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Crossed Extensions of Lie Algebras

Apurba Das

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Abstract. It is known that Hochschild cohomology groups are represented by crossed extensions of associative algebras. In this paper, we introduce crossed n-fold extensions of a Lie algebra $\mathfrak{g}$ by a module M, for $n \geq 2$. We show that such extensions represent elements in the $(n+1)$-th Chevalley-Eilenberg cohomology group $H^{n+1}_{CE}(\mathfrak{g}, M)$.

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Key Words: Lie algebras, Chevalley-Eilenberg cohomology, crossed modules, crossed extensions.

1. Introduction

Group cohomology of a group G with coefficients in a G-module M is related to crosses extensions of G by M. More precisely, crossed n-fold extensions of G by M represent elements in the $(n+1)$-th group cohomology $H^{n+1}(G, M)$ [7]. A similar result for Hochschild cohomology was considered by Baues and Minian [2]. Namely, they introduce crossed n-fold extensions of an associative algebra A by an A-bimodule M and prove that such extensions represent elements in the $(n+1)$-th Hochschild cohomology $HH^{n+1}(A, M)$ as abelian groups. A further generalization of this result has been obtained in [3].

vsp1 The cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module M is given by the Chevalley-Eilenberg cohomology. This cohomology theory controls the deformation of a given Lie algebra structure. The Chevalley-Eilenberg cohomology groups $H^{\bullet}_{CE}(\mathfrak{g}, M)$ are also related to special type of L_∞-algebras [1, Theorem 6.7]. In this paper, we give another interpretation of Chevalley-Eilenberg cohomology in terms of crossed extensions. The idea is the same as Baues and Minian for associative algebra case. We introduce crossed n-fold extensions of a Lie algebra $\mathfrak{g}$ by a $\mathfrak{g}$-module M. We show that such extensions represent elements in $H^{n+1}_{CE}(\mathfrak{g}, M)$. The only explanation for the truth of this result given in Gerstenhaber's article [6] is the sentense: "This follows immediately from the fact that both [cohomology theories] satisfy the axioms and vanish on injectives." Our aim in this paper is to give a full proof of this result.

We also sketch the similar result for Leibniz algebras. All vector spaces, linear maps are over a field $\mathbb{K}$.

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2. Chevalley-Eilenberg cohomology

Let $(\mathfrak{g}, [\ ,\ ])$ a Lie algebra. A *module* over $\mathfrak{g}$ consists of a vector space M together with a $\mathbb{K}$-bilinear map $[\ ,\ ] : \mathfrak{g} \times M \to M$ satisfying

$$[[x, y], m] = [x, [y, m]] - [y, [x, m]],$$

for $x, y \in \mathfrak{g}$ and $m \in M$. It is clear that $\mathfrak{g}$ is a module over $\mathfrak{g}$ with respect to the Lie bracket. It is called the *adjoint module*.

Given a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ and a module M, the corresponding Chevalley-Eilenberg cochain groups $\{C^n_{CE}(\mathfrak{g}, M)\}_{n \geq 0}$ are given by $C^0_{CE}(\mathfrak{g}, M) = M$ and $C^n_{CE}(\mathfrak{g}, M) = \mathrm{Hom}_{\mathbb{K}}(\wedge^n \mathfrak{g}, M)$, for $n \geq 1$. The coboundary map

$$\delta : C^n_{CE}(\mathfrak{g}, M) \to C^{n+1}_{CE}(\mathfrak{g}, M)$$

is given by

$$\begin{aligned}
(\delta m)(x) &= [x, m], \quad \text{for } \ m \in M \text{ and } x \in \mathfrak{g}, \\
(\delta f)(x_1, \ldots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} [x_i, f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \\
&\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1}),
\end{aligned}$$

for $f \in C^n_{CE}(\mathfrak{g}, M)$ and $x_1, \ldots, x_{n+1} \in \mathfrak{g}$. The corresponding cohomology groups are denoted by $H^\bullet_{CE}(\mathfrak{g}, M)$ and called the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in the module M [13].

It is easy to see that $H^0_{CE}(\mathfrak{g}, M)$ is the submodule of invariants of M:

$$H^0_{CE}(\mathfrak{g}, M) = \{m \in M |\ [x, m] = 0, \forall x \in \mathfrak{g}\}.$$

The first cohomology group $H^1_{CE}(\mathfrak{g}, M)$ can be seen as the space of outer derivations

$$H^1_{CE}(\mathfrak{g}, M) = \mathrm{OutDer}(\mathfrak{g}, M) := \frac{\mathrm{Der}(\mathfrak{g}, M)}{\mathrm{InnDer}(\mathfrak{g}, M)},$$

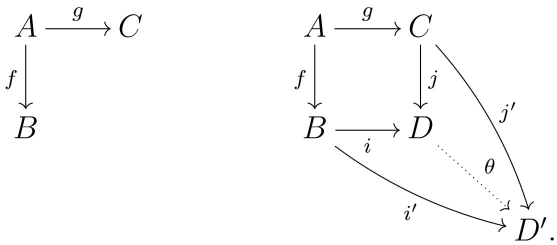
where a derivation is a map $f : \mathfrak{g} \to M$ satisfying $f[x, y] = [x, fy] - [y, fx]$, for all $x, y \in \mathfrak{g}$ and is called inner if $f(x) = [x, m]$, for some $m \in M$.

The second cohomology group $H^2_{CE}(\mathfrak{g}, M)$ can be described as the space of equivalence classes of Lie algebra extensions of $\mathfrak{g}$ by the module M [13].

Pushout of $\mathfrak{g}$-modules

Pushout in a category is an important tool to study some nice properties of the category.

