

Orthogonal Toroidal Lie Algebras, Vertex Algebras and Skew Howe Duality

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Abstract. We generalize the skew Howe dualities $(\mathfrak{so}_{2\nu}, \mathcal{O}(\ell))$ and $(\mathfrak{so}_{2\nu+1}, \text{Pin}(\ell))$ to the toroidal Lie algebra and vertex algebra setups.

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1. Introduction

In representation theory, one of the prominent notions is that of dual pair first studied by Howe for reductive Lie groups (algebras) (see [10, 11]). Let N and ℓ be two positive integers such that $N\ell$ is even. The classical skew Howe duality for complex orthogonal Lie algebras states that $(\mathfrak{so}_N, \mathfrak{so}_\ell)$ form a dual pair on the spinor representation $\Lambda(\mathbb{C}^{\frac{N\ell}{2}})$ of $\mathfrak{so}_{N\ell}$, in the sense that the restriction of $\Lambda(\mathbb{C}^{\frac{N\ell}{2}})$ on $\mathfrak{so}_N \oplus \mathfrak{so}_\ell(\subset \mathfrak{so}_{N\ell})$ is (strong) multiplicity free (cf. [11]). When $N = 2\nu$ is even (resp. $N = 2\nu + 1$ is odd), the \mathfrak{so}_ℓ -action on $\Lambda(\mathbb{C}^{\frac{N\ell}{2}})$ can be integrated to an action of the complex orthogonal group $\mathcal{O}(\ell)$ (resp. the complex Pin group $\text{Pin}(\ell)$). The main goal of this paper is to extend the $(\mathfrak{so}_{2\nu}, \mathcal{O}(\ell))$ and $(\mathfrak{so}_{2\nu+1}, \text{Pin}(\ell))$ dualities to the orthogonal toroidal Lie algebra and infinite rank orthogonal affine vertex algebra setups.

Let $\mathbb{C}_q = \mathbb{C}_q[t_0^{\pm 1}, t_1^{\pm 1}]$ be the quantum torus such that $t_1 t_0 = q t_0 t_1$, where q is a generic complex number. The orthogonal toroidal Lie algebra $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ by definition is a one-dimensional central extension of the orthogonal Lie algebra over \mathbb{C}_q (see Section 2 for details), which was first studied by Chen-Gao in [2]. It is remarkable that the spinor construction of $\mathfrak{so}_{N\ell}$ can be extended to the orthogonal affine Lie algebra $\widehat{\mathfrak{so}}_{N\ell} = \mathfrak{so}_{N\ell} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$, which affords an explicit realization of the basic representations of $\widehat{\mathfrak{so}}_{N\ell}$ on the exterior space $\mathcal{F}_N^\ell = \Lambda(\mathbb{C}^{\frac{N\ell}{2}} \otimes \mathbb{C}[t, t^{-1}])$ (see [5, 6, 7]). Furthermore, it was proved in [2] that there is an $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$ -module structure on the underlying space of the spinor representation \mathcal{F}_N^ℓ of $\widehat{\mathfrak{so}}_{N\ell}$.

In this paper, we prove that $(\widehat{\mathfrak{so}}_{2\nu}(\mathbb{C}_q), \mathcal{O}(\ell))$ and $(\widehat{\mathfrak{so}}_{2\nu+1}(\mathbb{C}_q), \text{Pin}(\ell))$ are Howe dual pairs on \mathcal{F}_N^ℓ (see Theorem), and that the restriction of the $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$ -module \mathcal{F}_N^ℓ

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on $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ is completely reducible with finite multiplicities. We remark that this is different from that in the affine case, where the multiplicities of the restriction of the $\widehat{\mathfrak{so}}_{N\ell}$ -module \mathcal{F}_N^ℓ on $\widehat{\mathfrak{so}}_N$ are in general infinite.

It was proved in [8] (see also [14]) that affine Kac-Moody algebras can be associated with affine vertex algebras through their modules. However, the generating functions for the Lie algebra $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ are non-local as \mathbb{C}_q is non-commutative, one needs to use the theory of equivariant quasi module for vertex algebras developed by Li in [15, 16] to associate $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ with vertex algebra. We prove that for any complex number ℓ , the category of restricted $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -modules of level ℓ is isomorphic to the category of equivariant quasi modules for the infinite rank orthogonal affine vertex algebra $V_{\widehat{\mathfrak{so}}_{N \times \infty}}(\ell, 0)$ (see Section 3). When ℓ is a positive integer, this implies that $(V_{\widehat{\mathfrak{so}}_{2\nu \times \infty}}(\ell, 0), \mathcal{O}(\ell))$ and $(V_{\widehat{\mathfrak{so}}_{2\nu+1 \times \infty}}(\ell, 0), \text{Pin}(\ell))$ are Howe dual pairs on \mathcal{F}_N^ℓ .

Note that the skew Howe duality for the simple orthogonal affine vertex algebras was studied by Jiang-Lam in [12], and the skew Howe duality for the completed infinite rank orthogonal Lie algebras was established by Wang in [17] (see also [3]). In [1], the skew Howe duality for general linear Lie algebras was generalized to the toroidal Lie algebra and vertex algebra setups. For this paper, we work on the field \mathbb{C} of complex numbers, and we use symbols \mathbb{C}^\times , \mathbb{Z} and \mathbb{N} for the sets of nonzero complex numbers, integers and nonnegative integers, respectively.

2. The toroidal Lie algebra $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$

Let N be a positive integer, and set

$$\mathbb{I}_N = \begin{cases} \{\pm 1, \dots, \pm \nu\} & \text{if } N = 2\nu \\ \{\pm 1, \dots, \pm \nu\} \cup \{0\} & \text{if } N = 2\nu + 1, \end{cases}$$

where $\nu = \lfloor \frac{N}{2} \rfloor$ is the largest integer less than or equal to $\frac{N}{2}$. Let \mathcal{A} be a unital associative algebra with anti-involution $\bar{}$. We denote by $\mathfrak{gl}_N(\mathcal{A})$ the Lie algebra of all $N \times N$ matrices $(a_{ij})_{i,j \in \mathbb{I}_N}$ with entries from \mathcal{A} . Then we have the following orthogonal Lie subalgebra of $\mathfrak{gl}_N(\mathcal{A})$:

$$\mathfrak{so}_N(\mathcal{A}) = \mathfrak{so}_N(\mathcal{A}, \bar{}) = \{A = (a_{ij}) \in \mathfrak{gl}_N(\mathcal{A}) \mid G_N A + \bar{A}^t G_N = 0\}, \tag{1}$$

where \bar{A}^t stands for the transport of $\bar{A} = (\bar{a}_{ij})$ and $G_N = \sum_{i \in \mathbb{I}_N} E_{i,-i}$. Here and henceforth, for $i, j \in \mathbb{I}_N$, $a \in \mathcal{A}$, we write $E_{i,j}a$ for the matrix whose only nonzero entry is the (i, j) -entry which is a . When $\mathcal{A} = \mathbb{C}$ with the trivial anti-involution, the $\mathfrak{so}_N(\mathbb{C})$ is the usual orthogonal Lie algebra \mathfrak{so}_N .

