Maximal Antipodal Sets of F_4 and FI

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Abstract. We explicitly classify congruent classes of maximal antipodal sets of F_4 by using the Jordan algebra $H_3(\mathbb{O})$. Moreover, we give a realization of the compact symmetric space of type FI as a totally geodesic submanifold in a Grassmannian $G_{15}(H_3(\mathbb{O}))$, where $G_{15}(H_3(\mathbb{O}))$ is the set of all subspaces of dimension 15 in $H_3(\mathbb{O})$. In this realization, we explicitly classify congruent classes of maximal antipodal sets of FI.

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1. Introduction

Let M be a compact Riemannian symmetric space and s_p be the geodesic symmetry at $p \in M$. Then, $p, q \in M$ are antipodal if $s_p(q) = q$. A subset $S \subset M$ is called an antipodal set, if any two points of S are antipodal. We call an antipodal set S is maximal, if S is maximal with respect to the inclusion relation among antipodal sets. A maximal antipodal set S is called great if the cardinality of S is the maximum and we call the cardinality of great antipodal sets the 2-number of M. We denote the 2-number of M by $\#_2 M$. Antipodal sets are introduced by Chen-Nagano [2].

In this paper, we classify congruent classes of maximal antipodal sets of the exceptional compact Lie group F_4 and the compact symmetric space FI explicitly. Chen-Nagano already decided the 2-number of F_4 and FI [2]. However, they did not classify congruent classes of maximal antipodal sets. Moreover, Griess classified congruent classes of maximal antipodal sets of F_4 by using an algebraic method [3]. However, he did not describe maximal antipodal sets explicitly. It is known that F_4 is the set of all automorphisms of the exceptional Jordan algebra.

In the present paper, we classify congruent classes of maximal antipodal sets of F_4 by a different approach from [3] and describe these congruent classes as a set of automorphisms of the exceptional Jordan algebra explicitly (Theorem 3.6). Moreover, we give a new geometric characterization of FI by using some subalgebra of the exceptional Jordan algebra (Theorem 4.15). By using this characterization, we give the classification of congruent classes of maximal antipodal sets of FI explicitly (Theorem 4.18). We remark that FII is a symmetric R-space. Congruent classes of maximal antipodal sets of symmetric R space are completely classified by Tanaka-Tasaki [8].

The present paper is organized as follows. In Section 2, we introduce a strategy to classify congruent classes of maximal antipodal sets of F_4 and FI. Moreover, we recall some fundamental results about G_2 and F_4 from [11]. We recall a concept of polars of compact symmetric spaces from [1]. In Section 3, we study maximal antipodal sets of F_4 and prove Theorem 3.6. In Section 4, we consider the characterization of FI and prove Theorem 4.15. Further, we study maximal antipodal sets of FI and prove Theorem 4.18. Moreover, we observe an application with respect to maximal antipodal sets of FI.

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2. Preliminaries

2.1. Strategy

If for any subsets A, B of a Riemannian symmetric space M there is an element g of the identity component of the isometry gorup of M such that $g(A) = B$, we say A is congruent to B. In this Subsection, we see a strategy used in the present paper to classify the congruent class of maximal antipodal sets of F_4 and FI. Let G be a connected compact Lie group and e be the unit element of G. Then, it is known that there is a biinvariant metric on G and G is a compact symmetric space with respect to this metric. In particular, the symmetry s_g at $g \in G$ is given by $s_g : G \to G; h \mapsto gh^{-1}g$. It is known that any maximal antipodal sets containing the unit element of G is an maximal elementary abelian 2-subgroup. We call this subgroup a maximal antipodal subgroup. Therefore, classifying congruent classes of maximal antipodal sets of G reduces to classifying conjugate classes of maximal antipodal subgroups of G.

We recall some fundamental results of polars introduced by Chen-Nagano [1]. For this purpose let M be a compact symmetric space. For any isometry h of M, we set $F(h, M) = \{x \in M; \ h(x) = x\}$.

Definition 2.1. Let M be a compact symmetric space and $o \in M$. A connected component of $F(s_o, M)$ is called a *polar* of o. The polar containing p ($p \in F(s_o, M)$) is denoted by $M_o^+(p)$. If a polar is a one-point set, then we call this polar a *pole*. We call $\{o\}$ the trivial pole.

It is known that the number of connected componets of $F(s_o, M)$ is finite. Therefore, the number of all polars is finite. Proposition 2.2 about polars is known.

Proposition 2.2. ([1] Theorem 2.8) *Let K be the identity component of the istoropy subgroup at $o \in M$ of the isometry group of M. Then, $M_o^+(p) = K(p)$. Set $K_z = \{k \in K \ ; \ k|_{M_o^+(p)} = \mathrm{Id}_{M_o^+(p)}\}$. Then, K/K_z is the identity component of the isometry group of $M_o^+(p)$.*

In G, the action of isotropy group of the isometry group of the unit element is given by the G-conjugation. We see $F(s_e, G) = \{g \in G \ ; \ g^{-1} = g\}$. If $p \in F(s_e, G)$, the polar of e containing p is given by $\bigcup_{g \in G} gpg^{-1}$. Let M_k^+ ($0 \le k \le m, M_0^+ = \{e\}$) be all polars of e in G and $F(s_e, G) = \bigsqcup_{k=0}^m M_k^+$. Let A be a maximal antipodal subgroup of G.

Then, since $A \subset F(s_e, G)$, the following decomposition of A is obtained.

$$A = \{e\} \sqcup (A \cap M_1^+) \sqcup \cdots \sqcup (A \cap M_m^+).$$

For each $g \in F(s_e, G)$, we set $\tau_g : G \to G$; $h \mapsto ghg^{-1}$. τ_g is involutive by $g = s_e(g) = g^{-1}$. Set $G^g = \{h \in G \; ; \; \tau_g(h) = h\}$. If G is simply connected, it is known that G^g is connected. Let σ_k $(1 \leq k \leq m)$ be an element of M_k^+.

Proposition 2.3.   *Let A be a maximal antipodal subgroup of G and $A \cap M_k^+ \neq \phi$ for some k $(1 \leq k \leq m)$. Then, A is conjugate to a maximal antipodal subgroup of G^{σ_k}.*

Proof.   Let $x \in A \cap M_k^+$. Then, there is $g \in G$ such that $x = g^{-1}\sigma_k g$. gAg^{-1} is a maximal antipodal subgroup of G and $\sigma_k \in gAg^{-1}$. In particular, since $p = s_{\sigma_k}(p) = \sigma_k p \sigma_k$ for any $p \in gAg^{-1}$, it is true that $p\sigma_k = \sigma_k p$ for any $p \in gAg^{-1}$. Thus, gAg^{-1} is a maximal antipodal subgroup of G^{σ_k} since $gAg^{-1} \subset G^{\sigma_k}$. ∎

Proposition 2.4.   *For any $1 \leq k \leq m$, any maximal antipodal subgroup of G^{σ_k} is a maximal antipodal subgroup of G.*

Proof.   Let A be a maximal antipodal subgroup of G^{σ_k}. We see $\sigma_k \in A$ as follows. It is true that $\sigma_k^2 = e$ by $s_e(\sigma_k) = \sigma_k$. Moreover, $g\sigma_k = \sigma_k g$ for any $g \in G^{\sigma_k}$, since σ_k is an element of the center in G^{σ_k}. Thus, $A \cup \{\sigma_k\}$ is an antipodal set of G^{σ_k}. By the maximality of A, it is true that $\sigma_k \in A$.

Let B be a maximal antipodal subgroup of G containing A. Then, $B \subset G^{\sigma_k}$ because $\sigma_k \in A \subset B$ and $\sigma_k g = g\sigma_k$ for any $g \in B$. Thus, B is an antipodal subgroup of G^{σ_k}. By the maximality of A in G^{σ_k}, it is true that $A = B$. Hence, A is a maximal antipodal subgroup of G. ∎

By the above arguments, in order to classify conjugate classes of maximal antipodal subgroups A of G such that $A \cap M_k^+ \neq \phi$, we classify conjugate classes of maximal antipodal subgroups of G^{σ_k} firstly. Nextly, we study whether there is $g \in G$ such that $g(A_1) \subset A_2$ or $g(A_2) \subset A_1$ for any representatives A_1, A_2 of two different conjugate classes of maximal antipodal subgroups of G^{σ_k}. In particular, we consider the case where any maximal antipodal subgroup A of G satisfies $A \cap M_k^+ \neq \phi$ and the number of conjugate classes of maximal antipodal subgroups of G^{σ_k} is 1. In this assumption, the number of conjugate classes of maximal antipodal subgroups of G is 1 and the conjugate class is given by the G-conjugate class of maximal antipodal subgroups of G^{σ_k}.

We study conjugate classes of maximal antipodal subgroups of F_4 by using this strategy. In F_4, the number of all polars is 3 (Proposition 2.20). Let M_0^+, M_1^+, M_2^+ $(M_0^+ = \{I\})$ be all polars of the unit element $I \in F_4$ and $\sigma_k \in M_k^+$ $(k = 1, 2)$. Then, one of $G^{\sigma_1}, G^{\sigma_2}$ is isomorphic to $Spin(9)$ and the other is isomorphic to $Sp(1) \cdot Sp(3)$ (see page 10)[11]. We set $G^{\sigma_1} \cong Sp(1) \cdot Sp(3)$ and $G^{\sigma_2} \cong Spin(9)$. Later, we will see that any maximal antipodal subgroup of F_4 is contained in G^{σ_1} and the number of conjugate classes of maximal antipodal subgroups of $Sp(1) \cdot Sp(3)$ is 1. Hence, the classification of conjugate classes of maximal antipodal subgroups of F_4 is reduced to the classification of conjugate classes of maximal antipodal subgroups of $Sp(1) \cdot Sp(3)$. By using this method, we explicitly give a representative of the conjugate class of maximal antipodal subgroup of F_4 as a set of automorphisms of exceptional Jordan algebra.

Proposition 2.10 is useful to classify congruent classes of maximal antipodal sets of a polar M^+ of a compact connected Lie group G.

