

Affine Structures, Wreath Products and Free Affine Actions on Linear Non-Archimedean Trees

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Abstract. Let Λ be an ordered abelian group, $\text{Aut}^+(\Lambda)$ the group of order-preserving group automorphisms of Λ , G a group and $\alpha: G \rightarrow \text{Aut}^+(\Lambda)$ a homomorphism. An α -affine action of G on a Λ -tree X is one that satisfies $d(gx, gy) = \alpha_g d(x, y)$ ($x, y \in X$, $g \in G$). We consider classes of groups that admit a free affine action in the case where $X = \Lambda$. Such groups form a much larger class than in the isometric case. We show in particular that unitriangular groups $\text{UT}(n, \mathbb{R})$ and groups $T^*(n, \mathbb{R})$ of upper triangular matrices over \mathbb{R} with positive diagonal entries admit free affine actions. Our proofs involve left symmetric structures on the respective Lie algebras and the associated affine structures on the groups in question. We also show that given ordered abelian groups Λ_0 and Λ_1 and a free order-preserving affine action of G on Λ_0 , we obtain another such action of the wreath product $G \wr \Lambda_1$ on a suitable Λ' .

It follows that all free soluble groups, residually free groups and locally residually torsion-free nilpotent groups admit free affine actions on some Λ' .

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1. Introduction

Let Λ be an ordered abelian group. A Λ -metric space is defined in an entirely analogous way to a conventional (\mathbb{R} -)metric space. Just as one has \mathbb{R} -trees and group actions by isometries on \mathbb{R} -trees, group actions on Λ -trees by isometries have been studied in the literature, with a particular emphasis on free actions. This strand of geometric group theory has its origins in the notion of a length function on a group as developed by Lyndon [15]. Groups admitting a free isometric action (without inversions) on a Λ -tree can be thought of as a generalisation of free groups where the maximal abelian subgroups are stabilisers of lines in the Λ -tree and are naturally embedded in Λ . One of the principal novelties in contrast with free groups is that the maximal abelian subgroups are not necessarily cyclic. See [13] for a survey of results in this area.

There is a more general concept of group actions by affine automorphisms on Λ -trees. A permutation g of a Λ -tree X is an *affine automorphism with dilation factor* $\alpha_g \in \text{Aut}^+(\Lambda)$ if $d(gx, gy) = \alpha_g d(x, y)$ for all $x, y \in X$. Here $\text{Aut}^+(\Lambda)$ denotes the group of order-preserving group automorphisms of Λ . If $\alpha: G \rightarrow \text{Aut}^+(\Lambda)$ ($g \mapsto \alpha_g$) is a homomorphism, an α -affine action of G on a Λ -tree X is an action on X such

that each g is an affine automorphism of X with dilation factor α_g . We will often suppress the prefix α in practice and speak of an affine action. The study of such actions was initiated by I. Lioussé in [14], who considered the case $\Lambda = \mathbb{R}$, and was continued by the author in [16] and [17] where the case of a general ordered abelian group was studied.

As a special case one can consider group actions on linear Λ -trees: these are Λ -trees isometric to subtrees of Λ itself. In fact, in this paper the only Λ -trees we need to consider are of the form $X = \Lambda$ itself. Departing from standard usage, we will use the term ‘linear Λ -tree’ in this paper to mean a Λ -tree of the form $X = \Lambda$. Any reference to an affine action ‘on Λ ’ should also be taken to mean an affine action on Λ viewed as a Λ -tree.

In the isometric case, the analysis of groups that act on linear Λ -trees is easily done: a group admitting such an action factors through the group of isometries of Λ , which has the form $\Lambda \rtimes C_2$, where C_2 is a cyclic group of order 2 arising from a map of the form $\lambda \mapsto \lambda_0 - \lambda$. It follows that groups admitting order-preserving free actions on linear Λ -trees by isometries are embeddable in Λ .

In this paper we will consider exclusively actions where the order on Λ is preserved by all group elements. In this case an affine automorphism g is said to be *rigid* if either $g\lambda = \lambda$ for all $\lambda \in \Lambda$, or $g^\epsilon\lambda > \lambda$ for all $\lambda \in \Lambda$ where $\epsilon = \pm 1$. An order-preserving affine automorphism g is *hyperbolic* if it fixes no point, and the affine action of G is *free* if every non-trivial element is hyperbolic. We call g *essentially hyperbolic* if g is rigid and hyperbolic. The affine action of G is *essentially free* if each non-trivial g is essentially hyperbolic. (In the case of a non-linear Λ -tree, more general definitions are required – see [17, §2.1].)

The importance of essential freeness for our purposes is that it is robust under certain important operations, notably the base change functor and the Λ -fulfilment: see [17, §2.1] for further details. We note that free isometric actions without inversions are essentially free.

The main purpose of the current paper is to show that in the affine case groups admitting essentially free actions on a linear Λ -tree (for some Λ) form a much richer class than might be predicted from the isometric case. Our first main result is the following.

Theorem 1.1. *Every finitely generated torsion-free nilpotent group admits an essentially free affine action on \mathbb{Z}^m with the lexicographic order for some m .*

We will use the shorthand ITF for groups that admit a free isometric action without inversions on a Λ -tree for some Λ , or $\text{ITF}(\Lambda)$ if we wish to specify the ordered abelian group in question. Similarly ATF^e and $\text{ATF}^e(\Lambda)$ will be used for groups admitting an essentially free affine action on a Λ -tree. (In [17] we also use the notation ATF but we will not need this shorthand in the current paper.)

Note that locally fully residually ATF^e groups are ATF^e (see [16, Theorem 3.4]), and residually torsion-free nilpotent implies fully residually torsion-free nilpotent (see [4, §3] for example). Since an ultraproduct of groups with essentially free affine actions on linear Λ_i -trees itself admits an essentially free affine action on a linear Λ -tree (for some Λ) it follows that:

Corollary 1.2. *Every locally residually torsion-free nilpotent group G admits an essentially free affine action on some Λ .*

In particular, right-angled Artin groups, residually free groups and free products of residually free groups admit such an action.

A long-standing question of Baumslag asked whether free \mathbb{Q} -groups are residually torsion-free nilpotent. This has been answered recently in the affirmative by A. Jaikin-Zapirain [12]. It follows that free \mathbb{Q} -groups have an essentially free affine action on some linear Λ -tree.

Moreover, the free action given by Corollary 1.2 is ‘shift-free’ in the sense that each dilation factor α_g stabilises all convex subgroups of Λ .

The interplay between residually free, fully residually free and ITF groups is an interesting one. Note that a free product of residually free groups is itself residually free if and only if it is fully residually free, in which case the constituent groups are fully residually free. Moreover a residually free group is fully residually free if and only if it contains no subgroup of the form $F \times C_\infty$ where F is a free group of rank 2 and C_∞ is an infinite cyclic group (see [1]). Fully residually free groups are known to be ITF – see [5, §5.5]. Moreover an ITF group is fully residually free if and only if it is residually free. Restricting to $\Lambda = \mathbb{R}$, a surface group is ITF(\mathbb{R}) provided it is residually free. There are three ‘exceptional’ surface groups, namely the cyclic group of order two ($\pi_1(\mathbb{R}P)$), the Klein bottle group ($\pi_1(\mathbb{R}P \sharp \mathbb{R}P)$), and the fundamental group

$$\pi_1(\mathbb{R}P \sharp \mathbb{R}P \sharp \mathbb{R}P) = \langle x, y, z \mid x^2 y^2 z^2 = 1 \rangle$$

of the connected sum of three projective planes; these three groups are not residually free and do not act freely by isometries on an \mathbb{R} -tree. On the other hand the group $\pi_1(\mathbb{R}P \sharp \mathbb{R}P \sharp \mathbb{R}P)$ is ITF($\mathbb{Z} \times \mathbb{Z}$).

Our next main result concerns the groups $T^*(n, \mathbb{R})$ of upper triangular matrices with real entries and positive diagonal entries.

Theorem 1.3. *The group $T^*(n, \mathbb{R})$ admits an essentially free affine action on \mathbb{R}^k (with the lexicographic order) for some k .*

Our other main result concerns wreath products.