The quantum torus associated to a $q \in \mathbb{C}^\times$ by definition is an associative \mathbb{C} -algebra \mathbb{C}_q with generators $t_0^{\pm 1}, t_1^{\pm 1}$ and satisfies

$$(t_0^m t_1^n)(t_0^p t_1^s) = q^{np} t_0^{m+p} t_1^{n+s} \quad \text{for } m, n, p, s \in \mathbb{Z}.$$

Throughout this paper, we always assume that q is not a root of unity. Let $\bar{}$ be the anti-involution of \mathbb{C}_q defined by

$$\overline{t_0^m t_1^n} = q^{-mn} t_0^m t_1^{-n} \quad \text{for } m, n \in \mathbb{Z}.$$

Associated to (\mathbb{C}_q, \cdot) , we have an orthogonal Lie algebra $\mathfrak{so}_N(\mathbb{C}_q)$ defined in (1). For $i, j \in \mathbb{I}_N$ and $m, n \in \mathbb{Z}$, we set

$$f_{i,j}(m, n) = E_{i,j}t_0^m t_1^n - E_{-j,-i}\overline{t_0^m t_1^n}. \tag{2}$$

Then the Lie algebra $\mathfrak{so}_N(\mathbb{C}_q)$ is spanned by the elements $f_{i,j}(m, n)$ for $i, j \in \mathbb{I}_N$ and $m, n \in \mathbb{Z}$, such that

$$f_{i,j}(m, n) + q^{-mn} f_{-j,-i}(m, -n) = 0. \tag{3}$$

Let $\widehat{\mathfrak{gl}}_N(\mathbb{C}_q) = \mathfrak{gl}_N(\mathbb{C}_q) \oplus \mathbb{C}\mathbf{c}$ be a one-dimensional central extension of $\mathfrak{gl}_N(\mathbb{C}_q)$ with Lie product

$$\begin{aligned} & [E_{i,j}t_0^m t_1^n, E_{k,l}t_0^p t_1^s] \\ &= q^{np} \delta_{j,k} E_{i,l} t_0^{m+p} t_1^{n+s} - q^{ms} \delta_{i,l} E_{k,j} t_0^{m+p} t_1^{n+s} + \frac{1}{2} \delta_{j,k} \delta_{i,l} \delta_{m+p,0} \delta_{n+s,0} q^{np} m\mathbf{c} \end{aligned}$$

for $i, j, k, l \in \mathbb{I}_N, m, n, p, s \in \mathbb{Z}$, and \mathbf{c} is central. We define the orthogonal toroidal Lie algebra over the quantum torus \mathbb{C}_q to be the following one-dimensional central extension of $\mathfrak{so}_N(\mathbb{C}_q)$ ([2]):

$$\widehat{\mathfrak{so}}_N(\mathbb{C}_q) = \mathfrak{so}_N(\mathbb{C}_q) \oplus \mathbb{C}\mathbf{c} \subset \widehat{\mathfrak{gl}}_N(\mathbb{C}_q).$$

It is easy to check that

$$\begin{aligned} & [f_{i,j}(m, n), f_{k,l}(p, s)] \\ &= \delta_{j,k} q^{np} f_{i,l}(m+p, n+s) - \delta_{j,-l} q^{p(n-s)} f_{i,-k}(m+p, n-s) \\ & \quad + \delta_{i,-k} q^{-(m+p)s} f_{-l,j}(m+p, n-s) - \delta_{i,l} q^{ms} f_{k,j}(m+p, n+s) \\ & \quad + \delta_{j,k} \delta_{i,l} \delta_{m+p,0} \delta_{n+s,0} q^{np} m\mathbf{c} - \delta_{j,-l} \delta_{i,-k} \delta_{m+p,0} \delta_{n,s} m\mathbf{c} \end{aligned} \tag{4}$$

for $i, j, k, l \in \mathbb{I}_N, m, n, p, s \in \mathbb{Z}$. Let ℓ be a positive integer such that $N\ell$ is even and set $d = \lfloor \frac{\ell}{2} \rfloor$. For convenience, we define a map π from $\mathbb{I}_N \times \mathbb{I}_\ell$ to $\mathbb{I}_{N\ell}$ as follows:

(i) if $N = 2\nu$ and $\ell = 2d + 1$, then

$$\pi(i, r) = \begin{cases} (r+d)\nu + i & \text{if } i > 0, \\ (r-d)\nu + i & \text{if } i < 0. \end{cases}$$

(ii) if $N = 2\nu$ and $\ell = 2d$, then

$$\pi(i, r) = \begin{cases} (r+d)\nu + i & \text{if } i > 0, r < 0, \\ (r+d-1)\nu + i & \text{if } i > 0, r > 0, \\ (r-d)\nu + i & \text{if } i < 0, r > 0, \\ (r-d+1)\nu + i & \text{if } i < 0, r < 0. \end{cases}$$

(iii) if $N = 2\nu + 1$ and $\ell = 2d$, then

$$\pi(i, r) = \begin{cases} r & \text{if } i = 0, \\ (r+d)\nu + i + d & \text{if } i > 0, r < 0, \\ (r+d-1)\nu + i + d & \text{if } i > 0, r > 0, \\ (r-d)\nu + i - d & \text{if } i < 0, r > 0, \\ (r-d+1)\nu + i - d & \text{if } i < 0, r < 0. \end{cases}$$

One can check that π is a bijective map from $\mathbb{I}_N \times \mathbb{I}_\ell$ to $\mathbb{I}_{N\ell}$, and such that $\pi(-i, -r) = -\pi(i, r)$. Furthermore, we have the following result.