Definition 2.1. Let $\mathcal{C}$ be a category. Given two morphisms $f : A \to B$ and $g : A \to C$, a *pushout* (or *fibered sum*) is a triple (D, i, j) where $D \in \mathrm{Ob}(\mathcal{C})$ and $i : B \to D$ and $j : C \to D$ are morphisms in $\mathcal{C}$ such that $jg = if$ that satisfies the following universal property: for any triple (D', i', j') with $j'g = i'f$, there is a unique morphism $\theta : D \to D'$ making the following diagram commute



Let $\mathfrak{g}$ be a Lie algebra. Consider the category **$\mathfrak{g}$-mod** of $\mathfrak{g}$-modules and $\mathfrak{g}$-module morphisms. Then the pushout of two maps $f : A \to B$ and $g : A \to C$ in **$\mathfrak{g}$-mod** exists.

Let $S = \{(f(a), -g(a)) |\ a \in A\}$. Then it is easy to see that S is a $\mathfrak{g}$-submodule of $B \oplus C$. Take $D = \frac{B \oplus C}{S}$ and define $i : B \to D$ by $i(b) = (b, 0) + S$ and $j : C \to D$ by $j(c) = (0, c) + S$. It is easy to see that $jg = if$. Moreover, if (D', i', j') is another triple with $j'g = i'f$, we define $\theta : D \to D'$ by $\theta((b, c) + S) = i'(b) + j'(c)$. It is also easy to check that θ is unique. Hence the claim.

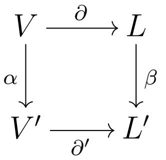
3. Crossed modules of Lie algebras

In this section, we give an interpretation of $H^3_{CE}(\mathfrak{g}, M)$ in terms of crossed module over Lie algebras. Our definition and terminologies are motivated by the one given by Baues and Minian for associative algebras [2].

Definition 3.1. A *crossed module* over a Lie algebra is a triple (V, L, ∂) in which L is a Lie algebra, V is a L-module and $\partial : V \to L$ is a map of L-modules such that $[\partial v, w] = -[\partial w, v]$, for $v, w \in V$.

The above notion of crossed module is equivalent to the usual notion of crossed module [12] in the following way. Note that the vector space V can be equipped with a bracket $[v, w]_V := [\partial v, w]$, i.e. the action of ∂v on w. This bracket is skew-symmetric, satisfies the Jacobi identity and renders ∂ a crossed module of Lie algebras in the usual sense. The other direction is obvious.

Definition 3.2. Let (V, L, ∂) and (V', L', ∂') be two crossed modules. A map between them consists of a linear map $\alpha : V \to V'$ and a Lie algebra map $\beta : L \to L'$ such that the following diagram commute



and satisfying $\alpha[x, v] = [\beta(x), \alpha(v)]$, for $x \in L,\ v \in V$.

Given a crossed module (V, L, ∂), we consider $\mathfrak{g} = \mathrm{coker}(\partial)$ and $M = \ker(\partial)$. The Lie algebra structure on L induces a Lie algebra structure on $\mathfrak{g}$ by $[\pi(x), \pi(y)] = \pi[x, y]$, where $\pi : L \to \mathfrak{g}$ is the projection map. Moreover, the action of L on V induces an action of $\mathfrak{g}$ on M via $[\pi(x), m] = [x, m]$, for $m \in M$.

Hence a crossed module yields an exact sequence

$$0 \to M \xrightarrow{i} V \xrightarrow{\partial} L \xrightarrow{\pi} \mathfrak{g} \to 0.$$

We call (V, L, ∂) a crossed module over the Lie algebra $\mathfrak{g}$ with kernel a $\mathfrak{g}$-module M.

Definition 3.3. Two crossed modules (V, L, ∂) and (V', L', ∂') over $\mathfrak{g}$ with kernel M are said to be *equivalent* if there is a morphism of crossed modules

$$(V, L, \partial) \to (V', L', \partial')$$

which induces identity maps on $\mathfrak{g}$ and M.

Let $\mathrm{Cross}(\mathfrak{g}, M)$ be the category of such crossed modules and morphisms between them. Let $\pi_0\mathrm{Cross}(\mathfrak{g}, M)$ be the class of connected components in the category $\mathrm{Cross}(\mathfrak{g}, M)$. In the next theorem, we see that elements of $\pi_0\mathrm{Cross}(\mathfrak{g}, M)$ are in one-to-one correspondence with the third Chevalley-Eilenberg cohomology group. See also [12].

Theorem 3.4. *There is a bijection* $\psi : \pi_0\mathrm{Cross}(\mathfrak{g}, M) \to H^3_{CE}(\mathfrak{g}, M)$.

Proof. Let $\mathcal{E} : 0 \to M \xrightarrow{i} V \xrightarrow{\partial} L \xrightarrow{\pi} \mathfrak{g} \to 0$ be an element in $\pi_0\mathrm{Cross}(\mathfrak{g}, M)$. Choose $\mathbb{K}$-linear sections $s : \mathfrak{g} \to L$ with $\pi s = \mathrm{id}$ and $q : \mathrm{Im}(\partial) \to V$ with $\partial q = \mathrm{id}$. For any $x, y \in \mathfrak{g}$, we have

$$\pi([s(x), s(y)] - s[x, y]) = 0.$$

This shows that $[s(x), s(y)] - s[x, y]$ is in $\ker(\pi) = \mathrm{Im}(\partial)$. Take

$$g(x, y) = q([s(x), s(y)] - s[x, y]) \in V.$$

Define a map $\theta_{\mathcal{E},s,q} : \mathfrak{g}^{\otimes 3} \to M$ by

$$\begin{aligned}\theta_{\mathcal{E},s,q}(x, y, z) = {} & [s(x), g(y, z)] - [s(y), g(x, z)] + [s(z), g(x, y)] \\ & - g([x, y], z) + g([x, z], y) - g([y, z], x).\end{aligned}$$

Since ∂ is a map of L-modules, it follows that $\partial(\theta_{\mathcal{E},s,q}(x, y, z)) = 0$. Therefore, $\theta_{\mathcal{E},s,q}(x, y, z) \in \ker(\partial) = M$. The map $\theta_{\mathcal{E},s,q} : \mathfrak{g}^{\otimes 3} \to M$ is skew-symmetric in x, y, z. Hence $\theta_{\mathcal{E},s,q} : \wedge^3\mathfrak{g} \to M$. The map $\theta_{\mathcal{E},s,q}$ defines a 3-cocycle in the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in M.

