Journal of Lie Theory
Volume **32** (2022) 327–382

On Extensions, Lie-Poisson Systems, and Dissipation

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Communicated by P. J. Olver

Abstract. Lie-Poisson systems on the dual spaces of *unified products* are studied. Having been equipped with a twisted 2-cocycle term, the *extending structure* framework allows not only to study the dynamics on 2-cocycle extensions, but also to (de)couple mutually interacting Lie-Poisson systems. On the other hand, symmetric brackets; such as the double bracket, the Cartan-Killing bracket, the Casimir dissipation bracket, and the Hamilton dissipation bracket are worked out in detail. Accordingly, the collective motion of two mutually interacting irreversible dynamics, as well as the mutually interacting metriplectic flows, are obtained. The theoretical results are illustrated in three examples. As an infinite-dimensional physical model, decompositions of the BBGKY hierarchy are presented. As for the finite-dimensional examples, the coupling of two Heisenberg algebras, and the coupling of two copies of $3D$ dynamics are studied.

Mathematics Subject Classification: 53D17, 37J37.

Key Words: Lie-Poisson equation, metriplectic system, unified product.

1. Introduction

It is a well known fact that two mutually interacting dynamical/mechanical systems, when coupled, cannot preserve their individual motions in the collective system. This is manifested in the equation of motion of the collective system as the existence of additional terms, to those belonging to the individual subsystems. It is the Hamiltonian realization of this collective motion, of two mutually interacting physical systems, that we study in the present paper. In order to be able to present the mutual interactions as Lie group / Lie algebra actions, we shall consider the systems whose configuration spaces are Lie groups, [5, 7, 77, 85].

If the configuration space of a physical system admits a Lie group structure, then the reduced Hamiltonian dynamics can be achieved on the dual space of the Lie algebra [1, 4, 53, 62] since it carries a natural Poisson structure, called the Lie-Poisson bracket. Many physical systems fit into this geometry; such as the rigid body, fluid and plasma theories [40, 42, 76].

The early examples of coupled systems were motivated by coupling two different characters of a physical system; such as the fluid motion under the EM field [60] or the rigid body motion under the gravity [84]. Such systems have been first studied, in literature, from the point of view of the semidirect product theory, which has been successfully established both in Lagrangian dynamics, see for example [13, 14, 15, 41, 59], and Hamiltonian dynamics, see for example [64, 65], as well

ISSN 0949–5932 / \$2.50

as [59, 91]. The semidirect product theory, later, upgraded in [90] to a theory that accommodates 2-cocycle extensions, though, within the framework of Leibniz algebras.

From the extension point of view, both semi-direct products and 2-cocycle extensions fall into the category of extensions that allows only one way of interaction between the individual spaces that give rise to the extension. Therefore, as far as the (de)coupling of physical systems are concerned, such underlying algebraic structures can support only the examples that involve two systems with only one effecting the other.

On the other extreme, there are the double cross sum Lie algebras [47, 54, 55, 56, 57, 58, 86, 87, 88, 95], which are the extensions built on mutual interactions of two Lie (sub)algebras. As a result, they provide the suitable ground to study the dynamics of physical systems which are built on two mutually interacting subsystems [24, 25, 26, 27].

The Hamiltonian (Lie-Poisson) theory over double cross sum Lie algebras has been developed only recently in [24]. The theory, then, has immediately found applications in kinetic theory [23], and fluid theories [19]. Its Lagrangian counterpart, on the other hand, has been developed for the first order Euler-Poincaré theories in [24], and for the higher order theories in [22]. The applications to discrete Lagrangian dynamics, in the realm of Lie groupoids and Lie algebroids, were considered only recently in [27]. As for the infinite dimensional setting, it has been shown in [26] that non-relativistic collisionless Vlasov's plasma may be expressed as a non-trivial double cross sum of two subdynamics, one of which being the compressible Euler fluid, while the other is the kinetic moments of order > 2 of the plasma density function.

Although the latter decomposition is in perfect harmony with the nature of physics, it hints a direction for the further study; namely, to find a proper algebraic/geometric framework in order to decompose the dynamics of Vlasov's kinetic moments at any order (see [30, 31] for the geometry of kinetic moments).

The graded character of the Lie algebra of kinetic moments allows only two Lie subalgebra decompositions. The cut by the zeroth moment, and the cut by the first moment yield the only two decompositions to realize (and hence study) the Lie algebra of kinetic moments as a double cross sum.

The higher order cuts, on the other hand, attract deep attention. We refer the reader to [19] for the 10-moment kinetic theory which paves the way towards the whole Grad hierarchy [32] including the entropic moments [35], and, in addition, to [52, 82, 83] for an incomplete literature list related to the kinetic moments.

However, any higher cut yields a Lie subalgebra with a complementary space, which is merely a subspace. Such decompositions go beyond the double cross product/sum constructions, and demand a more general class of extensions.

The two generalizations of the semi-direct products; namely, the 2-cocycle extensions and the double cross products/sums, may be collected under the single roof of *unified products*, which were studied extensively in [2] on the level of Lie algebras, and in [3] on the level of Lie groups. By a slight abuse of language, a unified product of Lie algebras, having equipped with a twisted 2-cocycle, becomes a double cross sum in the triviality of the twisted cocycle, and a 2-cocycle extension in the case the twisted cocycle is an ordinary Lie algebra cohomology cocycle.

Although the formal algebraic treatment of such *extending structures* was achieved in [2, 3], they are subsumed in the extensions of [89], at least, in case the total Lie algebra consists of a number of direct sum copies of a fixed Lie algebra. It worths to note that, the full generality of the type of extensions in [89] goes beyond the present paper, and falls into those introduced very recently in [21].

The first goal of the present work

The main objective of the present paper is to develop Hamiltonian (Lie-Poisson) dynamics on *unified products*. This is the task we achieve in Section 4. The rich geometry offered by such *extending structures* allows us to couple a Lie-Poisson bracket with external variables interacting in all possible ways. From the decomposition point of view, this corresponds to the decomposition of Lie-Poisson dynamics to one of its Lie-Poisson subdynamics, and a complementary subsystem, which is not necessarily Lie-Poisson. The Hamiltonian dynamics on double cross products/sums, however, accommodates only Lie-Poisson subsystems. So, the Lie-Poisson bracket and the Lie-Poisson equations presented here are applicable to any Lie-Poisson model.

In association with this task, we shall address all possible decompositions of (3 particles) BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy [39]. It was proved in [61] that BBGKY hierarchy can be recast as a Lie-Poisson equation for $n > 3$ particles. Focusing on the case $n = 3$, which is missing in [61], we shall present two decompositions of the BBGKY dynamics; the double cross sum decomposition, and the unified product decomposition. This way, we shall be able to compare the two approaches, which, together, determine the relationship between the moments of the 3-particle plasma density function.

The second goal of the present work

Along the lines of the above paragraphs we have noted that the Lie-Poisson theory on 2-cocycle extensions may accommodate interesting examples, see for instance[90], and that double cross product/sum decomposition is an efficient strategy to analyze a system through simpler subsystems. Motivated by these facts, we shall apply the double cross product/sum strategy to Lie-Poisson theory on 2-cocycle extensions. More precisely, we shall derive in Proposition 6.1 the conditions for the double cross sum decomposition of a 2-cocycle extension into two 2-cocycle extensions with compatible cocycles. We shall, furthermore, present explicitly the Lie-Poisson dynamics on the coupled system. The geometric framework, and the dynamical equations, will be illustrated in Subsection 8.2 through two copies of the Heisenberg algebra.

Coupling of dissipative systems

If a dynamical system is in the Hamiltonian form, then as a result of the skew-symmetry of the Poisson bracket, the Hamiltonian function is a conserved quantity [62]. This geometric fact corresponds to the conservation of energy when applied to some physical problems. The time-reversal character of the Hamiltonian dynamics depends basically on this observation. Those systems violating the time-reversal property, therefore, could not be put into the Hamiltonian formulation. Nevertheless, one may achieve to add a (Rayleigh type) dissipative term to the Lie-Poisson dynamics by means of a linear operator from the dual space to the Lie algebra [8].

This naive strategy works very well for many physical problems. A more geometric approach is to add an additional feature to the manifold. There are methods, in the literature, to achieve this.

Coupling of reversible and irreversible dynamics

In the early '80s, extensions of the Poisson geometry were introduced independently in order to add dissipative terms into Hamiltonian formulations (see Subsection 2.2). Following the first initiation [46], a coupling of a dissipative and a Hamiltonian dynamics was introduced in [45, 72] which is called *metriplectic* in [74]. Addressing the same problem, a similar construction was presented in [33], while investigating geometrical framework of Boltzmann equation. Later, this structure is called *GENERIC (General Equation for Non-Equilibrium Reversible Irreversible Coupling)* in [36]. In metriplectic systems, the geometry is determined by two compatible brackets; namely, a Poisson bracket and a (possibly semi-Riemannian) symmetric bracket. In GENERIC, a dissipation potential is employed in order to arrive at the irreversible part of the dynamics. Accordingly, the Legendre transformation of dissipation potential determines the time irreversible part of the dynamics, [81, 34, 79]. If the dissipation potential is quadratic, then one arrives at a bracket for the irreversible motion.

One of the problems in this coupling is to determine a proper symmetric bracket, or a dissipation potential compatible with the Poisson geometry (see Subsection 2.3). In the present work, we shall refer to geometric ways to obtain symmetric brackets; such as the double bracket in [10, 11, 12] (the same formalism was published independently in [93]), Cartan-Killing bracket [75], and Casimir dissipation bracket [29]. We are interested in these geometries since, in the Lie-Poisson framework, they may be defined by the Lie algebra bracket directly in an algorithmic way. In the present work, we shall study extensions/couplings of the symmetric brackets as well as the dissipative systems. The latter will be the third goal here.

The third goal of the present work

In order to present a complete picture, we shall study in Section 7 the couplings of the dissipative terms which are added to the Lie-Poisson dynamics. In other words, our third goal is to couple two metriplectic systems under mutual actions. First, we aim to provide a way to couple two mutually interacting systems involving Rayleigh type dissipative terms. We shall then present couplings of Double brackets, Cartan-Killing brackets, and Casimir dissipation brackets.

As far as the coupling problem is concerned, our emphasis will be laid on $3D$ systems. Accordingly, we shall present two illustrations. On the one hand, we shall provide couplings of both reversible and irreversible rigid body dynamics under mutual interactions, which, from the decomposition point of view, corresponds to the Iwasava decomposition of $SL(2,\mathbb{C})$, while on the other hand, will shall continue with the Heisenberg algebra in Subsection 8.2, endowing the geometry with dissipations.