Lemma 2.1. *The assignment $(i, j \in \mathbb{I}_N, m, n \in \mathbb{Z})$*

$$f_{i,j}(m, n) \mapsto \sum_{r \in \mathbb{I}_\ell} f_{\pi(i,r), \pi(j,r)}(m, n), \quad \mathbf{c} \mapsto \ell \mathbf{c} \quad (5)$$

determines an embedding from $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ to $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$, and the assignment $(r, s \in \mathbb{I}_\ell)$

$$f_{r,s} := E_{r,s} - E_{-s,-r} \mapsto \sum_{i \in \mathbb{I}_N} f_{\pi(i,r), \pi(i,s)}(0, 0) \quad (6)$$

determines an embedding from \mathfrak{so}_ℓ to $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$. Furthermore, $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ commutes with \mathfrak{so}_ℓ as subalgebras of $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$.

Proof. It is easy to check that the assignment (5) (resp. (6)) gives rise to an embedding from $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ (resp. \mathfrak{so}_ℓ) to $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$. Furthermore, we have

$$\begin{aligned} & \left[\sum_{k \in \mathbb{I}_\ell} f_{\pi(i,k), \pi(j,k)}(m, n), \sum_{t \in \mathbb{I}_N} f_{\pi(t,r), \pi(t,s)}(0, 0) \right] \\ &= f_{\pi(i,r), \pi(j,s)}(m, n) - f_{\pi(i,-s), \pi(j,-r)}(m, n) + f_{\pi(i,-s), \pi(j,-r)}(m, n) - f_{\pi(i,r), \pi(j,s)}(m, n) \\ &= 0 \end{aligned}$$

for $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$ and $r, s \in \mathbb{I}_\ell$. This completes the proof of lemma. \blacksquare

3. Vertex algebra associated with $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$

Let $\mathcal{M}_{N \times \infty} = \{(a_{i,j}^{m,n}) \mid i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}, \text{ and } a_{i,j}^{m,n} \in \mathbb{C}\}$

be the space of infinite complex matrices, and set

$$\mathfrak{gl}_{N \times \infty} = \{(a_{i,j}^{m,n}) \in \mathcal{M}_{N \times \infty} \mid a_{i,j}^{m,n} = 0 \text{ for all but finitely many } a_{i,j}^{m,n}\}.$$

For $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$, write $E_{i,j}^{m,n}$ for the unit matrix in $\mathfrak{gl}_{N \times \infty}$ whose only nonzero entry is the $((i, m), (j, n))$ -entry 1. These unit matrices form a basis of $\mathfrak{gl}_{N \times \infty}$ with

$$[E_{i,j}^{m,n}, E_{k,l}^{p,s}] = \delta_{j,k} \delta_{n,p} E_{i,l}^{m,s} - \delta_{i,l} \delta_{m,s} E_{k,j}^{p,n}$$

for $i, j, k, l \in \mathbb{I}_N, m, n, p, s \in \mathbb{Z}$. We define the following infinite rank orthogonal Lie algebra:

$$\mathfrak{so}_{N \times \infty} = \{A = (a_{i,j}^{m,n}) \in \mathfrak{gl}_{N \times \infty} \mid A^t J_N + J_N A = 0\}, \quad (7)$$

where $A^t = (a_{j,i}^{n,m})$ and $J_N = \sum_{n \in \mathbb{Z}} \sum_{i \in \mathbb{I}_N} E_{i,-i}^{n,n} \in \mathfrak{gl}_{N \times \infty}$. It is straightforward to check that the Lie algebra $\mathfrak{so}_{N \times \infty}$ is spanned by the following elements

$$F_{i,j}^{m,n} = E_{i,j}^{m,n} - E_{-j,-i}^{n,m} \quad (8)$$

for $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$, and such that $F_{i,j}^{m,n} + F_{-j,-i}^{n,m} = 0$. Furthermore, one has

$$[F_{i,j}^{m,n}, F_{k,l}^{p,s}] = \delta_{j,k} \delta_{n,p} F_{i,l}^{m,s} - \delta_{j,-l} \delta_{n,s} F_{i,-k}^{m,p} + \delta_{i,-k} \delta_{m,p} F_{-l,j}^{s,n} - \delta_{i,l} \delta_{m,s} F_{k,j}^{p,n}$$

for $i, j, k, l \in \mathbb{I}_N$ and $m, n, p, s \in \mathbb{Z}$.

We define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{so}_{N \times \infty}$ by

$$\langle F_{i,j}^{m,n}, F_{k,l}^{p,s} \rangle = \delta_{j,k} \delta_{i,l} \delta_{m,s} \delta_{n,p} - \delta_{i,-k} \delta_{j,-l} \delta_{m,p} \delta_{n,s}, \tag{9}$$

which is nondegenerate, symmetric and invariant. For $r \in \mathbb{Z}$, we define a linear map σ_r on $\mathfrak{so}_{N \times \infty}$ by

$$\sigma_r(F_{i,j}^{m,n}) = F_{i,j}^{m+r,n+r} \tag{10}$$

for $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$. It is easy to show that $\sigma_r \in \text{Aut}(\mathfrak{so}_{N \times \infty}, \langle \cdot, \cdot \rangle)$, the group of automorphisms on $\mathfrak{so}_{N \times \infty}$ that preserve the bilinear form $\langle \cdot, \cdot \rangle$. Then we have a group isomorphism

$$\rho : \mathbb{Z} \rightarrow \text{Aut}(\mathfrak{so}_{N \times \infty}, \langle \cdot, \cdot \rangle), \quad r \mapsto \sigma_r. \tag{11}$$

One can check that for $a, b \in \mathfrak{g}$,

$$[\sigma_r(a), b] = 0 \quad \text{and} \quad \langle \sigma_r(a), b \rangle = 0 \tag{12}$$

for all but finitely many $r \in \mathbb{Z}$.