We first show that the class of $\theta_{\mathcal{E},s,q}$ in $H^3_{CE}(\mathfrak{g}, M)$ does not depend on the section s. Suppose $\overline{s} : \mathfrak{g} \to L$ is another section of π and let $\theta_{\mathcal{E},\overline{s},q}$ be the corresponding 3-cocycle defined by using $\overline{s}$ instead of s. Then there exists a linear map $h : \mathfrak{g} \to V$ with $s - \overline{s} = \partial h$. Observe that

$$\begin{aligned}[s(x), g(y, z)] - [\overline{s}(x), \overline{g}(y, z)] &= [s(x) - \overline{s}(x), g(y, z)] + [\overline{s}(x), (g - \overline{g})(y, z)] \\ &= [\partial h(x), q([s(y), s(z)] - s[y, z])] + [\overline{s}(x), (g - \overline{g})(y, z)] \\ &= -[([s(y), s(z)] - s[y, z]), h(x)] + [\overline{s}(x), (g - \overline{g})(y, z)].\end{aligned}$$

Therefore we have

$$
\begin{aligned}
&(\theta_{\mathcal{E},s,q} - \theta_{\mathcal{E},\overline{s},q})(x,y,z) \\
&= -[([s(y), s(z)] - s[y,z]), h(x)] + [([s(x), s(z)] - s[x,z]), h(y)] \\
&\quad - [([s(x), s(y)] - s[x,y]), h(z)] + [\overline{s}(x), (g-\overline{g})(y,z)] \\
&\quad - [\overline{s}(y), (g-\overline{g})(x,z)] + [\overline{s}(z), (g-\overline{g})(x,y)] \\
&\quad - (g-\overline{g})([x,y],z) + (g-\overline{g})([x,z],y) - (g-\overline{g})([y,z],x). \qquad (1)
\end{aligned}
$$

Define a map $b : \wedge^2\mathfrak{g} \to V$ by

$$b(x,y) = [s(x), h(y)] - h([x,y]) - [s(y), h(x)] - [\partial h(x), h(y)].$$

Then a easy calculation shows that $\partial b = \partial(g - \overline{g})$. Hence $(g - \overline{g} - b) : \wedge^2\mathfrak{g} \to M$. It follows from (1) that

$$
\begin{aligned}
&(\theta_{\mathcal{E},s,q} - \theta_{\mathcal{E},\overline{s},q})(x,y,z) \\
&= -[([s(y), s(z)] - s[y,z]), h(x)] + [([s(x), s(z)] - s[x,z]), h(y)] \\
&\quad - [([s(x), s(y)] - s[x,y]), h(z)] + (\delta(g-\overline{g}-b))(x,y,z) \\
&\quad + [\overline{s}(x), b(y,z)] - [\overline{s}(y), b(x,z)] + [\overline{s}(z), b(x,y)] \\
&\quad - b([x,y],z) + b([x,z],y) - b([y,z],x).
\end{aligned}
$$

In the right hand side of the above equation, we substitute the definition of b in last six terms. After many cancellations on the right hand side (one has to use the fact that (V, L, ∂) is a crossed module for some cancellations), we are only left with the term $(\delta(g - \overline{g} - b))(x,y,z)$. Hence the class of $\theta_{\mathcal{E},s,q}$ does not depend on the section s.

Next consider a map

$$
\begin{array}{lccccccccccc}
\mathcal{E} := & 0 & \longrightarrow & M & \xrightarrow{i} & V & \xrightarrow{\partial} & L & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\
\mathcal{E}' := & 0 & \longrightarrow & M & \xrightarrow[i']{} & V' & \xrightarrow[\partial']{} & L' & \xrightarrow[\pi']{} & \mathfrak{g} & \longrightarrow & 0
\end{array}
$$

of crossed modules. Let $s' : \mathfrak{g} \to L'$ and $q' : \mathrm{Im}(\partial') \to V'$ be sections of π' and ∂', respectively. Note that $(\pi'\beta s)(x) = (\pi s)(x) = x$, for all $x \in \mathfrak{g}$. Therefore, $\beta s : \mathfrak{g} \to L'$ is another section of π'. Thus, we have

$$
\begin{aligned}
(\theta_{\mathcal{E},s,q} - \theta_{\mathcal{E}',\beta s,q'})(x,y,z) &= [s(x), g(y,z)] - [s(y), g(x,z)] + [s(z), g(x,y)] \\
&= -g([x,y],z) + g([x,z],y) - g([y,z],x) \\
&\quad - [\beta s(x), g'(y,z)] + [\beta s(y), g'(x,z)] - [\beta s(z), g'(x,y)] \\
&= +g'([x,y],z) - g'([x,z],y) - g'([y,z],x),
\end{aligned}
$$

where $g'(x,y) = q'([\beta s(x), \beta s(y)] - \beta s[x,y])$. Here we have used the same notation $[\ ,\ ]$ to denote the action of L on V and the action on L' on V'.

Hence, we have

$$\begin{aligned}(\theta_{\mathcal{E},s,q}-\theta_{\mathcal{E}',\beta s,q'})(x,y,z) &= [\beta s(x),(\alpha q-q'\beta)([s(y),s(z)]-s[y,z])]\\ &\quad -[\beta s(y),(\alpha q-q'\beta)([s(x),s(z)]-s[x,z])]\\ &\quad +[\beta s(z),(\alpha q-q'\beta)([s(x),s(y)]-s[x,y])]\\ &\quad -(\alpha q-q'\beta)\big([s[x,y],s(z)]-s[[x,y],z]\big)\\ &\quad +(\alpha q-q'\beta)\big([s[x,z],s(y)]-s[[x,z],y]\big)\\ &\quad -(\alpha q-q'\beta)\big([s[y,z],s(x)]-s[[y,z],x]\big).\end{aligned}$$

