

Partial Classification of Irreducible Modules for Loop-Witt Algebras

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Abstract. Consider the Lie algebra of the group of diffeomorphisms of a n -dimensional torus which is also known as the derivation algebra of the Laurent polynomial algebra A over n commuting variables, denoted by $DerA$. In this paper we consider the Lie algebra $(A \rtimes DerA) \otimes B$ for some commutative associative unital algebra B over \mathbb{C} and classify all irreducible modules for $(A \rtimes DerA) \otimes B$ with finite dimensional weight spaces under some natural conditions. In particular, we show that Larsson's constructed modules of tensor fields exhaust all such irreducible modules for $(A \rtimes DerA) \otimes B$.

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1. Introduction

Importance of the representation theory of infinite dimensional Lie algebras are well known in both mathematics and physics. Many mathematicians studied representations for the Lie algebra of the derivation algebra of $A = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$, so called Witt algebra denoted by W_n or $DerA$, for reference [1, 7, 12]. In particular, for $n = 1$ the universal central extension for the derivation algebra of A is known as the Virasoro algebra, generally denoted by Vir . Representations of Virasoro algebras were well studied in [6, 15, 16] and references therein. In [15], O.Mathieu proved that a simple module for Vir with finite dimensional weight spaces is either a highest weight module, a lowest weight module or a uniformly bounded module of intermediate series. Moreover from an interesting paper ([18]) this result comes out that $DerA$ has no non-trivial central extensions for $n \geq 2$. In [14], Larsson constructed a large class of modules for $DerA$, so called modules of tensor fields. In [7], the second author proved that for $n \geq 2$, modules of tensor fields for $DerA$ exhausts all irreducible weight modules for $A \rtimes DerA$ with finite dimensional weight spaces. In [1], complete classification of irreducible modules for $DerA$ has been done by Y. Billig and V. Futorny, which was conjectured by the second author.

On the other hand, in present days mathematicians are showing interest to study representations of loop algebras of some well known Lie algebras. For instance, representation of current algebras (also called loop algebras) for finite dimensional simple Lie algebras has been studied in [5, 2]. In [17], representation of current

algebras were generalized for finite dimensional perfect Lie algebras. In [8], loop Affine Lie algebras were studied, in fact any irreducible module for the well known Toroidal Lie algebras has been reduced to an irreducible module for the loop Affine Lie algebras. Further, in [9, 10] modules for loop Affine-Virasoro algebras and in [4] modules for loop toroidal Lie algebras has been studied. For $B = \mathbb{C}[t^{\pm 1}]$, complete classification of irreducible $Vir \otimes B$ -modules has been done in [11]. Later this work was generalized by A. Savage for $Vir \otimes B$, where B is a commutative associative finitely generated unital algebra over \mathbb{C} in [19]. In [19], A.Savage proved that a simple module for $Vir \otimes B$ with finite dimensional weight spaces is either highest weight module, lowest weight module or uniformly bounded module.

Broadly speaking, irreducible modules for $Vir \otimes B$ have same kind of behaviour as Vir modules. Classification of irreducible modules for loop-Witt algebras is a challenging open problem. Our results is one step towards that. In this paper we study irreducible weight modules for $(A \rtimes Der A) \otimes B$ with finite dimensional weight spaces, where B is a commutative associative unital algebra over \mathbb{C} . It is natural to ask whether modules of tensor fields exhausts all irreducible modules for $(A \rtimes Der A) \otimes B$ or not. Finally in Theorem 4.5, we prove that under some natural conditions modules of tensor fields exhausts all irreducible modules for $(A \rtimes Der A) \otimes B$.

This paper has been organised as follows. In section 2, we start with basic definitions and preliminaries. At the end of this section we recall the Theorem 2.1, proved in [7] regarding classification of irreducible $A \rtimes Der A$ modules for all $n \geq 2$. In section 3, we define weight modules for $\tau = (A \rtimes Der A) \otimes B$ and construct an irreducible weight module for τ . Then we consider an irreducible weight module V for τ with finite dimensional weight spaces. In Theorem 3.6, we prove that under some natural conditions V becomes an irreducible module for $A \rtimes Der A$. In section 4, we prove that Theorem 2.1 holds for $n = 1$ too. At the end of this section, in Theorem 4.5, we prove our final classification theorem.

We would like to point out one difference between loop of finite dimensional simple Lie algebras and loop of any extension of Witt algebras. For the loop of simple finite dimensional Lie algebras we get irreducible modules as evaluation modules at multiple points, see [2, 5, 17]. But for the loop of Witt algebras which has been studied in [11, 19] and the current paper, irreducible modules comes from a single points evaluation modules. So we conjecture that for the classification of loop-Witt algebras with uniformly bounded weight spaces we get a single point evaluation module.

2. Notations and Preliminaries

2.1. Throughout the paper all the vector spaces, algebras, tensor products are taken over the field of complex numbers \mathbb{C} . Let \mathbb{Z} , \mathbb{N} , \mathbb{Z}_+ denote the sets of integers, natural numbers and non-negative integers respectively. Let \mathbb{C}^n denote the direct sum of n copies of the vector space \mathbb{C} . Let e_i , $1 \leq i \leq n$ be the standard basis of \mathbb{C}^n and $(,)$ be the form on \mathbb{C}^n such that $(e_i, e_j) = \delta_{ij}$, for $1 \leq i, j \leq n$. For any Lie algebra G , let $U(G)$ denote the universal enveloping algebra of G .