Associated to $(\mathfrak{so}_{N \times \infty}, \langle \cdot, \cdot \rangle)$, we have the affine Lie algebra ([13])

$$\widehat{\mathfrak{so}_{N \times \infty}} = \mathfrak{so}_{N \times \infty} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k},$$

where \mathbf{k} is central, and for $a, b \in \mathfrak{so}_{N \times \infty}, m, n \in \mathbb{Z}$,

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m \delta_{m+n,0} \langle a, b \rangle \mathbf{k}. \tag{13}$$

We write $a(m) = a \otimes t^m$ for $a \in \mathfrak{so}_{N \times \infty}$ and $m \in \mathbb{Z}$, and define a linear character on \mathbb{Z} by

$$\chi_q : \mathbb{Z} \rightarrow \mathbb{C}^\times, \quad r \mapsto q^r. \tag{14}$$

With this character χ_q , we lift the \mathbb{Z} -action ρ on $\mathfrak{so}_{N \times \infty}$ to the affine Lie algebra $\widehat{\mathfrak{so}_{N \times \infty}}$ by

$$\rho(r)(a(m) + \mu \mathbf{k}) = \chi_q(r)^m (\sigma_r(a)(m)) + \mu \mathbf{k} \tag{15}$$

for $m, r \in \mathbb{Z}, a \in \mathfrak{so}_{N \times \infty}$ and $\mu \in \mathbb{C}$. From [16], we obtain the (\mathbb{Z}, χ_q) -covariant algebra $\widehat{\mathfrak{so}_{N \times \infty}}[\mathbb{Z}]$ of $\widehat{\mathfrak{so}_{N \times \infty}}$, where

$$\widehat{\mathfrak{so}_{N \times \infty}}[\mathbb{Z}] = \widehat{\mathfrak{so}_{N \times \infty}} / \text{span}\{\rho(r)(u) - u \mid r \in \mathbb{Z}, u \in \widehat{\mathfrak{so}_{N \times \infty}}\}$$

as vector spaces, and the Lie bracket is given by

$$[\bar{u}, \bar{v}] = \sum_{r \in \mathbb{Z}} [\overline{\rho(r)(u)}, \bar{v}] \tag{16}$$

for $u, v \in \widehat{\mathfrak{so}_{N \times \infty}}$, where \bar{u} stands for the image of u in $\widehat{\mathfrak{so}_{N \times \infty}}[\mathbb{Z}]$ under the natural quotient map. By definition we have

$$\overline{F_{i,j}^{m,n}(r)} = q^{-mr} \overline{F_{i,j}^{0,n-m}(r)} \tag{17}$$

for $i, j \in \mathbb{I}_N, m, n, r \in \mathbb{Z}$. Moreover, we have the following result (cf. [1, Lemma 2.3]).

Lemma 3.1. *The (\mathbb{Z}, χ_q) -covariant algebra $\widehat{\mathfrak{so}}_{N \times \infty}[\mathbb{Z}]$ is spanned by the set*

$$\{\bar{\mathbf{k}}, \overline{F_{i,j}^{0,n}(m)} \mid i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}\},$$

and subject to the relations $\overline{F_{i,j}^{0,n}(m)} = -q^{-mn} \overline{F_{-j,-i}^{0,-n}(m)}$.

Let ℓ be a fixed complex number. View \mathbb{C} as a $(\mathfrak{so}_{N \times \infty} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k})$ -module with $\mathfrak{so}_{N \times \infty} \otimes \mathbb{C}[t]$ acts trivially, and \mathbf{k} acts as the scalar ℓ . Set

$$V_{\widehat{\mathfrak{so}}_{N \times \infty}}(\ell, 0) = \mathcal{U}(\widehat{\mathfrak{so}}_{N \times \infty}) \otimes_{\mathcal{U}(\mathfrak{so}_{N \times \infty} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k})} \mathbb{C} \tag{18}$$

an induced $\widehat{\mathfrak{so}}_{N \times \infty}$ -module. There is a natural \mathbb{Z} -grading on $\widehat{\mathfrak{so}}_{N \times \infty}$ such that

$$\deg a(n) = -n \quad \text{and} \quad \deg \mathbf{k} = 0 \quad \text{for } a \in \mathfrak{so}_{N \times \infty}, n \in \mathbb{Z},$$

and this induces an \mathbb{N} -graded on $V_{\widehat{\mathfrak{so}}_{N \times \infty}}$ by defining $\deg \mathbb{C} = 0$. We define a linear operator $L(0)$ on $V_{\widehat{\mathfrak{so}}_{N \times \infty}}(\ell, 0)$ by $L(0)v = nv$, for $v \in V_{\widehat{\mathfrak{so}}_{N \times \infty}}(\ell, 0)$ with $\deg v = n$. Denote $\mathbf{1} = 1 \otimes 1 \in V_{\widehat{\mathfrak{so}}_{N \times \infty}}(\ell, 0)$, and identify $\mathfrak{so}_{N \times \infty}$ as the degree-one subspace of $V_{\widehat{\mathfrak{so}}_{N \times \infty}}(\ell, 0)$ by setting $a = a(-1)\mathbf{1}$, for $a \in \mathfrak{so}_{N \times \infty}$. It is well-known that there exists a vertex algebra structure on $V_{\widehat{\mathfrak{so}}_{N \times \infty}}(\ell, 0)$, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector, and the vertex operators

$$Y(a, x) = a(x) := \sum_{m \in \mathbb{Z}} a(m)x^{-m-1} \quad \text{for } a \in \mathfrak{so}_{N \times \infty}.$$

Definition 3.2. Let Γ be a group equipped with a linear character $\chi : \Gamma \rightarrow \mathbb{C}^\times$. A *vertex Γ -algebra* is a vertex algebra V equipped with a group homomorphism

$$R : \Gamma \rightarrow \text{GL}(V); \quad \gamma \mapsto R_\gamma,$$

satisfying the condition $R_\gamma(\mathbf{1}) = \mathbf{1}$, and

$$R_\gamma Y(v, x) R_\gamma^{-1} = Y(R_\gamma(v), \chi(\gamma)^{-1}x) \tag{19}$$

for $\gamma \in \Gamma, v \in V$. Moreover, an equivariant quasi module for a vertex Γ -algebra V is a vector space W equipped with a linear map

$$Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x))); \quad v \mapsto Y_W(v, x),$$

satisfying the condition $Y_W(\mathbf{1}, x) = \text{Id}_W$, and

$$Y_W(R_\gamma v, x) = Y_W(v, \chi(\gamma)x)$$

for $\gamma \in \Gamma, v \in V$, and for $u, v \in V$, the quasi Jacobi identity

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) p(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) p(x_1, x_2) Y_W(v, x_2) Y_W(u, x_1) \\ = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) p(x_1, x_2) Y_W(Y(u, x_0)v, x_2) \end{aligned} \tag{20}$$

holds on W for some polynomial $p(x_1, x_2)$ of the form

$$p(x_1, x_2) = (x_1 - \chi(\gamma_1)x_2) \cdots (x_1 - \chi(\gamma_k)x_2) \tag{21}$$

for $\gamma_1, \dots, \gamma_k \in \Gamma$. ■

The notion of vertex Γ -algebra and its equivariant quasi module were first introduced by Li in [15, 16]. From [16], it is known that the action ρ of \mathbb{Z} on $\mathfrak{so}_{N \times \infty}$ can be uniquely lifted to an action of \mathbb{Z} on $V_{\widehat{\mathfrak{so}_{N \times \infty}}}(\ell, 0)$, which will be also denoted as ρ . Furthermore, $V_{\widehat{\mathfrak{so}_{N \times \infty}}}(\ell, 0)$ becomes a vertex Γ -algebra with $\Gamma = \mathbb{Z}$, $\chi = \chi_q$ and $R_r = \chi_q(r)^{-L(0)}\rho(r)$ for $r \in \mathbb{Z}$, where χ_q is defined in (14). We say that an $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$ -module W is restricted if

$$f_{i,j}(n, x) := \sum_{m \in \mathbb{Z}} f_{i,j}(m, n)x^{-m-1} \in \text{Hom}(W, W((x))) \tag{22}$$

for any $i, j \in \mathbb{I}_N$ and $n \in \mathbb{Z}$, and is of level $\ell \in \mathbb{C}$ if \mathbf{c} acts as the scalar ℓ . As the main result of this section, we have the following correspondence theorem.