Contents

In the following section we shall present, for the sake of the completeness, a brief summary of the preliminary material which includes Hamiltonian dynamics and metriplectic dynamics. Section 3 is reserved for unified products, as well as the two extreme constructions the theory accommodates; the double cross sums of Lie algebras and 2-cocycle extensions. In Section 4, we study Lie-Poisson dynamics on unified products, while in Section 5, we consider the possible decompositions of BBGKY dynamics as an illustration of the theoretical results obtained in the previous sections. The conditions for the double cross sum decomposition of a 2-cocycle extension are determined in Section 6. Couplings of the symmetric brackets, on the other hand, are given in Section 7. Finally, in Sections 8 and 9, $3D$ examples are provided.

2. Fundamentals: Lie-Poisson dynamics, and dissipation

2.1. Hamiltonian dynamics

Consider a Poisson manifold $(P, \{\bullet, \bullet\})$ [50, 92]. On this geometry, Hamilton's equation generated by a Hamiltonian function(al) $\mathcal{H}$ is defined to be

$$\dot{\mathbf{z}} = \{\mathbf{z}, \mathcal{H}\}, \tag{2.1}$$

where $\mathbf{z}$ is in P. Define the Hamiltonian vector field $X_{\mathcal{H}}$, for a Hamiltonian function(al) $\mathcal{H}$, through

$$X_{\mathcal{H}}(\mathcal{F}) = \{\mathcal{F}, \mathcal{H}\}. \tag{2.2}$$

A function(al) $\mathcal{C}$ is called a Casimir function(al) if it commutes with all other function(al)s that is, $\{\mathcal{F}, \mathcal{C}\} = 0$ for all $\mathcal{F}$. If there does not exist any non-constant Casimir function(al) for a Poisson bracket, then we say that the Poisson bracket is non-degenerate. It should be noted that the Hamiltonian vector field generated by a Casimir function(al) $\mathcal{C}$ is identically zero. The characteristic distribution, that is the image space of all Hamiltonian vector fields is integrable. This reads a foliation of P as a collection of symplectic leaves [94]. That is, on each leaf, the Poisson bracket turns out to be non-degenerate. If the bracket is already non-degenerate on P then there exists only one leaf, and P turns out to be a symplectic manifold.

Skew-symmetry of Poisson bracket verifies that Hamiltonian function(al) is preserved throughout the motion. Since the Hamiltonian function(al) is taken as the total energy in classical systems, we may call this property as the conservation of energy. This manifests the reversible character of Hamiltonian dynamics.

Referring to Poisson bracket, we define a bivector field Λ as follows

$$\Lambda(d\mathcal{F}, d\mathcal{H}) := \{\mathcal{F}, \mathcal{H}\} \tag{2.3}$$

for all $\mathcal{F}$ and $\mathcal{H}$, [17]. Here, $d\mathcal{F}$ and $d\mathcal{H}$ denote the de-Rham exterior derivatives. Hence, we may alternatively introduce a Poisson manifold by a tuple (P, Λ) consisting of a manifold and a bivector field. Recall that there exists a Schouten-Nijenhuis algebra on bivector fields [6]. In this picture, the Jacobi identity turns out to be the commutation of Λ with itself under the Schouten-Nijenhuis bracket; that is,

$$[\Lambda, \Lambda] = 0. \tag{2.4}$$

Lie-Poisson systems

Consider a Lie algebra $\mathfrak{K}$, equipped with a Lie bracket $[\bullet, \bullet]$, [43]. The dual $\mathfrak{K}^*$ admits a Poisson bracket, called Lie-Poisson bracket [40, 42, 51, 53, 62].

For two function(al)s $\mathcal{F}$ and $\mathcal{H}$, the (plus/minus) Lie-Poisson bracket is defined as

$$\{\mathcal{F},\mathcal{H}\}(\mathbf{z}) = \pm\Big\langle \mathbf{z}, \Big[\frac{\delta\mathcal{F}}{\delta\mathbf{z}}, \frac{\delta\mathcal{H}}{\delta\mathbf{z}}\Big] \Big\rangle \tag{2.5}$$

where $\delta\mathcal{F}/\delta\mathbf{z}$ is the partial derivative (for infinite dimensional cases, the Fréchet derivative) of the function(al) $\mathcal{F}$. Here, the pairing on the right hand side is the duality between $\mathfrak{K}^*$ and $\mathfrak{K}$ whereas the bracket is the Lie algebra bracket on $\mathfrak{K}$. Note that, we assume the reflexivity condition on $\mathfrak{K}$, that is the double dual $\mathfrak{K}^{**} = \mathfrak{K}$. The dynamics of an observable $\mathcal{F}$, governed by a Hamiltonian function(al) $\mathcal{H}$, is then computed to be

$$\begin{aligned}\dot{\mathcal{F}} = \{\mathcal{F},\mathcal{H}\}(\mathbf{z}) &= \pm\Big\langle \mathbf{z}, \Big[\frac{\delta\mathcal{F}}{\delta\mathbf{z}}, \frac{\delta\mathcal{H}}{\delta\mathbf{z}}\Big] \Big\rangle \\ &= \pm\Big\langle \mathbf{z}, -\operatorname{ad}_{\delta\mathcal{H}/\delta\mathbf{z}} \frac{\delta\mathcal{F}}{\delta\mathbf{z}} \Big\rangle = \pm\Big\langle \operatorname{ad}^*_{\delta\mathcal{H}/\delta\mathbf{z}} \mathbf{z}, \frac{\delta\mathcal{F}}{\delta\mathbf{z}} \Big\rangle. \end{aligned} \tag{2.6}$$

Here, $\operatorname{ad}_{\mathbf{x}} \mathbf{x}' := [\mathbf{x}, \mathbf{x}']$ for all $\mathbf{x}$ and $\mathbf{x}'$ in $\mathfrak{K}$ is the (left) adjoint action of the Lie algebra $\mathfrak{K}$ on itself whereas ad^* is the (left) coadjoint action of the Lie algebra $\mathfrak{K}$ on the dual space $\mathfrak{K}^*$. Notice that $\operatorname{ad}^*_{\mathbf{x}}$ is defined to be minus of the linear algebraic dual of $\operatorname{ad}_{\mathbf{x}}$. Then, we obtain the equation of motion governed by a Hamiltonian function(al) $\mathcal{H}$ as

$$\dot{\mathbf{z}} \mp \operatorname{ad}^*_{\delta\mathcal{H}/\delta\mathbf{z}} \mathbf{z} = 0. \tag{2.7}$$

Remark 2.1. There are a plus/minus notations in (2.5) and (2.6). A plus sign appears if the reduction (Lie-Poisson reduction) is performed referring to a right symmetry whereas a minus sign appears if the reduction is performed referring to a left symmetry. For the plasma dynamics (see Section 5), we shall refer plus Lie-Poisson bracket since Vlasov's plasma has a right (called relabelling) symmetry. For finite-dimensional rigid body motion (see Section 8), we shall employ minus Lie-Poisson bracket.

Coordinate realizations

Assume a (local) coordinate chart (z_i) (we prefer subscripts since we only focus on the dual spaces) around a point $\mathbf{z}$ in P. Then the Poisson bivector can be represented by a set of coefficient functions Λ_{ij} determining a Poisson bracket as

$$\{\mathcal{F},\mathcal{H}\} = \Lambda_{ij}\frac{\partial\mathcal{F}}{\partial z_i}\frac{\partial\mathcal{H}}{\partial z_i}. \tag{2.8}$$

Then the equation of motion generated by a Hamiltonian function $\mathcal{H}$ becomes

$$\dot{z}_i = \Lambda_{ij}\frac{\partial\mathcal{H}}{\partial z_j}. \tag{2.9}$$

Let us examine the Lie-Poisson structure which is defined on the dual of a finite dimensional Lie algebra. For this, let $\mathfrak{K}$ be a K dimensional Lie algebra with a basis $\{\mathbf{k}_i\} = \{\mathbf{k}_1, \ldots, \mathbf{k}_K\}$. The Lie algebra bracket on $\mathfrak{K}$ determine a set of scalars C^l_{ij}, called structure constants, satisfying

$$[\mathbf{k}_i, \mathbf{k}_j] = C^m_{ij}\mathbf{k}_m, \tag{2.10}$$

where the summation convention is assumed over the repeated indices.

Note that, after fixing a basis, the structure constants define a Lie bracket in a unique way. One has the dual basis $\{\mathbf{k}^i\} = \{\mathbf{k}^1, \dots, \mathbf{k}^K\}$ on the dual space $\mathfrak{K}^*$. We denote an element of $\mathfrak{K}^*$ by $\mathbf{z} = z_i \mathbf{k}^i$ where the coordinates $\{z_1, \dots, z_N\}$ are being real numbers. The (plus/minus) Lie-Poisson bracket (2.5) can be computed in this picture as

$$\{\mathcal{F}, \mathcal{H}\} = \pm C_{ij}^m z_n \frac{\partial \mathcal{F}}{\delta z_i} \frac{\partial \mathcal{H}}{\delta z_j}. \tag{2.11}$$

The calculation in (2.11) means that the coefficients Λ_{ij} of the (plus/minus) Poisson bivector (2.8) are determined through the linear relations

$$\Lambda_{ij} = \pm C_{ij}^m z_m. \tag{2.12}$$

In this case, Lie-Poisson equations (2.7) are computed to be

$$\dot{z}_j \mp C_{ij}^n z_n \frac{\partial \mathcal{H}}{\delta z_i} = 0. \tag{2.13}$$

Rayleigh dissipation

Let us present a simply way to add dissipation to Lie-Poisson dynamics. Define a linear transformation Υ from $\mathfrak{K}^*$ to $\mathfrak{K}$. We equip a dissipative term to the right hand side of the Lie-Poisson system (2.7) by simply adding $\mp ad^*_{\Upsilon(\mathbf{z})}\mathbf{z}$, that is,

$$\dot{z} \mp ad^*_{\delta\mathcal{H}/\delta\mathbf{z}}\mathbf{z} = \mp ad^*_{\Upsilon(\mathbf{z})}\mathbf{z}, \tag{2.14}$$

see for instance [8]. We ask Υ to be a gradient relative to a certain metric at least on adjoint orbits. In the upcoming subsection, we introduce a geometric framework for obtaining dissipation in the Lie-Poisson setting.

2.2. Metriplectic Systems

As discussed in the introduction, in order to add dissipative terms to Hamiltonian dynamics, two geometric models are addressed in the literature, namely metriplectic systems [45, 72, 74] and GENERIC (an acronym for General Equation for Non-Equilibrium Reversible-Irreversible Coupling), [33, 36, 34]. In this work, we are interested in dissipative dynamics defined through symmetric brackets [80], hence metriplectic dynamics. Let us describe this geometry in detail.