It follows from the above expression that $(\theta_{\mathcal{E},s,q}-\theta_{\mathcal{E}',\beta s,q'})(x,y,z)=(\delta\phi)(x,y,z)$ for some $\phi:\wedge^2\mathfrak{g}\to M$. Hence, $[\theta_{\mathcal{E},s,q}]=[\theta_{\mathcal{E}',\beta s,q'}]$ in $H^3_{CE}(\mathfrak{g},M)$. Moreover, from the first part, we have $[\theta_{\mathcal{E}',\beta s,q'}]=[\theta_{\mathcal{E}',s',q'}]$. Hence the class $[\theta_{\mathcal{E},s,q}]$ does not depend on the sections s and q. We denote the corresponding class by $[\theta_{\mathcal{E}}]$. Therefore, the map

$$\psi:\pi_0\mathrm{Cross}(\mathfrak{g},M)\to H^3_{CE}(\mathfrak{g},M),\ \mathcal{E}\to[\theta_{\mathcal{E}}]$$

is well-defined. The injectivity of ψ follows similarly to the case of Shukla cohomology [4]. See also [12]. The surjectivity of ψ follows from the next observation. ■

Let $\mathfrak{g}$ be a Lie algebra and $0\to M\to M'\xrightarrow{\pi}M''\to 0$ an exact sequence of $\mathfrak{g}$-modules. Given an abelian extension

$$0\to M''\to\mathfrak{e}\to\mathfrak{g}\to 0 \tag{2}$$

of $\mathfrak{g}$ by M'', we consider the Yoneda product

$$0\to M\to M'\xrightarrow{\mu}\mathfrak{e}\to\mathfrak{g}\to 0. \tag{3}$$

Writing $\mathfrak{e}=M''\oplus\mathfrak{g}$ as a vector space, we have $\mu(m')=(\pi(m'),0)$. An $\mathfrak{e}$-action on M' is induced from the $\mathfrak{g}$-action on M', namely,

$$[(m'',x),m']=[x,m'],\qquad \text{for } (m'',x)\in\mathfrak{e} \text{ and } m'\in M'.$$

It is easy to see that (3) defines a crossed module of $\mathfrak{g}$ by M. It has been shown in [12] that the corresponding third cohomology class in $H^3_{CE}(\mathfrak{g},M)$ as constructed above is the image of the second cohomology class in $H^2_{CE}(\mathfrak{g},M'')$ (defined by the abelian extension (2)) under the connecting homomorphism ∂ in the cohomology long exact sequence

$$\cdots\to H^2_{CE}(\mathfrak{g},M)\to H^2_{CE}(\mathfrak{g},M')\to H^2_{CE}(\mathfrak{g},M'')\xrightarrow{\partial}H^3_{CE}(\mathfrak{g},M)\to\cdots.$$

Observation. *Surjectivity of the map ψ (of Theorem 3.4).* The surjectivity of ψ can be shown as follows [12]. As the category $\mathfrak{g}$-mod possesses enough injectives, we can choose an injective $\mathfrak{g}$-module I and a monomorphism $i:M\to I$. Consider the short exact sequence of $\mathfrak{g}$-modules

$$0\to M\xrightarrow{i}I\to M''\to 0 \tag{4}$$

where M'' is the cokernel of the map i. Since I is injective, the cohomology long exact sequence yields $H^2_{CE}(\mathfrak{g},M'')\cong H^3_{CE}(\mathfrak{g},M)$. This isomorphism is given by the connecting homomorphism in the long exact sequence of cohomology.

Thus, a cohomology class $[\gamma] \in H^3_{CE}(\mathfrak{g}, M)$ corresponds to a class $[\alpha] \in H^2_{CE}(\mathfrak{g}, M'')$. We now consider an abelian extension $0 \to M'' \to \mathfrak{e} \to \mathfrak{g} \to 0$ corresponding to the cohomology class $[\alpha] \in H^2_{CE}(\mathfrak{g}, M'')$. It follows from the above discussion that the Yoneda product of (4) and the above abelian extension gives rise to a crossed module whose associated third cohomology class is given by $[\gamma]$.

4. Crossed n-fold extensions of Lie algebras

Using the notion of crossed modules of the previous section, we introduce crossed n-fold extensions of a Lie algebra $\mathfrak{g}$ by a $\mathfrak{g}$-module M. Finally, we show that such extensions represent cohomology classes in $H^{n+1}_{CE}(\mathfrak{g}, M)$.

Let $\mathfrak{g}$ be a Lie algebra and M be a $\mathfrak{g}$-module. Let $n \geq 2$.

Definition 4.1. A crossed n-fold extension of $\mathfrak{g}$ by M is an exact sequence

$$0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \cdot \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0$$

of $\mathbb{K}$-vector spaces with the following properties:

- (M_1, L, ∂_1) is a crossed module with $\operatorname{coker}(\partial_1) = \mathfrak{g}$,
- for $1 < i \leq n-1$, M_i is a $\mathfrak{g}$-module and ∂_i, f are morphisms of $\mathfrak{g}$-modules.

Definition 4.2. Let

$$\mathcal{E} := \big(0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \cdot \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0\big)$$

and

$$\mathcal{E}' := \big(0 \to M' \xrightarrow{f'} M'_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \cdot \xrightarrow{\partial'_2} M'_1 \xrightarrow{\partial'_1} L' \xrightarrow{\pi'} \mathfrak{g} \to 0\big)$$

be two crossed n-fold extensions of $\mathfrak{g}$ by M and M', respectively. A morphism between them consists of maps $\alpha : M \to M'$, $\beta : L \to L'$ and $\delta_i : M_i \to M'_i$ for $1 \leq i \leq n-1$ such that

- all squares commute (see Figure 4.1),
- α, δ_i $(2 \leq i \leq n-1)$ are morphism of $\mathfrak{g}$-modules,
- $(\delta_1, \beta) : (M_1, L, \partial_1) \to (M'_1, L', \partial'_1)$ is a map of crossed modules inducing the identity map on $\mathfrak{g}$.