Theorem 1.4. *Let H be a group that admits an essentially free affine action on some Λ_0 , and let Λ_1 be another ordered abelian group. Then the wreath product $H \wr \Lambda_1$ has an essentially free affine action on $\Lambda_1 \times \Lambda'$ where Λ' is the subgroup of the product $\prod_{\omega \in \Lambda_1} \Lambda_0$ consisting of those elements with well-ordered support.*

It follows inductively that an iterated wreath product of torsion-free abelian groups $G = A_1 \wr \cdots \wr A_k$ admits an essentially free affine action on a linear Λ -tree. In case $k = 2$ it follows from a result of Baumslag [2] that G is residually torsion-free nilpotent, so one can also deduce that G admits a free affine action on a linear Λ -tree using Corollary 1.2.

Corollary 1.5. *Every free soluble group (i.e. every free group in the variety of soluble groups of given derived length) admits an essentially free affine action on a linear Λ -tree for some Λ .*

This corollary follows from Theorem 1.4 together with a well-known result of Šmel'kin [18]. We note also that an alternative proof can be given using Corollary 1.2. For a result of Gruenberg asserts that free polynilpotent groups (of given class row), and in particular free soluble groups (of given derived length), are residually torsion-free nilpotent (see [10, Theorem 7.1], [3, Theorem 1.3]). Combining this result with Corollary 1.2 gives Corollary 1.5. One notable difference is the action obtained via Corollary 1.2 is shift-free as already observed while the wreath product action arising from Theorem 1.4 is not shift-free.

It follows from our results that the group $\text{Aut}^+(\mathbb{Q}^n)$ of order-preserving automorphisms of \mathbb{Q}^n is ATF^e though it is not clear whether it can be shown that $\text{Aut}^+(\mathbb{R}^n)$ is ATF^e . The main obstacle to using the arguments in this paper in the latter case is the fact that the ring of additive endomorphisms of \mathbb{R} is non-commutative.

In §2 we show that the groups $\text{UT}(n, \mathbb{Z})$ are $\text{ATF}^e(\mathbb{Z}^m)$ for some m . There is a natural action of $\text{UT}(n+1, \mathbb{Z})$ on \mathbb{Z}^n by affine automorphisms. While this action is far from free, one can easily characterise those matrices that correspond to affine automorphisms that are rigid and have no fixed point – we use the term essentially hyperbolic to refer to these automorphisms and matrices. The problem then reduces to showing that $\text{UT}(n, \mathbb{Z})$ admits an embedding in $\text{UT}(m+1, \mathbb{Z})$ whose image consists of essentially hyperbolic matrices. The key idea here is to consider affine structures on $\text{UT}(n, \mathbb{K})$ (where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$) starting from a left symmetric structure on $\mathfrak{ut}(n, \mathbb{K})$. These give rise to matrices with a natural block decomposition which can then be shown to be essentially hyperbolic. From this we obtain an embedding $\bar{\gamma} : \text{UT}(n, \mathbb{K}) \rightarrow \text{UT}(m+1, \mathbb{K})$ whose image consists of essentially hyperbolic matrices, and thence the desired embedding $\text{UT}(n, \mathbb{Z}) \rightarrow \text{UT}(m+1, \mathbb{Z})$.

Taking $\mathbb{K} = \mathbb{R}$ we then show in §3 how to extend the embedding $\bar{\gamma}$ to one of $T^*(n, \mathbb{R})$ in an appropriate $T^*(k, \mathbb{R})$. It is necessary to increase the dimension of the codomain by n and incorporate the log of the diagonal entries of $T^*(n, \mathbb{R})$.

Finally in §4 we consider wreath products of groups that admit essentially free affine actions on some Λ_0 with ordered abelian groups and prove Theorem 1.4. In fact we prove a somewhat stronger result concerning a more general wreath product which we call the lexicographic wreath product which contains the restricted wreath product. This enables us to apply Šmel'kin's theorem as outlined above.

2. Free affine actions of unitriangular groups

2.1. Affine actions on linear Λ -trees

We refer to [5] for the basic theory of Λ -trees and isometric group actions thereon, and to [16, §1] for the basic theory of affine actions on Λ -trees. Let Λ be a linearly ordered abelian group written additively.

A Λ -metric space is a set X equipped with a Λ -valued distance function d satisfying the usual metric space axioms (with \mathbb{R} replaced by Λ). Note that Λ is itself a Λ -metric space where $d(x, y) = |x - y| = \max\{x - y, y - x\}$. In fact, (X, d) is a Λ -tree. As noted in the introduction, the only Λ -trees that we will need to consider in this paper have this form.

We will make use of the lexicographic order in a somewhat general context. Let Ω be a linearly ordered set, and let a group G_ω be given for each $\omega \in \Omega$. We let

$\mathcal{L}_{\omega \in \Omega} G_\omega$ denote the subgroup of the Cartesian product $\prod_{\omega \in \Omega} G_\omega$ consisting of those $(g_\omega)_{\omega \in \Omega}$ for which $\{\omega \in \Omega : g_\omega \neq 1\}$ is a well-ordered subset of Ω . Suppose in particular that $G_\omega = \Lambda_\omega$ is an ordered abelian group for each ω . Note that

$$\bigoplus_{\omega \in \Omega} \Lambda_\omega \leq \mathcal{L}_{\omega \in \Omega} \Lambda_\omega \leq \prod_{\omega \in \Omega} \Lambda_\omega.$$

We define a partial order on the Cartesian product by declaring $(\lambda_\omega)_{\omega \in \Omega} < (\mu_\omega)_{\omega \in \Omega}$ if $\lambda_{\omega_0} < \mu_{\omega_0}$ where $\omega_0 = \min\{\omega \in \Omega : \lambda_\omega \neq \mu_\omega\}$. This restricts to a linear order on $\mathcal{L}_{\omega \in \Omega} \Lambda_\omega$ making it an ordered abelian group, which we will call the *lexicographic product of the Λ_ω* . If $\Omega = \{1, \dots, n\}$ these three groups coincide, and are typically written $\Lambda_1 \times \dots \times \Lambda_n$, or Λ_0^n if the Λ_ω are equal to a common Λ_0 . In §2 and §3 the indexing set Ω will be finite, though the wreath product construction in §4 will require an infinite linearly ordered Ω .

If A is a subset of an ordered abelian group Λ such that $x, z \in A$ and $x \leq y \leq z$ imply $y \in A$, we say that A is a convex subset; in particular one may speak of a convex subgroup. The convex subgroups of an ordered abelian group are linearly ordered by inclusion. In the case of a lexicographic product as above, each ω determines a convex subgroup. If $x, y \in \Lambda$ and $nx < |y|$ for all $n \in \mathbb{Z}$ we write $x \ll |y|$; this is equivalent to the proper inclusion $[x] \subset [y]$ where $[x]$ denotes the convex subgroup spanned by x .

The group of o -automorphisms (that is, order-preserving group automorphisms) of an ordered abelian group Λ will be denoted $\text{Aut}^+(\Lambda)$. Note that for subgroups Λ_0 of the real additive group, o -automorphisms of Λ_0 have the form $x \mapsto \alpha^*x$ for some positive α^* . Now consider the ordered abelian group Λ_0^n (with the lexicographic order). In this context, it is useful to view elements of Λ_0^n as column vectors $(x_n, \dots, x_1)^\top$, which we will do throughout this paper. (Here \mathbf{x}^\top denotes the transpose.) Thus, for example,

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ll \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ll \dots \ll \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

This convention will enable o -automorphisms to be represented by upper triangular matrices acting on the left: elements of $\text{Aut}^+(\Lambda_0^n)$ have the form

$$\begin{pmatrix} \alpha_n^* & h_{n(n-1)} & \dots & h_{n1} \\ 0 & \alpha_{n-1}^* & \dots & h_{(n-1)1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_1^* \end{pmatrix} \tag{1}$$

where each $\alpha_i^* \in \text{Aut}^+(\Lambda_0)$ and h_{ij} is an additive group endomorphism of Λ_0 ; see [16, §1.1].