2.2. Let $n \geq 1$ be a fixed natural number and $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ denote the Laurent polynomial algebra. Let us define $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. We denote elements of \mathbb{Z}^n by m, r, s, t etc. For $r = \sum_{i=1}^n r_i e_i \in \mathbb{Z}^n$, let $t^r = t_1^{r_1} t_2^{r_2} \dots t_n^{r_n} \in A$ and $D^i(r) = t^r t_i \frac{d}{dt_i}$ be a derivation on A . Let $DerA$ be the Lie algebra of derivations on A . It is well known that $D^i(r)$ for $1 \leq i \leq n$ and $r \in \mathbb{Z}^n$ forms a basis for $DerA$. For $u = \sum_{i=1}^n u_i e_i \in \mathbb{C}^n$, let $D(u, r) = \sum_{i=1}^n u_i D^i(r)$. Then $DerA$ is a Lie algebra with respect to the bracket operation

$$[D(u, r), D(v, s)] = D(w, r + s), \tag{1}$$

where $w = (u, s)v - (v, r)u$ for all $r, s \in \mathbb{Z}^n$ and $u, v \in \mathbb{C}^n$.

Let \mathfrak{h} be the subspace of $DerA$ spanned by $D^i(0)$ for $1 \leq i \leq n$. Then \mathfrak{h} is a maximal abelian sub-algebra of $DerA$.

In particular when $n = 1$, we consider $A = \mathbb{C}[t^{\pm 1}]$ and denote $d_n = t^n d$ for all $n \in \mathbb{Z}$, where $d = t \frac{d}{dt}$. Then we have $[d_m, d_n] = (n - m)d_{m+n}$ for all $m, n \in \mathbb{Z}$.

2.3. Let \mathfrak{g}_1 and \mathfrak{g}_2 be two Lie algebras and $\phi : \mathfrak{g}_2 \rightarrow Der(\mathfrak{g}_1)$ be a Lie algebra homomorphism from \mathfrak{g}_2 to the Lie algebra of derivations of \mathfrak{g}_1 . Then the semi-direct product of the Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 is denoted by

$$\mathfrak{g}_1 \rtimes \mathfrak{g}_2 = \{g_1 + g_2 : g_1 \in \mathfrak{g}_1, g_2 \in \mathfrak{g}_2\}$$

with the bracket operation

$$[g_1 + g_2, h_1 + h_2] = [g_1, h_1] + [g_2, h_2] + \phi(g_2)h_1 - \phi(h_2)g_1. \tag{2}$$

2.4. Consider the Lie algebra $A \rtimes DerA$ with the Lie bracket given by

$$[t^r, t^s] = 0, \tag{3}$$

$$[D(u, r), t^s] = (u, s)t^{r+s} \tag{4}$$

for all $r, s \in \mathbb{Z}^n$ and $u \in \mathbb{C}^n$. Note that A is an ideal in $A \rtimes DerA$. Let $H = \mathbb{C} \oplus \mathfrak{h}$ be a Cartan sub-algebra of $A \rtimes DerA$.

2.5. Let B be a commutative associative unital algebra over \mathbb{C} . Consider the algebra $\tau = (A \rtimes DerA) \otimes B$ and define a Lie algebra structure on τ by

$$[X \otimes b, Y \otimes b'] = [X, Y] \otimes bb',$$

for all $X, Y \in A \rtimes DerA$, $b, b' \in B$. We will identify $H \otimes 1$ with H . For convenience, we denote $D(u, r) \otimes b = D(u, r)b$ and $t^r \otimes b = t^r b$ for all $u \in \mathbb{C}^n$, $r \in \mathbb{Z}^n$, $b \in B$.

2.6. Let \mathfrak{gl}_n be the Lie algebra of $n \times n$ matrices over \mathbb{C} . Then $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}I$, where \mathfrak{sl}_n be the subalgebra of \mathfrak{gl}_n consisting of trace zero matrices and I be the $n \times n$ identity matrix. Let $V(\mu)$ be a finite dimensional irreducible module for \mathfrak{sl}_n , where μ is a dominant integral weight. Let I act by a scalar c on $V(\mu)$ and denote the resultant finite dimensional irreducible \mathfrak{gl}_n module by $V(\mu, c)$. For $\alpha \in \mathbb{C}^n$, we denote $F^\alpha(\mu, c) = V(\mu, c) \otimes A$. We first recall $A \rtimes DerA$ modules which have been studied in [7].

Theorem 2.1. (Theorem 2.9; [7]) *Let $n \geq 2$ and V be an irreducible module for $A \rtimes \text{Der}A$ which is also a weight module for H with finite dimensional weight spaces. Also assume*

- (1) *V is an A -module as an associative algebra and Lie module structure of A comes from associative algebra.*
- (2) *$1.v = v$ for all $v \in V$.*

Then $V \simeq F^\alpha(\mu, c)$ for some α, c, μ and the actions of $A \rtimes \text{Der}A$ on $F^\alpha(\mu, c)$ are given by

$$D(u, r)v \otimes t^m = (u, m + \alpha)v \otimes t^{m+r} + \sum_{i,j} (u_i r_j E_{ji}v) \otimes t^{m+r}, \tag{5}$$

$$t^r v \otimes t^m = v \otimes t^{m+r}. \tag{6}$$

3. Main results

Definition 3.1. A module V for τ is said to be a *weight module* if the action of H on V is diagonalizable, i.e. V can be decomposed as $V = \bigoplus_{\lambda \in H^*} V_\lambda$, where $H^* = \text{Hom}_{\mathbb{C}}(H, \mathbb{C})$ and $V_\lambda = \{v \in V : hv = \lambda(h)v \text{ for all } h \in H\}$. The space V_λ is called the *weight space* with respect to the weight λ of V . ■

Let $\eta : B \rightarrow \mathbb{C}$ be an algebra homomorphism such that $\eta(1) = 1$. Define a τ -module action on $F^\alpha(\mu, c)$ by

$$D(u, r)bv \otimes t^m = \eta(b)\{(u, m + \alpha)v \otimes t^{m+r} + \sum_{i,j} (u_i r_j E_{ji}v) \otimes t^{m+r}\}, \tag{7}$$

$$t^r b.v \otimes t^m = \eta(b)v \otimes t^{m+r}. \tag{8}$$

It is easy to see that $F^\alpha(\mu, c)$ is a weight module for τ with finite dimensional weight spaces $V(\mu) \otimes t^m$. By Theorem 2.1, $F^\alpha(\mu, c)$ is an irreducible module for τ .

