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A Note on the Fusion Product Decomposition of Demazure Modules

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Abstract. We settle the fusion product decomposition theorem for higher level affine Demazure modules for the cases $E^{(1)}_{6,7,8}, F^{(1)}_4$ and $E^{(2)}_6$, thus completing the main theorems of V. Chari et al. [J. Algebra 455 (2016) 314–346] and D. Kus et al. [Representation Theory 20 (2016) 94–127]. We obtain a new combinatorial proof for the key fact, that was used in Chari et al. (op. cit.), to prove this decomposition theorem. We give a case free uniform proof for this key fact.

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Key Words: Current algebras, Demazure modules, Steinberg decomposition, Affine Weyl groups.

1. Introduction

Affine Demazure modules of higher level have been the subject of intensive study due to their connection with the representation theory of quantum affine algebras [5], crystal bases [13], the $X = M$ conjecture [11], Macdonald polynomials [1] and more. One of the key structural results concerning these modules is their decomposition into a fusion product of smaller Demazure modules of the same level. This *Steinberg type decomposition theorem*, proved in [4, 7, 14] (see also Theorem 4.1 below) plays a vital role in understanding their graded structure and has many applications. For instance, this is used in [2] to prove that certain level two affine Demazure modules in type A appear as graded limits of prime representations of the quantum affine algebra introduced by Hernandez-Leclerc [9] in the context of monoidal categorification of cluster algebras. The decomposition theorem also serves as a base case for the study of fusion products of Demazure modules of unequal levels, and can be used to obtain defining relations for special modules of this kind [12].

However, the decomposition theorem was only proved for the untwisted affine algebras other than $E^{(1)}_{6,7,8}$ and $F^{(1)}_4$ in [4] and for twisted affine algebras other than $E^{(2)}_6$ in [7]. The obstruction lay in a technical result concerning the action of the affine Weyl group on weights, which was established in [4, Proposition 3.5] first for the untwisted affines of type C and then by separate root sytem arguments in each of the types A, B, D and G; additionally an appendix tabulated computational evidence in types E and F.

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In this note, we formulate a stronger version of this technical result (Proposition 3.1) and give a short, uniform proof for all affine types. We also point out in section 4 the new cases of the decomposition theorem and other key results of [4] which now follow as consequences.

2. Preliminaries

2.1. We assume that the base field is complex numbers throughout the paper. We refer to [10] for the general theory of affine Lie algebras. We denote by A an indecomposable affine Cartan matrix, and by S the corresponding Dynkin diagram with the labeling of vertices as in Table Aff 1–3 from [10, pp. 54–55]. Let $\mathring{S}$ be the Dynkin diagram obtained from S by dropping the 0^{th} node and let $\mathring{A}$ be the Cartan matrix, whose Dynkin diagram is $\mathring{S}$. Let $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ be the affine Lie algebra and the finite–dimensional simple Lie algebra associated to A and $\mathring{A}$, respectively. We shall realize $\mathring{\mathfrak{g}}$ as a subalgebra of $\mathfrak{g}$. We let $\mathfrak{h}$ and $\mathring{\mathfrak{h}}$ denote the Cartan subalgebras of $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ respectively, and let $\mathfrak{h}^*_{\mathbb{R}}$ and $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ denote their real forms.

We let Φ and $\mathring{\Phi}$ denote the sets of roots of $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ respectively. We fix a basis $\Pi = \{\alpha_0, \dots, \alpha_n\}$ for Φ such that $\mathring{\Pi} = \{\alpha_1, \dots, \alpha_n\}$ is a basis for $\mathring{\Phi}$. The weight lattice (resp. coweight lattice) of Φ is denoted by P (resp. $P^\vee$) and the set of dominant integral weights is denoted by P^+. Similarly, the weight lattice (resp. coweight lattice) of $\mathring{\Phi}$ is denoted by $\mathring{P}$ (resp. $\mathring{P}^\vee$) and the set of dominant integral weights by $\mathring{P}^+$.