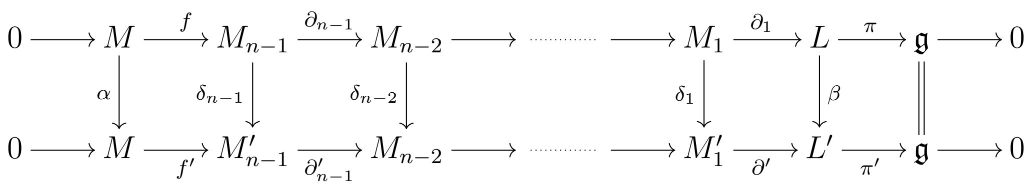


Figure 4.1

Let $\mathcal{E}^n(\mathfrak{g}, M)$ be the category whose objects are n-fold extensions of $\mathfrak{g}$ by M and the morphisms are the maps between such n-fold extensions that induce the identity map on M.

We take $\mathrm{Opext}^n(\mathfrak{g}, M) = \pi_0\mathcal{E}^n(\mathfrak{g}, M)$ the class of connected components in the category $\mathcal{E}^n(\mathfrak{g}, M)$. Then it follows that $\mathrm{Opext}^2(\mathfrak{g}, M) = \pi_0\mathrm{Cross}(\mathfrak{g}, M)$. Next we show that $\mathrm{Opext}^n(\mathfrak{g}, M)$ has a natural abelian group structure. To do that, we start with the following proposition.

Proposition 4.3. *Let $\mathcal{E} \in \mathrm{Opext}^n(\mathfrak{g}, M)$ be an n-fold extension of $\mathfrak{g}$ by M and $\alpha : M \to M'$ be an $\mathfrak{g}$-module map. Then there exists an n-fold extension $\alpha\mathcal{E} \in \mathrm{Opext}^n(\mathfrak{g}, M')$ and a morphism of the form $(\alpha, \delta_{n-1}, \ldots, \delta_1, \beta)$ from $\mathcal{E}$ to $\alpha\mathcal{E}$. Moreover, $\alpha\mathcal{E} \in \mathrm{Opext}^n(\mathfrak{g}, M')$ is the unique n-fold extension with this property.*

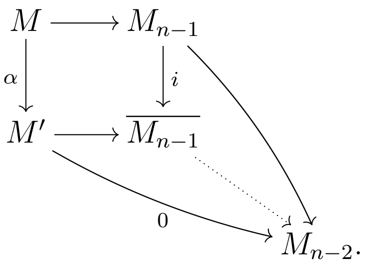
Proof. The proof is based on pushout of $\mathfrak{g}$-modules. Let

$$\mathcal{E} := (0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0) \in \mathrm{Opext}^n(\mathfrak{g}, M)$$

be a crossed n-fold extension. We consider the n-fold extension $\alpha\mathcal{E} \in \mathrm{Opext}^n(\mathfrak{g}, M)$ as

$$\alpha\mathcal{E} := (0 \to M' \to \overline{M_{n-1}} \to M_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0)$$

where $M' \to \overline{M_{n-1}} \to M_{n-2}$ is obtained from the following pushout of $\mathfrak{g}$-modules



Moreover $(\alpha, i, \mathrm{id}, \ldots, \mathrm{id})$ defines a morphism from $\mathcal{E}$ to $\alpha\mathcal{E}$.

Finally, let $\mathcal{E}' \in \mathrm{Opext}^n(\mathfrak{g}, M')$ be an n-fold extension and there is a morphism $(\alpha, \delta_{n-1}, \ldots, \delta_1, \beta) : \mathcal{E} \to \mathcal{E}'$. Then by properties of pushout, one obtains a map $(1, j, \delta_{n-2}, \ldots, \delta_1, \beta) : \alpha\mathcal{E} \to \mathcal{E}'$. This shows that the class of $\alpha\mathcal{E}$ and $\mathcal{E}'$ is same in $\mathrm{Opext}^n(\mathfrak{g}, M)$. ∎

Thus, it follows from the above proposition that a $\mathfrak{g}$-module map $\alpha : M \to M'$ induces a well-defined map

$$\alpha_* : \mathrm{Opext}^n(\mathfrak{g}, M) \to \mathrm{Opext}^n(\mathfrak{g}, M'),\ \mathcal{E} \mapsto \alpha\mathcal{E}.$$

Definition 4.4.

Let $\quad \mathcal{E} = \big(0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \cdot \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0\big) \in \mathrm{Opext}^n(\mathfrak{g}, M)$

and $\quad \mathcal{E}' = \big(0 \to M' \xrightarrow{f'} M'_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \cdot \xrightarrow{\partial'_2} M'_1 \xrightarrow{\partial'_1} L' \xrightarrow{\pi'} \mathfrak{g} \to 0\big) \in \mathrm{Opext}^n(\mathfrak{g}, M)$

be two crossed n-fold extensions of $\mathfrak{g}$ by M and M', respectively. The sum of $\mathcal{E}$ and $\mathcal{E}'$ over $\mathfrak{g}$ is denoted by $\mathcal{E} \oplus_{\mathfrak{g}} \mathcal{E}'$ and is given by the following n-fold extension

$$0 \to M \oplus M' \to M_{n-1} \oplus M'_{n-1} \to \cdots \cdot \to M_1 \oplus M'_1 \xrightarrow{(\partial_1, \partial'_1)} L \times_{\mathfrak{g}} L' \xrightarrow{q} \mathfrak{g} \to 0.$$

Here the Lie algebra structure on $L \times_{\mathfrak{g}} L'$ is given by

$$[(x, x'), (y, y')] = ([x, y], [x', y']), \qquad \text{for } (x, x'), (y, y') \in L \times_{\mathfrak{g}} L'.$$

The projection $q : L \times_{\mathfrak{g}} L' \to \mathfrak{g}$ is the obvious one. The action of the Lie algebra $L \times_{\mathfrak{g}} L'$ on $M_1 \oplus M_1'$ is defined coordinatewise. It is easy to check that

$$(M_1 \oplus M_1', L \times_{\mathfrak{g}} L', (\partial_1, \partial_1'))$$

defines a crossed module.