The main purpose of this paper is to prove that under some natural conditions $F^\alpha(\mu, c)$ exhausts all irreducible weight modules for τ with finite dimensional weight spaces.

Let V be an irreducible weight module for τ with finite dimensional weight spaces. Also assume that the action of $A \otimes B$ on V is associative and 1 acts as identity on V . Throughout this section we consider V as a τ -module with these properties.

Let us assume that there exists a weight $\lambda \in H^*$ such that $V_\lambda \neq 0$. Then due to irreducibility of V we have $V = \bigoplus_{m \in \mathbb{Z}^n} V_{\alpha+m}$, where

$$V_{\alpha+m} = \{v \in V : D(u, 0)v = (u, m + \alpha)v$$

for all $u \in \mathbb{C}^n\}$ and $\alpha = (\lambda(D^1(0)), \lambda(D^2(0)), \dots, \lambda(D^n(0))) \in \mathbb{C}^n$. Note that $V_\lambda = V_\alpha$ and t^k is injective, hence $t^k V_\alpha = V_{\alpha+k}$ for all $k \in \mathbb{Z}^n$. In particular, all $V_{\alpha+m}$ have equal dimension. Therefore we can identify V with $V_\alpha \otimes A$ as vector spaces. Since $1 \otimes B$ is central in τ there exists an algebra homomorphism $\psi : B \rightarrow \mathbb{C}$ such that $1 \otimes b.v = \psi(b)v$ for all $v \in V$. Let M be the kernel of this homomorphism. Then M is a maximal ideal in B , since $\psi(1) = 1$.

Let U be the universal enveloping algebra of τ . Then $U = \bigoplus_{m \in \mathbb{Z}^n} U_m$, where $U_m = \{X \in U : [D(u, 0), X] = (u, m)X \text{ for all } u \in \mathbb{C}^n\}$. Let L be the two sided

ideal in U generated by $t^r b.t^s b' - t^{r+s} b b'$ and $t^0 \otimes 1 - 1$. Since $A \otimes B$ acts associatively on V so L acts trivially on V and hence V is an irreducible U/L -module.

Let us consider the elements $T(u, r, b_1, b_2) = t^{-r} b_1 D(u, r) b_2 \in U/L$ for all $r \in \mathbb{Z}^n$, $u \in \mathbb{C}^n$, $b_1, b_2 \in B$ and define

$$D = \text{span}\{T(u, r, b_1, b_2) : r \in \mathbb{Z}^n, u \in \mathbb{C}^n, b_1, b_2 \in B\}.$$

Lemma 3.2. (1) D is a Lie algebra with the Lie bracket

$$\begin{aligned} & [T(u, r, b_1, b_2), T(v, s, b_3, b_4)] \\ &= T(w, r + s, b_1 b_3, b_2 b_4) - (u, s)T(v, s, b_1 b_2 b_3, b_4) + (v, r)T(u, r, b_1 b_3 b_4, b_2), \end{aligned}$$

where $w = (u, s)v - (v, r)u$.

(2) V_α is an irreducible D -module.

Proof. (1) $[t^{-r} b_1 D(u, r) b_2, t^{-s} b_3 D(v, s) b_4]$

$$\begin{aligned} &= [t^{-r} b_1 D(u, r) b_2, t^{-s} b_3] D(v, s) b_4 + t^{-s} b_3 [t^{-r} b_1 D(u, r) b_2, D(v, s) b_4] \\ &= -(u, s) t^{-s} b_1 b_2 b_3 D(v, s) b_4 + (v, r) t^{-r} b_1 b_4 b_3 D(u, r) b_2 + t^{-r-s} b_1 b_3 D(w, r + s) b_2 b_4 \\ &= T(w, r + s, b_1 b_3, b_2 b_4) - (u, s)T(v, s, b_1 b_2 b_3, b_4) + (v, r)T(u, r, b_1 b_3 b_4, b_2), \end{aligned}$$

where $w = (u, s)v - (v, r)u$.

(2) Check that $D(w, 0)$ commutes with D for all $w \in \mathbb{C}^n$, consequently V_α is a D -module. Let $v, w \in V_\alpha$ be two arbitrary non-zero vectors. Then by weight arguments and the irreducibility of V implies that there exists $X \in U_0$ such that $X.v = w$. By the PBW theorem we have $U(D) = U_0$ and hence V_α is an irreducible D -module. ■

Lemma 3.3. ([13]) (1) Let $L \subset \mathfrak{gl}(V)$ (V is finite dimensional) be a non-zero Lie algebra and acting irreducibly on V . Then L is reductive and its center is at most one-dimensional.

(2) Let L be a finite dimensional reductive Lie algebra. Then $L = [L, L] \oplus Z(L)$ and $[L, L]$ is semi-simple.

Let us consider the elements $T_1(u, r, b_1, b_2) = t^{-r} b_1 D(u, r) b_2 - D(u, 0) b_1 b_2$ and define

$$D_1 = \text{span}\{T_1(u, r, b_1, b_2) : r \in \mathbb{Z}^n, u \in \mathbb{C}^n, b_1, b_2 \in B\}.$$

Also let $I(u, r, b_1, b_2) = \psi(b_1) D(u, r) b_2 - D(u, 0) b_1 b_2$ and define

$$D_2 = \text{span}\{I(u, r, b_1, b_2) : r \in \mathbb{Z}^n, u \in \mathbb{C}^n, b_1, b_2 \in B\}.$$

Moreover consider the subspace W of V , defined by

$$W = \text{span}\{t^r v - v : v \in V_\alpha, r \in \mathbb{Z}^n\} = \text{span}\{t^r v - v : v \in V, r \in \mathbb{Z}^n\}.$$

Note that $\dim(V/W) = \dim(V_\alpha)$.

Proposition 3.4. (1) D_1 is a Lie algebra with the Lie bracket

$$\begin{aligned} & [T_1(u, r, b_1, b_2), T_1(v, s, b_3, b_4)] \\ &= T_1(w, r + s, b_1b_3, b_2b_4) - (u, s)T_1(v, s, b_3, b_1b_2b_4) + (v, r)T_1(u, r, b_1, b_2b_3b_4), \end{aligned}$$

where $w = (u, s)v - (v, r)u$. Moreover $[D, D] \subseteq D_1$.