Consider a Poisson manifold $(P, \{\bullet, \bullet\})$ and assume, additionally, a symmetric bracket $(\bullet, \bullet)$ on the space of smooth functions on P. The metriplectic bracket $[|\bullet, \bullet|]$ on the manifold P is defined by the addition of the Poisson bracket and the symmetric bracket, that is, for two function(al)s $\mathcal{H}$ and $\mathcal{F}$,

$$[|\mathcal{H}, \mathcal{F}|] = \{\mathcal{H}, \mathcal{F}\} + a(\mathcal{H}, \mathcal{F}), \tag{2.15}$$

where a is a scalar. Note that, a metriplectic bracket is an example of a Leibniz bracket [78]. There is no unique way to define a symmetric bracket. One way is to introduce a (possibly semi-)Riemannian metric $\mathcal{G}$ on M. After a bracket is established, the next task is to determine the generating function(al)s. In accordance with this, we determine two different kinds of metriplectic systems [38].

In the first kind of metriplectic systems, one refers a single function(al) $\mathcal{F}$ to generate the equations of motion, see [72]. Accordingly, the dynamics is given by

$$\dot{\mathbf{z}} = [|\mathbf{z}, \mathcal{F}|] = \{\mathbf{z}, \mathcal{F}\} + a(\mathbf{z}, \mathcal{F}) \quad \text{for } \mathbf{z} \in P. \tag{2.16}$$

In particular, by choosing the metric $\mathcal{G}$ positive definite, and by letting a be equal to -1, one arrives at the dissipation of the generating function $\mathcal{F}$ in time. In the second kind of metriplectic systems, there exist two function(al)s, namely a Hamiltonian function(al) $\mathcal{H}$ and an entropy-type function(al) $\mathcal{S}$. In this case, see [45], the dynamics is written as

$$\dot{\mathbf{z}} = \{\mathbf{z}, \mathcal{H}\} + a(\mathbf{z}, \mathcal{S}). \tag{2.17}$$

If the following degeneracies, see [73],

$$\{\mathcal{S}, \mathcal{H}\} = 0, \qquad (\mathcal{H}, \mathcal{S}) = 0 \tag{2.18}$$

hold, that is if the entropy is a Casimir, and energy is a dissipative invariant then the metriplectic dynamics (2.17) can be generated by a single function (free energy function) $\mathcal{F} = \mathcal{H} - \mathcal{S}$ defined as the difference of the Hamiltonian and entropy-type function(al)s. For such kind of systems, Hamiltonian function(al) $\mathcal{H}$ is a conserved quantity whereas the dissipative behavior of the system is interpreted as the increase of entropy along trajectories assuming that a is positive. This case is possible if the Poisson structure is degenerate and the symmetric tensor G is at most semi-definite.

There is an extensive list of studies on metriplectic systems. Let us mention some recent papers. In [9], metriplectic dynamics is discussed in the realm of triple brackets. A metriplectic realization of dissipative magneto-hydrodynamics is studied in [67]. More general theories on dissipative fluid models are more recently given in [16] and with a counterpart variation approach in [18], and a discussion of time-dependent unreduced states in [67]. In [48], metriplectic integrator is proposed for a kinetic theory.

In the following subsection, we introduce some examples of symmetric brackets that can be attached to the Lie-Poisson bracket.

2.3. Some symmetric brackets on the duals of Lie algebras

In this subsection, we list some symmetric brackets available on the dual $\mathfrak{K}^*$ of a Lie algebra $\mathfrak{K}$. After a symmetric bracket is determined, say $(\bullet, \bullet)$ the irreversible dynamics governed by a generating function, say $\mathcal{S}$, is computed to be

$$\dot{\mathbf{z}} = a(\mathbf{z}, \mathcal{S}), \tag{2.19}$$

where a is a real number. Assuming a basis $\{\mathbf{k}_i\}$, and the accordingly the local coordinates $\{z^i\}$ on $\mathfrak{K}$, the primary goal in this subsection is to define a symmetric tensor field

$$\mathcal{G} = \mathcal{G}_{ij} dz^i \otimes dz^j \tag{2.20}$$

on $\mathfrak{K}$. In the dual space $\mathfrak{K}^*$, we employ the dual basis $\{\mathbf{k}^i\}$ and the coordinates $\{z_i\}$. There are two distinguished functions on the Lie-Poisson picture. Here is a list:

Double bracket

Recall the structure of a Lie algebra $\mathfrak{K}$ given in (2.10). In the Lie-Poisson setting, the coefficients of the Poisson bivector are determined by the structure constants of the Lie algebra as (2.12) that is, $\Lambda_{ij} = C_{ij}^l z_l$ [75]. For two functions $\mathcal{F}$ and $\mathcal{H}$, we define a symmetric bracket, literarily called double bracket see [11]:

$$(\mathcal{F}, \mathcal{H})^{(D)} = \sum_j \Lambda_{ij}\Lambda_{lj} \frac{\partial \mathcal{F}}{\partial z_i}\frac{\partial \mathcal{H}}{\partial z_l} = \sum_j C_{ij}^r C_{lj}^s z_r z_s \frac{\partial \mathcal{F}}{\partial z_i}\frac{\partial \mathcal{H}}{\partial z_l}. \tag{2.21}$$

Hence, we can write the coefficients of the symmetric bracket in terms of the structure constants of the Lie algebra as

$$\mathcal{G}_{ij} = \sum_{l} C_{il}^{r} C_{jl}^{s} z_r z_s. \tag{2.22}$$

Now we define a metriplectic bracket on $\mathfrak{K}^*$ by adding Lie-Poisson bracket (2.11) and double bracket (2.21) that is

$$\dot{\mathcal{F}} = [|\mathcal{F}, \mathcal{H}|]^{(D)} = \{\mathcal{F}, \mathcal{H}\} + a(\mathcal{F}, \mathcal{H})^{(D)}. \tag{2.23}$$

So that according to the definition (2.16) we compute the equation of motion as

$$\dot{z}_j \mp C_{ij}^{m} z_m \frac{\partial \mathcal{H}}{\partial z_i} = a \sum_{i} C_{ji}^{r} C_{mi}^{n} z_r z_n \frac{\partial \mathcal{H}}{\partial z_m} \tag{2.24}$$

where on the left hand side we have the reversible Hamiltonian dynamics, while the dissipative term is located at the right hand side.

Cartan-Killing bracket

Consider a Lie algebra given in coordinates as (2.10). Referring to skew-symmetry of the structure constants the Cartan-Killing metric is defined as

$$\mathcal{G}_{ij} = C_{im}^{n} C_{jn}^{m}. \tag{2.25}$$

It can be easily shown that the scalars $\mathcal{G}_{ij}$ define a symmetric and bilinear covariant tensor [75]. We define a symmetric bracket for functions $\mathcal{F}$ and $\mathcal{H}$ in terms of the metric as follows

$$(\mathcal{F}, \mathcal{H})^{(CK)} = \frac{\partial \mathcal{F}}{\partial z_i} \mathcal{G}_{ij} \frac{\partial \mathcal{H}}{\partial z_j} = C_{ij}^{n} C_{mn}^{j} \frac{\partial \mathcal{F}}{\partial z_i} \frac{\partial \mathcal{H}}{\partial z_m}. \tag{2.26}$$

To arrive at a metriplectic bracket on $\mathfrak{K}^*$, we add the Lie-Poisson bracket (2.11) and the symmetric bracket (2.26) that is,

$$\dot{\mathcal{F}} = [|\mathcal{F}, \mathcal{H}|]^{(CK)} = \{\mathcal{F}, \mathcal{H}\} + a(\mathcal{F}, \mathcal{H})^{(CK)}. \tag{2.27}$$

Accordingly, the metriplectic dynamics is computed to be

$$\dot{z}_j \mp C_{ij}^{m} z_m \frac{\partial \mathcal{H}}{\partial z_i} = a C_{ji}^{n} C_{mn}^{i} \frac{\partial \mathcal{H}}{\partial z_m} \tag{2.28}$$

where on the left hand side we have the reversible Hamiltonian dynamics, while on the right hand side the dissipative term is presented.

Casimir dissipation bracket

Define a symmetric bilinear operator ψ on Lie algebra $\mathfrak{K}$. Referring to any Casimir function $\mathcal{C}$ of the Lie-Poisson bracket, define a symmetric bracket [29] of two functions $\mathcal{F}$ and $\mathcal{H}$ as

$$(\mathcal{F}, \mathcal{H})^{(CD)} = -\psi \left(\left[\frac{\delta \mathcal{F}}{\delta \mathbf{z}}, \frac{\delta \mathcal{H}}{\delta \mathbf{z}} \right], \left[\frac{\delta C}{\delta \mathbf{z}}, \frac{\delta \mathcal{H}}{\delta \mathbf{z}} \right] \right). \tag{2.29}$$

This bracket is suitable for the second type of metriplectic bracket. Note that the change of Hamiltonian function over time is constant and the change of Casimir function can be given as

$$\dot{\mathcal{C}} = -\psi \left(\left[\frac{\delta \mathcal{C}}{\delta \mathbf{z}}, \frac{\delta \mathcal{H}}{\delta \mathbf{z}} \right], \left[\frac{\delta \mathcal{C}}{\delta \mathbf{z}}, \frac{\delta \mathcal{H}}{\delta \mathbf{z}} \right] \right) = - \left\| \left[\frac{\delta \mathcal{C}}{\delta \mathbf{z}}, \frac{\delta \mathcal{H}}{\delta \mathbf{z}} \right] \right\|^2 < 0.$$

The dynamics of an arbitrary observable $\mathcal{F}$ governed by a Hamiltonian function $\mathcal{H}$ is deduced by this bracket as

$$\begin{aligned}\dot{\mathcal{F}} = -\psi\left(\left[\frac{\delta\mathcal{F}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right],\left[\frac{\delta\mathcal{C}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]\right) &= \left\langle\left[\frac{\delta\mathcal{C}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]^{\flat}, ad_{\delta H/\delta\mathbf{z}}\frac{\delta\mathcal{F}}{\delta\mathbf{z}}\right\rangle \\ &= \left\langle -ad^{*}_{\delta\mathcal{H}/\delta\mathbf{z}}\left[\frac{\delta C}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]^{\flat},\frac{\delta\mathcal{F}}{\delta\mathbf{z}}\right\rangle .\end{aligned} \tag{2.30}$$

Here, the musical mapping $\flat$, from $\mathfrak{K}$ to $\mathfrak{K}^*$, is defined through the symmetric operator ψ, satisfying the identity $\langle\mathbf{x}^{\flat},\mathbf{x}'\rangle = \psi(\mathbf{x},\mathbf{x}')$ for two elements $\mathbf{x}$ and $\mathbf{x}'$ in $\mathfrak{K}$. In this case, the dissipative equation of motion can be written as

$$\dot{\mathbf{z}} = -ad^{*}_{\frac{\delta\mathcal{H}}{\delta\mathbf{z}}}\left[\frac{\delta C}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]^{\flat}. \tag{2.31}$$

We collect the Lie-Poisson bracket (2.11) and the Casimir Dissipation bracket (2.29) together to arrive at the following metriplectic bracket

$$\dot{\mathcal{F}} = [|\mathcal{F},\mathcal{H}|]^{(CD)} = \{\mathcal{F},\mathcal{H}\} + a(\mathcal{F},\mathcal{H})^{(CD)}. \tag{2.32}$$