For a subfield \mathbb{K} of \mathbb{R} , we write $\text{UT}(n, \mathbb{K})$ to denote the group of upper triangular matrices with off-diagonal entries belonging to \mathbb{K} , and diagonal entries equal to 1. Note that $\text{UT}(n, \mathbb{K})$ embeds naturally in $\text{Aut}^+(\mathbb{K}^n)$. Since $\text{Aut}^+(\mathbb{Z})$ is trivial and endomorphisms of \mathbb{Z} are given by integer multiplication it follows from (1) that

$$\text{Aut}^+(\mathbb{Z}^n) \cong \text{UT}(n, \mathbb{Z}). \tag{2}$$

Moreover, $\text{UT}(n, \mathbb{K}) \cong \text{UT}(n-1, \mathbb{K}) \rtimes \mathbb{K}^{n-1}$. In general, matrices of the form (1) with diagonal entries equal to 1 correspond to o -automorphisms of the form $\lambda \mapsto \lambda + \zeta(\lambda)$ where $\zeta: \Lambda_0^n \rightarrow \Lambda_0^n$ is an additive homomorphism satisfying $\zeta(\lambda) \ll |\lambda|$ for $\lambda \neq 0$. Since additive group endomorphisms $\mathbb{Q} \rightarrow \mathbb{Q}$ are \mathbb{Q} -linear, it follows that for each $i > j$ the h_{ij} act by scalar multiplication. Thus $\text{Aut}^+(\mathbb{Q}^n) \cong T^*(n, \mathbb{Q})$, the group of rational upper triangular matrices with positive diagonal entries. However, while $T^*(n, \mathbb{R})$ embeds in $\text{Aut}^+(\mathbb{R}^n)$, the embedding is not surjective since additive endomorphisms of \mathbb{R} are not necessarily \mathbb{R} -linear.

Now let G be a group and $\alpha: G \rightarrow \text{Aut}^+(\Lambda)$ ($g \mapsto \alpha_g$) a homomorphism. An α -affine action of G on a Λ -tree X is an action on X such that $d(gx, gy) = \alpha_g d(x, y)$ for all $x, y \in X$; of course isometric actions correspond to the case where α is trivial.

Note that if g is an order-preserving affine automorphism of Λ with dilation factor $\alpha_g \in \text{Aut}^+(\Lambda)$, then there exists $\nu_g \in \Lambda$ such that $g \cdot \lambda = \alpha_g \lambda + \nu_g$ for all λ . This equation can be written in matrix form as

$$\begin{pmatrix} g\mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_g & \nu_g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}; \quad (3)$$

moreover the matrix γ_{gh} corresponding to gh is the product of the matrices γ_g and γ_h corresponding to g and h . It follows that the group of order-preserving affine automorphisms of Λ has the form $\text{Aut}^+(\Lambda) \rtimes \Lambda$. If $\Lambda = \mathbb{K}^n$ then $\text{UT}(n+1, \mathbb{K})$ has a natural affine action on \mathbb{K}^n .

Henceforth we will assume that *all affine actions on Λ (as a linear Λ -tree) preserve the order*. For a subfield \mathbb{K} of \mathbb{R} we will identify elements of $\text{UT}(n+1, \mathbb{K})$ with the corresponding affine automorphism of \mathbb{K}^n arising from (1) and (3).

The following fact can be deduced from the observations made so far.

Lemma 2.1. *The group $\text{UT}(n+1, \mathbb{Z})$ is isomorphic to the group of all affine automorphisms of \mathbb{Z}^n .*

An affine automorphism g of Λ is *rigid* if either g fixes all points of Λ , or $g\lambda > \lambda$ for all λ , or $g\lambda < \lambda$ for all $\lambda \in \Lambda$.

We say that an (order-preserving) automorphism g of Λ is *hyperbolic* if no point is fixed by g . We note that affine hyperbolic automorphisms of non-linear Λ -trees are not as well-behaved as one might expect from the isometric case; see [16, §1] for further discussion. Here we will consider affine actions on Λ -trees that are essentially free; see [17, §3.1] for the general definition and basic properties of essentially free actions. As stated in the introduction, in our case where the Λ -tree is equal to Λ an affine action is essentially free if each non-trivial group element g is rigid and hyperbolic (or *essentially hyperbolic*). This amounts to the requirement that in (3) we have $(1 - \alpha_g)(\lambda) \ll |\nu_g|$ for all λ . If $\Lambda = \mathbb{K}^n$ this amounts to the requirement that the lowest non-zero entry of $\gamma_g - 1$ appears in the last column, and this entry is strictly lower than every other non-zero entry of $\gamma_g - 1$. (Here 1 denotes the identity matrix.) That is, a non-trivial automorphism g is essentially hyperbolic if the following implication holds for $\gamma_g = (a_{ij})$:

If, for some $i < n+1$, either $a_{ii} \neq 1$ or $a_{ij} \neq 0$ for some $i < j < n+1$,
then $a_{k(n+1)} \neq 0$ for some $i < k < n+1$.

Note that a product of essentially hyperbolic automorphisms is not necessarily essentially hyperbolic.

The promised free affine action of $UT(n, \mathbb{Q})$ on \mathbb{Q}^m for a suitable m will be obtained as a special case of an embedding $\bar{\gamma}: UT(n, \mathbb{K}) \rightarrow UT(m + 1, \mathbb{K})$ and showing that non-trivial elements of the image are essentially hyperbolic with respect to the natural affine action on \mathbb{K}^m . Note that most of our arguments in this section work for any ordered field \mathbb{K} . However, in §3 we restrict to $\mathbb{K} = \mathbb{R}$ since we need $\log x \in \mathbb{K}$ for positive $x \in \mathbb{K}$.

2.2. Affine structures and left symmetric structures

Let \mathbb{K} be a subfield of \mathbb{R} . We will proceed by considering the standard left symmetric structure on the Lie algebra $\mathfrak{g} = \mathfrak{ut}(n, \mathbb{K})$ consisting of all $n \times n$ matrices with entries belonging to \mathbb{K} and zeroes on and below the main diagonal, and using it to obtain a complete affine structure $d\bar{\gamma}$ on \mathfrak{g} , and thence the promised free action of $UT(n, \mathbb{K})$. We refer to [8, §2] for an account of the requisite Lie algebra material (including an example covering the case $n = 3$). Essentially we follow the steps (1)–(4) as described in that paper in reverse order. We note however that not all simply transitive actions arising in this way are essentially hyperbolic in our sense: the final example of [7] is not essentially hyperbolic.

So observe first that the Lie algebra $\mathfrak{g} = \mathfrak{ut}(n, \mathbb{K})$ is graded: that is, \mathfrak{g} decomposes as a direct sum of $n - 1$ vector subspaces σ_i consisting of matrices S_i^* whose non-zero entries appear only on the i th superdiagonal. Crucially, one has $[\sigma_i, \sigma_j] \subseteq \sigma_{i+j}$ where, as usual, $[A, B] = AB - BA$, and σ_k is taken to be the zero subalgebra if $k \geq n$. In fact in our case we have $\sigma_i\sigma_j \subseteq \sigma_{i+j}$.

Arising from any grading, one can define a left symmetric structure that is compatible with the Lie bracket using the following formula and extending the operation to \mathfrak{g} by bilinearity:

$$S_i^* \cdot S_j^* = \frac{j}{i+j} [S_i^*, S_j^*] \tag{4}$$

Next, define a linear map $\lambda': \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ via

$$\lambda'(\mathbf{x}) : \mathbf{y} \mapsto \mathbf{x} \cdot \mathbf{y}. \tag{5}$$

In order to represent the $\lambda'(\mathbf{x})$ as matrices we now represent \mathfrak{g} as a set of column vectors as follows. For $\mathbf{x}, \mathbf{y} \in \mathfrak{g}$, we write $\mathbf{x} = (x_{ij})$ and $\mathbf{y} = (y_{ij})$ with $x_{ij} = y_{ij} = 0$ for $i \geq j$. We now use the following notation for the superdiagonals of \mathbf{x} and \mathbf{y} ,

$$\begin{array}{ll} S_1 & = (x_{12}, x_{23}, \dots, x_{n-1 \ n}) & T_1 & = (y_{12}, y_{23}, \dots, y_{n-1 \ n}) \\ S_2 & = (x_{13}, x_{24}, \dots, x_{n-2 \ n}) & T_2 & = (y_{13}, y_{24}, \dots, y_{n-2 \ n}) \\ S_3 & = (x_{14}, x_{25}, \dots, x_{n-3 \ n}) & T_3 & = (y_{14}, y_{25}, \dots, y_{n-3 \ n}) \\ & \vdots & & \vdots \\ S_{n-1} & = (x_{1n}) & T_{n-1} & = (y_{1n}), \end{array} \tag{6}$$

and write \mathbf{x} in the form $\mathbf{x} = \langle S_1, S_2, \dots, S_{n-1} \rangle$.