(2) D_2 is a Lie sub-algebra of $\text{Der}A \otimes B$ with the Lie bracket

$$\begin{aligned} & [I(u, r, b_1, b_2), I(v, s, b_3, b_4)] \\ &= I(w, r + s, b_1b_3, b_2b_4) - (u, s)I(v, s, b_3, b_1b_2b_4) + (v, r)I(u, r, b_1, b_2b_3b_4), \end{aligned}$$

where $w = (u, s)v - (v, r)u$.

(3) $\pi : D_1 \rightarrow D_2$ defined by $\pi(T_1(u, r, b_1, b_2)) = I(u, r, b_1, b_2)$ is a surjective Lie algebra homomorphism.

(4) V_α is an irreducible D_1 -module.

(5) W is a D_2 -sub-module. In particular V/W is a D_2 -module and a D_1 -module via π .

(6) $V_\alpha \simeq V/W$ as D_1 -module.

Proof. (1) From Lemma 3.2 we have

$$\begin{aligned} & [T(u, r, b_1, b_2), T(v, s, b_3, b_4)] \\ &= T(w, r + s, b_1b_3, b_2b_4) - (u, s)T(v, s, b_1b_2b_3, b_4) + (v, r)T(u, r, b_1b_3b_4, b_2) \\ &= \{T(w, r + s, b_1b_3, b_2b_4) - D(w, 0)b_1b_2b_3b_4\} - (u, s)\{T(v, s, b_1b_2b_3, b_4) - D(v, 0)b_1b_2b_3b_4\} \\ &\quad + (v, r)\{T(u, r, b_1b_3b_4, b_2) - D(u, 0)b_1b_2b_3b_4\} \\ &= T_1(w, r + s, b_1b_3, b_2b_4) - (u, s)T_1(v, s, b_1b_2b_3, b_4) + (v, r)T_1(u, r, b_1b_3b_4, b_2). \end{aligned}$$

From the relations above we have $[D, D] \subseteq D_1$. Now

$$\begin{aligned} & [T_1(u, r, b_1, b_2), T_1(v, s, b_3, b_4)] = [t^{-r}b_1D(u, r)b_2, t^{-s}b_3D(v, s)b_4] - \\ & \quad - [D(u, 0)b_1b_2, t^{-s}b_3D(v, s)b_4] + [D(v, 0)b_3b_4, t^{-r}b_1D(u, r)b_2] \\ &= [T(u, r, b_1b_2), T(v, s, b_3b_4)] - \\ & \quad - \{[D(u, 0)b_1b_2, t^{-s}b_3]D(v, s)b_4 + t^{-s}b_3[D(u, 0)b_1b_2, D(v, s)b_4]\} \\ & \quad + \{[D(v, 0)b_3b_4, t^{-r}b_1]D(u, r)b_2 + t^{-r}b_1[D(v, 0)b_3b_4, D(u, r)b_2]\} \\ &= [T(u, r, b_1b_2), T(v, s, b_3b_4)] + (u, s)T(v, s, b_1b_2b_3, b_4) - (u, s)T(v, s, b_3, b_1b_2b_4) \\ & \quad - (v, r)T(u, r, b_1b_3b_4, b_2) + (v, r)T(u, r, b_1, b_2b_3b_4) \\ &= T(w, r + s, b_1b_3, b_2b_4) - (u, s)T(v, s, b_3, b_1b_2b_4) + (v, r)T(u, r, b_1, b_2b_3b_4) \\ & \quad \text{(using Lemma 3.2(1))} \\ &= T_1(w, r + s, b_1b_3, b_2b_4) - (u, s)T_1(v, s, b_3, b_1b_2b_4) + (v, r)T_1(u, r, b_1, b_2b_3b_4). \end{aligned}$$

(2) $[I(u, r, b_1, b_2), I(v, s, b_3, b_4)]$

$$\begin{aligned} &= [\psi(b_1)D(u, r)b_2 - D(u, 0)b_1b_2, \psi(b_3)D(v, 0)b_4 - D(v, 0)b_3b_4] \\ &= \psi(b_1b_3)[D(u, r)b_2, D(v, s)b_4] + (v, r)\psi(b_1)D(u, r)b_2b_3b_4 - (u, s)\psi(b_3)D(v, s)b_1b_2b_4 \\ &= \psi(b_1b_3)D(w, r + s)b_2b_4 + (v, r)\psi(b_1)D(u, r)b_2b_3b_4 - (u, s)\psi(b_3)D(v, s)b_1b_2b_4 \\ &= I(w, r + s, b_1b_3, b_2b_4) + (v, r)I(u, r, b_1, b_2b_3b_4) - (u, s)I(v, s, b_3, b_1b_2b_4). \end{aligned}$$

(3) Follows from (1) and (2).

(4) Check that $D(w, 0)$ commutes with D_1 for all $w \in \mathbb{C}^n$. Therefore V_α is a D_1 -module. By Lemma 3.2, V_α is an irreducible D -module. Let $\phi: D \rightarrow \text{End}(V_\alpha)$ be the representation. Then by Lemma 3.3, we have $\phi(D)$ is reductive and $\phi(D) = Z(\phi(D)) \oplus \phi([D, D])$. Since V_α is irreducible, $Z(\phi(D))$ acts as scalar on V_α . Consequently V_α is an irreducible module for $[D, D] \subset D_1$.