Then we compute the equation of motion as

$$\dot{\mathbf{z}} \mp ad^{*}_{\frac{\delta\mathcal{H}}{\delta\mathbf{z}}} z = (-a)ad^{*}_{\frac{\delta H}{\delta\mathbf{z}}}\left[\frac{\delta\mathcal{C}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]^{\flat}. \tag{2.33}$$

Hamilton dissipation bracket

We start with assuming a symmetric (semi-positive definite) bilinear operator ψ defined on a Lie algebra $\mathfrak{K}$. We fix a Casimir function $\mathcal{C}$ of the Lie-Poisson bracket (2.5), and then introduce the following symmetric bracket on the dual space $\mathfrak{K}^*$, for two function(al)s $\mathcal{F}$ and $\mathcal{H}$, given by

$$(\mathcal{F},\mathcal{H})^{(HD)} = -\psi\Big(\Big[\frac{\delta\mathcal{F}}{\delta\mathbf{z}},\frac{\delta\mathcal{C}}{\delta\mathbf{z}}\Big],\Big[\frac{\delta\mathcal{H}}{\delta\mathbf{z}},\frac{\delta\mathcal{C}}{\delta\mathbf{z}}\Big]\Big), \tag{2.34}$$

where the brackets on the right sides are Lie algebra brackets on $\mathfrak{K}$. An interesting feature of this symmetric bracket is to see that the Casimir function(al) $\mathcal{C}$ is a conserved quantity for the dynamics determined by the bracket (2.34) since $\dot{\mathcal{C}} = 0$ due to the skew-symmetry of the Lie-bracket. On the other hand the generating function $\mathcal{H}$ dissipates, that is,

$$\dot{\mathcal{H}} = (\mathcal{H},\mathcal{H})^{(HD)} = -\psi\Big(\Big[\frac{\delta\mathcal{H}}{\delta\mathbf{z}},\frac{\delta\mathcal{C}}{\delta\mathbf{z}}\Big],\Big[\frac{\delta\mathcal{C}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\Big]\Big) = -\left\|\Big[\frac{\delta\mathcal{H}}{\delta\mathbf{z}},\frac{\delta\mathcal{C}}{\delta\mathbf{z}}\Big]\right\|^{2} \leq 0.$$

More generally, the gradient flow of an observable $\mathcal{F}$ generated by a function(al) $\mathcal{H}$ is computed to be

$$\begin{aligned}\dot{\mathcal{F}} = -\psi\left(\left[\frac{\delta\mathcal{F}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right],\left[\frac{\delta\mathcal{C}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]\right) &= \left\langle\left[\frac{\delta\mathcal{C}}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]^{\flat}, ad_{\delta H/\delta\mathbf{z}}\frac{\delta\mathcal{F}}{\delta\mathbf{z}}\right\rangle \\ &= \left\langle -ad^{*}_{\delta\mathcal{H}/\delta\mathbf{z}}\left[\frac{\delta C}{\delta\mathbf{z}},\frac{\delta\mathcal{H}}{\delta\mathbf{z}}\right]^{\flat},\frac{\delta\mathcal{F}}{\delta\mathbf{z}}\right\rangle .\end{aligned} \tag{2.35}$$

Accordingly, the equation of motion generated by $\mathcal{S}$ may be given by

$$\dot{\mathbf{z}} = -ad^*_{\delta C/\delta \mathbf{z}}\left[\frac{\delta \mathcal{C}}{\delta \mathbf{z}}, \frac{\delta \mathcal{S}}{\delta \mathbf{z}}\right]^\flat. \tag{2.36}$$

Now we are ready to add the Lie-Poisson bracket (2.11) and the symmetric bracket exhibited in (2.34) in order to define a metriplectic (Leibniz) bracket on $\mathfrak{K}^*$. By assuming that both the Lie-Poisson and the gradient dynamics generated by a single function $\mathcal{H}$ we have that

$$\dot{\mathcal{F}} = [|\mathcal{F}, \mathcal{H}|]^{(HD)} = \{\mathcal{F}, \mathcal{H}\} + a(\mathcal{F}, \mathcal{H})^{(HD)}. \tag{2.37}$$

Then we compute the equation of motion as

$$\dot{\mathbf{z}} \mp ad^*_{\delta \mathcal{H}/\delta \mathbf{z}} z = (-a) ad^*_{\delta C/\delta \mathbf{z}}\left[\frac{\delta \mathcal{C}}{\delta \mathbf{z}}, \frac{\delta \mathcal{H}}{\delta \mathbf{z}}\right]^\flat. \tag{2.38}$$

3. Extensions of Lie algebras

In this section, we shall introduce a construction called *unified product* of Lie algebras. We shall then examine its particular instances such as the double cross sum Lie algebras and 2-cocycle extensions.

3.1. Extending structures

Let $(\mathfrak{g}, [\bullet, \bullet])$ be a Lie algebra and, assume that, it acts on a vector space $\mathfrak{h}$ from the right that is

$$\triangleleft : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{h}, \qquad \eta \otimes \xi \mapsto \eta \triangleleft \xi. \tag{3.1}$$

Our goal in this subsection is to construct the most general extension of $\mathfrak{g}$ by $\mathfrak{h}$. To have this, we permit existence of the following maps

$$\begin{aligned} \Phi : \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathfrak{g}, \qquad (\eta, \eta') \mapsto \Phi(\eta, \eta') \\ \kappa : \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathfrak{h}, \qquad (\eta, \eta') \mapsto \kappa(\eta, \eta') \end{aligned} \tag{3.2}$$

along with a linear map

$$\triangleright : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}, \qquad \eta \otimes \xi \mapsto \eta \triangleright \xi. \tag{3.3}$$

Note here that (3.3) is not an action since $\mathfrak{h}$ is a mere vector space. The need of the operations (3.2) and (3.3) will be evident in the sequel where we examine this in the point of view of decomposition. The following theorem determines the conditions to define a Lie algebra structure on the direct sum $\mathfrak{K} = \mathfrak{g} \oplus \mathfrak{h}$, see also [2].

Theorem 3.1. *The direct sum $\mathfrak{K} = \mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra via*

$$\begin{aligned} &[(\xi \oplus \eta), (\xi' \oplus \eta')]_{\Phi\bowtie} \\ &\quad = \big([\xi, \xi'] + \eta \triangleright \xi' - \eta' \triangleright \xi + \Phi(\eta, \eta')\big) \oplus \big(\kappa(\eta, \eta') + \eta \triangleleft \xi' - \eta' \triangleleft \xi\big), \end{aligned} \tag{3.4}$$

where the mappings are the ones in (3.1), (3.2) *and* (3.3), *if and only if, for any* $\eta, \eta', \eta'' \in \mathfrak{h}$, *and any* $\xi, \xi' \in \mathfrak{g}$,

$\kappa(\eta, \eta) = 0, \qquad \Phi(\eta, \eta) = 0,$

$\kappa(\eta, \eta') \triangleleft \xi = \kappa(\eta, \eta' \triangleleft \xi) - \kappa(\eta', \eta \triangleleft \xi) + \eta \triangleleft (\eta' \triangleright \xi) - \eta' \triangleleft (\eta \triangleright \xi),$

$\kappa(\eta, \eta') \triangleright \xi = [\xi, \Phi(\eta, \eta')] + \Phi(\eta, \eta' \triangleleft \xi) + \Phi(\eta \triangleleft \xi, \eta') + \eta \triangleright (\eta' \triangleright \xi) - \eta' \triangleright (\eta \triangleright \xi),$

$$\eta \triangleright [\xi,\xi'] = [\xi,\eta \triangleright \xi'] - [\xi',\eta \triangleright \xi] + (\eta \triangleleft \xi) \triangleright \xi' - (\eta \triangleleft \xi') \triangleright \xi,$$
$$\eta \triangleleft [\xi,\xi'] = (\eta \triangleleft \xi) \triangleleft \xi' - (\eta \triangleleft \xi') \triangleleft \xi,$$
$$\circlearrowright \Phi(\eta,\kappa(\eta',\eta''))+ \circlearrowright \eta \triangleright \Phi(\eta',\eta'') = 0,$$
$$\circlearrowright \kappa(\eta,\kappa(\eta',\eta''))+ \circlearrowright \eta \triangleleft \Phi(\eta',\eta'') = 0, \tag{3.5}$$

where $\circlearrowright$ *refers to the cyclic sum over the indicated elements.*

Proof. We first observe that

$$[\eta,\eta] = \big(\Phi(\eta,\eta),\kappa(\eta,\eta)\big) = 0 \tag{3.6}$$

if and only if
$$\Phi(\eta,\eta) = 0, \qquad \kappa(\eta,\eta) = 0 \tag{3.7}$$

for any $\eta \in \mathfrak{h}$. Next, we shall consider the mixed Jacobi identities. Let us begin with

$$[\xi,[\eta,\eta']] + [\eta,[\eta',\xi]] + [\eta',[\xi,\eta]] = 0, \tag{3.8}$$

where

$$\begin{aligned}
[\xi,[\eta,\eta']] &= [\xi,(\Phi(\eta,\eta'),\,\kappa(\eta,\eta'))] = \big(- \kappa(\eta,\eta') \triangleright \xi + [\xi,\Phi(\eta,\eta')]_{\mathfrak{g}}, -\kappa(\eta,\eta') \triangleleft \xi\big),\\
[\eta,[\eta',\xi]] &= [\eta,(-\eta' \triangleright \xi,-\eta' \triangleleft \xi)] = \big(- \eta \triangleright (\eta' \triangleright \xi) - \Phi(\eta' \triangleleft \xi,\eta),\\
&\qquad - \kappa(\eta' \triangleleft \xi,\eta) - \eta \triangleleft (\eta' \triangleright \xi)\big),\\
[\eta',[\xi,\eta]] &= [\eta',(\eta \triangleright \xi,\eta \triangleleft \xi)] = \big(\eta' \triangleright (\eta \triangleright \xi) + \Phi(\eta \triangleleft \xi,\eta'),\\
&\qquad \kappa(\eta \triangleleft \xi,\eta') + \eta' \triangleleft (\eta \triangleright \xi)\big).
\end{aligned} \tag{3.9}$$

Hence, (3.8) is satisfied if and only if

$$\begin{aligned}
\kappa(\eta,\eta') \triangleleft \xi &= \kappa(\eta \triangleleft \xi,\eta') - \kappa(\eta' \triangleleft \xi,\eta) + \eta' \triangleleft (\eta \triangleright \xi) - \eta \triangleleft (\eta' \triangleright \xi)\\
[\xi,\Phi(\eta,\eta')]_{\mathfrak{g}} &= \kappa(\eta,\eta') \triangleright \xi + \eta \triangleright (\eta' \triangleright \xi) - \eta' \triangleright (\eta \triangleright \xi)\\
&\qquad + \Phi(\eta' \triangleleft \xi,\eta) - \Phi(\eta \triangleleft \xi,\eta')
\end{aligned} \tag{3.10}$$

for any $\eta,\eta' \in \mathfrak{h}$, and any $\xi \in \mathfrak{g}$.