Proposition 2.5. *Let G be a connected compact Lie group and e be the unit element. Let M^+ be a polar of e. If the number of conjugate classes of maximal antipodal subgroups of G is 1, then the number of congruent classes of maximal antipodal sets of M^+ is 1.*

Proof. Let A_1, A_2 be two maximal antipodal sets of M^+. Then, $\{e\} \cup A_1$ and $\{e\} \cup A_2$ is antipodal sets in G. Let B_1, B_2 be maximal antipodal subgroups of G such that $\{e\} \cup A_i \subset B_i$ ($i = 1, 2$). Remark that $B_i \cap M^+ = A_i$ ($i = 1, 2$) by the maximality of A_i ($i = 1, 2$) in M^+. Then, there is $g \in G$ such that $B_2 = gB_1g^{-1}$. Since $gM^+g^{-1} \subset M^+$, $g(A_1)g^{-1} \subset A_2$. By the maximality, it is true that $gA_1g^{-1} = A_2$. In particular, $\tau_g : M^+ \to M^+; p \mapsto gpg^{-1}$ is contained in the identity component of the isometry group of M^+ by Proposition 2.2. Thus, any two maximal antipodal sets in M^+ are congruent. ∎

In F_4, $M_1^+ \cong F_4/Sp(1) \cdot Sp(3)$ and $M_2^+ \cong F_4/Spin(9)$ because

$$M_1^+ = \cup_{g \in F_4} g\sigma_1 g^{-1} \cong F_4/F_4^{\sigma_1}$$

and $M_2^+ = \cup_{g \in F_4} g\sigma_2 g^{-1} \cong F_4/F_4^{\sigma_2}$. In particular, M_1^+ is the compact symmetric space of type FI and M_2^+ is the compact symmetric space of type FII in the Cartan's classification. It is known that FII is a symmetric R-space and congruent classes of maximal antipodal sets of symmetric R-spaces are classified by Tanaka-Tasaki [8]. Hence, congruent classes of maximal antipodal sets of FII are already classified . On the other hand, congruent classes of maximal antipodal sets of FI is not classified. By the classification of conjugate classes of maximal antipodal subgroups of F_4 and Proposition 2.10, we will classify congruent classes of maximal antipodal sets of FI explicitly.

2.2. Octonions and G_2

In this subsection, we recall the exceptional Jordan algebra and F_4 from [11] Let $\mathbb{O} = \bigoplus_{i=0}^{7} \mathbb{R}e_i$ be the octonions, where $\{e_0, \cdots, e_7\}$ is a basis. We define the multiplication between two elements of $\mathbb{O}$ as follows. e_0 is the unit element of this multiplication and denote e_0 by 1. For any $1 \leq i \neq j \leq 7$, $e_i^2 = -1$ and $e_i e_j = -e_j e_i$. Assume the distributive law. In the figure below the multiplications between e_1, e_2, e_3 are defined as

$$e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2,$$

and the multiplications between three elements on each of other lines are defined similarly. We refer for the properties of the octonions to [11].

Let $x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$. We define the conjugate $\bar{x}$ and the real part $R(x)$ of x as follows:

$$\bar{x} = x_0 - \sum_{i=1}^{7} x_i e_i, \quad R(x) = x_0.$$

Moreover, set the inner product $(\ ,\ )$ and the norm $|\cdot|$ as follows. Let $y = \sum_{i=0}^{7} y_i e_i$.

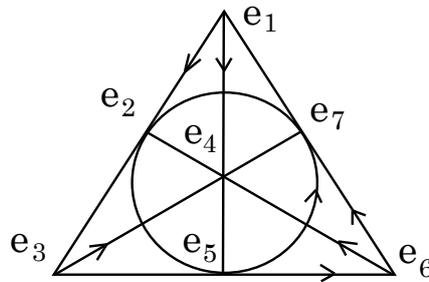Then, $$(x, y) = \sum_{i=0}^{7} x_i y_i, \quad |x| = \sqrt{(x, x)}.$$

Figure 1: Octonions

In $\mathbb{O}$, the associative law does not follow, but the following Moufang's formula is true. Let $a, x, y \in \mathbb{O}$. Then, the following is true.

$$(ax)(ya) = a(xy)a.$$

If $f : \mathbb{O} \to \mathbb{O}$ is a linear automorphism and satisfies $f(xy) = f(x)f(y)$ for any $x, y \in \mathbb{O}$, we say that f is an automorphism of $\mathbb{O}$.

Definition 2.6.   ([11] subsection 1.2) We define G_2 as the group of all automorphisms of $\mathbb{O}$.

Proposition 2.7.   ([11] subsection 1.2) *For any $g \in G_2$, the following are true.*

(1) $(g(x), g(y)) = (x, y)$ *for any* $x, y \in \mathbb{O}$.

(2) $g(1) = 1$.

(3) $g(\bar{x}) = \overline{g(x)}$ *for any* $x \in \mathbb{O}$.

It is known that G_2 with induced topology from $GL(\mathbb{O})$ is connected , so

$$G_2 \subset SO(7) \cong \{g \in SO(\mathbb{O}) \cong SO(8) \ ; \ g(1) = 1\},$$

where $SO(\mathbb{O})$ is the set of all linear orthogonal automorphism f of $\mathbb{O}$ such that $\det f = 1$. Let $\kappa : SO(\mathbb{O}) \to SO(\mathbb{O}); g \mapsto \kappa g$, where $\kappa g(x) = \overline{g(\bar{x})}$ $(x \in \mathbb{O})$.

Proposition 2.8. (Principle of triality in $SO(\mathbb{O})$) *In $SO(\mathbb{O})$, the following are true.*

(1) *Let $g_1 \in SO(\mathbb{O})$. Then, there is $g_2, g_3 \in SO(\mathbb{O})$ such that*

$$g_1(x)g_2(y) = (\kappa g_3)(xy)$$

for any $x, y \in \mathbb{O}$. Moreover, if $\alpha, \beta \in SO(\mathbb{O})$ satisfy $g_1(x)\alpha(y) = (\kappa\beta)(xy)$ for any $x, y \in \mathbb{O}$, then $(\alpha, \beta) = \pm(g_2, g_3)$.

(2) *If $g_1, g_2, g_3 \in SO(\mathbb{O})$ satisfy $g_1(x)g_2(y) = (\kappa g_3)(xy)$ for any $x, y \in \mathbb{O}$, then the following is true:*

$$g_2(x)g_3(y) = (\kappa g_1)(xy), \quad and \quad g_3(x)g_1(y) = (\kappa g_2)(xy).$$

By using the principle of triality in $SO(\mathbb{O})$, we consider the following subgroup $\tilde{D}_4$ of $SO(\mathbb{O}) \times SO(\mathbb{O}) \times SO(\mathbb{O})$.

$$\tilde{D}_4 = \{(g_1, g_2, g_3) \in SO(\mathbb{O}) \times SO(\mathbb{O}) \times SO(\mathbb{O}) \ ; \ g_1(x)g_2(y) = (\kappa g_3)(xy) \ (x, y \in \mathbb{O})\}.$$

It is known that $\tilde{D}_4$ is isomorphic to $Spin(8)$. For any $g \in G_2$, it is true that $g(x)g(y) = (\kappa g)(xy)$ for any $x, y \in \mathbb{O}$, so $(g, g, g) \in \tilde{D}_4$. Let $\phi_0 : G_2 \to \tilde{D}_4$; $g \mapsto (g, g, g)$ and $D(SO(\mathbb{O})^3)$ be the diagonal set of $SO(\mathbb{O})^3$.

Then we get $\phi_0(G_2) = \tilde{D}_4 \cap D(SO(\mathbb{O})^3)$. Moreover, we set ϕ_i $(i=1,2,3)$ as follows:

$$\phi_1 : G_2 \to \tilde{D}_4; \ g \mapsto (g, -g, -g),$$
$$\phi_2 : G_2 \to \tilde{D}_4; \ g \mapsto (-g, g, -g),$$
$$\phi_3 : G_2 \to \tilde{D}_4; \ g \mapsto (-g, -g, g).$$

Let $I_{\mathbb{O}}$ be a the unit element of G_2. Then, $Z(\tilde{D}_4) = \{\phi_i(I_{\mathbb{O}}) \ ; \ i = 0, 1, 2, 3\}$ is the center of $\tilde{D}_4$. In particular, for each $i \neq j$ $(0 \leq i, j \leq 3)$ there is $g \in Z(\tilde{D}_4)$ such that $\phi_i(G_2) = g\phi_j(G_2)$.

Maximal antipodal subgroup of G_2 is studied by Tanaka-Tasaki-Yasukura [10]. For each $x \in \mathbb{O}$, we set $C_x : \mathbb{O} \to \mathbb{O} \ ; \ a \mapsto xa\bar{x}$. Let $p_0 = I_{\mathbb{O}}$ and $p_1, \cdots, p_7 \in G_2$ as follows:

$$p_1 = C_{e_1} \circ C_{e_2} \circ C_{e_3}, \ p_2 = C_{e_1} \circ C_{e_4} \circ C_{e_5}, \ p_3 = C_{e_2} \circ C_{e_4} \circ C_{e_6}, \ p_4 = C_{e_3} \circ C_{e_5} \circ C_{e_6},$$
$$p_5 = C_{e_1} \circ C_{e_6} \circ C_{e_7}, \ p_6 = C_{e_2} \circ C_{e_5} \circ C_{e_7}, \ p_7 = C_{e_3} \circ C_{e_4} \circ C_{e_7}.$$

Proposition 2.9. ([10]) $X = \{p_i \ ; \ i = 0, \cdots, 7\}$ *is a maximal antipodal subgroup of G_2. Moreover, any maximal antipodal subgroup of G_2 is conjugate to X.*

2.3. $G_2/SO(4)$

We often denote p_1 by σ_1. Then, $\sigma_1^2 = I_{\mathbb{O}}$. We define $G_2^{\sigma_1} = \{g \in G_2 \ ; \ g\sigma_1 = \sigma_1 g\}$. It is known that $G_2^{\sigma_1} \cong SO(4)$ [11], so we denote $G_2^{\sigma_1}$ by $SO(4)$ simply. We observe Proposition 2.10 about polars of G_2.