(5) Since the action of $A \otimes B$ is associative, so $t^s b.v = t^s \otimes 1.1 \otimes b.v = \psi(b)t^s v$, for all $s \in \mathbb{Z}^n, b \in B$ and $v \in V$. Now,

$$\begin{aligned} & (\psi(b_1)D(u, r)b_2 - D(u, 0)b_1b_2).(t^s v - v) \\ &= \psi(b_1)[t^s(D(u, r)b_2.v + (u, s)t^{s+r}b_2.v - D(u, r)b_2.v) - \\ & \quad - [t^s D(u, 0)b_1b_2.v + (u, s)t^s b_1b_2.v - D(u, 0)b_1b_2.v]] \\ &= \psi(b_1)[t^s(D(u, r)b_2.v - D(u, r)b_2.v) + (u, s)\psi(b_1)[t^{s+r}b_2.v - t^s b_2.v] - \\ & \quad - [t^s D(u, 0)b_1b_2.v - D(u, 0)b_1b_2.v]] \in W. \end{aligned}$$

(6) Define $\phi: V \rightarrow V/W$ by $\phi(v) = \bar{v}$ for all $v \in V$. Now restrict ϕ to V_α and denote by $\phi|_{V_\alpha}$. Note that $\overline{t^s v} = \bar{v}$ for all $s \in \mathbb{Z}^n, v \in V$.

Claim: $\phi(T_1(u, r, b_1, b_2).v) = \pi(T_1(u, r, b_1, b_2)).\bar{v}$, for all $v \in V_\alpha, u \in \mathbb{C}^n, r \in \mathbb{Z}^n, b_1, b_2 \in B$. Now,

$$\begin{aligned} & \phi(T_1(u, r, b_1, b_2).v) \\ &= \overline{t^{-r}b_1D(u, r)b_2.v - D(u, 0)b_1b_2.v} = \overline{\psi(b_1)t^{-r}D(u, r)b_2.v - D(u, 0)b_1b_2.v} \\ &= \overline{\psi(b_1)D(u, r)b_2.v - D(u, 0)b_1b_2.v} \quad (\text{since } \overline{t^s v} = \bar{v}) \\ &= \overline{\psi(b_1)D(u, r)b_2.v - D(u, 0)b_1b_2.v} = \pi(T_1(u, r, b_1, b_2)).\bar{v}. \end{aligned}$$

Thus $\phi|_{V_\alpha}$ is a D_1 -module map. Now $\phi|_{V_\alpha}$ is non-zero and V_α is an irreducible D_1 -module, so $\phi|_{V_\alpha}$ is injective. $\phi|_{V_\alpha}$ is surjective because of equal dimension and hence $V_\alpha \simeq V/W$ as D_1 -module. ■

Let us consider the space

$$\tilde{D} = \text{span}\{\psi(b_1)D(u, r)b_2 - D(u, r)b_1b_2 : r \in \mathbb{Z}^n, u \in \mathbb{C}^n, b_1, b_2 \in B\}.$$

Observe that $\tilde{D} \subset D_2$ and $\tilde{D} = \text{Der}A \otimes M$, where M is the maximal ideal $\ker\psi$. Therefore \tilde{D} is an ideal in $\text{Der}A \otimes B$ and hence an ideal in D_2 .

Lemma 3.5. $D(u, 0)b$ acts as scalar on V_α for all $u \in \mathbb{C}^n, b \in B$.

Proof. By Proposition 3.4, D_2 acts on V/W irreducibly. Let $\phi: D_2 \rightarrow \text{End}(V/W)$ be the finite dimensional irreducible representation and $\rho = \phi|_{\tilde{D}}$. We prove this lemma in two cases.

Case 1: Let $M^k = 0$ for some $k \in \mathbb{N}$.

We have $[\text{Der}A \otimes M^i, \text{Der}A \otimes M^j] = \text{Der}A \otimes M^{i+j}$ for all $i, j \in \mathbb{N}$, since $\text{Der}A$ is a simple Lie algebra. This implies that $\text{Der}A \otimes M$ is a solvable ideal in D_2 , thus $\phi(\tilde{D})$ is a solvable ideal in the Lie algebra $\phi(D_2)$. By Lemma 3.3, $\phi(D_2)$ is a reductive Lie algebra and hence $\phi(\tilde{D})$ acts as a scalar on V/W .

In particular, $\psi(b)D(u, 0) - D(u, 0)b$ acts as scalar on V/W . Now by Proposition 3.4(6,3), we have $T_1(u, 0, b, 1)$ acts as scalar on V_α . This gives us $D(u, 0)b$ acts as scalar on V_α .

Case 2: Let $M^k \neq 0$ for all $k \in \mathbb{N}$. In this case, it is sufficient to show that $DerA \otimes M^k$ acts trivially on V/W for some $k \in \mathbb{N}$. Because

$$\left[\frac{DerA \otimes M^i}{ker\phi}, \frac{DerA \otimes M^j}{ker\phi} \right] = \frac{DerA \otimes M^{i+j}}{ker\phi}$$

for all $i, j \in \mathbb{N}$ implies that $\frac{DerA \otimes M^2}{ker\phi}$ is a solvable ideal contained in the semi-simple Lie algebra $[\frac{D_2}{ker\phi}, \frac{D_2}{ker\phi}]$. Thus $DerA \otimes M^2$ acts trivially on V/W . Therefore $\phi(DerA \otimes M)$ is an abelian ideal in $\phi(D_2)$. Now the result follows by same argument as case 1.

Claim: $DerA \otimes M^{2N+6}.V/W = 0$, where $N = \dim (DerA \otimes M/J)$, $J = ker\rho$.

Sub-claim 1: $D(v, 0) \otimes M^{2N+5} \subset J$ for all $v \in \mathbb{C}^n$, for all $n \geq 2$.

Let $b = b_1b_2\dots, b_{2N+5}$ be a non-zero element of M^{2N+5} . Consider the set of vectors

$$\{D(u_i, r_i)b_1 + J : u_i \in \mathbb{C}^n - \{0\}, r_i \in \mathbb{Z}^n - \{0\}, r_i \neq r_j, \forall i \neq j, 1 \leq i \leq N + 1\}.$$

Since we have the relation $\dim(DerA \otimes M/ker\rho) = N$, there exists a non-zero vector $X = \sum_{i=1}^k D(u_i, r_i)b_1 \in J$ for some $k \leq N + 1$. Now,

$$[X, D(v, -r_1)b_2] = D(w_1, 0)b_1b_2 + \sum_{i=2}^k D(w_i, r_i - r_1)b_1b_2,$$

where $w_1 = -(u_1, r_1)v - (v, r_1)u_1$ and $w_i = -(u_i, r_1)v - (v, r_i)u_i$ for all $i \neq 1$.