Next, we consider the Jacobi identity of an arbitrary $\eta \in \mathfrak{h}$, and any $\xi,\xi' \in \mathfrak{g}$, namely:

$$[[\xi,\xi'],\eta] + [[\xi',\eta],\xi] + [[\eta,\xi],\xi'] = 0. \tag{3.11}$$

In this case,
$$[[\xi,\xi'],\eta] = \big(\eta \triangleleft [\xi,\xi']_{\mathfrak{g}},\, \eta \triangleright [\xi,\xi']_{\mathfrak{g}})\big),$$

together with

$$\begin{aligned}
[[\xi',\eta],\xi] &= \big((\eta \triangleleft \xi') \triangleleft \xi,\, -(\eta \triangleleft \xi) \triangleright \xi' + [\xi,\eta \triangleright \xi']_{\mathfrak{g}}\big),\\
[[\eta,\xi],\xi'] &= \big(- (\eta \triangleleft \xi) \triangleleft \xi',\, (\eta \triangleleft \xi') \triangleright \xi - [\eta \triangleright \xi,\xi']_{\mathfrak{g}}\big).
\end{aligned} \tag{3.12}$$

Hence, (3.11) is satisfied if and only if

$$\begin{aligned}
\eta \triangleleft [\xi,\xi']_{\mathfrak{g}} &= -(\eta \triangleleft \xi) \triangleleft \xi' + (\eta \triangleleft \xi) \triangleleft \xi',\\
\eta \triangleright [\xi,\xi']_{\mathfrak{g}} &= [\eta \triangleright \xi,\xi']_{\mathfrak{g}} + [\xi,\eta \triangleright \xi']_{\mathfrak{g}} + (\eta \triangleleft \xi) \triangleright \xi' - (\eta \triangleleft \xi') \triangleright \xi
\end{aligned} \tag{3.13}$$

for any $\eta \in \mathfrak{h}$, and any $\xi,\xi' \in \mathfrak{g}$. Finally we consider the Jacobi identity

$$[[\eta,\eta'],\eta''] + [[\eta',\eta''],\eta] + [[\eta'',\eta],\eta'] = 0 \tag{3.14}$$

for any $\eta,\eta',\eta'' \in \mathfrak{h}$.

We have,

$$
\begin{aligned}
[[\eta,\eta'],\eta''] &= \big((\Phi(\eta,\eta'),\kappa(\eta,\eta')),\,\eta''\big)\\
&= \big(\eta'',\kappa(\kappa(\eta,\eta')) + \eta''\triangleleft\Phi(\eta,\eta'),\,\eta''\triangleright(\Phi(\eta,\eta') + \Phi(\eta'',\kappa(\eta,\eta')))\big),
\end{aligned}
$$

as well as

$$
\begin{aligned}
[[\eta',\eta''],\eta] &= [(\Phi(\eta',\eta''),\kappa(\eta',\eta'')),\,\eta]\\
&= \big(\eta,\kappa(\kappa(\eta',\eta'')) + \eta\triangleleft\Phi(\eta',\eta''),\,\eta''\triangleright\Phi(\eta',\eta) + \Phi(\eta,\kappa(\eta',\eta''))\big)\\
[[\eta'',\eta],\eta'] &= [(\Phi(\eta,\eta''),\kappa(\eta,\eta'')),\,\eta'] \qquad (3.15)\\
&= \big(\eta',\kappa(\kappa(\eta,\eta'')) + \eta'\triangleleft\Phi(\eta,\eta''),\,\eta'\triangleright\Phi(\eta,\eta'') + \Phi(\eta',\kappa(\eta,\eta''))\big).
\end{aligned}
$$

Accordingly, (3.14) is satisfied if and only if

$$
\begin{aligned}
&\sum_{(\eta,\eta',\eta'')}\kappa(\eta'',\kappa(\eta,\eta')) + \sum_{(\eta,\eta',\eta'')}\eta''\triangleleft\Phi(\eta,\eta') = 0,\\
&\sum_{(\eta,\eta',\eta'')}\eta''\triangleright\Phi(\eta,\eta') + \sum_{(\eta,\eta',\eta'')}\Phi(\eta'',\kappa(\eta,\eta')) = 0
\end{aligned}
\qquad (3.16)
$$

for any $\eta,\eta',\eta''\in\mathfrak{h}$. ■

We denote the direct product space $\mathfrak{K}=\mathfrak{g}\oplus\mathfrak{h}$ equipped with the Lie algebra bracket (3.4) by $\mathfrak{K}=\mathfrak{g}_{\Phi}\bowtie\mathfrak{h}$. In [2], the realization presented in Theorem 3.1 has been introduced under the name of *extending structures*, or more precisely the *unified product* of $\mathfrak{g}$ and $\mathfrak{h}$. We shall follow this terminology as well. We remark that the last two identities in (3.5) are called twisted cocycle identity for Φ and twisted Jacobi identity for κ, respectively. In the following subsection we shall exploit that extended structure realizes both matched (double cross sum) Lie algebra and 2-cocycle extension of a Lie algebra as particular instances.

We refer the reader to [44] for extensions of Hamiltonian vector fields and to [49] for extensions of Poisson algebras.

Decomposing a Lie algebra

Instead of extending a Lie algebra with its representation space, one can decompose a Lie algebra into the internal direct sum of one of its Lie subalgebra and its complement. The latter manifests the inverse of the statement in Theorem 3.1. Let us detail the process.

We start with a Lie algebra $\mathfrak{K}$, and assume a subalgebra, say $\mathfrak{g}$, of it. It is always possible to define a complementary subspace $\mathfrak{h}\subset\mathfrak{K}$ so that $\mathfrak{K}=\mathfrak{g}\oplus\mathfrak{h}$. In most general case, for η,η' in $\mathfrak{h}$, under the Lie algebra bracket of $\mathfrak{K}$, we have

$$
[\eta,\eta'] = \Phi(\eta,\eta')\oplus\kappa(\eta,\eta')\in\mathfrak{g}\oplus\mathfrak{h}. \qquad (3.17)
$$

Here, we define the mappings

$$
\Phi:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{g},\qquad \Phi(\eta,\eta') := proj_{\mathfrak{g}}[\eta,\eta'] \qquad (3.18)
$$

$$
\kappa:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{h},\qquad \kappa(\eta,\eta') := proj_{\mathfrak{h}}[\eta,\eta'] \qquad (3.19)
$$

where *proj* denotes the projection operator. We note that if Φ is identically zero then $\mathfrak{h}$ becomes a Lie subalgebra of $\mathfrak{K}$. In this case, κ becomes the Lie algebra bracket on $\mathfrak{h}$.

Proposition 3.2. *Given a decomposition of Lie algebra $\mathfrak{K} = \mathfrak{g} \oplus \mathfrak{h}$, where $\mathfrak{g}$ is a Lie subalgebra, the mappings Φ and κ in* (3.2) *may be recovered from* (3.17), *and the mutual* actions *recovered from*

$$[0 \oplus \eta, \xi \oplus 0] = \eta \triangleright \xi \oplus \eta \triangleleft \xi \tag{3.20}$$

satisfy the conditions in (3.5). *That is, $\mathfrak{K}$ can be identified to the unified product $\mathfrak{g}_{\Phi}\bowtie\mathfrak{h}$. In other words,* (3.4) *determines a decomposition of the Lie bracket on $\mathfrak{K}$.*

Coordinate realizations

Choose a basis $\{\mathbf{e}_\alpha\}$ on an N-dimensional Lie algebra $\mathfrak{g}$ and a basis $\{\mathbf{f}_a\}$ on M-dimensional vector space $\mathfrak{h}$. We shall reserve the Greek scripts to denote the basis of the Lie algebra $\mathfrak{g}$, whereas we shall make use of the Latin scripts for the basis for the vector space $\mathfrak{h}$. Accordingly, we have

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = C_{\alpha\beta}^{\theta}\mathbf{e}_\theta, \qquad [\mathbf{f}_a, \mathbf{f}_b] = \Phi_{ab}^{\alpha}\mathbf{e}_\alpha + \kappa_{ab}^{d}\mathbf{f}_d, \tag{3.21}$$

where the set $C_{\alpha\beta}^{\theta}$ determines the structure constants of the Lie subalgebra $\mathfrak{g}$ whereas the sets of constants Φ_{ab}^{α} and κ_{ab}^{d} are coordinate realizations of the mappings Φ and κ determined in (3.17). We identify the mappings (3.1) and (3.3) in terms of the basis $\mathbf{e}_\alpha$, $\mathbf{f}_a$ as

$$\mathbf{f}_a \triangleleft \mathbf{e}_\alpha = R_{a\alpha}^{b}\mathbf{f}_b, \qquad \mathbf{f}_a \triangleright \mathbf{e}_\alpha = L_{a\alpha}^{\beta}\mathbf{e}_\beta. \tag{3.22}$$

Needless to say, the scalars $L_{a\alpha}^{\beta}$ and $R_{a\alpha}^{b}$ determine the mappings in a unique way. In the present finite case, the unified product $\mathfrak{g}_{\Phi}\bowtie\mathfrak{h}$ is an $N+M$ dimensional vector space. Referring to the basis of the constitutive subspaces, one can define a basis $\{\bar{\mathbf{e}}_1, \dots, \bar{\mathbf{e}}_{N+M}\}$ on $\mathfrak{g}_{\Phi}\bowtie\mathfrak{h}$ as

$$\{\bar{\mathbf{e}}_\alpha, \bar{\mathbf{e}}_a\} \subset \mathfrak{g}_{\Phi}\bowtie\mathfrak{h}, \qquad \bar{\mathbf{e}}_\alpha = \mathbf{e}_\alpha \oplus 0, \qquad \bar{\mathbf{e}}_a = 0 \oplus \mathbf{f}_a. \tag{3.23}$$