2.2. Affine Weyl group. We recall the key facts about affine Weyl groups following [10] and [3]. Let W and $\mathring{W}$ be the Weyl groups of $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ respectively. We denote by s_i the reflection associated to the simple root α_i for $0 \le i \le n$.

Then we have $W = \langle s_0, s_1, \cdots, s_n \rangle$ and $\mathring{W} = \langle s_1, \cdots, s_n \rangle$. Let $\langle\ ,\ \rangle$ be the standard non-degenerate symmetric invariant form on $\mathfrak{g}$. This determines a bijection $\mathfrak{h} \to \mathfrak{h}^*$. There is an action of W on $\mathfrak{h}$ and $\mathfrak{h}^*$, compatible with this bijection.

For each $\alpha \in \mathring{\mathfrak{h}}^*$, we define $t_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$t_\alpha(\Lambda) = \Lambda + \Lambda(c)\alpha - (\langle \Lambda, \alpha \rangle + \frac{\langle \alpha, \alpha \rangle}{2}\Lambda(c))\delta$$

for $\Lambda \in \mathfrak{h}^*$. Here c denotes the canonical central element of $\mathfrak{g}$ and δ the unique indivisible positive imaginary root of Φ.

Let θ denote the highest root of $\mathring{\Phi}$ and define $a_0 = 2$ if $\mathfrak{g}$ is of type $A^{(2)}_{2n}$ $(n \ge 1)$ and $a_0 = 1$ otherwise. Then $W = \mathring{W} \ltimes t_M$, where $t_M = \{t_\mu : \mu \in M\}$ and M is the sublattice of $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ generated by the elements $w(\frac{1}{a_0}\theta)$ for all $w \in \mathring{W}$. The explicit description of the lattice M can be found in [3, page 414]. The affine Weyl group W acts on the set $\mathfrak{h}^*_{\mathbb{R},1} = \{\Lambda \in \mathfrak{h}^*_{\mathbb{R}} : \Lambda(c) = 1\}$ and this induces an action on the orbit space $\mathfrak{h}^*_{\mathbb{R},1}/\mathbb{R}\delta \cong \mathring{\mathfrak{h}}^*_{\mathbb{R}}$. The element $t_\mu \in M$ acts on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ as the translation $x \mapsto x + \mu$. Thus, W acts as affine transformations on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$.

Let $H_{\alpha^\vee,k} = \{\lambda \in \mathring{\mathfrak{h}}^* : \lambda(\alpha^\vee) = k\}$; this defines an affine hyperplane in $\mathring{\mathfrak{h}}^*$ for $\alpha \in \mathring{\Phi}, k \in \mathbb{Z}$, where $\alpha^\vee$ is the coroot corresponding to the root α. Let $\mathring{\mathcal{H}}$ be the set of hyperplanes $\{H_{\alpha^\vee,0} : \alpha \in \mathring{\Phi}\}$. The connected components of $\mathring{\mathfrak{h}}^*_{\mathbb{R}} - \bigcup_{H \in \mathring{\mathcal{H}}} H$ are the Weyl chambers of $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$. The elements of $\mathring{W}$ permute the hyperplanes in $\mathring{\mathcal{H}}$ and hence act on the set of Weyl chambers.

The set $C = \left\{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} : \lambda(\alpha_i^\vee) > 0 \text{ for } 1 \le i \le n\right\}$ is called the *fundamental Weyl chamber*. The map $w \mapsto w(C)$ gives a bijection from $\mathring{W}$ to the set of Weyl chambers and the closure $\bar{C}$ is a fundamental region for the $\mathring{W}$ action on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$.
Let $\mathcal{H}$ be a set of affine hyperplanes $\left\{H_{\alpha^\vee,k} : \alpha \in \mathring{\Phi}, k \in Z_\alpha\right\}$ in $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$, where the sets Z_α are defined as follows (see [3, Page 414]):