In the opposite direction we associate an element S^* of \mathfrak{g} to a vector S of length $n - i$ via

$$S^* = \langle 0, \dots, 0, \underbrace{S}_{i\text{th position}}, 0, \dots, 0 \rangle.$$

Then $S_i^* \cdot T_j^* = (W_{i,j})^*$ where $W_{i,j} = (w_1, \dots, w_{n-(i+j)})$ and

$$w_k = \frac{j}{i+j} (x_{k,i+k} y_{i+k,i+j+k} - y_{k,j+k} x_{j+k,i+j+k}). \quad (7)$$

Of course if $n \leq i+j$, we take $(W_{i,j})^*$ to be the zero matrix.

We now set $\lambda'(\mathbf{x})(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, and writing \mathbf{x} and \mathbf{y} as linear combinations of S_i^* and T_j^* respectively ($1 \leq i, j \leq n-1$), we obtain

$$\lambda'(\mathbf{x})(\mathbf{y}) = \sum_{r=2}^{n-1} \sum_{i=1}^{r-1} (W_{i,r-i})^* = \sum_{r=2}^{n-1} \bar{W}_r^*$$

where $\bar{W}_r = (\bar{w}_1, \dots, \bar{w}_{n-r})$ and

$$\bar{w}_k = \sum_{i=1}^{r-1} \frac{r-i}{r} (x_{k,i+k} y_{i+k,r+k} - y_{k,r-i+k} x_{r-i+k,r+k}). \quad (8)$$

Note that all entries of \bar{W}_r^* are zero apart from the entries on the r th superdiagonal.

Taking $m = \frac{n(n-1)}{2}$ we define a linear isomorphism $t : \mathfrak{g} \rightarrow \mathbb{K}^m$ via

$$\prec S_1, S_2, \dots, S_{n-1} \succ \mapsto (S_{n-1}, S_{n-2}, \dots, S_1)^\top. \quad (9)$$

Now $\lambda'(\mathbf{x})$ can be represented as an $m \times m$ matrix. More precisely, $\lambda'(\mathbf{x})$ determines a matrix $\lambda(\mathbf{x}) \in \mathbf{ut}(m, \mathbb{K})$ where $\lambda(\mathbf{x}) = t \cdot \lambda'(\mathbf{x}) \cdot t^{-1}$. Thus

$$\begin{aligned} \lambda(\mathbf{x})(T_{n-1}, \dots, T_1)^\top &= \lambda(\mathbf{x}) \cdot t \prec T_1, \dots, T_{n-1} \succ = t \cdot \lambda'(\mathbf{x}) \prec T_1, \dots, T_{n-1} \succ \\ &= t \cdot \lambda'(\mathbf{x})(\mathbf{y}) = t \sum_{r=2}^{n-1} \bar{W}_r^* = t \sum_{r=2}^{n-1} \prec 0, \dots, 0, \underbrace{\bar{W}_r}_{r}, 0, \dots, 0 \succ \\ &= \sum_{r=2}^{n-1} \prec 0, \dots, 0, \underbrace{\bar{W}_r}_{n-r}, 0, \dots, 0 \top = (\bar{W}_{n-1}, \bar{W}_{n-2}, \dots, \bar{W}_2, 0)^\top. \end{aligned}$$

Now observe that T_{n-j} is a vector of length j consisting of entries of the form $y_{k,k+(n-j)}$. Moreover we can write $\lambda(\mathbf{x})$ in the block form

$$\lambda(\mathbf{x}) = \begin{pmatrix} M_{n-1,n-1} & M_{n-1,n-2} & \cdot & \cdot & \cdot & M_{n-1,1} \\ M_{n-2,n-1} & M_{n-2,n-2} & \cdot & \cdot & \cdot & M_{n-2,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{2,n-1} & M_{2,n-2} & \cdot & \cdot & \cdot & M_{2,1} \\ M_{1,n-1} & M_{1,n-2} & \cdot & \cdot & \cdot & M_{1,1} \end{pmatrix} \quad (10)$$

where each $M_{n-i,n-j}$ is an $i \times j$ matrix. It follows that

$$\sum_{j=1}^{n-1} M_{n-i,n-j} T_{n-j}^\top = \bar{W}_{n-i}^\top. \quad (11)$$

In the next section we will need a somewhat detailed description of the entries of $\lambda(\mathbf{x})$, so we will derive it here. First, for $\nu \in \mathbb{N}$ put $k_\nu = \min\{k \in \mathbb{N} : \nu \leq \sum_{\eta=1}^k \eta\}$, and $r_\nu = \nu - \sum_{\eta=1}^{k_\nu-1} \eta$. Thus $1 \leq r_\nu \leq k_\nu$ and

$$\rho = \sum_{\eta=1}^{k_\rho-1} \eta + r_\rho \quad \text{and} \quad \sigma = \sum_{\eta=1}^{k_\sigma-1} \eta + r_\sigma. \tag{12}$$

Lemma 2.2. *The (ρ, σ) entry of $\lambda(\mathbf{x})$ is*

$$m_{\rho, \sigma} = \begin{cases} \frac{n - k_\sigma}{n - k_\rho} x_{r_\rho, r_\rho + (k_\sigma - k_\rho)} & \text{if } k_\rho < k_\sigma \text{ and } r_\rho < r_\sigma \text{ and } r_\sigma - r_\rho = k_\sigma - k_\rho \\ -\frac{n - k_\sigma}{n - k_\rho} x_{n - k_\sigma + r_\rho, n - k_\rho + r_\rho} & \text{if } k_\rho < k_\sigma \text{ and } r_\rho = r_\sigma \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Proof. Observe that the (ρ, σ) entry of $\lambda(\mathbf{x})$ is the (r_ρ, r_σ) entry of the (k_ρ, k_σ) block $M_{n-k_\rho, n-k_\sigma}$ of M : recall that this block has dimensions $k_\rho \times k_\sigma$. Using equation (11) we now obtain

$$\sum_{j=1}^{n-1} M_{n-k_\rho, n-j} \cdot \begin{pmatrix} y_{1, n-j+1} \\ y_{2, n-j+2} \\ \vdots \\ y_{j, n} \end{pmatrix} = \bar{W}_{n-k_\rho}^\top = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{k_\rho} \end{pmatrix} \tag{14}$$

where by (8)

$$z_p = \sum_{l=1}^{n-k_\rho-1} \frac{n - k_\rho - l}{n - k_\rho} (x_{p, p+l} y_{p+l, n-k_\rho+p} - y_{p, n-k_\rho+p-l} x_{n-k_\rho+p-l, n-k_\rho+p}). \tag{15}$$

Suppose that T_{n-k_σ} is the only non-zero superdiagonal of \mathbf{y} and that all entries of this superdiagonal are non-zero. Then (14) simplifies to $M_{n-k_\rho, n-k_\sigma} T_{n-k_\sigma}^\top = \bar{W}_{n-k_\rho}^\top$. Suppose that the (r_ρ, r_σ) entry of $M_{n-k_\rho, n-k_\sigma}$ is non-zero. Then equating the r_ρ row of (14) and the expression (15) with $p = r_\rho$ we see that the coefficient of $y_{r_\sigma, r_\sigma + (n-k_\sigma)}$ is non-zero whence $n - k_\rho - l = n - k_\sigma$ giving $l = k_\sigma - k_\rho$. Moreover we have either

- (1) $r_\rho < r_\sigma$, $k_\sigma - k_\rho = r_\sigma - r_\rho$ and the coefficient of $y_{r_\sigma, r_\sigma + (n-k_\sigma)} \neq 0$ has the form

$$\frac{n - k_\sigma}{n - k_\rho} x_{r_\rho, r_\rho + (k_\sigma - k_\rho)}, \quad \text{or}$$

- (2) $r_\rho = r_\sigma$ and the coefficient of $y_{r_\sigma, r_\sigma + (n-k_\sigma)} \neq 0$ has the form

$$-\frac{n - k_\sigma}{n - k_\rho} x_{n - k_\sigma + r_\rho, n - k_\rho + r_\rho}.$$