In view of (3.21) and (3.22), one can calculate the structure constants of the bracket (3.4) via

$$\begin{aligned}
[\bar{\mathbf{e}}_\beta, \bar{\mathbf{e}}_\alpha]_{\Phi\bowtie} &= \bar{C}_{\beta\alpha}^{\gamma}\bar{\mathbf{e}}_\gamma + \bar{C}_{\beta\alpha}^{a}\bar{\mathbf{e}}_a = [\mathbf{e}_\beta \oplus 0, \mathbf{e}_\alpha \oplus 0]_{\Phi\bowtie} = C_{\beta\alpha}^{\gamma}\mathbf{e}_\gamma \oplus 0,\\
[\bar{\mathbf{e}}_\beta, \bar{\mathbf{e}}_a]_{\Phi\bowtie} &= \bar{C}_{\beta a}^{\gamma}\bar{\mathbf{e}}_\gamma + \bar{C}_{\beta a}^{d}\bar{\mathbf{e}}_d = [\mathbf{e}_\beta \oplus 0, 0 \oplus \mathbf{f}_a]_{\Phi\bowtie} = -L_{a\beta}^{\gamma}\mathbf{e}_\gamma \oplus -R_{a\beta}^{d}\mathbf{f}_d,\\
[\bar{\mathbf{e}}_b, \bar{\mathbf{e}}_a]_{\Phi\bowtie} &= \bar{C}_{ba}^{\gamma}\bar{\mathbf{e}}_\gamma + \bar{C}_{ba}^{d}\bar{\mathbf{e}}_d = [0 \oplus \mathbf{f}_b, 0 \oplus \mathbf{f}_a]_{\Phi\bowtie} = \Phi_{ba}^{\gamma}\mathbf{e}_\gamma \oplus \kappa_{ba}^{d}\mathbf{f}_d.
\end{aligned} \tag{3.24}$$

As a result, the structure constants of $\mathfrak{g}_{\Phi}\bowtie\mathfrak{h}$ may be written as

$$\begin{aligned}
&\bar{C}_{\beta\alpha}^{\gamma} = C_{\beta\alpha}^{\gamma}, \qquad \bar{C}_{\beta\alpha}^{a} = 0, \qquad \bar{C}_{\beta a}^{\gamma} = -L_{a\beta}^{\gamma},\\
&\bar{C}_{\beta a}^{d} = -R_{a\beta}^{d}, \qquad \bar{C}_{ba}^{\gamma} = \Phi_{ba}^{\gamma}, \qquad \bar{C}_{ba}^{d} = \kappa_{ba}^{d}.
\end{aligned} \tag{3.25}$$

3.2. Double cross sum Lie algebras

We shall recall the matched pair construction from [57, 56]. Let $(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\bullet,\bullet]_{\mathfrak{h}})$ be two Lie algebras admitting mutual actions

$$\triangleright\colon \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}, \qquad \triangleleft\colon \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{h}. \tag{3.26}$$

That is, the following identities hold, for any $\xi,\xi'\in\mathfrak{g}$, and any $\eta,\eta'\in\mathfrak{h}$:

$$\begin{aligned}[\eta,\eta']\triangleright\xi &= \eta\triangleright(\eta'\triangleright\xi)-\eta'\triangleright(\eta\triangleright\xi),\\ \eta\triangleleft[\xi,\xi'] &= (\eta\triangleleft\xi)\triangleleft\xi'-(\eta\triangleleft\xi')\triangleleft\xi.\end{aligned} \tag{3.27}$$

The direct sum $\mathfrak{K}=\mathfrak{g}\oplus\mathfrak{h}$ may be endowed with a Lie algebra structure, provided (3.26) obey certain compatibility conditions. More precisely, we have the following.

Theorem 3.3. *The direct sum of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ under mutual actions* (3.26) *is a Lie algebra through the bracket*

$$[(\xi\oplus\eta),\,(\xi'\oplus\eta')]_{\bowtie}=\big([\xi,\xi']+\eta\triangleright\xi'-\eta'\triangleright\xi\big)\oplus\big([\eta,\eta']+\eta\triangleleft\xi'-\eta'\triangleleft\xi\big) \tag{3.28}$$

if the mutual actions satisfy the compatibility conditions

$$\begin{aligned}[\eta,\eta']\triangleleft\xi &= [\eta,\eta'\triangleleft\xi]-[\eta',\eta\triangleleft\xi]+\eta\triangleleft(\eta'\triangleright\xi)-\eta'\triangleleft(\eta\triangleright\xi),\\ \eta\triangleright[\xi,\xi'] &= [\xi,\eta\triangleright\xi']-[\xi',\eta\triangleright\xi]+(\eta\triangleleft\xi)\triangleright\xi'-(\eta\triangleleft\xi')\triangleright\xi\end{aligned} \tag{3.29}$$

for any $\eta,\eta'\in\mathfrak{h}$, and any $\xi,\xi'\in\mathfrak{g}$.

If a Lie algebra $\mathfrak{K}$ is constructed in the realm of Proposition 3.3, then the Lie algebra $\mathfrak{K}$ is said to be the *double cross sum* of $\mathfrak{g}$ and $\mathfrak{h}$, and is denoted by $\mathfrak{K}=\mathfrak{g}\bowtie\mathfrak{h}$, see also [58]. In this case, the pair $(\mathfrak{g},\mathfrak{h})$ of Lie algebras is called a *matched pair* of Lie algebras. Let us remark that, if one of the actions in (3.26) is trivial then we arrive at a semidirect product Lie algebra. In other words, the double cross sum construction is a generalization of the semidirect product construction.

From unified products to double cross sums

Keeping the extending structures of Subsection 3.1 in mind, let us consider the particular case of that structure where the mapping Φ in (3.2) is taken to be identically zero. In the realm of Theorem 3.1, for the case of $\Phi\equiv 0$, the last condition in the list (3.5) gives that κ mapping satisfies the Jacobi identity that is

$$\circlearrowright\kappa(\eta,\kappa(\eta',\eta''))=0. \tag{3.30}$$

This reads that the vector space $\mathfrak{h}$ turns out to be a Lie algebra:

$$[\eta,\eta']:=\kappa(\eta,\eta'). \tag{3.31}$$

Further, for $\Phi\equiv 0$, the third line and the fifth line in the compatibility list (3.5) reduce to the action conditions (3.27), and the second and fourth lines in (3.5) become the matched pair compatibility conditions in (3.29). This observation says that Theorem 3.3 is a particular case of Theorem 3.1. That is, every double cross sum Lie algebra is a unified product. For the Lie algebra double cross sums, Proposition 3.2 takes the following particular form, see [58, Prop. 8.3.2].

Proposition 3.4. *Let $\mathfrak{K}$ be a Lie algebra with two Lie subalgebras $\mathfrak{g}$ and $\mathfrak{h}$ such that $\mathfrak{K}$ is isomorphic to the direct sum of $\mathfrak{g}$ and $\mathfrak{h}$ as vector spaces through the vector addition in $\mathfrak{K}$. Then $\mathfrak{K}$ is isomorphic to the matched pair $\mathfrak{g}\bowtie\mathfrak{h}$ as Lie algebras, and the mutual actions* (3.26) *are derived from*

$$[\eta,\xi]=\eta\triangleright\xi\oplus\eta\triangleleft\xi. \tag{3.32}$$

Here, the inclusions of the subalgebras are defined to be

$$\mathfrak{g}\longrightarrow\mathfrak{K}:\xi\to(\xi\oplus 0),\qquad \mathfrak{h}\longrightarrow\mathfrak{K}:\eta\to(0\oplus\eta). \tag{3.33}$$

Coordinate realizations

Assume that the coordinates are chosen as in Subsection 3.1. To have the local characterization of a double cross sum Lie algebra, we first examine the structure constants given in (3.24). Accordingly we note that the first and the second lines remain the same, however, since Φ^{α}_{ab} are all zero, the constants κ^{d}_{ba} now turn out to be the structure constants of the Lie algebra $\mathfrak{h}$. In order to highlight this, we denote the structure constants by D^{d}_{ba}. We thus have

$$[\mathbf{e}_{\alpha}, \mathbf{e}_{\beta}] = C^{\gamma}_{\alpha\beta}\mathbf{e}_{\gamma}, \qquad [\mathbf{f}_a, \mathbf{f}_b] = \Phi^{\alpha}_{ab}\mathbf{e}_{\alpha} + \kappa^{d}_{ab}\mathbf{f}_d, \tag{3.34}$$

and therefore

$$[\bar{\mathbf{e}}_b, \bar{\mathbf{e}}_a] = \bar{C}^{\gamma}_{ba}\bar{\mathbf{e}}_{\gamma} + \bar{C}^{d}_{ba}\bar{\mathbf{e}}_d = [0 \oplus \mathbf{f}_b, 0 \oplus \mathbf{f}_a]_{\bowtie} = 0 \oplus D^{d}_{ba}\mathbf{f}_d. \tag{3.35}$$

Hence, the structure constants of the double cross sum Lie algebra are the same as (3.25), except in the present case, $\bar{C}^{\gamma}_{ba} = 0$, and $\bar{C}^{d}_{ba} = D^{d}_{ba}$. More precisely, we have

$$\begin{aligned} &\bar{C}^{\gamma}_{\beta\alpha} = C^{\gamma}_{\beta\alpha}, \qquad \bar{C}^{a}_{\beta\alpha} = 0, \qquad \bar{C}^{\gamma}_{\beta a} = -L^{\gamma}_{a\beta}, \\ &\bar{C}^{d}_{\beta a} = -R^{d}_{a\beta}, \qquad \bar{C}^{\gamma}_{ba} = 0, \qquad \bar{C}^{d}_{ba} = D^{d}_{ba}. \end{aligned} \tag{3.36}$$

3.3. 2-cocycle Extensions

In this subsection, we shall discuss a second particular instance of the unified products discussed in Subsection 3.1. In this case, we assume that the right action (3.1) of $\mathfrak{g}$ on $\mathfrak{h}$ and the Lie bracket on $\mathfrak{g}$ are trivial, while all the other geometric ingredients of Theorem 3.1 remain the same. In other words, we let

$$\eta \triangleright \xi = 0, \qquad [\xi, \xi'] = 0 \tag{3.37}$$

for all ξ and ξ' in $\mathfrak{g}$, and for all η in $\mathfrak{h}$. As a result, one observes that the Lie bracket (3.4) and the list of conditions (3.5) will reduce to particular forms. Let us examine them from the bottom to the top. Since the right action is trivial, the last condition in (3.5) turns out to be the Jacobi identity (3.30) for κ, which indicates that the two-tuple $(\mathfrak{h}, \kappa)$ becomes a Lie algebra. Accordingly, in this section, we denote κ by a bracket notation $[\bullet, \bullet]$ as in (3.31). On the other hand, the penultimate condition, namely the twisted 2-cocycle condition, in (3.5) takes the particular form

$$\circlearrowright \Phi(\eta, [\eta', \eta''])+ \circlearrowright \eta \triangleright \Phi(\eta', \eta'') = 0. \tag{3.38}$$

This determines Φ as a $\mathfrak{g}$-valued 2-cocycle on $\mathfrak{h}$, see for instance [28]. The second, the fourth and the fifth conditions in (3.5) are identically satisfied whereas the third line

$$[\eta, \eta'] \triangleright \xi = \eta \triangleright (\eta' \triangleright \xi) - \eta' \triangleright (\eta \triangleright \xi) \tag{3.39}$$

reads as $\triangleright$ is a left action of $\mathfrak{h}$ on $\mathfrak{g}$. Eventually, we arrive at the following reduced form

$$[\xi \oplus \eta, \xi' \oplus \eta']_{{}_{\Phi}\rtimes} = \big(\eta \triangleright \xi' - \eta' \triangleright \xi + \Phi(\eta, \eta')\big) \oplus [\eta, \eta'] \tag{3.40}$$

of the Lie algebra bracket (3.4). In this case, we denote the total space by $\mathfrak{g}_{\Phi}\!\rtimes\mathfrak{h}$, and call it the 2-cocycle extension of $\mathfrak{h}$ by the vector space $\mathfrak{g}$.