Theorem 3.3. *Let ℓ be a complex number. For any restricted $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$ -module W of level ℓ , there exists an equivariant quasi $V_{\widehat{\mathfrak{so}_{N \times \infty}}}(\ell, 0)$ -module structure on W , where the vertex operator is uniquely determined by*

$$Y_W(F_{i,j}^{m,n}, x) = q^m f_{i,j}(n - m, q^m x) \tag{23}$$

for $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$. On the other hand, for any equivariant quasi $V_{\widehat{\mathfrak{so}_{N \times \infty}}}(\ell, 0)$ -module (W, Y_W) , W is a restricted $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$ -module of level ℓ with the action given by

$$f_{i,j}(n, x) = Y_W(F_{i,j}^{0,n}, x) \quad \text{for } i, j \in \mathbb{I}_N, n \in \mathbb{Z}. \tag{24}$$

Proof. We first give a realization of $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$ as the covariant algebra of $\widehat{\mathfrak{so}_{N, \infty}}$. From Lemma 3.1 and (3), we see that there is a linear isomorphism, say θ , from $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$ to the (\mathbb{Z}, χ_q) -covariant algebra $\widehat{\mathfrak{so}_{N, \infty}}[\mathbb{Z}]$ determined by

$$\theta(f_{i,j}(m, n)) = \overline{F_{i,j}^{0,n}(m)}, \quad \theta(\mathbf{c}) = \bar{\mathbf{k}} \tag{25}$$

for $i, j \in \mathbb{I}_N$ and $m, n \in \mathbb{Z}$. Furthermore, from (13),(16) and (17) we have

$$\begin{aligned} & \overline{[F_{i,j}^{0,n}(m), F_{k,l}^{0,s}(p)]} \\ &= \sum_{r \in \mathbb{Z}} q^{mr} \overline{([F_{i,j}^{r,n+r}, F_{k,l}^{0,s}](m+p) + m\delta_{m+p,0} \langle F_{i,j}^{r,n+r}, F_{k,l}^{0,s} \rangle \bar{\mathbf{k}})} \\ &= q^{-mn} \delta_{j,k} \overline{F_{i,l}^{-n,s}(m+p)} - q^{m(s-n)} \delta_{j,-l} \overline{F_{i,-k}^{s-n,0}(m+p)} + \delta_{i,-k} \overline{F_{-l,j}^{s,n}(m+p)} \\ &\quad - q^{ms} \delta_{i,l} \overline{F_{k,j}^{0,n+s}(m+p)} + q^{np} \delta_{j,k} \delta_{i,l} \delta_{n+s,0} \delta_{m+p,0} m \bar{\mathbf{k}} - \delta_{j,-l} \delta_{i,-k} \delta_{n,s} \delta_{m+p,0} m \bar{\mathbf{k}} \\ &= q^{np} \delta_{j,k} \overline{F_{i,l}^{0,s+n}(m+p)} - q^{p(n-s)} \delta_{j,-l} \overline{F_{i,-k}^{0,n-s}(m+p)} + q^{-s(m+p)} \delta_{i,-k} \overline{F_{-l,j}^{0,n-s}(m+p)} \\ &\quad - q^{ms} \delta_{i,l} \overline{F_{k,j}^{0,n+s}(m+p)} + q^{np} \delta_{j,k} \delta_{i,l} \delta_{n+s,0} \delta_{m+p,0} m \bar{\mathbf{k}} - \delta_{j,-l} \delta_{i,-k} \delta_{n,s} \delta_{m+p,0} m \bar{\mathbf{k}}. \end{aligned}$$

This, together with (4), proves that θ is a Lie isomorphism.

Similar to the algebra $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$, we say that an $\widehat{\mathfrak{so}_{N, \infty}}[\mathbb{Z}]$ -module W is restricted if, for $i, j \in \mathbb{I}_N$ and $m, n \in \mathbb{Z}$,

$$\overline{F_{i,j}^{m,n}(x)} := \sum_{r \in \mathbb{Z}} \overline{F_{i,j}^{m,n}(r)} x^{-r-1} \in \mathcal{E}(W),$$

and is said of level ℓ if $\bar{\mathbf{k}} = \ell$ on W . Assume now that W is a restricted $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$ -module of level ℓ . Via the isomorphism θ , it follows from (17) that W becomes a restricted $\widehat{\mathfrak{so}_{N, \infty}}[\mathbb{Z}]$ -module of level ℓ with

$$\overline{F_{i,j}^{m,n}(x)} = q^m f_{i,j}(n - m, q^m x) \quad \text{for } i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}.$$

Note that the character χ_q is injective, in view of [16, Theorem 4.9], there is an equivariant quasi $V_{\widehat{\mathfrak{so}_{N \times \infty}}}(\ell, 0)$ -module on W with

$$Y_W(F_{i,j}^{m,n}, x) = \overline{F_{i,j}^{m,n}}(x) \quad \text{for } i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}.$$

This proves the first assertion in theorem. On the other hand, let W be an equivariant quasi $V_{\widehat{\mathfrak{so}_{N \times \infty}}}(\ell, 0)$ -module. Then again by [16, Theorem 4.9] there is a natural $\widehat{\mathfrak{so}_{N, \infty}}[\mathbb{Z}]$ -module structure on W such that $\overline{\mathbf{k}} = \ell$ and

$$\overline{F_{i,j}^{m,n}}(x) = Y_W(F_{i,j}^{m,n}, x) \quad \text{for } i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}.$$