$$Z_\alpha = \begin{cases} 2\mathbb{Z} & \text{if } \mathfrak{g} \text{ is of type } B_n^{(1)}\ (n \ge 2), C_n^{(1)}\ (n \ge 3), F_4^{(1)} \text{ and } \alpha \text{ is short} \\ 3\mathbb{Z} & \text{if } \mathfrak{g} \text{ is of type } G_2^{(1)} \text{ and } \alpha \text{ is short} \\ \frac{1}{2}\mathbb{Z}, & \text{if } \mathfrak{g} \text{ is of type } A_{2n}^{(2)}\ (n \ge 1) \text{ and } \alpha \text{ is long} \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

The connected components of $\mathring{\mathfrak{h}}^*_{\mathbb{R}} - \bigcup_{H\in\mathcal{H}} H$ are called the *alcoves* of $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$. The elements of W permute the affine hyperplanes in $\mathcal{H}$ and hence act on the set $\mathcal{A}$ of alcoves. The set

$$A = \left\{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} : \lambda(\alpha_i^\vee) > 0 \text{ for } 1 \le i \le n, \lambda(\theta^\vee) < 1/a_0\right\}$$

is called the *fundamental alcove*. The map $w \mapsto w(A)$ gives a bijection from W onto the set of alcoves $\mathcal{A}$. Moreover, the closure

$$\bar{A} = \left\{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} : \lambda(\alpha_i^\vee) \ge 0 \text{ for } 1 \le i \le n, \lambda(\theta^\vee) \le 1/a_0\right\}$$

is a fundamental region for the W action on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$. Finally, let $\Lambda_0 \in P^+$ be the fundamental weight corresponding to the 0^{th} vertex of the Dynkin diagram of $\mathfrak{g}$ and let w_0 denote the unique longest element in $\mathring{W}$.

2.3. We will need the following well-known elementary lemma.

Lemma 2.1. *Let $\mathcal{F}$ denote the set of alcoves contained in the fundamental Weyl chamber. Then $\bar{C} = \bigcup_{B\in\mathcal{F}} \bar{B}$.*

3. The technical result

3.1. Set $M^+ = M \cap \bar{C}$. The following is our main technical result which is crucial in proving Theorem 4.1. We refer to [4, Section 4] and [7, Section 6] for more details.

Proposition 3.1. *Let $\mathfrak{g}$ be an affine Lie algebra. Given $\ell \in \mathbb{N}$ and $\lambda \in \mathring{P}^+$, there exists $\mu \in M^+$ and $w \in \mathring{W}$ such that $wt_\mu(\ell\Lambda_0 - \lambda) \in P^+$.*

Proof. Let $\lambda' = -w_0(\lambda)$. Since $\lambda \in \mathring{P}^+$ and $-w_0(\mathring{P}^+) = \mathring{P}^+$, we have $\lambda' \in \mathring{P}^+$. Thus, the element $\lambda'/\ell \in \bar{C}$. There exists $B \in \mathcal{F}$ such that $\lambda'/\ell \in \bar{B}$ by Lemma 2.1. Since W acts simply transitively on the set of alcoves $\mathcal{A}$, we have $\bar{B} = t_{\mu'}\, u\, \bar{A}$ for some $\mu' \in M$ and $u \in \mathring{W}$. Since $0 \in \bar{A}$, we have $\mu' = t_{\mu'}\, u(0) \in \bar{B}$. This implies that $\mu' \in \bar{B} \cap M \subseteq \bar{C} \cap M = M^+$. Set $\mu = -w_0\, \mu' \in M^+$ and $w = u^{-1}w_0 \in \mathring{W}$ and consider

$$wt_\mu(\ell\Lambda_0 - \lambda) \equiv \ell\Lambda_0 + w(\ell\mu - \lambda) \bmod \mathbb{Z}\delta.$$

We claim that $wt_\mu(\ell\Lambda_0 - \lambda) \in P^+$. This will follow if we prove that (i) $w(\ell\mu - \lambda) \in \mathring{P}^+$ and (ii) $\ell\Lambda_0(\alpha_0^\vee) + w(\ell\mu - \lambda)(\alpha_0^\vee) \ge 0$.