Once again, as a manifestation of Proposition 3.2 we can discuss the decomposition point of view as follows. Assume a Lie algebra $\mathfrak{K}$ and one of its nontrivial centers, say $\mathfrak{g}$. Consider the decomposition $\mathfrak{g}\oplus\mathfrak{h}$ inducing nontrivial Φ and κ mappings as in (3.18) and (3.19) and a left action (3.3) then Proposition 3.2 reads that, $\mathfrak{K}$ can be decomposed into a 2-cocycle extension of $\mathfrak{h}$ by $\mathfrak{g}$ that is $\mathfrak{K} = \mathfrak{g}_{\Phi}\rtimes\mathfrak{h}$.

Coordinate realizations

Assume once more that the coordinates are chosen as in Subsection 3.1. In this case, the Lie bracket on $\mathfrak{g}$, and the right action of $\mathfrak{g}$ on $\mathfrak{h}$ are trivial. Therefore, the structure constants $C^{\gamma}_{\beta\alpha}$ of $\mathfrak{g}$ given in (3.21) will all vanish, while the constants Φ^{α}_{ab} and D^{d}_{ab} determining Φ and κ remain the same. Since the right action is zero, $R^{d}_{a\beta}$ in (3.22) will be zero, whereas the scalars $L^{\beta}_{a\alpha}$ are determine the left action. Applying the aforementioned changes to the system of equations (3.24), one obtains the structure constants of a 2-cocycle extension.

4. Lie-Poisson dynamics on extensions

Dual of a Lie algebra admits Lie-Poisson bracket according to the definition in (2.5). In the present section, following the order of the extensions and couplings in Section 3, we compute the associated Lie-Poisson brackets.

4.1. Lie-Poisson systems on duals of extended structures

Assuming the Lie algebraic framework in Subsection 3.1, we let all the conditions in Theorem 3.1 hold. Now we start with the left action $\triangleright$ in (3.3), and then freezing an element η in $\mathfrak{h}$ in this operation we obtain a linear mapping $\eta\triangleright$ on the subalgebra $\mathfrak{g}$. This linear mapping and the dual action $\overset{*}{\triangleleft}\eta$ are

$$\begin{aligned} &\eta\triangleright:\mathfrak{g}\longrightarrow\mathfrak{g}, \qquad \xi\mapsto\eta\triangleright\xi \\ &\overset{*}{\triangleleft}:\mathfrak{g}^*\otimes\mathfrak{h}\longrightarrow\mathfrak{g}^*, \qquad \langle\mu\overset{*}{\triangleleft}\eta,\xi\rangle=\langle\mu,\eta\triangleright\xi\rangle. \end{aligned} \tag{4.1}$$

This dual mapping is a right representation of $\mathfrak{h}$ on the dual space $\mathfrak{g}^*$. Later, by freezing $\xi\in\mathfrak{g}$ in the left action $\triangleright$ in (3.3), we define a linear mapping $\mathfrak{b}_\xi:\mathfrak{h}\mapsto\mathfrak{g}$. We record here this linear mapping $\mathfrak{b}_\xi$ and the dual mapping $\mathfrak{b}^*_\xi$ as

$$\mathfrak{b}_\xi:\mathfrak{h}\longrightarrow\mathfrak{g}, \qquad \mathfrak{b}_\xi(\eta)=\eta\triangleright\xi, \tag{4.2}$$

$$\mathfrak{b}^*_\xi:\mathfrak{g}^*\longrightarrow\mathfrak{h}^*, \qquad \langle\mathfrak{b}^*_\xi\mu,\eta\rangle=\langle\mu,\mathfrak{b}_\xi\eta\rangle=\langle\mu,\eta\triangleright\xi\rangle. \tag{4.3}$$

Consider the right action $\triangleleft$ in (3.1). In this operation we freeze ξ in $\mathfrak{g}$ to get an automorphism on $\mathfrak{h}$, denoted by $\triangleleft\,\xi$. Accordingly we record $\triangleleft\xi$ and its dual $\xi\overset{*}{\triangleright}$ in

$$\begin{aligned} &\triangleleft\,\xi:\mathfrak{h}\longrightarrow\mathfrak{h}, \qquad \eta\mapsto\eta\triangleleft\xi \\ &\overset{*}{\triangleright}:\mathfrak{g}\times\mathfrak{h}^*\longrightarrow\mathfrak{h}^*, \qquad \langle\xi\overset{*}{\triangleright}\nu,\eta\rangle=\langle\nu,\eta\triangleleft\xi\rangle, \end{aligned} \tag{4.4}$$

where $\overset{*}{\triangleright}$ is a left representation of $\mathfrak{g}$ on the dual $\mathfrak{h}^*$. Further, we freeze an element, say η in $\mathfrak{h}$, in the right action (3.1). This enables us to define a linear mapping $\mathfrak{a}_\eta$ from $\mathfrak{g}$ to $\mathfrak{h}$. Here are the mapping $\mathfrak{a}_\eta$ and its dual $\mathfrak{a}^*_\eta$ in a respective order

$$\mathfrak{a}_\eta:\mathfrak{g}\mapsto\mathfrak{h}, \qquad \mathfrak{a}_\eta(\xi)=\eta\triangleleft\xi \tag{4.5}$$

$$\mathfrak{a}^*_\eta:\mathfrak{h}^*\mapsto\mathfrak{g}^*, \qquad \langle\mathfrak{a}^*_\eta\nu,\xi\rangle=\langle\nu,\mathfrak{a}_\eta\xi\rangle=\langle\nu,\eta\triangleleft\xi\rangle. \tag{4.6}$$

Let us recall the mappings Φ and κ displayed in (3.18) and (3.19), respectively. Define two functions κ_η and Φ_η as

$$\kappa_\eta : \mathfrak{h} \to \mathfrak{h} \qquad \kappa_\eta(\eta') := \kappa(\eta, \eta') \tag{4.7}$$

$$\Phi_\eta : \mathfrak{h} \to \mathfrak{g} \qquad \Phi_\eta(\eta') := \Phi(\eta, \eta') \tag{4.8}$$

where $\eta, \eta' \in \mathfrak{h}$, $\nu \in \mathfrak{h}^*$ and $\mu \in \mathfrak{g}^*$. According to these definitions, the dual mappings are calculated as

$$\kappa_\eta^* : \mathfrak{h}^* \to \mathfrak{h}^* \qquad \langle \kappa_\eta^* \nu, \eta' \rangle = \langle \nu, -\kappa_\eta(\eta') \rangle = -\langle \nu, \kappa(\eta, \eta') \rangle, \tag{4.9}$$

$$\Phi_\eta^* : \mathfrak{g}^* \to \mathfrak{h}^* \qquad \langle \Phi_\eta^* \mu, \eta' \rangle = \langle \mu, -\Phi_\eta(\eta') \rangle = -\langle \mu, \Phi(\eta, \eta') \rangle, \tag{4.10}$$

respectively.

Proposition 4.1. *The coadjoint action on of an element $\xi \oplus \eta$ in the unified product $\mathfrak{g}\,{}_\Phi\!\bowtie \mathfrak{h}$ onto an element $\mu \oplus \nu$ in the dual space $\mathfrak{g}^* \oplus \mathfrak{h}^*$ is computed to be*

$$\mathrm{ad}^*_{(\xi\oplus\eta)}(\mu \oplus \nu) = \underbrace{\big(ad^*_\xi \mu - \mu \overset{*}{\triangleleft} \eta - \mathfrak{a}^*_\eta \nu\big)}_{\in\ \mathfrak{g}^*} \oplus \underbrace{\big(\kappa^*_\eta \nu + \Phi^*_\eta \mu + \xi \overset{*}{\triangleright} \nu + \mathfrak{b}^*_\xi \mu\big)}_{\in\ \mathfrak{h}^*}, \tag{4.11}$$

where (the italic) ad^* denotes the infinitesimal coadjoint action of $\mathfrak{g}$ on its dual $\mathfrak{g}^*$.

Using the equations (4.9) and (4.10), the (plus/minus) extended Lie-Poisson bracket is computed to be

$$\begin{aligned} \{\mathcal{H}, \mathcal{F}\}_{\Phi\bowtie}(\mu \oplus \nu) &= \pm \left\langle \mu \oplus \nu, \left[\left(\frac{\delta \mathcal{H}}{\delta \mu} \oplus \frac{\delta \mathcal{H}}{\delta \nu}\right), \left(\frac{\delta \mathcal{F}}{\delta \mu} \oplus \frac{\delta \mathcal{F}}{\delta \nu}\right)\right]_{\Phi\bowtie} \right\rangle \\ &\pm \left\langle \mu, \left[\frac{\delta \mathcal{H}}{\delta \mu}, \frac{\delta \mathcal{F}}{\delta \mu}\right] \right\rangle \pm \left\langle \nu, \kappa\left(\frac{\delta \mathcal{H}}{\delta \nu}, \frac{\delta \mathcal{F}}{\delta \nu}\right) \right\rangle \pm \underbrace{\left\langle \mu, \Phi\left(\frac{\delta \mathcal{H}}{\delta \nu}, \frac{\delta \mathcal{F}}{\delta \nu}\right) \right\rangle}_{\text{A: from twisted cocycle}} \\ &\pm \underbrace{\left\langle \mu, \frac{\delta \mathcal{H}}{\delta \nu} \triangleright \frac{\delta \mathcal{F}}{\delta \mu} \right\rangle \mp \left\langle \mu, \frac{\delta \mathcal{F}}{\delta \nu} \triangleright \frac{\delta \mathcal{H}}{\delta \mu} \right\rangle}_{\text{B: action of } \mathfrak{h} \text{ on } \mathfrak{g} \text{ from the left}} \\ &\pm \underbrace{\left\langle \nu, \frac{\delta \mathcal{H}}{\delta \nu} \triangleleft \frac{\delta \mathcal{F}}{\delta \mu} \right\rangle \mp \left\langle \nu, \frac{\delta \mathcal{F}}{\delta \nu} \triangleleft \frac{\delta \mathcal{H}}{\delta \mu} \right\rangle}_{\text{C: action of } \mathfrak{g} \text{ on } \mathfrak{h} \text{ from the right}} \end{aligned} \tag{4.12}$$

for two functions $\mathcal{H}, \mathcal{F}$. In view of the reflexivity $\delta\mathcal{H}/\delta\mu$ and $\delta\mathcal{F}/\delta\mu$ are elements of $\mathfrak{g}$, whereas $\delta\mathcal{H}/\delta\nu$ and $\delta\mathcal{F}/\delta\nu$ are elements of $\mathfrak{h}$. The Lie bracket on the first line in (4.12) is the extended Lie bracket $[\bullet, \bullet]_{\Phi\bowtie}$ in (3.4). In the Poisson bracket, the term labelled by A is a manifestation of the existence of twisted cocycle Φ. The terms labelled by B are due to the *left action* of $\mathfrak{h}$ on $\mathfrak{g}$, while the terms labelled by C are due to the right action of $\mathfrak{g}$ on $\mathfrak{h}$.