Proposition 2.10. ([6] Section 4) *The number of polars of G_2 is two. In particular, $\{I_{\mathbb{O}}\}$ and $N^+ := \bigcup_{g \in G_2} g\sigma_1 g^{-1}$ are all polars of $I_{\mathbb{O}}$.*

We see that $N^+ \cong G_2/SO(4)$. Hence, N^+ is the compact symmetric space of type G in the Cartan's classification [5]. Congruent classes of maximal antipodal sets of N^+ is studied by Tanaka-Tasaki-Yasukura [10]. Let $Y = X - \{p_0\} = \{p_1, \cdots, p_7\}$.

Proposition 2.11. ([10]) *Y is a maximal antipodal set of N^+. Moreover, any maximal antipodal set of N^+ is congruent to Y.*

We will express each element $p \in Y$ as $p = g\sigma_1 g^{-1}$ $(g \in G_2)$. We define $g_1 = I_{\mathbb{O}}$ and $g_2, \cdots, g_7 \in G_2$ as follows.

$$g_2 : \mathbb{O} \to \mathbb{O}; \ \sum_{i=0}^{7} x_i e_i \mapsto x_0 + x_1 e_1 + x_4 e_2 + x_5 e_3 + x_2 e_4 + x_3 e_5 - x_6 e_6 - x_7 e_7,$$

$$g_3 : \mathbb{O} \to \mathbb{O}; \ \sum_{i=0}^{7} x_i e_i \mapsto x_0 + x_4 e_1 + x_2 e_2 + x_6 e_3 + x_1 e_4 - x_5 e_5 + x_3 e_6 - x_7 e_7,$$

$$g_4 : \mathbb{O} \to \mathbb{O}; \ \sum_{i=0}^{7} x_i e_i \mapsto x_0 + x_5 e_1 + x_6 e_2 + x_3 e_3 - x_4 e_4 + x_1 e_5 + x_2 e_6 - x_7 e_7,$$

$$g_5 : \mathbb{O} \to \mathbb{O}; \ \sum_{i=0}^{7} x_i e_i \mapsto x_0 + x_1 e_1 + x_6 e_2 + x_7 e_3 - x_4 e_4 - x_5 e_5 + x_2 e_6 + x_3 e_7,$$

$$g_6 : \mathbb{O} \to \mathbb{O}; \quad \sum_{i=0}^{7} x_i e_i \mapsto x_0 + x_5 e_1 - x_2 e_2 + x_7 e_3 - x_4 e_4 + x_1 e_5 + x_6 e_6 + x_3 e_7,$$

$$g_7 : \mathbb{O} \to \mathbb{O}; \quad \sum_{i=0}^{7} x_i e_i \mapsto x_0 + x_4 e_1 + x_7 e_2 + x_3 e_3 + x_1 e_4 - x_5 e_5 - x_6 e_6 + x_2 e_7.$$

By direct calculation we obtain Proposition 2.12.

Proposition 2.12. $p_i = g_i p_1 g_i^{-1}$ *for each* $1 \le i \le 7$.

By Proposition 2.12, we obtain Corollary 2.13 immediately. Set

$$Y' = \{g_i(SO(4)) \; ; \; 1 \le i \le 7\} \subset G_2/SO(4).$$

Corollary 2.13. Y' *is a maximal antipodal set of* $G_2/SO(4)$. *Moreover, any maximal antipodal set of* $G_2/SO(4)$ *is congruent to* Y'.

If a 4-dimensional subspace $V \subset \mathbb{O}$ satisfies $(ab)c = a(bc)$ for any $a, b, c \in V$, we say that V is an associative subspace. For example, $\mathbb{H} = \bigoplus_{i=0}^{3} \mathbb{R}e_i$ is an associative subspace. Set $G_{\mathbb{H}}(\mathbb{O}) := \{V \; ; \; V \text{ is an associative subspace.}\}$. $G_{\mathbb{H}}(\mathbb{O})$ is called the associative Grassmannian. We see that G_2 acts on $G_{\mathbb{H}}(\mathbb{O})$ easily. In particular, it is known that this action is transitive and the isotropy subgroup at $\mathbb{H}$ of G_2 is $SO(4)$ ([4], Section IV, Theorem 1.4). Thus, $G_{\mathbb{H}}(\mathbb{O}) \cong G_2/SO(4)$. Moreover, the correspondence is given by $G_{\mathbb{H}}(\mathbb{O}) \ni g(\mathbb{H}) \mapsto g(SO(4)) \in G_2/SO(4)$ for any $g \in G_2$. Let $V_1 = \mathbb{H}$ and $V_2, \cdots, V_7 \in G_{\mathbb{H}}(\mathbb{O})$ be as follows:

$$V_2 = \bigoplus_{i \in \{0,1,4,5\}} \mathbb{R}e_i, \quad V_3 = \bigoplus_{i \in \{0,2,4,6\}} \mathbb{R}e_i, \quad V_4 = \bigoplus_{i \in \{0,3,5,6\}} \mathbb{R}e_i,$$

$$V_5 = \bigoplus_{i \in \{0,1,6,7\}} \mathbb{R}e_i, \quad V_6 = \bigoplus_{i \in \{0,2,5,7\}} \mathbb{R}e_i, \quad V_7 = \bigoplus_{i \in \{0,3,4,7\}} \mathbb{R}e_i.$$

We see that $g_i(V_1) = V_i$ for any $1 \le i \le 7$. Thus, we obtain Corollary 2.14 immediately by Corollary 2.13. Set $Y'' := \{V_i \; ; \; 1 \le i \le 7\} \subset G_{\mathbb{H}}(\mathbb{O})$.

Corollary 2.14. ([10]) Y'' *is a maximal antipodal set of* $G_{\mathbb{H}}(\mathbb{O})$. *Moreover, any maximal antipodal set of* $G_{\mathbb{H}}(\mathbb{O})$ *is congruent to* Y''.

Remark that each element of Y'' is corresponding to each line of Figure 1. For example, $V_1 = \mathbb{H}$ is corresponding to the line through e_1, e_2, e_3.

2.4. Exceptional Jordan algebra and F_4

In this subsection, we recall fundamental results of F_4 from [11]. Let $M(3, \mathbb{O})$ be the set of all 3×3 matrices $A = (a_{ij})_{1 \le i,j \le 3}$ such that $a_{ij} \in \mathbb{O}$. Set

$$H_3(\mathbb{O}) = \{X \in M(3, \mathbb{O}) \; ; \; X^* = X\}, \quad \text{where } X^* = {}^t\overline{X}.$$

Then, each element X of $H_3(\mathbb{O})$ can be written as

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbb{R}, \; x_i \in \mathbb{O}.$$

We often denote the above X by $X(\xi, x)$. $H_3(\mathbb{O})$ is a 27-dimensional real vector space. We define a multiplication $X \circ Y$ for $X, Y \in H_3(\mathbb{O})$ as follows.

$$X \circ Y = \frac{1}{2}(XY + YX).$$

This multiplication is called the Jordan multiplication and the algebra $(H_3(\mathbb{O}), \circ)$ is called the exceptional Jordan algebra. By the definition, it is true that $X \circ Y = Y \circ X$ for any $X, Y \in H_3(\mathbb{O})$. We define the trace tr and an inner product $(,)$ as follows:

For $X = X(\xi, x)$, $Y \in H_3(\mathbb{O})$: $\operatorname{tr}(X) := \xi_1 + \xi_2 + \xi_3$, $(X, Y) = \operatorname{tr}(X \circ Y)$.

Let $E_{ij} \in M(3, \mathbb{O})$ be the matrix such that (i, j)-component is 1 and the others are 0. Then, we set $E_i, F_i(x) \in H_3(\mathbb{O})$ $(x \in \mathbb{O}, 1 \leq i \leq 3)$ as follows. For any $x \in \mathbb{O}$,

$$E_1 = E_{11}, \quad E_2 = E_{22}, \quad E_3 = E_{33},$$
$$F_1(x) = xE_{23} + \bar{x}E_{32}, \quad F_2(x) = \bar{x}E_{13} + xE_{31}, \quad F_3(x) = xE_{12} + \bar{x}E_{21}.$$

Set $J = \{E_i, F_i(e_j) \; ; \; 1 \leq i \leq 3, \; 0 \leq j \leq 7\}$. Then, J is a basis of $H_3(\mathbb{O})$. For any $X, Y \in J$, $X \circ Y$ is given by as follows:

$$E_i \circ E_i = E_i, \; E_i \circ E_j = 0 \; (i \neq j),$$

$$E_i \circ F_i(x) = 0, \; E_i \circ F_j(x) = \frac{1}{2}F_j(x) \; (i \neq j),$$

$$F_i(x) \circ F_i(y) = (x, y)(E_{i+1} + E_{i+2}), \; F_i(x) \circ F_{i+1}(y) = \frac{1}{2}F_{i+2}(\overline{xy}),$$

where the indices are considered as mod 3.

If an $\mathbb{R}$-linear automorphism $f : H_3(\mathbb{O}) \to H_3(\mathbb{O})$ satisfies $f(X \circ Y) = f(X) \circ f(Y)$ for any $X, Y \in H_3(\mathbb{O})$, then we call f an automorphism of $H_3(\mathbb{O})$.

Definition 2.15. ([11] Subsection 2.2) We define F_4 as the group of all automorphisms of $H_3(\mathbb{O})$.

In this article, by using the realization of F_4 as the automorphism group of $H_3(\mathbb{O})$ we explicitly give the conjugate classes of maximal antipodal subgroups of F_4. We denote the unit element of F_4 by I.

Proposition 2.16. ([11] Lemma 2.1) *For any $f \in F_4$, $(f(X), f(Y)) = (X, Y)$ for any $X, Y \in H_3(\mathbb{O})$.*

By Proposition 2.16, we see that F_4 is a subgroup of $SO(27) \cong SO(H_3(\mathbb{O}))$, where $SO(H_3(\mathbb{O}))$ is the set of all orthogonal transformations f of $H_3(\mathbb{O})$ such that $\det f = 1$, since it is known that F_4 is connected [11].