It now follows that $m_{\rho, \sigma}$ has the claimed form. ■

We now obtain a function $d\bar{\gamma}: \mathfrak{g} \rightarrow \mathfrak{ut}(m + 1, \mathbb{K})$ via

$$d\bar{\gamma}(\mathbf{x}) = \begin{pmatrix} \lambda(\mathbf{x}) & t(\mathbf{x}) \\ 0 & 0 \end{pmatrix}. \tag{16}$$

Note that $d\bar{\gamma}$ is a Lie algebra homomorphism into the affine algebra $\mathfrak{aff}(m)$ where elements of the image have linear part $\lambda(\mathbf{x})$ and translation part $t(\mathbf{x})$, where $\lambda(\mathbf{x})$ is a nilpotent matrix for all \mathbf{x} and where t is an isomorphism of vector spaces. That $d\bar{\gamma}$ preserves the Lie bracket follows from its definition in terms of the left symmetric structure. (It does not, however, preserve matrix multiplication.) It follows that $\bar{\gamma} = \exp \cdot d\bar{\gamma} \cdot \log$ is a (Lie) group homomorphism $\mathrm{UT}(n, \mathbb{K}) \rightarrow \mathrm{UT}(m+1, \mathbb{K})$; here \exp and \log denote the matrix exponential and logarithm functions, which can be evaluated as a finite sum in the case at hand since $d\bar{\gamma}(\mathbf{x})$ is a nilpotent matrix.

Lemma 2.3. (a) *If a matrix of the form $d\bar{\gamma}(\mathbf{x})$ is non-zero, then at least one block of its last column is non-zero. Moreover, if $S_{i_0} \neq 0$ and $S_i = 0$ for all $i < i_0$ then $M_{i,j} = 0$ for all i and j such that $i - j < i_0$.*

(b) *$\bar{\gamma}(g)$ is essentially hyperbolic for $g \neq 1$.*

Proof. (a) If all blocks of the last column of $d\bar{\gamma}(\mathbf{x})$ are zero then $\mathbf{x} = 0$ since t is a linear isomorphism and $d\bar{\gamma}(\mathbf{x}) = 0$. The stated condition on S_{i_0} implies that it is the i_0 th superdiagonal and the first non-zero superdiagonal of \mathbf{x} . Since any non-zero entries of $M_{n-i,n-j}$ are constant multiples of entries of the $(j-i)$ th superdiagonal of \mathbf{x} , we see that the blocks $M_{i,j}$ are zero for $i - j < i_0$.

(b) For $g \neq 1$ we put $\mathbf{x} = \log g$, take i_0 as in part (a), and note that the foregoing argument shows that all block superdiagonals (with respect to the block decomposition in (10)) of $N = d\bar{\gamma}(\mathbf{x})$ below the i_0 th are zero. Thus N^k is a matrix in which all block superdiagonals below the (ki_0) th are zero. It follows that the lowest non-zero superdiagonal of $\sum_{k=0}^{m-1} \frac{1}{k!} N^k - I$ remains the i_0 th, and the lowest non-zero entry in this matrix is in the last column and is strictly lower than any other non-zero entries.

Since $\bar{\gamma}(g) = \exp(N)$, this shows that $\bar{\gamma}(g)$ is essentially hyperbolic. \blacksquare

Example 2.4. Let us illustrate the constructions described by considering the case $n = 4$. Here $m = 6$. Consider generic elements \mathbf{x} and \mathbf{y} of $\mathfrak{ut}(4, \mathbb{Q})$; these have the form

$$\mathbf{x} = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 0 & y_{12} & y_{13} & y_{14} \\ 0 & 0 & y_{23} & y_{24} \\ 0 & 0 & 0 & y_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Expressing \mathbf{x} and \mathbf{y} as sums of superdiagonal matrices, applying the definition of the left symmetric structure (4) and using the bilinearity of this structure we obtain

$$\begin{aligned} \lambda'(\mathbf{x})(\mathbf{y}) &= \mathbf{x} \cdot \mathbf{y} \\ &= \begin{pmatrix} 0 & 0 & \frac{1}{2}(x_{12}y_{23} - y_{12}x_{23}) & \frac{2}{3}(x_{12}y_{24} - y_{13}x_{34}) + \frac{1}{3}(x_{13}y_{34} - y_{12}x_{24}) \\ 0 & 0 & 0 & \frac{1}{2}(x_{23}y_{34} - y_{23}x_{34}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{17}$$

Take $t : \mathfrak{ut}(4, \mathbb{Q}) \rightarrow \mathbb{Q}^6$ to be the map $\mathbf{y} \mapsto (y_{14}, y_{13}, y_{24}, y_{12}, y_{23}, y_{34})^\top$.

Then the map $\lambda'(\mathbf{x})$ can be represented as the matrix

$$\lambda(\mathbf{x}) = \left(\begin{array}{ccc|ccc} 0 & -\frac{2}{3}x_{34} & \frac{2}{3}x_{12} & -\frac{1}{3}x_{24} & 0 & \frac{1}{3}x_{13} \\ 0 & 0 & 0 & -\frac{1}{2}x_{23} & \frac{1}{2}x_{12} & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{2}x_{34} & \frac{1}{2}x_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \in \mathfrak{ut}(6, \mathbb{Q}).$$

One can easily verify that the formula in Lemma 2.2 is satisfied. The matrix $d\bar{\gamma}(\mathbf{x})$ is obtained by adjoining the column $t(\mathbf{x}) = (x_{14}, x_{13}, x_{24}, x_{12}, x_{23}, x_{34})^\top$ and a row of zeroes.

Recall that the exponential of an element N of $\mathfrak{ut}(n, \mathbb{K})$ has the form $\sum_{k=0}^{n-1} \frac{1}{k!} N^k$, while in a similar spirit $\log(I + B)$ may be similarly evaluated as $\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} B^k$ for $B \in \mathfrak{ut}(n, \mathbb{K})$.

Therefore for $A = \begin{pmatrix} 1 & c & e & f \\ 0 & 1 & b & d \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we have

$$\bar{\gamma}(A) = \exp \cdot d\bar{\gamma} \cdot \log(A) = \exp \cdot d\bar{\gamma} \left(\begin{array}{cccc} 0 & c & e - \frac{1}{2}bc & f - \frac{1}{2}cd - \frac{1}{2}ae + \frac{1}{3}abc \\ 0 & 0 & b & d - \frac{1}{2}ab \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$= \exp \left(\begin{array}{ccc|ccc} 0 & -\frac{2}{3}a & \frac{2}{3}c & -\frac{1}{3}d + \frac{1}{6}ab & 0 & \frac{1}{3}e - \frac{1}{6}bc & f - \frac{1}{2}cd - \frac{1}{2}ae + \frac{1}{3}abc \\ \hline 0 & 0 & 0 & -\frac{1}{2}b & \frac{1}{2}c & 0 & e - \frac{1}{2}bc \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a & \frac{1}{2}b & d - \frac{1}{2}ab \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & a \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & -\frac{2}{3}a & \frac{2}{3}c & -\frac{1}{3}d + \frac{1}{3}ab & -\frac{1}{3}ac & \frac{1}{3}e & f - \frac{1}{3}cd - \frac{2}{3}ae + \frac{1}{3}abc \\ \hline 0 & 1 & 0 & -\frac{1}{2}b & \frac{1}{2}c & 0 & e - \frac{1}{2}bc \\ 0 & 0 & 1 & 0 & -\frac{1}{2}a & \frac{1}{2}b & d - \frac{1}{2}ab \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 1 & a \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

The main facts established so far in this section can be summarised as follows.

Proposition 2.5. *Let \mathbb{K} be a subfield of \mathbb{R} , and set $m = n(n-1)/2$. There exists an injective group homomorphism $\bar{\gamma}: \text{UT}(n, \mathbb{K}) \rightarrow \text{UT}(m+1, \mathbb{K})$ for which the non-trivial elements of the image are essentially hyperbolic with respect to the natural affine action on \mathbb{K}^m .*

Note that in Example 2.4 the entries of $\bar{\gamma}(A)$ are not typically integers. We next show that if $\mathbb{K} = \mathbb{Q}$ then $\bar{\gamma}$ can be adjusted so that $\bar{\gamma}$ maps $\text{UT}(n, \mathbb{Z})$ into $\text{UT}(m+1, \mathbb{Z})$.