Via the isomorphism θ , W is a restricted $\widehat{\mathfrak{so}_N}(\mathbb{C}_q)$ -module of level ℓ with

$$f_{i,j}(n, x) = \overline{F_{i,j}^{0,n}}(x) = Y_W(F_{i,j}^{0,n}, x) \quad \text{for } i, j \in \mathbb{I}_N, n \in \mathbb{Z}.$$

This completes the proof of the theorem. ■

4. The $(\widehat{\mathfrak{so}_{2\nu}}(\mathbb{C}_q), \mathbf{O}(\ell))$ and $(\widehat{\mathfrak{so}_{2\nu+1}}(\mathbb{C}_q), \text{Pin}(\ell))$ dualities

Let N, ℓ be positive integers such that $N\ell$ is even. In this section we prove that $(\widehat{\mathfrak{so}_{2\nu}}(\mathbb{C}_q), \mathbf{O}(\ell))$ and $(\widehat{\mathfrak{so}_{2\nu+1}}(\mathbb{C}_q), \text{Pin}(\ell))$ are dual pairs (in the sense of Howe) on an fermionic Fock space \mathcal{F}_N^ℓ .

We begin by recalling the notion of dual pairs ([10, 11]). Let G be a complex reductive group, and

$$W = \bigoplus_{i \in I} \text{Hom}_G(W_i, W) \otimes W_i$$

a locally regular G -module of countable dimension, where $W_i, i \in I$, exhaust all non-isomorphic irreducible regular G -submodules of W .

Definition 4.1. Let \mathcal{L} be a Lie algebra and let V be a vertex Γ -algebra. We say that (\mathcal{L}, G) (resp. (V, G)) is a (Howe) *dual pair* on W if

- (i) W is an \mathcal{L} -module (resp. equivariant quasi V -module) such that the actions of \mathcal{L} (resp. V) and G commute.
- (ii) For every $i \in I$, the \mathcal{L} -submodule (resp. equivariant quasi V -submodule) $\text{Hom}_G(W_i, W)$ of W is irreducible.
- (iii) For any $i, j \in I$, $\text{Hom}_G(W_i, W) \cong \text{Hom}_G(W_j, W)$ if and only if $i = j$. ■

Next, we introduce the fermionic Fock space \mathcal{F}_N^ℓ . Let \mathcal{C}_N^ℓ be the Clifford algebra with generators $\psi_i^r(m)$, for $i \in \mathbb{I}_N, r \in \mathbb{I}_\ell, m \in \mathbb{Z}$, subject to the relation

$$\psi_i^r(m)\psi_j^s(n) + \psi_j^s(n)\psi_i^r(m) = \delta_{i+j,0}\delta_{r+s,0}\delta_{m+n,0}1$$

for $i, j \in \mathbb{I}_N, r, s \in \mathbb{I}_\ell$ and $m, n \in \mathbb{Z}$. For convenience, for $i \in \mathbb{I}_N$ and $r \in \mathbb{I}_\ell$, we define the following generating functions:

$$\psi_i^r(x) = \sum_{m \in \mathbb{Z}} \psi_i^r(m)x^{-m}.$$

As in [4], we define a normal ordering on \mathcal{C}_N^ℓ as follows:

$$: \psi_i^r(m)\psi_j^s(n) := \begin{cases} \psi_i^r(m)\psi_j^s(n) & \text{if } m < n, \\ \frac{1}{2}(\psi_i^r(m)\psi_j^s(n) - \psi_j^s(n)\psi_i^r(m)) & \text{if } m = n, \\ -\psi_j^s(n)\psi_i^r(m) & \text{if } m > n \end{cases} \quad (26)$$

for $i, j \in \mathbb{I}_N, r, s \in \mathbb{I}_\ell$ and $m, n \in \mathbb{Z}$.

Let \mathcal{F}_N^ℓ be the simple \mathcal{C}_N^ℓ -module generated by a vacuum vector $\mathbf{1}$ such that

$$\psi_i^r(m+1)\mathbf{1} = \psi_{-i}^r(m)\mathbf{1} = 0 \tag{27}$$

for $1 \leq i \leq \nu, r \in \mathbb{I}_\ell, m \in \mathbb{N}$, and

$$\psi_0^r(m+1)\mathbf{1} = \psi_0^{-r}(m)\mathbf{1} = 0 \tag{28}$$

for $1 \leq r \leq d, m \in \mathbb{N}$, where $d = \lfloor \frac{\ell}{2} \rfloor$. Note that the condition (28) is needed only when $N = 2\nu + 1$ is odd (and so $\ell = 2d$ is even).

Lemma 4.2. *There is an $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -module structure on \mathcal{F}_N^ℓ with $\mathbf{c} = \ell$ and*

$$f_{i,j}(n, x) = \begin{cases} \sum_{r \in \mathbb{I}_\ell} : \psi_i^r(x) \psi_{-j}^{-r}(x) : & \text{if } n = 0, \\ \sum_{r \in \mathbb{I}_\ell} \left(: \psi_i^r(x) \psi_{-j}^{-r}(q^n x) : + \frac{1}{2} \delta_{i,j} \frac{1+q^n}{1-q^n} \right) & \text{if } n \neq 0, \end{cases} \tag{29}$$

where $i, j \in \mathbb{I}_N$ and $n \in \mathbb{Z}$. On the other hand, there is an \mathfrak{so}_ℓ -module structure on \mathcal{F}_N^ℓ with

$$f_{r,s} = \sum_{m \in \mathbb{Z}} \sum_{i \in \mathbb{I}_N} : \psi_i^r(m) \psi_{-i}^{-s}(-m) :, \tag{30}$$

where $r, s \in \mathbb{I}_\ell$. Furthermore, the $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -action on \mathcal{F}_N^ℓ commutes with that of \mathfrak{so}_ℓ .