In the following we see some subgroups of F_4. Set

$$D_4' = \{g \in F_4 \; ; \; g(E_i) = E_i \; (i = 1, 2, 3)\}.$$

Proposition 2.17. ([11] Theorem 1.16.2) $D_4' \cong \tilde{D}_4 \cong Spin(8)$ *Moreover, the following map is an isomorphism.*

$$\tilde{D}_4 \to D_4' \; ; \; (g_1, g_2, g_3) \mapsto g,$$

$$g\left(\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}\right) = \begin{pmatrix} \xi_1 & g_3(x_3) & \overline{g_2(x_2)} \\ \overline{g_3(x_3)} & \xi_2 & g_1(x_1) \\ g_2(x_2) & \overline{g_1(x_1)} & \xi_3 \end{pmatrix} \quad \begin{matrix} x_i \in \mathbb{O}, \\ \xi_i \in \mathbb{R}. \end{matrix}$$

By Proposition 2.17, we identify $\tilde{D}_4$ with D_4'. In the following, we often denote D_4' by $Spin(8)$ simply. We already know G_2 is a subgroup of $Spin(8)$, so we see that G_2 is a subgroup of F_4. Recall $\sigma_1 \in G_2$. σ_1 is given by

$$\sigma_1\left(\sum_{i=0}^{7} x_i e_i\right) = \sum_{i=0}^{3} x_i e_i - \sum_{i=4}^{7} x_i e_i.$$

Since $\sigma_1^2 = I_{\mathbb{O}}$, we see $\phi_0(\sigma_1)^2 = I$ in F_4. We denote $\phi_0(\sigma_1)$ by σ_1 simply. We define a subgroup $F_4^{\sigma_1}$ of F_4 as follows.

$$(F_4)^{\sigma_1} = \{g \in F_4 \ ; \ g\sigma_1 = \sigma_1 g\}.$$

Since it is known that F_4 is simply connected, $F_4^{\sigma_1}$ is connected.

For $x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$, set $\alpha(x) = (m(x), a(x))$, where $m(x) = \sum_{i=0}^{3} x_i e_i$ and $a(x) = x_4 + x_5 e_1 - x_6 e_2 + x_7 e_3$. Then, $\mathbb{O} \ni x \mapsto \alpha(x) \in \mathbb{H} \oplus \mathbb{H}$ is an $\mathbb{R}$-linear isomorphism. Moreover, for any $X(x, \xi)$, set $\hat{\alpha}(X) = (m(X), a(X))$, where

$$m(X) = \begin{pmatrix} \xi_1 & m(x_3) & \overline{m(x_2)} \\ \overline{m(x_3)} & \xi_2 & m(x_1) \\ m(x_2) & \overline{m(x_1)} & \xi_3 \end{pmatrix}, \quad a(X) = (a(x_1), a(x_2), a(x_3)).$$

Then, $H_3(\mathbb{O}) \ni X \mapsto (m(X), a(X)) \in H_3(\mathbb{H}) \oplus \mathbb{H}^3$ is an $\mathbb{R}$-linear isomorphism, where $H_3(\mathbb{H}) = \{X \in M(3, \mathbb{H}) \ ; \ X^* = X\}$ and $M(3, \mathbb{H})$ is the set of all 3×3 matrices whose components are the quaternions. We consider a homomorphism $\eta : Sp(1) \times Sp(3) \to F_4$ as follows. For any $(p, Q) \in Sp(1) \times Sp(3)$ and $X \in H_3(\mathbb{O})$, set

$$\eta(p, Q)(X) = \hat{\alpha}^{-1}(Q(m(X))Q^{-1} + p(a(X))Q^{-1}).$$

Then, $\eta(Sp(1) \times Sp(3)) \subset F_4^{\sigma_1}$ [11]. In particular, Proposition 2.18 is true.

Proposition 2.18. ([11] Theorem 2.11.2) $\eta : Sp(1) \times Sp(3) \to F_4^{\sigma_1}$ *is a surjective homomorphism and* $\mathrm{Ker}\eta = \{\pm(1, I_3)\}$, *where* I_n *is the* $n \times n$ *identity matrix.*

We denote $Sp(1) \times Sp(3)/\{\pm(1, I_3)\}$ by $Sp(1) \cdot Sp(3)$. By Proposition 2.18, $F_4^{\sigma_1}$ is isomophic to $Sp(1) \cdot Sp(3)$, so we often denote $F_4^{\sigma_1}$ by $Sp(1) \cdot Sp(3)$ simply.

We define $\sigma_2 \in F_4$ as follows.

$$\sigma_2(X(\xi, x)) = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x_2} \\ -\overline{x_3} & \xi_2 & x_1 \\ -x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \quad (X(\xi, x) \in H_3(\mathbb{O})).$$

By the definition, we see $\sigma_2^2 = I$. We set $F_4^{\sigma_2} = \{g \in F_4 \ ; \ \sigma_2 g = g\sigma_2\}$ and $(F_4)_{E_1} = \{g \in F_4 \ ; \ g(E_1) = E_1\}$. We see that $F_4^{\sigma_2}$ is connected since F_4 is simply connected.

Proposition 2.19. ([11] Theorem 2.9.1) $F_4^{\sigma_2} = (F_4)_{E_1} \cong Spin(9)$

By the definition, $Spin(8)$ is a subgroup of $F_4^{\sigma_2}$. We often denote $F_4^{\sigma_2}$ by $Spin(9)$ simply.

2.5. Polars of F_4

In this section, we observe polars of F_4. Polars of F_4 is given by the following proposition.

Proposition 2.20. ([6] Section 4) *The number of polars of F_4 is 3. Let $M_0^+ = \{I\}$, $M_1^+ = \bigcup_{g \in F_4} g\sigma_1 g^{-1}$, $M_2^+ = \bigcup_{g \in F_4} g\sigma_2 g^{-1}$. Then, each M_i^+ $(i = 0, 1, 2)$ is polar of I and $F(s_I, F_4) = \bigsqcup_{i=0}^{2} M_i^+$.*

By the definition of $Sp(1) \cdot Sp(3)$ and $Spin(9)$, $M_1^+ \cong F_4/Sp(1) \cdot Sp(3)$ and $M_2^+ \cong F_4/Spin(9)$. In particular, M_1^+ is the compact symmetric space of type FI and M_2^+ is the compact symmetric space of type FII in the Cartan's classification [5]. It is known that FII is a symmetric R-space. Tanaka-Tasaki already classified congruent classes of maximal antipodal sets of symmetric R-spaces [8].
Define $\sigma_2', \sigma_2'' \in F_4$ as follows:

$$\sigma_2'(X(\xi, x)) = \begin{pmatrix} \xi_1 & -x_3 & \overline{x_2} \\ -\overline{x_3} & \xi_2 & -x_1 \\ x_2 & -\overline{x_1} & \xi_1 \end{pmatrix}, \quad \sigma_2''(X(\xi, x)) = \begin{pmatrix} \xi_1 & x_3 & -\overline{x_2} \\ \overline{x_3} & \xi_2 & -x_1 \\ -x_2 & -\overline{x_1} & \xi_1 \end{pmatrix}.$$

We see that $\sigma_2', \sigma_2'' \in M_2^+$ by calculating eigenvalues.

Proposition 2.21. ([8] Theorem 4.3) *$\{\sigma_2, \sigma_2', \sigma_2''\}$ is a maximal antipodal set of M_2^+. Moreover, any maximal antipodal set of M_2^+ is congruent to this maximal antipodal set.*

By Proposition 2.21, we obtain Proposition 2.22 about maximal antipodal subgroups of F_4 immediately.

Proposition 2.22. *Let S be any maximal antipodal subgroup of F_4. Then, $S \cap M_1^+ \neq \phi$.*

Proof.   We assume that $S \cap M_1^+ = \phi$. Then, $S \subset \{I\} \sqcup M_2^+$. We see that $S \cap M_2^+$ is a maximal antipodal set in M_2^+ by the maximality of S. Hence, there is $g \in F_4$ such that $gSg^{-1} = \{I, \sigma_2, \sigma_2', \sigma_2''\}$ by Proposition 2.2 and Proposition 2.21. We see that $\sigma_1 h = h\sigma_1$ for any $h \in gSg^{-1}$. Moreover, $\sigma_1^2 = I$ by the definition. Hence, $\{I, \sigma_1, \sigma_2, \sigma_2', \sigma_2''\}$ is an antipodal set in F_4 containing gSg^{-1}. However, this contradicts to the maximality of S. Thus, we showed that $S \cap M_1^+ \neq \phi$.   ∎

By the arguments in Subsection 2.1, to classify conjugate classes of maximal antipodal subgroups of F_4 it is sufficient to classify conjugate classes of maximal antipodal subgroups of $F_4^{\sigma_1} \cong Sp(1) \cdot Sp(3)$ and study whether there is $g \in F_4$ such that $gA_1 g^{-1} \subset A_2$ or $gA_2 g^{-1} \subset A_1$ for any representatives A_1, A_2 of two any different conjugate classes of maximal antipodal subgroups of $Sp(1) \cdot Sp(3)$.

3.   Maximal antipodal subgroups of F_4

In this section, we classify conjugate classes of maximal antipodal subgroups of F_4 and describe it explicitly using the exceptional Jordan algebra.

3.1. Maximal antipodal sets of $Sp(1) \cdot Sp(n)$

Let $G(n)$ be $O(n), U(n)$ or $Sp(n)$. In this subsection, we classify conjugate classes of maximal antipodal subgroups of $G(1) \cdot G(n) = G(1) \times G(n)/\{\pm(1, I_n)\}$ by using the method which is based on [9]. Let $\pi : G(1) \times G(n) \to G(1) \cdot G(n)$ be the natural projection. Let $1, i, j, k$ be the standard basis of the quaternions, that is they satisfy $i^2 = j^2 = k^2 = -1, ij = k, ji = -k, ik = j, ki = -j, jk = i, kj = -i$ and 1 is the unit element.

Let A be a maximal antipodal subgroup of $G(1) \cdot G(n)$ and $B = \pi^{-1}(A)$. Then, B is a subgroup of $G(1) \times G(n)$ and $a^2 = \pm(1, I_n)$ for any $a \in B$ since $\pi(a)^2 = \pi(1, I_n)$. If B is commutative, we consider the following two cases:

(1-1) $a^2 = (1, I_n)$ for any $a \in B$,

(1-2) there is $a \in B$ such that $a^2 = -(1, I_n)$.

If B is not commutative, then $ab = \pm ba$ for any $a, b \in B$, where $-g = -(1, I_n)g$ for any $g \in G(1) \times G(n)$ since $\pi(ab) = \pi(ba)$. In this case, we consider the following two cases.