Definition 4.5. (Baer sum) Let $\mathcal{E}, \mathcal{E}' \in \mathrm{Opext}^n(\mathfrak{g}, M)$ with $n \geq 3$. Then the *Baer sum* $\mathcal{E} + \mathcal{E}' \in \mathrm{Opext}^n(\mathfrak{g}, M)$ is defined by

$$\mathcal{E} + \mathcal{E}' = \nabla_M(\mathcal{E} \oplus_{\mathfrak{g}} \mathcal{E}')$$

where $\nabla_M : M \oplus M \to M,\ (m_1, m_2) \mapsto m_1 + m_2$ is the codiagonal map.

Definition 4.6. (Zero extension) Let $n \geq 3$. Then

$$0 \to M = M \to \underbrace{0 \to \cdots \to 0}_{n-2} \to \mathfrak{g} \to \mathfrak{g} \to 0$$

is a crossed n-fold extension of $\mathfrak{g}$ by M. We define $0 \in \mathrm{Opext}^n(\mathfrak{g}, M)$ to be the class of this extension.

Remark 4.7. Let

$$\mathcal{E} := (0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0) \in \mathrm{Opext}^n(\mathfrak{g}, M)$$

be a crossed n-fold extension, for $n \geq 3$. Suppose there is a map $g : M_{n-1} \to M$ of $\mathfrak{g}$-modules satisfying $gf = \mathrm{id}_M$. Then there is a morphism

$$\begin{array}{ccccccccccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & M_{n-1} & \xrightarrow{\partial_{n-1}} & M_{n-2} & \xrightarrow{\partial_{n-2}} & \cdots & \xrightarrow{\partial_2} & M_1 & \xrightarrow{\partial_1} & L & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \pi} & & \| & & \\
0 & \longrightarrow & M & = & M & \xrightarrow[0]{} & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \xrightarrow[0]{} & \mathfrak{g} & = & \mathfrak{g} & \longrightarrow & 0
\end{array}$$

of crossed n-fold extensions. This shows that the class of $\mathcal{E}$ defines $0 \in \mathrm{Opext}^n(\mathfrak{g}, M)$. Let M be an injective $\mathfrak{g}$-module and $\mathcal{E} \in \mathrm{Opext}^n(\mathfrak{g}, M)$ be any crossed n-fold extension defined above. Consider the inclusion $\mathfrak{g}$-module map $M \xrightarrow{f} M_{n-1}$ and the identity map $\mathrm{id}_M : M \to M$. Since M is an injective $\mathfrak{g}$-module, there exists a $\mathfrak{g}$-module map $g : M_{n-1} \to M$ such that $gf = \mathrm{id}_M$. Therefore, it follows from the previous paragraph that $\mathcal{E}$ defines $0 \in \mathrm{Opext}^n(\mathfrak{g}, M)$. In other words, we get $\mathrm{Opext}^n(\mathfrak{g}, M) = 0$, for $n \geq 3$.

Remark 4.8. Let

$$\mathcal{E} := (0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0) \in \mathrm{Opext}^n(\mathfrak{g}, M)$$

be a crossed n-fold extension, for $n \geq 3$. It follows from Proposition 4.3 that $f\mathcal{E} \in \mathrm{Opext}^n(\mathfrak{g}, M_{n-1})$ and is given by the bottom row of the following diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & M_{n-1} & \xrightarrow{\partial_{n-1}} & M_{n-2} & \xrightarrow{\partial_{n-2}} & \cdots & \xrightarrow{\partial_1} & L & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (\mathrm{id}, \partial_{n-1})} & & \| & & & & \| & & \| & & \\
0 & \longrightarrow & M_{n-1} & \xrightarrow[(\mathrm{id}, 0)]{} & M_{n-1} \oplus M_{n-2} & \xrightarrow[pr_2]{} & M_{n-2} & \longrightarrow & \cdots & \xrightarrow[\partial_1]{} & L & \xrightarrow[\pi]{} & \mathfrak{g} & \longrightarrow & 0.
\end{array}$$

Observe that, in $f\mathcal{E}$, there is a map $g = pr_1 : M_{n-1} \oplus M_{n-2} \to M_{n-1}$ which satisfies $g \circ (\mathrm{id}, 0) = \mathrm{id}_{M_{n-1}}$. Hence, it follows from Remark 4.7 that $f\mathcal{E} = 0$ in $\mathrm{Opext}^n(\mathfrak{g}, M_{n-1})$.

Definition 4.9. (Inverse of an extension) Let

$$\mathcal{E} := \big(0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0\big) \in \mathrm{Opext}^n(\mathfrak{g}, M)$$

be an extension. Then

$$-\mathcal{E} := \big(0 \to M \xrightarrow{-f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0\big) \in \mathrm{Opext}^n(\mathfrak{g}, M)$$

defines a new crossed n-extension of $\mathfrak{g}$ by M.

The proof of the following theorem is straightforward.

Theorem 4.10. *Let $n \geq 3$. Then the set $\mathrm{Opext}^n(\mathfrak{g}, M)$ eqipped with the Baer sum is an abelian group. The zero element and inverse elements of this group are given by Definitions 4.6 and 4.6, respectively.*
Moreover, if $\alpha : M \to M'$ is a morphism of $\mathfrak{g}$-modules, the map

$$\alpha_* : \mathrm{Opext}^n(\mathfrak{g}, M) \to \mathrm{Opext}^n(\mathfrak{g}, M')$$

is a morphism of groups.