Both these facts follow from the following observation:

$$w(\ell\mu - \lambda) = u^{-1}(\lambda' - \ell\mu') = \ell u^{-1}\left(\frac{\lambda'}{\ell} - \mu'\right) = \ell u^{-1} t_{-\mu'}\left(\frac{\lambda'}{\ell}\right) \in \ell\bar{A} \cap \mathring{P}.$$

Since this belongs to $\ell\bar{A}$, we have $w(\ell\mu-\lambda)(\theta^\vee) \leq \ell/a_0$. Recalling that $\alpha_0^\vee = c - a_0\theta^\vee$ [10, Chapter 6], we conclude $\ell + w(\ell\mu - \lambda)(\alpha_0^\vee) = \ell - a_0 w(\ell\mu - \lambda)(\theta^\vee) \geq 0$, proving (ii). Since $\ell\bar{A} \cap \mathring{P} \subseteq \mathring{P}^+$ we have $w(\ell\mu - \lambda) \in \mathring{P}^+$, proving (i). ∎

4. Remarks

In the interest of completeness, we briefly point out the new cases of the main results of [4] which hold, now that Proposition 3.1 has been established.

4.1. As before, let $\mathring{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra and $\mathfrak{g}$ the corresponding untwisted affine Lie algebra. We consider the Demazure modules of $\mathfrak{g}$ that are stable under the current algebra $\mathring{\mathfrak{g}} \otimes \mathbb{C}[t]$. These modules are parametrized by pairs (ℓ, λ) with ℓ a positive integer and λ a dominant integral weight of $\mathring{\mathfrak{g}}$. Let $D(\ell, \lambda)$ denote the Demazure module corresponding to the pair (ℓ, λ).

Theorem 4.1. *Let $\mathring{\mathfrak{g}}$ be one of E_6, E_7, E_8 or F_4. Let $\lambda \in \mathring{P}^+$, $\ell \in \mathbb{N}$ and suppose that $\lambda = \ell\,(\sum_{i=1}^k \lambda_i) + \lambda_0$ with $\lambda_0 \in \mathring{P}^+$ and $\lambda_i \in (\mathring{P}^\vee)^+$ for $1 \leq i \leq k$. Then there is an isomorphism of $\mathring{\mathfrak{g}}[t]$-modules*

$$D(\ell, \lambda) \cong D(\ell, \ell\lambda_1) * D(\ell, \ell\lambda_2) * \cdots * D(\ell, \ell\lambda_k) * D(\ell, \lambda_0).$$

Here $*$ denotes the fusion product, and we refer to [4, Section 2.7] for the definition of fusion products of $\mathring{\mathfrak{g}} \otimes \mathbb{C}[t]$–modules. As mentioned earlier, this was established in [4, Theorem 1] for all untwisted affines other than those of types E, F and in [7] for the twisted cases (modulo an extra hypothesis for $E_6^{(2)}$). It now also follows that [7, Theorem 7] holds without this extra condition.

4.2. Now let $\mathring{\mathfrak{g}}$ be simply-laced, so that $(\mathring{P}^\vee)^+ = \mathring{P}^+$. Let α_i and ω_i for $1 \leq i \leq n$ denote the simple roots and fundamental weights respectively of $\mathring{\mathfrak{g}}$. Given $\ell \geq 1$ and $\lambda \in \mathring{P}^+$, there is a unique decomposition $\lambda = \ell\,(\sum_{i=1}^n m_i\omega_i) + \lambda_0$ where $\lambda_0 = \sum_{i=1}^n r_i\omega_i$ with $0 \leq r_i < \ell$ and $m_i \geq 0$ for all i. Theorem 4.1 implies that for $\mathring{\mathfrak{g}}$ of type E, one has

$$D(\ell, \lambda) \cong_{\mathring{\mathfrak{g}}[t]} D(\ell, \ell\omega_1)^{*m_1} * D(\ell, \ell\omega_2)^{*m_2} * \cdots * D(\ell, \ell\omega_n)^{*m_n} * D(\ell, \lambda_0) \qquad (1)$$