(2-1) There is $(a, b) \in B \times B$ satisfying $ab = -ba$ such that there is $x \in \{a, b, ab\}$ which satisfies $x^2 = -(1, I_n)$.

(2-2) For any $(a, b) \in B \times B$ satisfying $ab = -ba$, $x^2 = -(1, I_n)$ for any $x \in \{a, b, ab\}$.

Define
$$\Delta_n = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \in O(n) \right\}.$$

Proposition 3.1. *Any maximal antipodal subgroup of $O(1) \cdot O(n)$ is conjugate to $\pi(\Delta_1 \times \Delta_n)$.*

Proof. Let A be a maximal antipodal subrgoup of $O(1) \cdot O(n)$ and $B = \pi^{-1}(A)$. In this case, it is sufficient to consider the case (1-1) since $O(1) = \{\pm 1\}$. In the case (1-1), B is a maximal antipodal subgroup of $O(1) \times O(n)$. It is known that any maximal antipodal subgroup of $O(n)$ is conjugate to Δ_n [9]. Hence, B is conjugate to $\Delta_1 \times \Delta_n$. ∎

Now we consider the conjugate classes of maximal antipodal subgroups of $U(1) \cdot U(n)$.

Proposition 3.2. *Any maximal antipodal subgroup of $U(1) \cdot U(n)$ is conjugate to $\pi\{1, i\}(\Delta_1 \times \Delta_n))$.*

Proof. Let A be a maximal antipodal subgroup of $U(1) \cdot U(n)$ and $B = \pi^{-1}(A)$. It is sufficient to consider the case (1-1) and (1-2) since $U(1)$ is commutative. In the case (1-1), B is a maximal antipodal subgroup of $U(1) \times U(n)$. It is known that any maximal antipodal subgroup of $U(n)$ is conjugate to Δ_n [9]. Hence, B is conjugate to $\Delta_1 \times \Delta_n$. In the case (1-2), since B is commutative, there is $g \in U(1) \times U(n)$ such that $gBg^{-1} \subset U(1) \times D(U(n))$, where $D(U(n))$ is the set of all diagonal matrices of $U(n)$. Since $a^2 = \pm(1, I_n)$ for any $a \in B$, we see $gBg^{-1} \subset \{1, i\}(\Delta_1 \times \Delta_n)$. By the maximality of B, it is true that $gBg^{-1} = \{1, i\}(\Delta_1 \times \Delta_n)$. Thus, B is conjugate to $\{1, i\}(\Delta_1 \times \Delta_n)$. It is obvious that $\Delta_1 \times \Delta_n \subset \{1, i\}(\Delta_1 \times \Delta_n)$. Thus, A is conjugate to $\pi(\{1, i\}(\Delta_1 \times \Delta_n))$. ∎

Before we consider conjugate classes of maximal antipodal subgroups of $Sp(1) \cdot Sp(n)$, we observe Lemma 3.3 and Lemma 3.4.

Lemma 3.3. ([2] Lemma 5.6) *Let $G(n)$ be $U(n)$ or $Sp(n)$ and assume that $\pi : G(1) \times G(n) \to G(1) \cdot G(n)$ is the natural projection. Set A a maximal antipodal subgroup of $G(1) \cdot G(n)$ and $B = \pi^{-1}(A)$. If $a, b, c \in B$ satisfy $ab = -ba$, then at least one of c, ac, bc, abc commutes with a and b.*

Lemma 3.4. *Let $M = \{g \in Sp(1) \times Sp(n) \; ; \; g^2 = -(1, I_n)\}$. If $g \in M$ satisfies $g(i(1, I_n)) = -(i(1, I_n))g$, then there is $h \in U(1) \times U(n)$ such that $hgh^{-1} = k(1, I_n)$.*

Proof. We see that $M = \bigcup_{g \in Sp(1) \times Sp(n)} g(i, iI_n)g$ since $U(1) \times DU(n)$ is a maximal torus of $Sp(1) \times Sp(n)$ and $j(-i)(-j) = i$. In particular we have that $M \cong Sp(1)/U(1) \times Sp(n)/U(n)$ and M is a totally geodesic submanifold of $Sp(1) \times Sp(n)$. The following $T \subset M$ is a maximal flat torus of M:

$$T = \left\{ t(i(1, I_n))t^{-1} \; ; \; t = (e^{j\theta}, \begin{pmatrix} e^{j\theta_1} & & \\ & \ddots & \\ & & e^{j\theta_n} \end{pmatrix}), \; \begin{matrix} \theta, \theta_a \in \mathbb{R}, \\ 1 \le a \le n \end{matrix} \right\}$$

$$= \left\{ i(e^{j\theta}, \begin{pmatrix} e^{j\theta_1} & & \\ & \ddots & \\ & & e^{j\theta_n} \end{pmatrix}) \; ; \; \begin{matrix} \theta, \theta_a \in \mathbb{R}, \\ 1 \le a \le n \end{matrix} \right\}.$$

Hence, for any $p \in M$ there is $q \in U(1) \times U(n)$ and $t \in T$ such that $p = qtq^{-1}$. If $p \in M$ satisfies $p(i(1, I_n)) = -i(1, I_n)p$, then $t(i(1, I_n)) = -i(1, I_n)t$ since $g(i(1, I_n)) = i(1, I_n)g$ for any $g \in U(1) \times U(n)$. Moreover,

$$\{t \in T \; ; \; t(i(1, I_n)) = -i(1, I_n)t\} = k(\Delta_1 \times \Delta_n).$$

Thus, since $i(-k)(-i) = k$, we showed the statement. ■

Proposition 3.5. *Any maximal antipodal subgroup of $Sp(1) \cdot Sp(n)$ is conjugate to $\pi(\{1, i, j, k\}(\Delta_1 \times \Delta_n))$.*

Proof. Let A be a maximal antipodal subgroup of $Sp(1) \cdot Sp(n)$ and $B = \pi^{-1}(A)$. In the case (1-1), B is a maximal antipodal subgroup of $Sp(1) \times Sp(n)$. It is known that any maximal antipodal subgroup of $Sp(n)$ is conjugate to Δ_n [9]. Hence, B is conjugate to $\Delta_1 \times \Delta_n$.

In the case (1-2), let $a \in B$ satisfy $a^2 = -(1, I_n)$. Then, there is $g \in Sp(1) \times Sp(n)$ such that $gag^{-1} = i(1, I_n)$. Hence, we may assume that $i(1, I_n) \in B$. Since B is commutative, $B \subset \{g \in Sp(1) \times Sp(n) \; ; \; i(1, I_n)g = gi(1, I_n)\} = U(1) \times U(n)$ and there is $h \in U(1) \times U(n)$ such that $hBh^{-1} \subset U(1) \times DU(n)$. Since $b^2 = \pm(1, I_n)$ for each $b \in B$, we see that $hBh^{-1} \subset \{1, i\}(\Delta_1 \times \Delta_n)$. By the maximality of A, we see that $hBh^{-1} = \{1, i\}(\Delta_1 \times \Delta_n)$. Thus, B is conjugate to $\{1, i\}(\Delta_1 \times \Delta_n)$.

In the case that B is not commutative, we assume that $a \in B$ satisfies $a^2 = (1, I_n)$. Since $U(1) \times DU(n)$ is a maximal torus of $Sp(1) \times Sp(n)$, a is conjugate to an element of $\Delta_1 \times \Delta_n$. We see that there are no $g \in Sp(1) \times Sp(n)$ such that $ag = -ga$. Thus, we only consider the case (2-2). In the case (2-2), let $a, b \in B$ satisfy $ab = -ba$ and $a^2 = b^2 = -(1, I_n)$. We see that a is conjugate to $i(1, I_n)$ and may assume that $a = i(1, I_n)$. Then, $b(i(1, I_n)) = -(i(1, I_n))b$, so there is $g \in U(1) \times U(n)$ such that $gbg^{-1} = k(1, I_n)$ by Lemma 3.4. Hence, we may assume that $b = k(1, I_n)$ since $gi(1, I_n)g^{-1} = i(1, I_n)$ for any $g \in U(1) \times U(n)$. Let Q be the group generated by $i(1, I_n)$ and $k(1, I_n)$, that is $Q = \{x(1, I_n) \; ; \; x \in \{\pm 1, \pm i, \pm j, \pm k\}\}$. Set $Z = \{g \in Sp(1) \times Sp(n) \; ; \; gx = xg \; (x \in Q)\}$. It is obvious that $Z = O(1) \times O(n)$. We see $B \subset QZ$ by Lemma 3.3. Then, there is a subgroup $B' \subset Z$ such that $\pi(B')$ is a maximal antipodal subgroup of $O(1) \cdot O(n)$ and $B \subset QB'$. From Proposition 3.1, B' is conjugate to $\Delta_1 \times \Delta_n$. By the maximality of A, we see that B is conjugate to $Q(\Delta_1 \times \Delta_n)$.

It is obvious that $\Delta_1 \times \Delta_n \subset \{1,i\}(\Delta_1 \times \Delta_n) \subset \{1,i,j,k\}(\Delta_1 \times \Delta_n)$. Thus, we showed that any maximal antipodal subgroup of $Sp(1) \cdot Sp(n)$ is conjugate to $\pi(\{1,i,j,k\}(\Delta_1 \times \Delta_n))$. ∎

3.2. Classification of maximal antipodal subgroups of F_4

By the arguments in Subsection 2.1 and Proposition 3.5, we obtain the following theorem immediately. We denote $\eta(\{1,i,j,k\}(\Delta_1 \times \Delta_3))$ by S.

Theorem 3.6. *Any maximal antipodal subgroup of F_4 is conjugate to S. Moreover, $\#_2 F_4 = 32$.*

We will describe each element of S as an automorphism of the exceptional Jordan algebra. By the definition of the isomorphism $\eta : Sp(1) \times Sp(3) \to F_4^{\sigma_1}$, we see that $g(E_i) = E_i$ for any $g \in S$ and $i = 1,2,3$. Thus, $S \subset Spin(8)$. By calculations, we obtain Proposition 3.7. We recall $\phi_i : G_2 \to Spin(8) \subset F_4$ $(0 \le i \le 3)$ and $p_j \in G_2$ $(0 \le j \le 7)$. Remark that $X = \{p_i \; ; \; 0 \le i \le 7\}$ is a maximal antipodal subgroup of G_2.