Remark 4.11. The set $\mathrm{Opext}^2(\mathfrak{g}, M)$ also inherits the structure of an abelian group. We denote $0 \in \mathrm{Opext}^2(\mathfrak{g}, M)$ to be the class of the crossed module

$$0 \to M = M \xrightarrow{0} \mathfrak{g} = \mathfrak{g} \to 0.$$

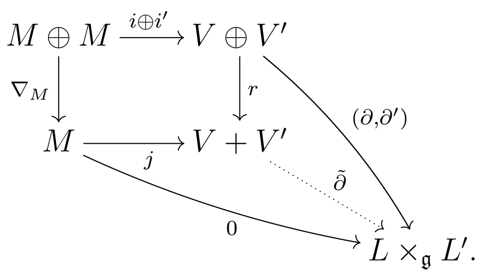
The addition in $\mathrm{Opext}^2(\mathfrak{g}, M)$ is given as follows. Let

$$\mathcal{E} := (0 \to M \xrightarrow{i} V \xrightarrow{\partial} L \xrightarrow{\pi} \mathfrak{g} \to 0)$$

and $\mathcal{E}' := (0 \to M \xrightarrow{i'} V' \xrightarrow{\partial'} L' \xrightarrow{\pi'} \mathfrak{g} \to 0)$ be two crossed modules in $\mathrm{Opext}^2(\mathfrak{g}, M)$. Then the Baer sum $\mathcal{E} + \mathcal{E}'$ is the class of the extension

$$0 \to M \xrightarrow{j} V + V' \xrightarrow{\tilde{\partial}} L \times_{\mathfrak{g}} L' \xrightarrow{q} \mathfrak{g} \to 0.$$

Here $V + V'$ is the pushout of $\mathbb{K}$-vector spaces



The $(L \times_{\mathfrak{g}} L')$-module structure on $V + V'$ is induced from the module structure on $V \oplus V'$ via the quotient map $r : V \oplus V' \to V + V'$. Namely, the action is given by $[(x, x'), r(v, v')] = r([x, v], [x', v'])$, for $(x, x') \in L \times_{\mathfrak{g}} L'$ and $(v, v') \in V \oplus V'$. With this module structure on $V + V'$, it is easy to show that $(V + V', L \times_{\mathfrak{g}} L', \tilde{\partial})$ is a crossed module.

Next we associate a long exact sequence to any short exact sequence of $\mathfrak{g}$-modules. This will help us to prove our main theorem. Let

$$0 \to M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \to 0$$

be a short exact sequence of $\mathfrak{g}$-modules. For any $n \geq 2$, we define a homomorphism

$$\delta : \mathrm{Opext}^n(\mathfrak{g}, M'') \to \mathrm{Opext}^{n+1}(\mathfrak{g}, M)$$

as follows. Given an extension

$$\mathcal{E} = \big(0 \to M'' \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \cdot \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0\big) \in \mathrm{Opext}^n(\mathfrak{g}, M''),$$

we define $\delta\mathcal{E} \in \mathrm{Opext}^{n+1}(\mathfrak{g}, M)$ to be the class of the extension

$$0 \to M \xrightarrow{\alpha} M' \xrightarrow{f\beta} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \cdot \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0.$$

Then δ is a well-defined homomorphism for $n \geq 2$.
The proof of the following theorem is similar to the proof of Baues and Minian for associative algebra case [2], hence we omit the details.

Theorem 4.12. *A short exact sequence*

$$0 \to M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \to 0$$

of $\mathfrak{g}$-modules induces a long exact sequence of abelian groups ($n \geq 2$)

$$\mathrm{Opext}^n(\mathfrak{g}, M) \xrightarrow{\alpha_*} \mathrm{Opext}^n(\mathfrak{g}, M') \xrightarrow{\beta_*} \mathrm{Opext}^n(\mathfrak{g}, M'') \xrightarrow{\delta} \mathrm{Opext}^{n+1}(\mathfrak{g}, M) \to \cdots.$$

Using the above result, we can now prove our main theorem.

Theorem 4.13. *For any $n \geq 2$, there exists an isomorphism of abelian groups*

$$\mathrm{Opext}^n(\mathfrak{g}, M) \cong H^{n+1}_{CE}(\mathfrak{g}, M).$$

Proof. The result is true for $n = 2$ (see Theorem 3.4). Let $n \geq 3$. We consider a short exact sequence

$$0 \to M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \to 0$$

of $\mathfrak{g}$-modules with M' injective. Then by Remark 4.7 and Theorem 4.12, it follows that

$$\mathrm{Opext}^n(\mathfrak{g}, M'') \cong \mathrm{Opext}^{n+1}(\mathfrak{g}, M).$$

Moreover, it follows from the cohomology long exact exact sequence that

$$H^{n+1}_{CE}(\mathfrak{g}, M'') \cong H^{n+2}_{CE}(\mathfrak{g}, M).$$

Therefore,

$$\begin{aligned} \mathrm{Opext}^3(\mathfrak{g}, M) \cong\ \mathrm{Opext}^2(\mathfrak{g}, M'') &\cong H^3_{CE}(\mathfrak{g}, M'') \qquad \text{(by Theorem 3.4)} \\ &\cong H^4_{CE}(\mathfrak{g}, M). \end{aligned}$$

We conclude the result by using the induction on n. ∎

Remark 4.14. Lie-Rinehart algebras are algebraic analog of Lie algebroids and closely related to Lie algebras. These algebras pay special attention due to its connection in Poisson geometry [8]. In [5] the authors studied crossed modules for Lie-Rinehart algebras and classify them by the third cohomology of Lie-Rinehart algebras. However, their crossed modules are similar to the traditional one for Lie algebras. It would be interesting to study crossed extensions of Lie-Rinehart algebras and their classification in terms of higher cohomologies of Lie-Rinehart algebras.

5. The case of a Leibniz algebra

The notion of Leibniz algebra was introduced by Loday in connection with cyclic homology and Hochschild homology of matrix algebras [10]. The cohomology theory of Leibniz algebras was introduced by the same author. In this section, we outline that Leibniz cohomology groups are also represented by crossed extensions of Leibniz algebras.