Lemma 2.6. *Let Γ be a finitely generated subgroup of $G = \text{UT}(n, \mathbb{Q})$. Then there exists a diagonal matrix P such that $P\Gamma P^{-1}$ is a subgroup of $\text{UT}(n, \mathbb{Z}) < \text{UT}(n, \mathbb{Q})$. In particular, Γ embeds in $\text{UT}(n, \mathbb{Z})$. If the elements of Γ are essentially hyperbolic, so are those of $P\Gamma P^{-1}$.*

Proof. For $\kappa \neq 0$ let $D_i(\kappa)$ be the diagonal matrix with i th diagonal entry equal to κ and other diagonal entries equal to 1. Conjugating a matrix A by $D_i(\kappa)$ has the effect of multiplying the i th row of A by κ and dividing the i th column by κ . Therefore, given a finite set of matrices $A \in \text{UT}(n, \mathbb{Q})$, such as an inverse-closed finite generating set of Γ , we can take d_i to be a common multiple of all denominators of entries in the i th rows of the matrices A . Put

$$\bar{d}_i = \prod_{j=i}^n d_j, \quad \text{and} \quad \bar{D}_i = D_i(\bar{d}_i).$$

Then conjugating the matrices A by $P = \bar{D}_n \bar{D}_{n-1} \cdots \bar{D}_1$ yields a set of upper triangular matrices with integer entries and each diagonal entry equal to 1. It is easy to see that essential hyperbolicity is preserved by this conjugation. ■

Theorem 2.7. *The following conditions are equivalent.*

- (1) G is finitely generated torsion-free nilpotent.
- (2) G embeds in $\text{UT}(n, \mathbb{Z})$ for some n .
- (3) G admits an essentially free affine action on \mathbb{Z}^n for some n .
- (4) G admits a faithful, order-preserving, affine action on \mathbb{Z}^n for some n .

Proof. Lemmas 2.3 and 2.6 show that $\text{UT}(n, \mathbb{Z})$ admits an embedding in $\text{UT}(m+1, \mathbb{Z})$ such that non-trivial elements of the image are essentially hyperbolic. The implication (2) \Rightarrow (3) follows. The implication (4) \Rightarrow (2) follows from Lemma 2.1, and the implication (1) \Rightarrow (2) is a well-known result; see [11, Theorem 7.5]. The remaining implications are clear. ■

3. Free affine actions of upper triangular groups

In this section we exhibit essentially free rigid affine actions of the group $T^*(n, \mathbb{R})$ on linear \mathbb{R}^k -trees. Our main task is to extend the actions of the unitriangular groups $U_n = \text{UT}(n, \mathbb{R})$ described in the previous section to the groups $T^*(n, \mathbb{R})$.

Note that embeddings of $\text{UT}(n, \mathbb{R})$ in $\text{UT}(m+1, \mathbb{R})$ do not necessarily extend to embeddings of the respective upper triangular groups in general, a fact brought to my attention by Yves de Cornulier and Florian Eisele in response to a question of mine on MathOverflow – see [6] and [9].

Taking φ to be the embedding of $UT(n, \mathbb{R})$ in $UT(m+1, \mathbb{R})$ of the previous section, and keeping $m = n(n-1)/2$, let us write

$$\varphi(u) = \begin{pmatrix} \varphi_0(u) & b(u) \\ 0 & 1 \end{pmatrix}$$

where $\varphi_0(u) \in UT(m, \mathbb{R})$ and $b(u) \in \mathbb{R}^m$.

Next, let d be a diagonal matrix with positive diagonal entries d_1, \dots, d_n respectively and consider the conjugation map $\chi_d: x \mapsto dx d^{-1}$. Now $t(\mathbf{x}) \mapsto t(\chi_d \cdot \mathbf{x})$ is a vector space automorphism of \mathbb{R}^m , which can therefore be represented as a matrix d^* . Direct calculation shows that χ_d has the effect of replacing the entry x_{ij} of \mathbf{x} by $\frac{d_i}{d_j} x_{ij}$, and therefore d^* can be seen to be a block diagonal matrix whose i th block is the $i \times i$ diagonal matrix with entries $d_1/d_{n-i+1}, d_2/d_{n-i+2}, \dots, d_i/d_n$ ($1 \leq i \leq n-1$).

We will find it useful to denote the column vector

$$\begin{pmatrix} \log d_1 \\ \vdots \\ \log d_n \end{pmatrix}$$

by $\log d$. We now define $\bar{\varphi}: T^*(n, \mathbb{R}) \rightarrow T^*(m+n+1, \mathbb{R})$ by the assignments

$$\bar{\varphi}(u) = \begin{pmatrix} \varphi_0(u) & 0 & b(u) \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{\varphi}(d) = \begin{pmatrix} d^* & 0 & 0 \\ 0 & I_n & \log d \\ 0 & 0 & 1 \end{pmatrix}$$

for $u \in U_n$ and $d \in D_n$ and setting $\bar{\varphi}(ud) = \bar{\varphi}(u)\bar{\varphi}(d)$. We next show that $\bar{T}^* = \langle \bar{\varphi}(U_n), \bar{\varphi}(D_n) \rangle$ is a group isomorphic to $T^*(n, \mathbb{R})$ whose natural action on \mathbb{R}^{m+n} is affine and essentially free.

For $d \in D_n$ recall that $d^* \in D_m$ as defined above.

We will also write \tilde{d} for $\begin{pmatrix} d^* & 0 \\ 0 & 1 \end{pmatrix} \in D_{m+1}$, and \bar{d} for $\bar{\varphi}(d) \in D_{m+n+1}$.

Lemma 3.1. (a) $\chi_{\tilde{d}} \cdot \exp = \exp \cdot \chi_{\tilde{d}}$;

(b) $\chi_d \cdot \log = \log \cdot \chi_d$;

(c) $d^* \cdot t = t \cdot \chi_d$;

(d) $\chi_{d^*} \cdot \lambda = \lambda \cdot \chi_d$;

(e) $\chi_{\tilde{d}} \cdot d\bar{\gamma} = d\bar{\gamma} \cdot \chi_d$;

(f) $\chi_{\tilde{d}} \cdot \varphi = \varphi \cdot \chi_d$;

(g) $\chi_{d^*} \cdot \varphi_0 = \varphi_0 \cdot \chi_d$;

(h) $d^* \cdot b = b \cdot \chi_d$;

(i) $\chi_{\tilde{d}} \cdot \bar{\varphi} = \bar{\varphi} \cdot \chi_d$;

(j) $T^*(n, \mathbb{R})$ is isomorphic to \bar{T}^* .

Proof. (a) and (b) are standard (and easily checked) facts, and (c) is immediate from the definition of d^* .

(d) Consider the (ρ, σ) entry of $\lambda \cdot \chi_d(\mathbf{x}) = \lambda(d\mathbf{x}d^{-1})$. By Lemma 2.2 the (ρ, σ) entry of $\lambda \cdot \chi_d\mathbf{x}$ is obtained by multiplying the corresponding entry of $\lambda(\mathbf{x})$ by

$$\left\{ \begin{array}{ll} \frac{d_{r_\rho}}{d_{r_\rho+(k_\sigma-k_\rho)}} & \text{if } k_\sigma > k_\rho \text{ and } r_\sigma > r_\rho \text{ and } k_\sigma - k_\rho = r_\sigma - r_\rho \\ \frac{d_{n-k_\sigma+r_\rho}}{d_{n-k_\rho+r_\rho}} & \text{if } k_\sigma > k_\rho \text{ and } r_\sigma = r_\rho \\ 1 & \text{otherwise.} \end{array} \right.$$

On the other hand, conjugation of $\lambda(\mathbf{x})$ by d^* has the effect of multiplying the (ρ, σ) entry by $\frac{d_{r_\rho}}{d_{r_\rho+n-k_\rho}} \frac{d_{r_\sigma+n-k_\sigma}}{d_{r_\sigma}}$. It is now clear that in each of the cases distinguished above, the (ρ, σ) entries of $\lambda(d\mathbf{x}d^{-1})$ and of $d^*\lambda(\mathbf{x})(d^*)^{-1}$ agree. That is, $\lambda \cdot \chi_d = \chi_{d^*} \cdot \lambda$, as claimed.

