Ricci scalar (i.e. the trace of the Riemann curvature) must be somewhere positive and somewhere negative [8, 18]. However, Corollary 4.1 indicates $\mathbf{R} = 0$ for any two-step nilpotent G (a Heisenberg group for example).

Finally, homogeneous Finsler spaces are all geodesically complete. However, we can construct many incomplete bi-invariant spray structures. For example,

$$\mathbf{G} = y^1 \partial_{x^1} + \cdots + y^n \partial_{x^n} + \sqrt{(y^1)^2 + \cdots + (y^n)^2}(y^1 \partial_{y^1} + \cdots + y^n \partial_{y^n})$$

is an incomplete bi-invariant spray structure on $\mathbb{R}^n$. Notice that any maximally extended integral curve $y(t)$ for

$$-\eta = \sqrt{(y^1)^2 + \cdots + (y^n)^2}(y^1 \partial_{y^1} + \cdots + y^n \partial_{y^n})$$

is a ray initiating from the origin, corresponding to $t \in (-\infty, C)$ for some $C \in \mathbb{R}$. By Theorem D, this spray structure is not complete.

4.4. Application to a special case of Landsberg Conjecture

Recently, B. Najafi and A. Tayebi proved the Landsberg Conjecture for a homogeneous Finsler surface (i.e., 2-dimensional homogeneous Finsler manifold) [17]. Their theorem claims

Theorem 4.3. *Any homogeneous Landsberg surface is either Riemannian or locally Minkowskian.*

Here we use Theorem D to give an alternative proof for it.

Proof of Theorem 4.3. Without loss of generality, we may assume the homogeneous Landsberg surface (M, F) is connected. Denote $G = I_0(M, F)$ the connected isometry group of (M, F). Its dimension $\dim G$ may only be 2 or 3. When $\dim G = 3$, (M, F) is a Riemannian surface with constant curvature. When $\dim G = 2$, F can be transferred to a left invariant Landsberg metric on G which

is locally isometric to (M, F). When G is Abelian, any left invariant Finsler metric on G is locally Minkowskian.

To summarize: to prove Theorem 4.3, we only need to prove any left invariant Landsberg metric F on a two-dimensional non-Abelian connected Lie group G is Riemannian.

For simplicity, we use the same F to denote the Minkowski norm $F(e, \cdot)$ on $\mathfrak{g} = \mathrm{Lie}(G)$, and η the spray vector field determined by (21), i.e.,

$$g_y(\eta(y), u) = g_y(y, [u, y]), \ \forall y \in \mathfrak{g} \backslash \{0\}, \ u \in \mathfrak{g}.$$

Notice that η is tangent to the indicatrix $S_F = \{y | F(y) = 1\} \subset \mathfrak{g}$.

We can find a basis $\{e_1, e_2\}$ for $\mathfrak{g}$ satisfying $[e_1, e_2] = e_2$. Denote $(g_{ij}(y))$ the fundamental tensor of the Minkowski norm $F : \mathfrak{g} \to [0, +\infty)$, with respect to the linear coordinates $y = y^i e_i$. Any point $y \in S_F$ satisfies $\eta(y) = 0$ iff

$$g_y(y, \mathbb{R}e_2) = g_y(y, [\mathfrak{g}, y]) = g_y(\eta(y), \mathfrak{g}) = 0.$$

By the strong convexity of F, there exists exactly two points $y', y'' \in S_F$ where η vanishes. The complement $S_F \backslash \{y', y''\}$ is the disjoint union of two integral curves $y_1(t)$ and $y_2(t)$ for $-\eta$, with $t \in (-\infty, \infty)$. By Theorem D, there exist F-unit speed geodesics $c_i(t)$ with $c_i(0) = e$ and $\dot{c}_i(t) = (L_{c_i(t)})_*(y_i(t))$ for all $t \in (-\infty, \infty)$. Let $w_i(t) \in \mathfrak{g} = T_eG$ be the smooth vector field along $y_i(t)$ satisfying

$$g_{y_i(t)}(y_i(t), w_i(t)) = 0, \ g_{y_i(t)}(w_i(t), w_i(t)) = 1, \quad \forall t. \tag{30}$$

After possibly replacing $w_1(t)$ with $-w_1(t)$, we may further assume that $w_1(t)$ and $w_2(t)$ can be smoothly glued and extended to the whole indicatrix S_F.

Notice that the left translations are isometries, so they preserve fundamental tensors. Then (30) implies that $W_i(t) = (L_{c_i(t)})_*(w_i(t))$ satisfies

$$g_{\dot{c}_i(t)}(\dot{c}_i(t), W_i(t)) = 0, \ g_{\dot{c}_i(t)}(W_i(t), W_i(t)) = 1, \quad \forall t. \tag{31}$$

As we have mentioned in Remark 2.8, linearly parallel translation along the geodesic $c_i(t)$ also preserves the fundamental tensor. Since $c_i(t)$ is a geodesic, the vector field $\dot{c}_i(t) = (L_{c_i(t)})_*(y_i(t))$ is linearly parallel along the geodesic $c_i(t)$. The speciality of dimension two and (31) tell us that $W_i(t) = (L_{c_i(t)})_*(w_i(t))$ is linearly parallel along the geodesic $c_i(t)$. Applying Lemma 2.7 to the Landsberg property of F, we see that the Cartan tensor of F satisfies

$$\mathbf{C}_{\dot{c}_i(t)}(W_i(t), W_i(t), W_i(t)) \equiv C_i, \ \ t \in (-\infty, \infty).$$

Using left translations again, we get

$$\mathbf{C}_{y_i(t)}(w_i(t), w_i(t), w_i(t)) = \mathbf{C}_{\dot{c}_i(t)}(W_i(t), W_i(t), W_i(t)) \equiv C_i, \ t \in (-\infty, \infty),$$

for $i = 1, 2$ respectively. When t approaches $\pm \infty$, we see $C_1 = C_2$ by continuity because the vector field $w_i(t)$ along $y_i(t)$ for $i = 1, 2$ can be smoothly glued and extended to S_F.

In sum we have $|\mathbf{C}_y(w, w, w)| = C = |C_i|$ for every pair (y, w) satisfying $y \in S_F$, $g_y(y, w) = 0$ and $g_y(w, w) = 1$. On the other hand, the speciality of dimension

two implies that $\mathbf{C}_y(w, w, w)$ coincides with the mean Cartan tensor $\mathbf{I}(w)$, i.e., the derivative of $f(y) = \frac{1}{2} \ln \det(g_{ij}(y))$ in the direction of w (see (7.3) in [16]). At $y \in S_F$ where f achieves its maximum or minimum, $C = \mathbf{C}_y(w, w, w) = \mathbf{I}(w) = 0$. For dimension two, this is enough for us to see that the Cartan tensor of F vanishes everywhere, i.e., (G, F) is Riemannian. ∎

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