Proposition 3.7. $S = \bigcup_{i=0}^3 \phi_i(X)$. *Let* $S_1 = \phi_0(X)$ *and* $S_2 = \{\phi_i(p_0) \,;\, 0 \le i \le 3\}$. *Then, S_1 and S_2 is a subgroup of S and $S = S_1 S_2$.*

We consider $Spin(8) \subset Spin(9) \subset F_4$. Since S is contained in $Spin(8)$, we obtain Corollary 3.8 immediately.

Corollary 3.8. *Any maximal antipodal subgroup of $Spin(8)$ is conjugate to S and $\#_2 Spin(8) = 32$. Moreover, Any maximal antipodal subgroup of $Spin(9)$ is conjugate to S and $\#_2 Spin(9) = 32$.*

4. Maximal antipodal sets of FI

In this section, we study maximal antipodal sets of FI. We will give a new realization of FI and will show that a maximal antipodal set in this realization is a special combinatorial object.

4.1. Classification of maximal antipodal sets of FI

By the arguments of Subsection 2.1, we obtain the classification of congruent classes of maximal anipodal sets of FI. In the following, we denote $S \cap M_1^+$ by T. Recall $Y = \{p_1, \cdots, p_7\}$. We see that $T = \bigcup_{i=0}^3 \phi_i(Y)$ by calculating eigenvalues of each element of S.

Theorem 4.1. *T is a maximal antipodal set of M_1^+. Moreover, any maximal antipodal set of M_1^+ is congruent to T.*

We see that $\phi_0(p_1) = \sigma_1$. We describe each element $t \in T$ as $t = g\sigma_1 g^{-1}$ $(g \in F_4)$. Define $L_{e_4}, R_{e_4}, T_{e_4} \in SO(\mathbb{O})$ as follows.

$$L_{e_4} : \mathbb{O} \to \mathbb{O} \; ; \; x \mapsto e_4 x, \quad R_{e_4} : \mathbb{O} \to \mathbb{O} \; ; \; x \mapsto x e_4$$

and $T_{e_4} := L_{e_4} \circ R_{e_4} = R_{e_4} \circ L_{e_4}$.

Proposition 4.2. $(L_{e_4}, R_{e_4}, T_{e_4}) \in Spin(8)$. *Moreover, we have* $(R_{e_4}, T_{e_4}, L_{e_4})$, $(T_{e_4}, L_{e_4}, R_{e_4}) \in Spin(8)$.

Proof. By Moufang's formula and some properties of the octonions [11], for any $x, y \in \mathbb{O}$, we obtain

$$(\kappa T_{e_4})(xy) = \overline{T_{e_4}(\overline{xy})} = \overline{e_4(\overline{xy})e_4} = (-e_4)\overline{e_4(\overline{xy})} = (-e_4)(\overline{\overline{xy}})(-e_4)$$
$$= e_4(xy)e_4 = (e_4x)(ye_4) = (L_{e_4}x)(R_{e_4}y).$$

By the definition of $Spin(8) \subset F_4$, we obtain $(L_{e_4}, R_{e_4}, T_{e_4}) \in Spin(8)$. ∎

We see that $L_{e_4}p_1L_{e_4}^{-1} = -p_1$, $R_{e_4}p_1R_{e_4}^{-1} = -p_1$ by calculation. Hence, we obtain

$$(T_{e_4}, L_{e_4}, R_{e_4})\phi_0(p_1)(T_{e_4}, L_{e_4}, R_{e_4})^{-1} = \phi_1(p_1),$$
$$(R_{e_4}, T_{e_4}, L_{e_4})\phi_0(p_1)(R_{e_4}, T_{e_4}, L_{e_4})^{-1} = \phi_2(p_1),$$
$$(L_{e_4}, R_{e_4}, T_{e_4})\phi_0(p_1)(L_{e_4}, R_{e_4}, T_{e_4})^{-1} = \phi_3(p_1).$$

Moreover, we see that $\phi_0(g_j)\phi_i(p_1)\phi_0(g_j)^{-1} = \phi_i(p_j)$ for any $0 \leq i \leq 3$ and $1 \leq j \leq 7$ since $g_jp_1g_j^{-1} = p_j$. Define $g_{ij} \in F_4$ for any $0 \leq i \leq 3$ and $1 \leq j \leq 7$ as follows:

$$g_{i1} = \phi_0(g_j)(T_{e_4}, L_{e_4}, R_{e_4}), \quad g_{i2} = \phi_0(g_j)(R_{e_4}, T_{e_4}, L_{e_4}), \quad g_{i3} = \phi_0(g_j)((L_{e_4}, R_{e_4}, T_{e_4}).$$

By these arguments, we obtain Proposition 4.3.

Proposition 4.3. *For any $0 \leq i \leq 3$ and $1 \leq j \leq 7$, $\phi_i(p_j) = g_{ij}\phi_0(p_1)g_{ij}^{-1}$.*

By Proposition 4.3 and Theorem 4.1, we obtain Corollary 4.4 immediately. Let $T' := \{g_{ij}(Sp(1) \cdot Sp(3)) \in F_4/Sp(1) \cdot Sp(3) \; ; \; 0 \leq i \leq 3, \; 1 \leq j \leq 7\}$.

Corollary 4.4. *T' is a maximal antipodal set of $F_4/Sp(1) \cdot Sp(3)$. Moreover, any maximal antipodal set of $F_4/Sp(1) \cdot Sp(3)$ is congruent to T'.*

4.2. Realization of $F_4/Sp(1) \cdot Sp(3)$

In this subsection, we give a new realization of $F_4/Sp(1) \cdot Sp(3)$. Let $H_3(\mathbb{H})$ be as follows:

$$H_3(\mathbb{H}) = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \; ; \; x_i \in \mathbb{H}, \; \xi_i \in \mathbb{R}, \; i = 1, 2, 3 \right\}.$$

$H_3(\mathbb{H})$ is a 15-dimensional real subspace of $H_3(\mathbb{O})$. Moreover, we see that we have $X \circ Y \in H_3(\mathbb{H})$ for any $X, Y \in H_3(\mathbb{H})$. Hence, $H_3(\mathbb{H})$ is a subspace and a subalgebra of $H_3(\mathbb{O})$ with respect to the Jordan multiplication $\circ$. We denote $\circ|_{H_3(\mathbb{H}) \times H_3(\mathbb{H})}$ simply by $\circ_{\mathbb{H}}$.

Let V be a 15-dimensional subspace of $H_3(\mathbb{O})$. If V satisfies that $X \circ Y \in V$ for any $X, Y \in V$ and there is a linear isomorphism $f : H_3(\mathbb{H}) \to V$ such that $f(X \circ_{\mathbb{H}} Y) = f(X) \circ f(Y)$, then we call V an $H_3(\mathbb{H})$-subspace. We define $M := \{V \; ; \; V \text{ is a } H_3(\mathbb{H})\text{-subspace.}\}$. We often call this set $H_3(\mathbb{H})$-Grassmannian. F_4 acts on M by the definition. We will show that this action is transitive and $M \cong F_4/Sp(1) \cdot Sp(3)$.

Let $o = H_3(\mathbb{H})$. Then, we see that the eigenspace of σ_1 corresponding to the eigenvalue 1 is o and the eigenspace of σ_1 corresponding to the eigenvalue -1 is the orthogonal complement $o^\perp$ of o with respect to the inner product $(,)$ of $H_3(\mathbb{O})$.

If $k \in Sp(1) \cdot Sp(3)$, then $k(o) \subset o$ and $k(o^\perp) \subset o^\perp$ since $k\sigma_1 = \sigma_1 k$. Thus, k is contained in the isotropy group at o of F_4. Conversely, if k is contained in the isotropy group at o of F_4, then for any $v \in o$ and $u \in o^\perp$ it is obvious $k\sigma_1(v + u) = k(v) - k(u) = \sigma_1 k(v) + \sigma_1 k(u) = \sigma_1 k(v + u)$. Thus, $k\sigma_1 = \sigma_1 k$, so $k \in Sp(1) \cdot Sp(3)$. Therefore, $Sp(1) \cdot Sp(3)$ is the isotropy group at o of F_4.

Next, we show that F_4 acts on M trasitively. We observe some lemmas.

Lemma 4.5. ([11] Proposition 2.8.2) *For any $X \in H_3(\mathbb{O})$, there are $g \in F_4$ and $\xi_i \in \mathbb{R}$ $(i = 1, 2, 3)$ such that*

$$g(X) = \begin{pmatrix} \xi_1 & & \\ & \xi_2 & \\ & & \xi_3 \end{pmatrix}.$$

Moreover, ξ_1, ξ_2, ξ_3 are determined (up to their permutation) independent of the choice of $g \in F_4$.

Let $E = E_1 + E_2 + E_3$, where E_i $(i = 1, 2, 3)$ are defined in Subsection 2.4. Then, $g(E) = E$ for any $g \in F_4$ since $E \circ X = X$ for any $X \in H_3(\mathbb{O})$. Set $N_1 := \bigcup_{g \in F_4} g(E_1)$.

Lemma 4.6. *Let $X_1, X_2, X_3 \in H_3(\mathbb{O})$ be such that $X_i \neq X_j$ and $X_i \neq 0$ for any $1 \leq i \neq j \leq 3$. If $X_i \circ X_i = X_i$ and $X_i \circ X_j = 0$ for any $1 \leq i \neq j \leq 3$, then $X_1, X_2, X_3 \in N_1$.*

Proof. By Lemma 4.5 , there are $g \in F_4$ and $\xi_i \in \mathbb{R}$ $(i = 1, 2, 3)$ such that

$$g(X) = \begin{pmatrix} \xi_1 & & \\ & \xi_2 & \\ & & \xi_3 \end{pmatrix}.$$

Since $g(X_1) \circ g(X_1) = g(X_1)$, $\xi_i = 0$ or $\xi_i = 1$ for each $1 \leq i \leq 3$. We consider the following four F_4-orbits:

$$\{0\}, \quad \{E\}, \quad N_1 = \bigcup_{g \in F_4} g(E_1), \quad N_2 = \bigcup_{g \in F_4} g(E_2 + E_3).$$

Then, X_1 is contained one of these four F_4-orbits. By the assumption, $X_1 \neq 0$. Since $E \circ X = X$ for any $X \in H_3(\mathbb{O})$, $X_1 \neq E$ because $X_1 \circ X_i = 0$ $(i = 2, 3)$. Hence, $X_1 \in N_1$ or $X_1 \in N_2$. We see that $X_i \in N_1$ or $X_i \in N_2$ $(i = 2, 3)$ in a similar way.