(e) follows from (c) and (d) applied to (16).

(f) follows from (a), (b) and (e) applied to $\varphi = \bar{\gamma} = \exp \cdot d\bar{\gamma} \cdot \log$.

(g), (h) We have on the one hand

$$\varphi \cdot \chi_d(u) = \left(\begin{array}{c|c} \varphi_0(dud^{-1}) & b(dud^{-1}) \\ \hline 0 & 1 \end{array} \right),$$

and on the other hand

$$\begin{aligned} \chi_{d^*} \cdot \varphi(u) &= \left(\begin{array}{c|c} d^* & 0 \\ \hline 0 & 1 \end{array} \right) \left(\begin{array}{c|c} \varphi_0(u) & b(u) \\ \hline 0 & 1 \end{array} \right) \left(\begin{array}{c|c} (d^*)^{-1} & 0 \\ \hline 0 & 1 \end{array} \right) \\ &= \left(\begin{array}{c|c} d^* \varphi_0(u)(d^*)^{-1} & d^* b(u) \\ \hline 0 & 1 \end{array} \right). \end{aligned}$$

From (f) these matrices are equal, giving $\varphi_0 \cdot \chi_d = \chi_{d^*} \cdot \varphi_0$ and $b \cdot \chi_d = d^*b$.

(i) This now follows from (g) and (h) applied to the definition of $\bar{\varphi}$.

(j) To show that $\bar{\varphi}$ is a monomorphism we note first that the restriction of $\bar{\varphi}$ to U_n is injective since φ is injective, and the restriction of $\bar{\varphi}$ to D_n is injective since $d \mapsto \log d$ is; both restrictions are clearly homomorphisms. The images of these restrictions have trivial intersection, and (i) implies that $\bar{\varphi}(U_n)$ is normalised by $\bar{\varphi}(D_n)$. Moreover, by (i), we have $\bar{\varphi}(dud^{-1}) = \bar{\varphi}(d)\bar{\varphi}(u)\bar{\varphi}(d)^{-1}$. The result follows. ■

We can now deduce

Theorem 3.2. *Let $n \in \mathbb{N}$ and $m = n(n-1)/2$. The group $T^*(n, \mathbb{R})$ admits an essentially free affine action on \mathbb{R}^{m+n} via $g \cdot \mathbf{r} = \mathbf{s}$ where $\begin{pmatrix} \mathbf{s} \\ 1 \end{pmatrix} = \bar{\varphi}(g) \begin{pmatrix} \mathbf{r} \\ 1 \end{pmatrix}$.*

Proof. By Lemma 3.1(j) $T^*(n, \mathbb{R})$ is isomorphic to \bar{T}^* . Moreover the bottom right entry of each $\bar{g} \in \bar{T}^*$ is equal to 1, so there is a natural affine action of \bar{T}^* on \mathbb{R}^{m+n} as described in §2.1.

Each \bar{g} has the form

$$\bar{\varphi}(u)\bar{\varphi}(d) = \begin{pmatrix} \varphi_0(u)d^* & 0 & b(u) \\ 0 & I_n & \log d \\ 0 & 0 & 1 \end{pmatrix}.$$

If $d \neq 1$ then $\log d \neq 0$, while if $d = 1$ then $\bar{g} = \bar{\varphi}(u)$. Provided \bar{g} is non-trivial it is seen to be essentially hyperbolic in either case. Therefore the action of \bar{T}^* , and with it the action of $T^*(n, \mathbb{R})$ as given in the theorem, are essentially free. ■

4. Actions of wreath products

Proposition 4.1. *Let Ω be a linearly ordered set, and for $\omega \in \Omega$ let Λ_ω be an ordered abelian group on which H_ω admits an order-preserving $\alpha^{(\omega)}$ -affine action. Set Λ equal to the lexicographic product $\mathcal{L}_{\omega \in \Omega} \Lambda_\omega$ and $H = \mathcal{L}_{\omega \in \Omega} H_\omega$ and for $h = (h_\omega)_{\omega \in \Omega} \in H$ and $\lambda = (\lambda_\omega)_{\omega \in \Omega}$ let $\alpha_h \lambda = (\alpha_{h_\omega}^{(\omega)} \lambda_\omega)_{\omega \in \Omega}$.*

The action of H on Λ given by $(h_\omega)_{\omega \in \Omega} \cdot (\lambda_\omega)_{\omega \in \Omega} = (h_\omega \cdot \lambda_\omega)_{\omega \in \Omega}$ is α -affine and order-preserving. If the given actions of H_ω are respectively

- (a) *free;*
- (b) *rigid*
- (c) *essentially free*

so is that of H .

(See §2.1 for the definition of the lexicographic product.)

Proof. We will outline the straightforward proof. One checks that α is a homomorphism by showing that

$$\alpha_g(\alpha_h(\lambda_\omega)_{\omega \in \Omega}) = (\alpha_{g_\omega}^{(\omega)} \alpha_{h_\omega}^{(\omega)} \lambda_\omega)_{\omega \in \Omega} = \alpha_{gh}(\lambda_\omega)_{\omega \in \Omega},$$

where $g = (g_\omega)_{\omega \in \Omega}$. Similarly $g \cdot h(\lambda_\omega)_{\omega \in \Omega} = (g_\omega h_\omega \lambda_\omega)_{\omega \in \Omega} = gh(\lambda_\omega)_{\omega \in \Omega}$.

It is easy to check that since the action of each H_ω is order-preserving then so is the action of H . To see that the action is α -affine, consider $\lambda' \geq \lambda'' \in \Lambda$, and note that

$$\begin{aligned} d(h(\lambda'_\omega)_{\omega \in \Omega}, h(\lambda''_\omega)_{\omega \in \Omega}) &= h(\lambda'_\omega)_{\omega \in \Omega} - h(\lambda''_\omega)_{\omega \in \Omega} \\ &= (\alpha_{h_\omega}^{(\omega)}(\lambda'_\omega - \lambda''_\omega))_{\omega \in \Omega} = \alpha_h((\lambda'_\omega)_{\omega \in \Omega} - (\lambda''_\omega)_{\omega \in \Omega}). \end{aligned}$$

It is routine to show that if the action of each H_ω is free then so is the action of H . Finally, suppose that the action is rigid and that $h(\lambda_\omega)_{\omega \in \Omega} > (\lambda_\omega)_{\omega \in \Omega}$. Then there exists $\omega_0 \in \Omega$ such that $h_\omega \lambda_\omega = \lambda_\omega$ for all $\omega < \omega_0$ and $h_{\omega_0} \lambda_{\omega_0} > \lambda_{\omega_0}$. If $(\lambda'_\omega)_{\omega \in \Omega}$ is another element of Λ then $h_\omega \lambda'_\omega = \lambda'_\omega$ for all $\omega < \omega_0$, since the actions of H_ω are assumed to be rigid. Similarly $h_{\omega_0} \lambda'_{\omega_0} > \lambda'_{\omega_0}$. Thus $h(\lambda'_\omega)_{\omega \in \Omega} > (\lambda'_\omega)_{\omega \in \Omega}$. ■

For a group H and an ordered abelian group Λ_1 , we denote by $H \wr_\omega \Lambda_1$ the subgroup of the unrestricted wreath product $H \wr \Lambda_1$ consisting of those $(\lambda^*, (h_\lambda)_{\lambda \in \Lambda_1})$ where $\{\lambda \in \Lambda_1 : h_\lambda \neq 1\}$ is well-ordered. We will call $H \wr_\omega \Lambda_1$ the *lexicographic wreath product* of H and Λ_1 . Note that the restricted wreath product is contained in the lexicographic wreath product.