Thus if $(u_1, r_1) = 0$ choose v such that $(v, r_1) \neq 0$ and if $(u_1, r_1) \neq 0$ choose v such that $(v, r_1) = 0$, in both cases $w_1 \neq 0$. If all $w_i = 0$ for all $i \neq 1$, then we have $D(w_1, 0)b_1b_2 \in J$. If $w_i \neq 0$ for some $i \neq 1$, say $w_j \neq 0$. Then choosing any v' such that $(v', r_j - r_1) \neq 0$ and considering $[[X, D(v, -r_1)b_2], D(v', 0)b_3]$ we get the sum of at most $k - 1$ terms. Continuing the above process we can conclude that $D(u, 0)b_1b_2\dots b_{2k+1} \in J$ for some non-zero $u \in \mathbb{C}^n$.

For any non-zero $v \in \mathbb{C}^n$ choose $s \in \mathbb{Z}^n$ such that $(u, s) \neq 0$. Then we have

$$[D(v, s)b_{2k+2}, D(u, 0)b_1b_2\dots b_{2k+1}] = (u, s)D(v, s)b_1b_2\dots b_{2k+2} \in J.$$

Again,

$$[D(w, -s)b_{2k+3}, D(v, s)b_1b_2\dots b_{2k+2}] = D((w, s)v + (v, s)w, 0)b_1\dots b_{2k+3} \in J.$$

Now, if $(v, s) = 0$ choose $(w, s) \neq 0$ and if $(v, s) \neq 0$ choose $w = v$, then with this choice we have for all $v(\neq 0) \in \mathbb{C}^n$, $D(v, 0)b_1b_2\dots b_{2k+3} \in J$. Since $k \leq N + 1$ we have $D(v, 0) \otimes M^{2N+5} \subset J$ for all $v \in \mathbb{C}^n$.

Sub-claim 2: $d_0 \otimes M^{2N+5} \subset J$ for $n = 1$.

Let $b = b_1b_2\dots b_{2N+5} \in M^{2N+5}$. Let us consider the set of vectors

$$\left\{ d_{r_i}b_1 + J : \begin{array}{l} r_i \in \mathbb{N}, r_i > r_j \text{ for } i > j \text{ and } 2r_i \neq r_j + r_k \\ \text{for distinct } i, j, k, 1 \leq i \leq N + 1 \end{array} \right\}$$

(In particular $2^i, i \in \mathbb{N}$ can be chosen as r_i). Since $\dim(Der A \otimes M)/J = N$, there exists a non-zero vector $X = \sum_{i=1}^k \lambda_i d_{r_i} b_1 \in J$ with some $\lambda_i (\neq 0) \in \mathbb{C}$, $k \leq N + 1$. Let us assume $\lambda_1 \neq 0$ and consider

$$[d_{-r_1} b_2, X] = 2r_1 \lambda_1 d_0 b_1 b_2 + \sum_{i=2}^k \lambda_i (r_i + r_1) d_{r_i - r_1} b_1 b_2 \in J.$$

If $\lambda_i = 0$ for all $i \geq 2$ we are done. If not, let $\lambda_2 \neq 0$ and consider

$$[d_{r_1 - r_2} b_4, [d_0 b_3, [d_{-r_1} b_2, X]]] = 2(r_2 - r_1)^2 (r_2 + r_1) \lambda_2 d_0 b_1 b_2 b_3 b_4 + \sum_{i=3}^k \lambda_i (r_i^2 - r_1^2) (r_i + r_2 - 2r_1) d_{r_i - r_2} b_1 b_2 b_3 b_4 \in J.$$

Continuing this process we can conclude that $d_0 \otimes M^{2N+5} \subset J$, since $k \leq N + 1$. Therefore sub-claim 1 is true for all $n \geq 1$.

Now for any $b \in M$, $b' \in M^{2N+5}$, $u (\neq 0) \in \mathbb{C}^n$ and $s (\neq 0) \in \mathbb{Z}^n$, choose v such that $(v, s) \neq 0$ and then consider $[D(u, s)b, D(v, 0)b']$ which gives $D(u, s)bb' \in J$.

Hence the claim follows. ■

Let $T(u, r)b = T_1(u, r, 1, b)$ for all u, r, b and consider T_B be the space spanned by $T(u, r)b$. It is easy to see that T_B is a sub-algebra of D_1 . In fact one can check that

$$[T(v, s)b, T(u, r)b'] = T(w, r + s)bb' + (u, s)T(v, s)bb' - (v, r)T(u, r)bb',$$

where $w = (v, r)u - (u, s)v$.

Theorem 3.6. *Let V be an irreducible weight module for τ with finite dimensional weight spaces. Also let the action of $A \otimes B$ on V be associative and $1.v = v$ for all $v \in V$. Then V is an irreducible module for $A \rtimes Der A$.*

Proof. By Lemma 3.5, $D(u, 0)b$ acts on V_α as a scalar, say $\lambda(u, b)$ for all u, b . Then we have,

$$D(u, 0)b(v_\alpha \otimes t^m) = (\lambda(u, b) + (u, m)\psi(b))v_\alpha \otimes t^m \tag{9}$$

$$t^m b.(v_\alpha \otimes t^k) = \psi(b)v_\alpha \otimes t^{m+k} = \psi(b)t^m.(v_\alpha \otimes t^k). \tag{10}$$

for all $m, k \in \mathbb{Z}^n$ and $v_\alpha \in V_\alpha$.