Recall that a $\mathring{\mathfrak{g}}[t]$-module is said to be *prime* if it is not isomorphic to a fusion product of non-trivial $\mathring{\mathfrak{g}}[t]$ modules. In conjunction with [4, Proposition 3.9], Theorem 4.1 yields the following corollary:

Corollary 4.2. *Let $\mathring{\mathfrak{g}}$ be one of E_6, E_7 or E_8. The decomposition* (1) *gives a prime factorization of the Demazure module $D(\ell, \lambda)$, i.e., an expression as a fusion product of prime $\mathring{\mathfrak{g}}[t]$-modules.*

This was established in [4] for types A and D.

4.3. A second corollary concerns the notion of a Q-system [8]. Roughly speaking a Q-system is a short exact sequence of $\mathring{\mathfrak{g}}$-modules:

$$0 \to \bigotimes_{j\sim i} V(\ell\omega_j) \to V(\ell\omega_i)\otimes V(\ell\omega_i) \to V((\ell+1)\omega_i)\otimes V((\ell-1)\omega_i) \to 0$$

where ω_i is a miniscule weight of $\mathring{\mathfrak{g}}$ and $j\sim i$ means that nodes i and j are connected by an edge in the Dynkin diagram. Generalizations of Q-systems considered in [4, 5, 6, 7] involve replacing the tensor products above by fusion products of certain $\mathring{\mathfrak{g}}[t]$-modules. The following result was established in [4, Section 5]:

Proposition 4.3. ([4]) *Let $\mathring{\mathfrak{g}}$ be simply-laced. Let $\ell \geq 1$, $\lambda \in \mathring{P}^+$ such that $\ell \geq \max\{\lambda(\alpha^\vee) : \alpha \in \mathring{\Phi}^+\}$ and suppose ω_i is a miniscule weight such that $\lambda(\alpha_i^\vee) > 0$. Let $\mu = \ell\omega_i + \lambda - \lambda(\alpha_i^\vee)\alpha_i$. Then there exists a short exact sequence of $\mathring{\mathfrak{g}}[t]$-modules:*

$$0 \to \tau^*_{\lambda(\alpha_i^\vee)} D(\ell,\mu) \to D(\ell, \lambda+\ell\omega_i) \to D(\ell+1, \lambda+\ell\omega_i) \to 0 \qquad (2)$$

For a graded module V, $\tau_d^* V$ denotes V with its grading shifted by d; we refer to [4] for a fuller explanation of the notations. Proposition 4.3 and Theorem 4.1 together imply the following generalized Q-system in type E:

Corollary 4.4. *Assume that $\mathring{\mathfrak{g}}$ is of type E_6 or E_7, and retain the other notations of Proposition* 4.3. *Write $\mu = \ell\mu_1 + \mu_0$ for some $\mu_1, \mu_0 \in \mathring{P}^+$. Then there exists a natural short exact sequence of $\mathring{\mathfrak{g}}[t]$-modules:*

$$\begin{aligned} 0 &\to \tau^*_{\lambda(\alpha_i^\vee)} \left(D(\ell,\ell\mu_1) * D(\ell,\mu_0)\right) \to D(\ell,\ell\omega_i) * D(\ell,\lambda) \\ &\to D(\ell+1,(\ell+1)\omega_i) * D(\ell+1,\lambda-\omega_i) \to 0. \end{aligned}$$

We note that $\mathring{\mathfrak{g}} = E_8$ is excluded since it does not have miniscule weights. This result was established in types A, D in [4, Theorem 2].

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