We now assume that $X_1 \in N_2$. Then, there is $h \in F_4$ such that

$$h(X_1) = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Since $h(X_1) \circ h(X_i) = 0$ $(i = 2, 3)$, it is true that

$$h(X_2) = \begin{pmatrix} \xi' & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad h(X_3) = \begin{pmatrix} \xi'' & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \xi', \xi'' \in \mathbb{R}.$$

Since $h(X_i) \circ h(X_i) = h(X_i)$ $(i = 2, 3)$ and $X_2, X_3 \neq 0$,

$$h(X_2) = h(X_3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}.$$

This contradicts to $X_2 \neq X_3$. Thus, $X_1 \in N_1$. By the similar way, we obtain $X_2, X_3 \in N_1$. ∎

By Proposition 2.19, we obtain Lemma 4.7 immediately.

Lemma 4.7.   ([11] Theorem 2.9.1) $N_1 \cong F_4/Spin(9)$.

Hence, N_1 is a compact symmetric space and the symmetry s_{E_1} at E_1 is given by $s_{E_1}(g(E_1)) = \sigma_2(g(E_1))$ for any $g \in N_1$.

Lemma 4.8.   If $X_2 \in N_1$ satisfies $E_1 \circ X_2 = 0$, then there is $g \in Spin(9)$ such that $g(X_2) = E_2$.

Proof.   Since $E_1 \circ X_2 = 0$, we have

$$X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & \overline{x_1} \\ 0 & x_1 & \xi_3 \end{pmatrix} \quad \xi_2, \xi_3 \in \mathbb{R}, \ x_1 \in \mathbb{O}.$$

In particular, $X_2 \in F(s_{E_1}, N_1)$. On the other hand, it is known that the number of polars of N_1 is 2 [6]. Let $X \in F(s_{E_1}, N_1)$ $(X \neq E_1)$. Set $N_1^+ := \bigcup_{g \in Spin(9)} g(X)$. Then, $\{E_1\}$ and N_1^+ are all polars of E_1 in N_1 [1]. We see easily $E_2 \in N_1$ and $s_{E_1}(E_2) = E_2$. Thus, there is $g \in Spin(9)$ such that $g(X_2) = E_2$. ∎

By using Lemma 4.6 and Lemma 4.8, we obtain Proposition 4.9.

Proposition 4.9.   Let $X_1, X_2, X_3 \in H_3(\mathbb{O})$ be $X_i \neq X_j$ and $X_i \neq 0$ for any $1 \leq i \neq j \leq 3$. If $X_i \circ X_i = X_i$ and $X_i \circ X_j = 0$ for $1 \leq i \neq j \leq 3$, then there is $g \in F_4$ such that $g(X_1) = E_1, g(X_2) = E_2, g(X_3) = E_3$.

Proof.   By Lemma 4.6, $X_1, X_2, X_3 \in N_1$. Hence, there is $h_1 \in F_4$ such that $h_1(X_1) = E_1$. Then, $E_1 \circ h_1(X_2) = 0$ since $X_1 \circ X_2 = 0$. Therefore, there exists $h_2 \in Spin(9)$ such that $h_2 h_1(X_2) = E_2$ by Lemma 4.8. Let $g = h_2 h_1$. By the definition of $Spin(9)$, $g(X_1) = E_1$. Then, since $g(X_3)$ satisfies $E_1 \circ g(X_3) = E_2 \circ g(X_3) = 0$,

$$g(X_3) = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \xi \end{pmatrix}, \quad \xi \in \mathbb{R}.$$

Moreover, because $g(X_3) \circ g(X_3) = g(X_3)$ and $g(X_3) \neq 0$, it is true that $g(X_3) = E_3$. Thus, we proved the statements. ∎

Let V be an $H_3(\mathbb{H})$-subspace and $f : H_3(\mathbb{H}) \to V$ be a linear isomorphism satisfying $f(X \circ_{\mathbb{H}} Y) = f(X) \circ f(Y)$ for any $X, Y \in H_3(\mathbb{H})$. Set $X_i = f(E_i)$ $(i = 1, 2, 3)$.

Then, X_i ($i = 1, 2, 3$) satisfy $X_i \neq X_j$, $X_i \neq 0$, $X_i \circ X_j = 0$, $X_i \circ X_i = X_i$ for any $1 \leq i \neq j \leq 3$. Hence, there is $g \in F_4$ such that $g(X_i) = E_i$ ($i = 1, 2, 3$) by Proposition 4.9. If necessary, by changing f to $g \circ f$ we may assume that $f(E_i) = E_i$ for each $1 \leq i \leq 3$. Set $Y_{ij} = f(F_i(e_j))$ for $1 \leq i \leq 3$, $0 \leq j \leq 3$. Let $F_i(\mathbb{O}) := \{ F_i(x) \ ; \ x \in \mathbb{O} \}$ for each $i = 1, 2, 3$.

Proposition 4.10. $Y_{ij} \in F_i(\mathbb{O})$ *for each $i = 1, 2, 3$.*

Proof.   We only prove the statement in $i = 1$. In $i = 2, 3$, we can prove the statement by the similar way. Since $E_1 \circ Y_{1j} = 0$,

$$Y_{1j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x \\ 0 & \bar{x} & \xi_3 \end{pmatrix}$$

for some $\xi_2, \xi_3 \in \mathbb{R}$ and $x \in \mathbb{O}$. Moreover, since $E_2 \circ_\mathbb{H} F_1(e_j) = \frac{1}{2} F_1(e_j)$ and $E_3 \circ_\mathbb{H} F_1(e_j) = \frac{1}{2} F_1(e_j)$, we obtain that $E_2 \circ Y_{1j} = \frac{1}{2} Y_{1j}$ and $E_3 \circ Y_{1j} = \frac{1}{2} Y_{1j}$. Hence, we see that $\xi_2 = \xi_3 = 0$. Thus, $Y_{1j} \in F_1(\mathbb{O})$. ∎

Set $(,)_\mathbb{H} = (,)|_{H_3(\mathbb{H}) \times H_3(\mathbb{H})}$ and $\mathrm{tr}_\mathbb{H} = \mathrm{tr}|_{H_3(\mathbb{H})}$. The definition directly implies $(X, Y)_\mathbb{H} = \mathrm{tr}_\mathbb{H}(X \circ_\mathbb{H} Y)$. It is obvious that $(,)_\mathbb{H}$ is an inner product of $H_3(\mathbb{H})$.

Proposition 4.11. $f : (H_3(\mathbb{H}), (,)_\mathbb{H}) \to (V, (,)|_{V \times V})$ *is isometric.*

Proof.   By the definition of $(,)$, it is sufficient to show that $\mathrm{tr}(f(X)) = \mathrm{tr}_\mathbb{H}(X)$ for any $X \in H_3(\mathbb{H})$. However, we already know that $f(E_i) = E_i$ and $Y_{ij} \in F_i(\mathbb{O})$ for each $i = 1, 2, 3$ and $0 \leq j \leq 3$. Thus, we showed the statement. ∎

By Proposition 4.10, there is $x_{ij} \in \mathbb{O}$ ($1 \leq i \leq 3, 0 \leq j \leq 3$) such that $Y_{ij} = F_i(x_{ij})$. In particular, $F_1(x_{1j})$ ($0 \leq j \leq 3$) is an orthonormal system of $F_1(\mathbb{O})$, since f is isometric by Proposition 4.11 and $F_1(e_j)$ ($0 \leq j \leq 3$) is an orthonormal basis of $F_i(\mathbb{H})$, where $F_1(\mathbb{H}) = \{ F_i(x) \ ; \ x \in \bigoplus_{i=0}^{3} \mathbb{R} e_i \}$. By the definition of $Spin(8) \subset F_4$, we obtain Proposition 4.12.

Proposition 4.12. *There is $g \in Spin(8)$ such that $g(F_1(x_{1j})) = F_1(e_j)$ ($0 \leq j \leq 3$).*

By Proposition 4.12, we may assume that $Y_{1j} = F_1(e_j)$ ($0 \leq j \leq 3$). Lemma 4.13 is obvious.

Lemma 4.13.   *If $x \in \mathbb{O}$ satisfies $x e_1 = (x e_2) e_3$, then $x \in \bigoplus_{i=0}^{3} \mathbb{R} e_i$.*

Proposition 4.14. $\bigoplus_{j=0}^{3} \mathbb{R} F_i(x_{ij}) = F_i(\mathbb{H})$ *for $i = 2, 3$.*

Proof.   Since

$$\frac{1}{2} F_3(\overline{x_{20}}) = F_1(1) \circ F_2(x_{20}) = Y_{10} \circ Y_{20} = f(F_1(1)) \circ f(F_2(1))$$

$$= f(\frac{1}{2} F_3(1)) = \frac{1}{2} Y_{30} = \frac{1}{2} F_3(x_{30}),$$

we obtain $\overline{x_{20}} = x_{30}$. Moreover, $\overline{x_{21}} = -x_{31}$ by replacing $F_3(\overline{x_{20}})$ with $F_3(\overline{x_{21}})$ and $F_2(x_{20})$ with $F_2(x_{21})$ in the above argument.