Consider the actions of $T(u, r)b$ on V , for all $r \neq 0, u, b$.

$$T(u, r)b.v \otimes t^k = (t^{-r}D(u, r)b - D(u, 0)b)v \otimes t^k.$$

Now using (9) and the fact that t^r commutes with $T(u, r)b$ we have,

$$D(u, r)b.v \otimes t^k = T(u, r)b.v \otimes t^{k+r} + (\lambda(u, b) + (u, k)\psi(b))v \otimes t^{k+r} \tag{11}$$

for all $v \in V_\alpha$ and $k \in \mathbb{Z}^n$. In particular for $b = 1$, (11) gives

$$D(u, r).v \otimes t^k = T(u, r).v \otimes t^{k+r} + (u, k + \alpha)\psi(b)v \otimes t^{k+r}, \tag{12}$$

since $\lambda(u, 1) = (u, \alpha)$.

For any $r \neq 0$ choose v such that $(v, r) \neq 0$ and consider

$$[D(v, 0)b, D(u, r)] = (v, r)D(u, r)b. \tag{13}$$

Then using the actions of (9), (11) and (12) on the both sides of (13) we have the following action of $T(u, r)b$ on V ,

$$T(u, r)b = \psi(b)T(u, r) + \psi(b)(u, \alpha) - \lambda(u, b) \tag{14}$$

for all $r \neq 0$.

Now using (11), (12) and (14) it is immediate to check that

$$D(u, r)b = \psi(b)D(u, r) \tag{15}$$

on V for all $r \neq 0$. For all non-zero u, v, r, s and b we have

$$[D(v, s)b, D(u, r)]w_0 \otimes t^k = D((v, r)u - (u, s)v, r + s)bw_0 \otimes t^k, \tag{16}$$

for all $w_0 \in V_\alpha$ and $k \in \mathbb{Z}^n$. Using (15) in the left hand side of (16) we have

$$\psi(b)D((v, r)u - (u, s)v, r + s)w_0 \otimes t^k = D((v, r)u - (u, s)v, r + s)b.w_0 \otimes t^k. \tag{17}$$

In (17), put $u = v$ and find r such that $(u, r) \neq 0$. Then put $s = -r$ and conclude that $D(u, 0)b = \psi(b)D(u, 0)$ on V . Hence $\lambda(u, b) = \psi(b)(u, \alpha)$ and $D(u, r)b = \psi(b)D(u, r)$ on V for all u, r . This completes the proof. ■

Remark 3.7. Theorem 3.6 shows that our module for τ is actually an evaluation module at a single point. For more details on evaluation module see [11, 19, 2] and references therein.

4. Final classification

We would like to take the opportunity to clarify that the method used in [7] also work for $n=1$. In [7], Theorem 2.1 was proved for $A = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$ with the assumption $n \geq 2$. In this section we show that the Theorem 2.1 holds for $A = \mathbb{C}[t^{\pm 1}]$ too and prove a final classification theorem.

Let V be a module for $A \rtimes Der A$ satisfying all properties of Theorem 2.1. Let us assume that there exists $\lambda \in H^*$ such that $V_\lambda \neq 0$. Then $V = \bigoplus_{m \in \mathbb{Z}} V_{m+\alpha}$, where

$$V_{m+\alpha} = \{v \in V : d_0v = (m + \alpha)v\}, \quad \alpha = \lambda(d) \in \mathbb{C}.$$

Note that $V_\lambda = V_\alpha$ and t^k is injective, hence $t^kV_\alpha = V_k$ for all $k \in \mathbb{Z}$. Therefore we can identify V with $V_\alpha \otimes A$ as vector space.

Let $I(r) = (t^r - 1)d$, for all $r \in \mathbb{Z}$ and denote $T' = \text{span}\{I(r) : r \in \mathbb{Z}\}$. Let \mathfrak{m} be the maximal ideal in A generated by $(t - 1)$. It is clear that $\mathfrak{m}d = T'$.

Lemma 4.1. $[I(r), I(s)] = (s - r)I(r + s) + rI(r) - sI(s)$, for all $s, r \in \mathbb{Z}$. In particular, T' is a sub-algebra of $A \rtimes Der A$.

Let $W = \text{span}\{t^r.v - v : v \in V_\alpha \text{ and } r \in \mathbb{Z}\}$. It is easy to see that W is a T' sub-module of V and hence $\bar{V} = V/W$ is a T' module. Now we define an action of $A \rtimes \text{Der}A$ on $\bar{V} \otimes A$ by

$$d_r.v \otimes t^s = (I(r).v) \otimes t^{r+s} + (s + \alpha)v \otimes t^{r+s}, \tag{18}$$

$$t^r.v \otimes t^s = v \otimes t^{r+s}, \tag{19}$$

for all $r, s \in \mathbb{Z}$, $v \in \bar{V}$ and for some $\alpha \in \mathbb{C}$.

It is easy to see that this action define a module structure on $\bar{V} \otimes A$. We will denote this module by $\bar{V}^\alpha \otimes A$.

Proposition 4.2. $V \simeq \bar{V}^\alpha \otimes A$ as $A \rtimes \text{Der}A$ -module and \bar{V}^α is a finite dimensional irreducible T' -module.

Proof. Define $\phi : V \rightarrow \bar{V}^\alpha \otimes A$ by $\phi(t^m v) = \overline{t^m v} \otimes t^m$ for all $v \in V_\alpha$ and $m \in \mathbb{Z}$. Then ϕ is a $A \rtimes \text{Der}A$ module homomorphism. ϕ is clearly surjective. Since $\phi(V_\alpha) = \bar{V}^\alpha$, hence ϕ is non-zero. So by irreducibility of V we have ϕ is injective.

Suppose \bar{W} be a T' sub-module of \bar{V}^α , then $\bar{W} \otimes A$ becomes a $A \rtimes \text{Der}A$ sub-module of $\bar{V}^\alpha \otimes A$, a contradiction. ■

Lemma 4.3. (1) For all $k, l \in \mathbb{N}$, $i, j \in \mathbb{Z}$ we have

$$[(t-1)^k d_i, (t-1)^l d_j] = (l-k+j-i)(t-1)^{k+l} d_{i+j} + (l-k)(t-1)^{k+l-1} d_{i+j}.$$

(2) For any polynomial $f(\lambda)$,

$$f((1-t^{-1})d_0 - k).(t-1)^{k+1} d_{-1}.v = (t-1)^{k+1} d_{-1}.f((1-t^{-1})d_0).v,$$

for all $v \in V$ and $k \in \mathbb{N}$.