Proof. When N is even and $\ell = 1$, it was proved in [2] that the action (29) affords an $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -module (of level 1) structure on \mathcal{F}_N^1 . In particular, we obtain an $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$ -module structure on $\mathcal{F}_{N\ell}^1$ such that

$$f_{i,j}(n, x) = \begin{cases} : \psi_i^0(x) \psi_{-j}^0(x) : & \text{if } n = 0, \\ : \psi_i^0(x) \psi_{-j}^0(q^n x) : + \frac{1}{2} \delta_{i,j} \frac{1+q^n}{1-q^n} & \text{if } n \neq 0 \end{cases}$$

for $i, j \in \mathbb{I}_{N\ell}, n \in \mathbb{Z}$. Note that the bijection $\pi : \mathbb{I}_N \times \mathbb{I}_\ell \rightarrow \mathbb{I}_{N\ell}$ induces an algebra isomorphism from \mathcal{C}_N^ℓ to $\mathcal{C}_{N\ell}^1$ determined by $\psi_i^r \mapsto \psi_{\pi(i,r)}^0$. Via this isomorphism, \mathcal{F}_N^ℓ becomes a $\mathcal{C}_{N\ell}^1$ -module. It is straightforward to prove that, as $\mathcal{C}_{N\ell}^1$ -modules, \mathcal{F}_N^ℓ is isomorphic to $\mathcal{F}_{N\ell}^1$. Thus \mathcal{F}_N^ℓ becomes an $\widehat{\mathfrak{so}}_{N\ell}(\mathbb{C}_q)$ -module with

$$f_{\pi(i,r), \pi(j,s)}(n, x) = \begin{cases} : \psi_i^r(x) \psi_{-j}^{-s}(x) : & \text{if } n = 0, \\ : \psi_i^r(x) \psi_{-j}^{-s}(q^n x) : + \frac{1}{2} \delta_{i,j} \delta_{r,s} \frac{1+q^n}{1-q^n} & \text{if } n \neq 0, \end{cases}$$

where $i, j \in \mathbb{I}_N, r, s \in \mathbb{I}_\ell$ and $n \in \mathbb{Z}$. Therefore the assertion of this lemma follows from Lemma 2.1. ■

Let $O(\ell) = \{A \in \text{GL}_\ell(\mathbb{C}) \mid G_\ell A^t G_\ell = A\}$

be the complex orthogonal group of rank ℓ , where $G_\ell = \sum_{i \in \mathbb{I}_\ell} E_{i,-i}$, and let

$$\text{SO}(\ell) = \{A \in O(\ell) \mid \det A = 1\}$$

be the complex special orthogonal group.

When $N = 2\nu$, we view $U_\nu^\ell := \mathbb{C}^\ell \otimes \mathbb{C}^\nu \otimes \mathbb{C}[t, t^{-1}]$ as an \mathfrak{so}_ℓ -module such that \mathfrak{so}_ℓ acts naturally on \mathbb{C}^ℓ and acts trivially on $\mathbb{C}^\nu \otimes \mathbb{C}[t, t^{-1}]$. Then we have a natural \mathfrak{so}_ℓ -module structure on the exterior algebra $\Lambda(U_\nu^\ell)$ of U_ν^ℓ . Fix a basis e_1, \dots, e_ν (resp. e^r , $r \in \mathbb{I}_\ell$) of \mathbb{C}^ν (resp. \mathbb{C}^ℓ). Then it is straightforward to see that the \mathfrak{so}_ℓ -module \mathcal{F}_N^ℓ (see (30)) is isomorphic to $\Lambda(U_\nu^\ell)$, and the isomorphism is given by

$$\psi_i^r(-n)\mathbf{1} = e_i \otimes e^r \otimes t^n, \quad \psi_{-i}^r(-n-1)\mathbf{1} = e_i \otimes e^r \otimes t^{-n-1}$$

for $1 \leq i \leq \nu, r \in \mathbb{I}_\ell$ and $n \in \mathbb{N}$. In view of this, the \mathfrak{so}_ℓ -action on \mathcal{F}_N^ℓ can be integrated to an $\mathrm{SO}(\ell)$ -module and extended to an $\mathrm{O}(\ell)$ -module.

We denote by $\mathrm{End}(\mathcal{F}_N^\ell)^{\mathrm{O}(\ell)}$ the subalgebra of $\mathrm{End}(\mathcal{F}_N^\ell)$ fixed by $\mathrm{O}(\ell)$. Then we have:

Lemma 4.3. *Assume that the positive integer N is even. Then for any subspace U of \mathcal{F}_N^ℓ , U is a $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -submodule if and only if it is an $\mathrm{End}(\mathcal{F}_N^\ell)^{\mathrm{O}(\ell)}$ -submodule. Furthermore, two $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -submodules of \mathcal{F}_N^ℓ are isomorphic if and only if they are isomorphic as $\mathrm{End}(\mathcal{F}_N^\ell)^{\mathrm{O}(\ell)}$ -modules.*

Proof. For each $i, j \in \mathbb{I}_N$ and $m, n \in \mathbb{Z}$, we choose an element from $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$:

$$g_{i,j}(m, n) = \begin{cases} f_{i,j}(m, n), & \text{if } n = 0, \\ f_{i,j}(m, n) - \frac{\ell}{2}\delta_{m,0}\delta_{i,j}\frac{1+q^n}{1-q^n}\mathbf{c}, & \text{if } n \neq 0. \end{cases}$$

Then from Lemma 4.2, the action of $g_{i,j}(m, n)$ on \mathcal{F}_N^ℓ is given as follows:

$$g_{i,j}(m, n) = \sum_{k \in \mathbb{Z}} \sum_{r \in \mathbb{I}_\ell} q^{-nk} : \psi_i^r(m-k)\psi_{-j}^{-r}(k) : .$$

Note that, for any $i, j \in \mathbb{I}_N, m \in \mathbb{Z}$ and $w \in \mathcal{F}_N^\ell$, there is a finite subset $J_{i,j}^{m,w}$ of \mathbb{Z} such that for $k \in \mathbb{Z}$,

$$\sum_{r \in \mathbb{I}_\ell} : \psi_i^r(m-k)\psi_{-j}^{-r}(k) : w \neq 0 \iff k \in J_{i,j}^{m,w}. \tag{31}$$

Recall that as $\mathrm{O}(\ell)$ -modules, \mathcal{F}_N^ℓ is isomorphic to $\Lambda(U_\nu^\ell)$. Then the classical (skew) invariant theory for orthogonal groups ([9, 11]) states that the algebra $\mathrm{End}(\mathcal{F}_N^\ell)^{\mathrm{O}(\ell)}$ is generated by the following elements

$$\sum_{r \in \mathbb{I}_\ell} : \psi_i^r(m)\psi_{-j}^{-r}(n) :$$

for $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$. This implies that if U is an $\mathrm{End}(\mathcal{F}_N^\ell)^{\mathrm{O}(\ell)}$ -submodule, then $g_{i,j}(m, n)U \subset U$, for any $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$, and hence U is an $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -submodule. Conversely, if U is an $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -submodule, then we need to show

$$\sum_{r \in \mathbb{I}_\ell} : \psi_i^r(m)\psi_{-j}^{-r}(n) : w \in U \tag{32}$$

for $i, j \in \mathbb{I}_N, m, n \in \mathbb{Z}$ and $w \in W$.