In a similar way we obtain that $\overline{x_{22}} = -x_{32}$ and $\overline{x_{23}} = -x_{33}$. Moreover, since

$$\frac{1}{2}F_3(\overline{e_1 x_{20}}) = F_1(e_1) \circ F_2(x_{20}) = Y_{11} \circ Y_{20} = f(F_1(e_1)) \circ f(F_2(1))$$
$$= f(-\frac{1}{2}F_3(e_1)) = -\frac{1}{2}F_3(x_{31}),$$

we obtain $e_1 x_{20} = -\overline{x_{31}}$. In particular $e_1 x_{20} = x_{21}$ because we have $-\overline{x_{31}} = x_{21}$. By considering $F_3(e_2 x_{20})$ and $F_3(e_3 x_{20})$ we obtain similarly that $e_2 x_{20} = x_{22}$ and $e_3 x_{20} = x_{23}$. Since

$$\frac{1}{2}F_1(\overline{x_{22} x_{31}}) = F_2(x_{22}) \circ F_3(x_{31}) = Y_{22} \circ Y_{31} = f(F_2(e_2)) \circ f(F_3(e_1))$$
$$= f(\frac{1}{2}F_1(e_3)) = \frac{1}{2}F_1(e_3),$$

we obtain $-e_3 = x_{22} x_{31}$. By these arguments,

$$-e_3 = x_{22} x_{31} = (e_2 x_{20})(-\overline{x_{21}}) = (e_2 x_{20})(-\overline{e_1 x_{20}}) = (e_2 x_{20})(\overline{x_{20}} e_1).$$

Then $(\overline{e_2 x_{20}})e_3 = -\overline{x_{20}} e_1$ since $a\bar{a} = |a|, |ab| = |a||b|$ for any $a, b \in \mathbb{O}$ [11]. Thus we obtain $(\overline{x_{20}} e_2)e_3 = \overline{x_{20}} e_1$ and $x_{20} \in \mathbb{H}$ by Lemma 4.13. Therefore, we obtain that $x_{2j}, x_{3j} \in \mathbb{H}$ for each $0 \leq j \leq 3$ by above arguments. In particular, $x_{j0}, \cdots, x_{j3}$ are linearly independent for each $j = 2, 3$. Thus, we proved the statement. ∎

By arguments of this subsection, we obtain the following theorem.

Theorem 4.15. *For any $H_3(\mathbb{H})$-subspace V, there is $g \in F_4$ such that $g(V) = o$. In particular, the action of F_4 on M is transitive and $M \cong F_4/Sp(1) \cdot Sp(3)$. The corresponding is given by $F_4/Sp(1) \cdot Sp(3) \to M; g(Sp(1) \cdot Sp(3)) \mapsto g(o)$.*

Remark 4.16. Set

$$\mathbb{R} \oplus H_2(\mathbb{O}) := \left\{ \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x_1} & \xi_3 \end{pmatrix} ; \ x_1 \in \mathbb{O}, \xi_1, \xi_2, \xi_3 \in \mathbb{R} \right\}.$$

Then, $\mathbb{R} \oplus H_2(\mathbb{R})$ is a subspace of $H_3(\mathbb{O})$ and satisfies $X \circ Y \in \mathbb{R} \oplus H_2(\mathbb{R})$ for any $X, Y \in \mathbb{R} \oplus H_2(\mathbb{R})$. As we define $H_3(\mathbb{H})$-subspaces, we define $\mathbb{R} \oplus H_2(\mathbb{O})$-subspaces and denote N be the set of all $\mathbb{R} \oplus H_2(\mathbb{O})$-subspaces. Then, we see that F_4 acts on N. As we showed that $M \cong F_4/Sp(1) \cdot Sp(3)$, we can show that $N \cong F_4/Spin(9)$.

4.3. Maximal antipodal sets of $H_3(\mathbb{H})$-Grassmannian

We denote the $H_3(\mathbb{H})$-Grassmannian by $G_{\mathbb{H}}(H_3(\mathbb{O}))$. For any subspace $V \subset \mathbb{O}$, set $F_i(V) := \{F_i(x) \in H_3(\mathbb{O}) ; x \in V\}$. We consider some embeddings of $G_{\mathbb{H}}(\mathbb{O})$ into $G_{\mathbb{H}}(H_3(\mathbb{O}))$. Define $o_0, \cdots, o_3 \in G_{\mathbb{H}}(H_3(\mathbb{O}))$ as follows:

$$o_0 = \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(\mathbb{H}) + F_2(\mathbb{H}) + F_3(\mathbb{H}),$$

$$o_1 = \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(\mathbb{H}^\perp) + F_2(\mathbb{H}) + F_3(\mathbb{H}),$$

$$o_2 = \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(\mathbb{H}) + F_2(\mathbb{H}^\perp) + F_3(\mathbb{H}),$$

$$o_3 = \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(\mathbb{H}) + F_2(\mathbb{H}) + F_3(\mathbb{H}^\perp).$$

The Proposition 4.17 is obvious by the definition of $SO(4) \subset G_2$.

Proposition 4.17. $\bigcup_{g \in G_2} g(o_j) \cong G_2/SO(4)$ *for each* $0 \le j \le 3$.

By using these G_2-orbit $\cup_{g \in G_2} g(o_j)$ $(j = 0, \cdots, 3)$, we obtain the following four embeddings of $G_{\mathbb{H}}(\mathbb{O})$ into $G_{\mathbb{H}}(H_3(\mathbb{O}))$.

$$f_0 : G_{\mathbb{H}}(\mathbb{O}) \to G_{\mathbb{H}}(H_3(\mathbb{O})) \; ; \; V \mapsto \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(V) + F_2(V) + F_3(V),$$

$$f_1 : G_{\mathbb{H}}(\mathbb{O}) \to G_{\mathbb{H}}(H_3(\mathbb{O})) \; ; \; V \mapsto \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(V^\perp) + F_2(V) + F_3(V),$$

$$f_2 : G_{\mathbb{H}}(\mathbb{O}) \to G_{\mathbb{H}}(H_3(\mathbb{O})) \; ; \; V \mapsto \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(V) + F_2(V^\perp) + F_3(V),$$

$$f_3 : G_{\mathbb{H}}(\mathbb{O}) \to G_{\mathbb{H}}(H_3(\mathbb{O})) \; ; \; V \mapsto \bigoplus_{i=1}^{3} \mathbb{R}E_i + F_1(V) + F_2(V) + F_3(V^\perp).$$

By Corollary 4.4 and calculations, we obtain the following theorem. Recall the notation $V_1, \cdots, V_7 \in G_{\mathbb{H}}(\mathbb{O})$ from Subsection 2.3.

Theorem 4.18. *Let* $T'' = \{f_j(V_k) \; ; \; 0 \le j \le 3, 1 \le k \le 7\}$. *Then,* T'' *is a maximal antipodal set of* $G_{\mathbb{H}}(H_3(\mathbb{O}))$. *Moreover, any maximal antipodal set of* $G_{\mathbb{H}}(H_3(\mathbb{O}))$ *is congruent to* T''.

4.4. Application

As an application of maximal antipodal sets of $G_{\mathbb{H}}(H_3(\mathbb{O}))$, we consider the following problem.

Problem 4.19. Let $\{U_1, \cdots, U_{15}\} \subset \{E_i, F_i(e_j) \; ; \; 1 \le i \le 3, \, 0 \le j \le 7\}$. Define all sets $\{U_1, \cdots, U_{15}\}$ such that $\bigoplus_{i=1}^{15} \mathbb{R}U_i$ is a $H_3(\mathbb{H})$-subspace.

Considering this problem directly is hard, but we can solve this problem easily by using maximal antipodal sets of $G_{\mathbb{H}}(H_3(\mathbb{O}))$.

Set $J = \{E_i, F_i(e_j) \; ; \; 1 \le i \le 3, \, 0 \le j \le 7\}$. Let $P_{15}(J)$ be the set of all subsets of J whose cardinalities is 15. For any $A \in P_{15}(J)$, we define $V(A) = \bigoplus_{U_i \in A} \mathbb{R}U_i$.

Denote the set of all $V(A)$ which is $H_3(\mathbb{H})$ by W. To solve Problem 4.19 it is sufficient to study W.

In $G_{\mathbb{H}}(H_3(\mathbb{O}))$, the geodesic symmetry at $V \in G_{\mathbb{H}}(H_3(\mathbb{O}))$ is induced by the reflection $I_V - I_{V^\perp} \in O(H_3(\mathbb{O}))$, where $O(H_3(\mathbb{O}))$ is the set of all orthogonal transformations of $H_3(\mathbb{O})$. Thus, we see that $s_{\alpha_1}(\alpha_2) = \alpha_2$ for any $\alpha_1, \alpha_2 \in W$. In particular, W is an anipodal set.

Moreover, we see that T'' is contained in W. In fact, each element T'' is given by $\sum_{i=1}^{3} \mathbb{R}E_i + \sum_{m=1}^{4} \mathbb{R}F_1(e_{i_m}) + \mathbb{R}F_2(e_{j_m}) + \mathbb{R}F_3(e_{k_m})$ for some $\{i_1, \cdots, i_4\}, \{j_1, \cdots, j_4\},$ $\{k_1, \cdots, k_4\} \subset \{0, \cdots, 7\}$. Hence, $T'' \subset W$. Since T'' is a maximal antipodal set, it is true that $T'' = W$. Thus, we solved Problem 4.19. Let $R = \{E_1, E_2, E_3\}$. Define A_{ij} and $\overline{A_{ij}}$ $(1 \leq i \leq 3, 1 \leq j \leq 7)$ as follows.

$$A_{i1} = \{F_i(e_k) \; ; \; k = 0, 1, 2, 3\}, \quad A_{i2} = \{F_i(e_k) \; ; \; k = 0, 1, 4, 5\},$$
$$A_{i3} = \{F_i(e_k) \; ; \; k = 0, 2, 4, 6\}, \quad A_{i4} = \{F_i(e_k) \; ; \; k = 0, 3, 5, 6\},$$
$$A_{i5} = \{F_i(e_k) \; ; \; k = 0, 1, 6, 7\}, \quad A_{i6} = \{F_i(e_k) \; ; \; k = 0, 2, 5, 7\}$$
$$A_{i7} = \{F_i(e_k) \; ; \; k = 0, 3, 4, 7\},$$
$$\overline{A_{ij}} = \{F_i(e_k) \; ; \; 0 \leq k \leq 7\} - A_{ij}.$$

Theorem 4.20. *Let $A \in P_{15}(J)$. $V(A)$ is a $H_3(\mathbb{H})$-subspace if and only if A is contained in the following set.*

$$\left\{ \begin{matrix} R \cup A_{1j} \cup A_{2j} \cup A_{3j}, & R \cup \overline{A_{1j}} \cup A_{2j} \cup A_{3j}, \\ R \cup A_{1j} \cup \overline{A_{2j}} \cup A_{3j}, & R \cup A_{1j} \cup A_{2j} \cup \overline{A_{3j}} \end{matrix} \; ; \; 1 \leq j \leq 7 \right\}.$$

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