Theorem 4.2. *Let H be a group, Λ_0 and Λ_1 ordered abelian groups, and $\theta : H \rightarrow \text{Aut}^+(\Lambda_0)$ a homomorphism. Fix a θ -affine action of H on Λ_0 that preserves the order. Let $\Lambda = \Lambda_1 \times \mathcal{L}_{\lambda_1 \in \Lambda_1} \Lambda_0$ and set $G = H \wr_{\omega} \Lambda_1$. Then Λ is an ordered abelian group. Define $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$ via*

$$\alpha_{(\lambda^*, (h_\lambda)_{\lambda \in \Lambda_1})}(\lambda', (\mu_\lambda)_{\lambda \in \Lambda_1}) = (\lambda', (\theta_{h_{\lambda+\lambda^*}} \mu_{\lambda+\lambda^*})_{\lambda \in \Lambda_1}).$$

The action of G on Λ defined by

$$(\lambda^*, (h_\lambda)_{\lambda \in \Lambda_1}) \cdot (\lambda', (\mu_\lambda)_{\lambda \in \Lambda_1}) = (\lambda' + \lambda^*, (h_{\lambda+\lambda^*} \mu_{\lambda+\lambda^*})_{\lambda \in \Lambda_1})$$

is α -affine. If the given action of H is respectively

- (a) free
- (b) rigid
- (c) essentially free

then so is that of G .

Proof. Recall that multiplication of elements of G is performed as follows:

$$(\lambda^+, (k_\lambda))(\lambda^*, (h_\lambda)) = (\lambda^+ + \lambda^*, (k_{\lambda-\lambda^*} h_\lambda))$$

One can check that

$$[(\lambda^+, (k_\lambda))(\lambda^*, (h_\lambda))] \cdot (\lambda', (\mu_\lambda)) \quad \text{and} \quad (\lambda^+, (k_\lambda)) [(\lambda^*, (h_\lambda)) \cdot (\lambda', (\mu_\lambda))]$$

are both equal to $(\lambda^+ + \lambda^* + \lambda', (k_{\lambda+\lambda^*} h_{\lambda+\lambda^*} \mu_{\lambda+\lambda^*}))$. A similar calculation shows that α is a homomorphism.

To see that the action of G on Λ is α -affine take $\bar{h} = (\lambda^*, (h_\lambda)) \in G$, $\bar{\lambda}' = (\lambda', (\mu'_\lambda))$ and $\bar{\lambda}'' = (\lambda'', (\mu''_\lambda)) \in \Lambda$ with $\bar{\lambda}' \geq \bar{\lambda}''$. Observe that $d_\Lambda(\bar{\lambda}', \bar{\lambda}'') = \bar{\lambda}' - \bar{\lambda}''$ while

$$\begin{aligned} d_\Lambda(\bar{h}\bar{\lambda}', \bar{h}\bar{\lambda}'') &= (\lambda^*, (h_\lambda)) \cdot (\lambda', (\mu'_\lambda)) - (\lambda^*, (h_\lambda)) \cdot (\lambda'', (\mu''_\lambda)) \\ &= (\lambda' + \lambda^*, (h_{\lambda+\lambda^*} \mu'_{\lambda+\lambda^*})) - (\lambda'' + \lambda^*, (h_{\lambda+\lambda^*} \mu''_{\lambda+\lambda^*})) \\ &= (\lambda' - \lambda'', \theta_{h_{\lambda+\lambda^*}}(\mu'_{\lambda+\lambda^*} - \mu''_{\lambda+\lambda^*})) \\ &= \alpha_{\bar{h}} d_\Lambda(\bar{\lambda}', \bar{\lambda}''). \end{aligned}$$

If $\bar{h}\bar{\lambda}' = \bar{\lambda}'$ then $\lambda^* = 0$ giving, for each $\lambda \in \Lambda_1$, $\mu_\lambda = h_{\lambda+\lambda^*} \mu_{\lambda+\lambda^*} = h_\lambda \mu_\lambda$. If the given action of H on Λ_0 is free this forces $h_\lambda = 1$ for all λ , whence $\bar{h} = 1$.

Suppose now that the given action of H is rigid and $\bar{h}\bar{\lambda}' > \bar{\lambda}'$. To establish rigidity of the action of G it suffices to show that $\bar{h}\bar{\lambda}'' > \bar{\lambda}''$. If $\lambda^* > 0$ this claim is obvious, so suppose that $\lambda^* = 0$. Then taking the least λ_1 for which h_{λ_1} does not fix Λ_0 pointwise we have $h_{\lambda_1} \mu_{\lambda_1} > \mu_{\lambda_1}$ for all $\mu_{\lambda_1} \in \Lambda_0$, since the action of H on Λ is assumed rigid. It follows that $(0, (h_\lambda)) \cdot (\lambda'', (\mu''_\lambda)) = (\lambda'', (h_\lambda \mu''_\lambda)) > (\lambda'', (\mu''_\lambda))$, as required. This establishes (a) and (b), and (c) is now immediate. ■

Taking the free isometric (and hence essentially free) action of an ordered abelian group on itself and using induction, one deduces the following corollary.

Corollary 4.3. *Let Λ_i ($1 \leq i \leq n$) be ordered abelian groups. Then the iterated wreath product $\Lambda_1 \wr_{\omega} \Lambda_2 \wr_{\omega} \cdots \wr_{\omega} \Lambda_n$ admits an essentially free affine action on Λ , viewed as a linear Λ -tree for some ordered abelian group Λ .*

By a theorem of Šmel'kin [18] a free group in a product variety \mathfrak{WU} embeds in the \mathfrak{V} -verbal wreath product of the free groups in the respective varieties. Note that in case \mathfrak{V} is the variety of abelian groups, the \mathfrak{V} -verbal wreath product of two groups coincides with the restricted wreath product, which embeds in the lexicographic wreath product. We therefore deduce the following result from Corollary 4.3.

Corollary 4.4. *Every free soluble group (of given derived length) admits an essentially free affine action on Λ for some ordered abelian group Λ .*

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References

- [1] B. Baumslag: *Residually free groups*, Proc. London Math. Soc. (3) 17 (1967) 402–418.
- [2] G. Baumslag: *A representation of the wreath product of two torsion-free abelian groups in a power series ring*, Proc. Amer. Math. Soc. 17 (1966) 1159–1165.
- [3] G. Baumslag: *Finitely generated residually torsion-free nilpotent groups I*, J. Austral. Math. Soc. Ser. A 67/3 (1999) 289–317.
- [4] G. Baumslag: *Some reflections on proving groups residually torsion-free nilpotent I*, Illinois J. Math. 54/1 (2010) 315–325.
- [5] I. Chiswell: *Introduction to Λ -Trees*, World Scientific, River Edge (2001).
- [6] Y. de Cornulier: *Does every embedding of one unipotent group (over R) in another extend to an embedding of the respective upper triangular matrix groups?*, MathOverflow, URL: <https://mathoverflow.net/q/103208> (version: 2016-03-27).
- [7] T. De Cat, K. Dekimpe: *Translations in simply transitive affine actions of free 2-step nilpotent Lie groups*, Int. J. Pure Appl. Math. 30/1 (2006) 49–70.
- [8] K. Dekimpe, W. Malfait: *Affine structures on a class of virtually nilpotent groups*, Topology Appl. 73/2 (1996) 97–119.
- [9] F. Eisele: *Does every embedding of one unipotent group (over R) in another extend to an embedding of the respective upper triangular matrix groups?*, MathOverflow, URL: <https://mathoverflow.net/q/102976> (version: 2012-07-24).
- [10] K. W. Gruenberg: *Residual properties of infinite soluble groups*, Proc. London Math. Soc. (3) 7 (1957) 29–62.
- [11] P. Hall: *The Edmonton Notes on Nilpotent Groups*, Queen Mary College Mathematics Notes, Mathematics Department, Queen Mary College, London (1969).
- [12] A. Jaikin-Zapirain: *Free \mathbb{Q} -groups are residually torsion-free nilpotent*, (2020) preprint.
- [13] O. Kharlampovich, A. Myasnikov, D. Serbin: *Actions, length functions, and non-Archimedean words*, Int. J. Algebra Comput. 23/2 (2013) 325–455.
- [14] I. Liousse: *Actions affines sur les arbres réels*, Math. Zeitschrift 238/2 (2001) 401–429.
- [15] R. C. Lyndon: *Length functions in groups*, Math. Scand. 12 (1963) 209–234.

- [16] S. O Rourke: *Affine actions on non-Archimedean trees*, Int. J. Algebra Comput. 23/2 (2013) 217–253.
- [17] S. O Rourke: *A combination theorem for affine tree-free groups*, Int. J. Algebra Comput. 26/7 (2016) 1283–1321.
- [18] A. L. Šmel'kin: *Wreath products and varieties of groups*, Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965) 149–170.

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