For this purpose, we assume that $\sum_{r \in \mathbb{I}_\ell} : \psi_i^r(m) \psi_{-j}^{-r}(n) : w \neq 0$. Consider the equations

$$g_{i,j}(m+n, p)w = \sum_{k \in J_{i,j}^{m+n,w}} q^{-pk} \left(\sum_{r \in \mathbb{I}_\ell} : \psi_i^r(m+n-k) \psi_{-j}^{-r}(k) : w \right) \in U$$

for $p = 0, 1, \dots, |J_{i,j}^{m,w}| - 1$. As q is not a root of unity, the coefficient matrix of the above equations is a Vandermonde matrix. This implies that

$$\sum_{r \in \mathbb{I}_\ell} : \psi_i^r(m+n-k) \psi_{-j}^{-r}(k) : w \in U \text{ for all } k \in J_{i,j}^{m+n,w}.$$

By taking $k = n$, we see that (32) holds. This proves the first assertion of the lemma. The second assertion can be proved by a similar argument, which is omitted. \blacksquare

Now we consider the case $N = 2\nu + 1$ (and so $\ell = 2d$). Let $\text{Pin}(\ell)$ be the double cover group of the orthogonal group $\text{O}(\ell)$ and let $\text{Spin}(\ell)$ be the pull back of $\text{SO}(\ell)$ under the projection from $\text{Pin}(\ell)$ to $\text{O}(\ell)$. It is well-known that there is a natural \mathfrak{so}_ℓ -module structure on the exterior algebra $\Lambda(\mathbb{C}^d)$, the so-called spinor module [9]. View $\mathbb{C}^\ell \otimes t\mathbb{C}[t]$ as an \mathfrak{so}_ℓ -module such that \mathfrak{so}_ℓ acts naturally on \mathbb{C}^ℓ and acts trivially on $t\mathbb{C}[t]$, and so we have an \mathfrak{so}_ℓ -module structure on the exterior algebra $\Lambda(\mathbb{C}^\ell \otimes t\mathbb{C}[t])$ of $\mathbb{C}^\ell \otimes t\mathbb{C}[t]$. It is easy to see that as \mathfrak{so}_ℓ -modules, \mathcal{F}_1^ℓ is isomorphic to $\Lambda(\mathbb{C}^d) \otimes \Lambda(\mathbb{C}^\ell \otimes t\mathbb{C}[t])$, and the isomorphism is given by

$$\psi_0^r(0)\mathbf{1} = e^r \otimes (1 \otimes 1), \quad \psi_0^s(-n-1)\mathbf{1} = 1 \otimes (e^s \otimes t^{n+1})$$

for $1 \leq r \leq d, s \in \mathbb{I}_\ell$ and $n \in \mathbb{N}$. Then we have the following \mathfrak{so}_ℓ -module isomorphism

$$\mathcal{F}_{2\nu+1}^\ell \cong \mathcal{F}_1^\ell \otimes \mathcal{F}_{2\nu}^\ell \cong \Lambda(\mathbb{C}^d) \otimes \Lambda(\mathbb{C}^\ell \otimes t\mathbb{C}[t]) \otimes \Lambda(U_\nu^\ell).$$

This implies that the \mathfrak{so}_ℓ -action on \mathcal{F}_N^ℓ can be integrated to an $\text{Spin}(\ell)$ -module and extended to a $\text{Pin}(\ell)$ -module.

We denote by $\text{End}(\mathcal{F}_N^\ell)^{\text{Pin}(\ell)}$ the subalgebra of $\text{End}(\mathcal{F}_N^\ell)$ fixed by $\text{Pin}(\ell)$. By a similar argument as that of Lemma 4.3, we have:

Lemma 4.4. *Assume that the positive integer N is odd. Then for any subspace U of \mathcal{F}_N^ℓ , U is a $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -submodule if and only if it is an $\text{End}(\mathcal{F}_N^\ell)^{\text{Pin}(\ell)}$ -submodule. Furthermore, two $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -submodules of \mathcal{F}_N^ℓ are isomorphic if and only if they are isomorphic as $\text{End}(\mathcal{F}_N^\ell)^{\text{Pin}(\ell)}$ -modules.*

Note that \mathcal{F}_N^ℓ is a restricted $\widehat{\mathfrak{so}}_N(\mathbb{C}_q)$ -module of level ℓ . From Theorem 3.3, we see that \mathcal{F}_N^ℓ is an equivariant quasi $V_{\widehat{\mathfrak{so}}_N \times \infty}(\ell, 0)$ -module. Then we have:

Theorem 4.5. *Let N, ℓ be positive integers such that $N\ell$ is even. If N is even, then $(\widehat{\mathfrak{so}}_N(\mathbb{C}_q), \text{O}(\ell))$ and $(V_{\widehat{\mathfrak{so}}_N \times \infty}(\ell, 0), \text{O}(\ell))$ are dual pairs on \mathcal{F}_N^ℓ . Moreover, if N is odd, then $(\widehat{\mathfrak{so}}_N(\mathbb{C}_q), \text{Pin}(\ell))$ and $(V_{\widehat{\mathfrak{so}}_N \times \infty}(\ell, 0), \text{Pin}(\ell))$ are dual pairs on \mathcal{F}_N^ℓ .*

Proof. If N is even, then $\mathcal{F}_N^\ell \cong \Lambda(U_\nu^\ell)$ as $\text{O}(\ell)$ -modules. By the general duality theorem ([9, Theorem 4.2.1]), we know that $(\text{End}(\mathcal{F}_N^\ell)^{\text{O}(\ell)}, \text{O}(\ell))$ is a dual pair on \mathcal{F}_N^ℓ . This together with Lemma 4.3 gives that $(\widehat{\mathfrak{so}}_N(\mathbb{C}_q), \text{O}(\ell))$ is a dual pair on \mathcal{F}_N^ℓ . Furthermore, by Theorem 3.3, we see that $(V_{\widehat{\mathfrak{so}}_N \times \infty}(\ell, 0), \text{O}(\ell))$ is also a dual pair on \mathcal{F}_N^ℓ . Similarly, when N is odd, it follows from Lemma 4.4 and Theorem 3.3 that $(\widehat{\mathfrak{so}}_N(\mathbb{C}_q), \text{Pin}(\ell))$ and $(V_{\widehat{\mathfrak{so}}_N \times \infty}(\ell, 0), \text{Pin}(\ell))$ are dual pairs on \mathcal{F}_N^ℓ . \blacksquare

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