A (right) Leibniz algebra is a $\mathbb{K}$-vector space $\mathfrak{h}$ together with a $\mathbb{K}$-bilinear map $[~,~] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ satisfying

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \text{for all } x, y, z \in \mathfrak{h}.$$

Let $(\mathfrak{h}, [~,~])$ be a Leibniz algebra. An $\mathfrak{h}$-module is a vector space M together with two bilinear maps (both of them denoted by $[~,~]$) $\mathfrak{h} \times M \to M$ and $M \times \mathfrak{h} \to M$ satisfying

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

whenever one of the variable is from M and the others two are from $\mathfrak{h}$.

The cohomology of the Leibniz algebra $\mathfrak{h}$ with coefficients in M is the cohomology of the complex $(C^n_{Leib}(\mathfrak{h}, M), \delta)_{n \geq 0}$ where $C^0_{Leib}(\mathfrak{h}, M) = M$ and $C^n_{Leib}(\mathfrak{h}, M) = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{h}^{\otimes n}, M)$, for $n \geq 1$. The differential δ is given by $(\delta m)(x) = [x, m]$, for $m \in M$, $x \in \mathfrak{h}$ and

$$\begin{aligned}(\delta f)(x_1, \ldots, x_{n+1}) = [x_1, f(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}), x_i] \\ + \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_n).\end{aligned}$$

See [10] for more details. Similar to the case of Chevalley-Eilenberg cohomology, the lower degree Leibniz cohomology has the following interpretations. The zero-th cohomology group $H^0_{Leib}(\mathfrak{h}, M)$ is the submodule of invariants on M and the first cohomology group

$$H^1_{Leib}(\mathfrak{h}, M) = \frac{\{f : \mathfrak{h} \to M |~ f[x, y] = [x, f(y)] + [f(x), y], \forall x, y \in \mathfrak{h}\}}{\{\delta m |~ m \in M\}}$$

is the space of outer derivations. The second cohomology group $H^2_{Leib}(\mathfrak{h}, M)$ classifies the equivalence classes of extensions of the Leibniz algebra $\mathfrak{h}$ by M.

Crossed modules over a Leibniz algebra can be defined similarly. One only needs to care about the non-skew-symmetric version of the identities used in the case of

Lie algebra. A crossed module over a Leibniz algebra is a triple (V, L, ∂) in which L is a Leibniz algebra, V is a L-module and $\partial : V \to L$ is a map of L-modules satisfying

$$[\partial v, w] = [v, \partial w], \qquad \text{for all } v, w \in L.$$

In a similar way, a crossed module yields a Leibniz algebra structure on $\mathfrak{h} = \operatorname{coker}(\partial)$ and there is an exact sequence

$$0 \to M \to V \xrightarrow{\partial} L \xrightarrow{\pi} \mathfrak{h} \to 0$$

where $M = \ker(\partial)$. Moreover, there is a Leibniz algebra action of $\mathfrak{h}$ on M. A morphism of crossed modules over Leibniz algebras can be defined in a similar way. Let $\operatorname{Cross}(\mathfrak{h}, M)$ be the category of such crossed extensions and morphisms between them. In this case, one can also prove that elements of $\pi_0\operatorname{Cross}(\mathfrak{h}, M)$ are in one-to-one correspondence with $H^3_{Leib}(\mathfrak{h}, M)$. The proof is similar to the Lie algebra case (Theorem 3.4). However, in this case, the Leibniz 3-cocycle $\theta_{\mathcal{E},s,q} : \mathfrak{h}^{\otimes 3} \to M$ is given by

$$\begin{aligned}\theta_{\mathcal{E},s,q}(x, y, z) = {} & [s(x), g(y, z)] + [g(x, z), s(y)] - [g(x, y), s(z)] \\ & - g([x, y], z) + g([x, z], y) + g(x, [y, z]).\end{aligned}$$

Associated to any third cohomology class in $H^3_{Leib}(\mathfrak{h}, M)$ the existence of the corresponding crossed module can be shown by the way that have been described in Theorem 3.4.

Moreover, one can define crossed n-fold extensions of a Leibniz algebra $\mathfrak{h}$ by a module M. We denote the category of crossed n-fold extensions and morphisms between them by $\mathcal{E}^n(\mathfrak{h}, M)$. Moreover, we have the following theorem for Leibniz algebras.

Theorem 5.1. *The set* $\operatorname{Opext}^n(\mathfrak{h}, M)$ *of connected components in the category* $\mathcal{E}^n(\mathfrak{h}, M)$ *forms an abelian group, for* $n \geq 2$. *Moreover, there exists an isomorphism of abelian groups* $\operatorname{Opext}^n(\mathfrak{h}, M) \cong H^{n+1}_{Leib}(\mathfrak{h}, M)$.

The proof is similar to the lines of Section 4. We only need to verify that the Leibniz cohomology vanishes on injective modules. Let $\mathfrak{h}$ be a Leibniz algebra. An $\mathfrak{h}$-module I is said to be injective if for any $\mathfrak{h}$-module M, any submodule $N \subset M$ and any morphism of $\mathfrak{h}$-modules $N \to I$, there exists an extension to an $\mathfrak{h}$-module morphism $M \to I$.

In [11] Loday-Pirashvili defined the universal enveloping algebra $UL(\mathfrak{h})$ of a Leibniz algebra $\mathfrak{h}$ and show that the Leibniz cohomology $H^*_{Leib}(\mathfrak{h}, M)$ can be expressed as

$$H^*_{Leib}(\mathfrak{h}, M) \cong \operatorname{Ext}^*_{UL(\mathfrak{h})}(U(\mathfrak{h}_{\text{Lie}}), M), \tag{5}$$

where $\mathfrak{h}_{\text{Lie}}$ is the Lie algebra obtained from $\mathfrak{h}$ quotienting by the relation $[x, x] = 0$, for all $x \in \mathfrak{h}$. If I is an injective $\mathfrak{h}$-module, one can take the injective resolution which is I in degree zero and 0 in higher degrees. Therefore, as a consequence of (5) we get that $H^i_{Leib}(\mathfrak{h}, I) = 0$, for $i > 0$.

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Apurba Das, Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Uttar Pradesh, India; apurbadas348@gmail.com.

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