Proof. (1) Follows from Lemma 2.4 of [3].

(2) Note that,

$$\begin{aligned} & ((1-t^{-1})d_0 - k).(t-1)^{k+1} d_{-1}.v \\ &= [(1-t^{-1})d_0, (t-1)^{k+1} d_{-1}].v + (t-1)^{k+1} d_{-1}.(1-t^{-1})d_0.v - k(t-1)^{k+1} d_{-1}.v \\ &= [(t-1)d_{-1}, (t-1)^{k+1} d_{-1}].v + (t-1)^{k+1} d_{-1}.(1-t^{-1})d_0.v - k(t-1)^{k+1} d_{-1}.v \\ &= k(t-1)^{k+2} d_{-2}.v + k(t-1)^{k+1} d_{-2}.v + (t-1)^{k+1} d_{-1}.(1-t^{-1})d_0.v - k(t-1)^{k+1} d_{-1}.v \\ &= k(t-1)^{k+1} [(t-1)t^{-1} + t^{-1}]d_{-1}.v + (t-1)^{k+1} d_{-1}.(1-t^{-1})d_0.v - k(t-1)^{k+1} d_{-1}.v \\ &= (t-1)^{k+1} d_{-1}.(1-t^{-1})d_0.v \end{aligned}$$

Now assume that,

$$((1-t^{-1})d_0 - k)^n.(t-1)^{k+1} d_{-1}.v = (t-1)^{k+1} d_{-1}.((1-t^{-1})d_0)^n.v, \tag{20}$$

for some $n \in \mathbb{N}$. Then we have,

$$\begin{aligned} & ((1 - t^{-1})d_0 - k)^{n+1} \cdot (t - 1)^{k+1} d_{-1} \cdot v \\ &= ((1 - t^{-1})d_0 - k) \cdot [(t - 1)^{k+1} d_{-1} \cdot ((1 - t^{-1})d_0)^n \cdot v], \text{ by (20)} \\ &= (t - 1)^{k+1} d_{-1} \cdot (1 - t^{-1})d_0 \cdot ((1 - t^{-1})d_0)^n \cdot v \\ &= (t - 1)^{k+1} d_{-1} \cdot ((1 - t^{-1})d_0)^{n+1} \cdot v \end{aligned}$$

Hence by induction we have the result for any polynomial $f(\lambda)$. ■

Proposition 4.4. (1) $\mathbf{m}^2 d \bar{V}^\alpha = 0$.

(2) V is an irreducible module for \mathbb{C} and satisfy the actions given in Theorem 2.1.

Proof. (1) It is sufficient to prove that $\mathbf{m}^k d \bar{V}^\alpha = 0$ for some $k \in \mathbb{N}$. Because then using same argument as Lemma 3.5 (Case 2) we have the result. Consider the operator $(1 - t^{-1})d_0 : \bar{V}^\alpha \rightarrow \bar{V}^\alpha$. Let $f(\lambda)$ be the characteristic polynomial of this operator. Since we are over the field \mathbb{C} , there exists a $k \in \mathbb{N}$ such that $\gcd(f(\lambda), f(\lambda - l)) = 1$ for all $l > k$.

Now using Lemma 4.3(2), we get $(t - 1)^{l+1} d_{-1} \cdot v = 0$ for all $l > k$ and $v \in \bar{V}^\alpha$. By Lemma 4.3(1), we have the relation

$$[(t - 1)^{l+2} d_{-1}, (t - 1)d_i] - [(t - 1)^{l+1} d_{-1}, (t - 1)^2 d_i] = -2(t - 1)^{l+2} d_i$$

for all $i \in \mathbb{Z}$, $l \in \mathbb{N}$. This implies that $(t - 1)^{l+2} t^i d \cdot v = 0$ for all $v \in \bar{V}^\alpha$, $i \in \mathbb{Z}$, $l > k$. Therefore there exists $k_0 \in \mathbb{N}$ such that $\mathbf{m}^{k_0} d \bar{V}^\alpha = 0$.

(2) Let us define a map $\phi : \mathbf{m}d/\mathbf{m}^2 d \rightarrow \mathbb{C}$ by $\phi((t - 1)f(t)d) = f(1)$. It is easy to see that ϕ is a Lie algebra isomorphism and image of $I(r)$ under this map is r for all $r \in \mathbb{Z}$. Now by Proposition 4.2 and the action defined by (18), we have

$$d_r \cdot v \otimes t^s = r(1 \cdot v) \otimes t^{r+s} + (s + \alpha)v \otimes t^{r+s} \quad \blacksquare$$

Theorem 4.5. Let V be an irreducible weight module for τ with finite dimensional weight spaces. Also let the action of $A \otimes B$ on V be associative and $1 \cdot v = v$ for all $v \in V$. Then $V \simeq F^\alpha(\mu, c)$ for some dominant integral weight μ , $c \in \mathbb{C}$ and $\alpha \in \mathbb{C}^n$. Actions of the elements of τ on $F^\alpha(\mu, c)$ are given by

$$\begin{aligned} D(u, r)bv \otimes t^m &= \psi(b)\{(u, m + \alpha)v \otimes t^{m+r} + \sum_{i,j} (u_i r_j E_{ji}v) \otimes t^{m+r}\}, \\ t^r b \cdot v \otimes t^m &= \psi(b)v \otimes t^{r+m}, \end{aligned}$$

for all $u \in \mathbb{C}^n, r \in \mathbb{Z}^n, b \in B$ and for some algebra homomorphism $\psi : B \rightarrow \mathbb{C}$.

Proof. Follows from Theorem 2.1, Theorem 3.6, Proposition 4.2 and Proposition 4.4. ■

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