Recall the (plus/minus) Lie-Poisson equation in (2.7) determined as a coadjoint flow. In view of the Lie-Poisson bracket (4.12), governed by a Hamiltonian function $\mathcal{H} = \mathcal{H}(\mu, \nu))$, for the present picture, the (plus/minus) Lie-Poisson equation is computed as

$$\begin{aligned} \dot{\mu} &= \underbrace{\pm ad^*_{\frac{\delta\mathcal{H}}{\delta\mu}}(\mu)}_{\text{Lie-Poisson Eq. on } \mathfrak{g}^*} \underbrace{\mp \mu \overset{*}{\triangleleft} \frac{\delta\mathcal{H}}{\delta\nu}}_{\text{action of } \mathfrak{h}} \underbrace{\mp \mathfrak{a}^*_{\frac{\delta\mathcal{H}}{\delta\nu}}\nu}_{\text{action of } \mathfrak{g}}, \\ \dot{\nu} &= \pm\kappa^*_{\frac{\delta\mathcal{H}}{\delta\nu}}(\nu) \pm \underbrace{\Phi^*_{\frac{\delta\mathcal{H}}{\delta\nu}}(\mu)}_{\text{twisted cocycle}} \underbrace{\pm \frac{\delta\mathcal{H}}{\delta\mu} \overset{*}{\triangleright} \nu}_{\text{action of } \mathfrak{g}} \pm \underbrace{\mathfrak{b}^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\mu}_{\text{action of } \mathfrak{h}}. \end{aligned} \tag{4.13}$$

Accordingly, one can easily follow how the Lie-Poisson dynamics on $\mathfrak{g}^*$ is extended, through additional terms coming from the mutual actions of $\mathfrak{h}$ and $\mathfrak{g}$ on each other, as well as from the twisted 2-cocycle term.

Coordinate realizations

We follow the notation in Subsection 3.1. Recall, $(N+M)$-dimensional extended structure $\mathfrak{K} = \mathfrak{g}_{\Phi} \bowtie \mathfrak{h}$. Denote the dual basis of $\mathfrak{g}^*$ and $\mathfrak{h}^*$ by $\{\mathbf{e}^\alpha\}$ and $\{\mathbf{f}^a\}$, respectively. Then, define the dual basis

$$\{\bar{\mathbf{e}}^\alpha, \bar{\mathbf{e}}^a\} \subset \mathfrak{g}^* \oplus \mathfrak{h}^*, \qquad \bar{\mathbf{e}}^\alpha = \mathbf{e}^\alpha \oplus 0, \qquad \bar{\mathbf{e}}^a = 0 \oplus \mathbf{f}^a \tag{4.14}$$

on the dual space $\mathfrak{g}^* \oplus \mathfrak{h}^*$. Using this basis, we can write an element of $\mathfrak{g}^* \oplus \mathfrak{h}^*$ as

$$(\mu, \nu) = \mu_\alpha \bar{\mathbf{e}}^\alpha + \nu_a \bar{\mathbf{e}}^a. \tag{4.15}$$

In this picture, the mappings (4.1) and (4.4) turn out to be

$$(\mu_\alpha \mathbf{e}^\alpha) \overset{*}{\triangleleft} (\eta^a \mathbf{f}_a) = \mu_\alpha \eta^a L^\alpha_{a\beta} \mathbf{e}^\beta, \qquad (\xi^\alpha \mathbf{e}_\alpha) \overset{*}{\triangleright} (\nu_a \mathbf{f}^a) = \nu_a \xi^\alpha R^a_{b\alpha} \mathbf{f}^b, \tag{4.16}$$

where $L^\alpha_{a\beta}$ and $R^a_{b\alpha}$ are the scalars in (3.22) determining the *actions*. We next compute the dual mappings in (4.3) and (4.6), as well as (4.9) and (4.10) in terms of the local coordinates as

$$\mathfrak{b}^*_{(\xi^\alpha \mathbf{e}_\alpha)}(\mu_\alpha \mathbf{e}^\alpha) = \mu_\alpha \xi^\beta L^\alpha_{a\beta} \mathbf{f}^a, \qquad \mathfrak{a}^*_{(\eta^a \mathbf{f}_a)}(\nu_a \mathbf{e}^a) = \nu_a \eta^b R^a_{b\alpha} \mathbf{e}^\alpha, \tag{4.17}$$

$$\kappa^*_\eta \nu = -\kappa^a_{bd} \nu_a \eta^b \mathbf{f}^d, \qquad \Phi^*_\eta \mu = -\Phi^\alpha_{bk} \mu_\alpha \eta^b \mathbf{f}^k. \tag{4.18}$$

Therefore, the (plus/minus) Lie-Poisson bracket (4.12) may be presented as

$$\begin{aligned} \{\mathcal{H}, \mathcal{F}\}_{\Phi\bowtie}(\mu \oplus \nu) &= \pm\mu_\alpha C^\alpha_{\beta\gamma} \frac{\partial\mathcal{H}}{\partial\mu_\beta}\frac{\partial\mathcal{F}}{\partial\mu_\gamma} \pm \nu_a \kappa^a_{bd} \frac{\partial\mathcal{H}}{\partial\nu_b}\frac{\partial\mathcal{F}}{\partial\nu_d} \pm \mu_\alpha \Phi^\alpha_{bk} \frac{\partial\mathcal{H}}{\partial\nu_b}\frac{\partial\mathcal{F}}{\partial\nu_k} \\ &\pm\mu_\alpha L^\alpha_{a\beta}\Big(\frac{\partial\mathcal{H}}{\partial\nu_a}\frac{\partial\mathcal{F}}{\partial\mu_\beta} - \frac{\partial\mathcal{F}}{\partial\nu_a}\frac{\partial\mathcal{H}}{\partial\mu_\beta}\Big) \pm \nu_a R^a_{b\beta}\Big(\frac{\partial\mathcal{H}}{\partial\nu_b}\frac{\partial\mathcal{F}}{\partial\mu_\beta} - \frac{\partial\mathcal{F}}{\partial\nu_b}\frac{\partial\mathcal{H}}{\partial\mu_\beta}\Big), \end{aligned} \tag{4.19}$$

whereas the (plus/minus) Lie-Poisson dynamics (4.13) as

$$\begin{aligned} \dot{\mu}_\beta &= \pm\mu_\rho C^\rho_{\beta\alpha} \frac{\partial\mathcal{H}}{\partial\mu_\alpha} \mp \mu_\alpha L^\alpha_{a\beta} \frac{\partial\mathcal{H}}{\partial\nu_a} \mp \nu_a R^a_{b\beta} \frac{\partial\mathcal{H}}{\partial\nu_b}, \\ \dot{\nu}_d &= \pm\mu_\alpha \Phi^\alpha_{db} \frac{\partial\mathcal{H}}{\partial\nu_b} \pm \nu_a \kappa^a_{db} \frac{\partial\mathcal{H}}{\partial\nu_b} \pm \nu_a R^a_{d\alpha} \frac{\partial\mathcal{H}}{\partial\mu_\alpha} \pm \mu_\alpha L^\alpha_{d\beta} \frac{\partial\mathcal{H}}{\partial\mu_\beta}. \end{aligned} \tag{4.20}$$

4.2. Matched Lie-Poisson systems

In Subsection 3.2, the double cross sum Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$ was realized as a particular instance of the unified product of $\mathfrak{g}$ and $\mathfrak{h}$, choosing the twisted 2-cocycle Φ to be trivial. Accordingly, both $\mathfrak{g}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g} \bowtie \mathfrak{h}$. Therefore, in this case, the duals of each of these subspaces, namely $\mathfrak{g}^*$ and $\mathfrak{h}^*$, admit Lie-Poisson flows. This lets us to claim that the Lie-Poisson dynamics on the dual $\mathfrak{g}^* \oplus \mathfrak{h}^*$ of matched pair can be considered as the collective motion of two Lie-Poisson subdynamics [24]. Algebraically, this corresponds for $(\mathfrak{g}^*, \mathfrak{h}^*)$ to be a matched pair of Lie coalgebras, where we refer the reader to [69] for the details and the terminology of Lie coalgebras. On the dual of a matched pair, both of the dual actions $\overset{*}{\triangleleft}$ and $\overset{*}{\triangleright}$, exhibited in (4.1) and (4.4) respectively, are equally valid. Notice that, for the present discussion $\triangleright$ is a true left action of $\mathfrak{h}$ on $\mathfrak{g}$ so that $\overset{*}{\triangleleft}$ is a true right dual action of $\mathfrak{h}$ on $\mathfrak{g}^*$, which is not the case for the unified product $\mathfrak{g}\,{}_{\Phi}\!\bowtie \mathfrak{h}$ since $\mathfrak{h}$ is not assumed to be a Lie subalgebra. It is immediate to observe that the dual mappings $\mathfrak{b}_\xi^*$ and $\mathfrak{a}_\eta^*$, in (4.3) and (4.6) respectively, remain to be the same. The difference between a double cross sum and a unified product is that the κ mapping, in (3.19), is a Lie bracket and that Φ mapping, in (3.18), is zero. As stated previously, we prefer to denote κ by a bracket, so we write the mapping (4.9) as

$$\kappa_\eta^* \nu = ad_\eta^* \nu. \tag{4.21}$$

These observations lead us to the following proposition as a particular case of Proposition 4.1, see also [23, 24].

Proposition 4.2. *The coadjoint action* ad^* *of an element* $\xi \oplus \eta$ *in* $\mathfrak{g} \bowtie \mathfrak{h}$ *onto an element* $\mu \oplus \nu$ *in the dual space* $\mathfrak{g}^* \oplus \mathfrak{h}^*$ *is computed to be*

$$\mathrm{ad}^*_{(\xi\oplus\eta)}(\mu\oplus\nu) = \underbrace{\left(ad_\xi^*\mu - \mu \overset{*}{\triangleleft} \eta - \mathfrak{a}_\eta^*\nu\right)}_{\in\ \mathfrak{g}^*} \oplus \underbrace{\left(ad_\eta^*\nu + \xi \overset{*}{\triangleright} \nu + \mathfrak{b}_\xi^*\mu\right)}_{\in\ \mathfrak{h}^*}, \tag{4.22}$$

while the adjoint action on $\mathfrak{g} \bowtie \mathfrak{h}$ *is given by* (3.28).

As a result, the (plus/minus) Lie-Poisson bracket on the dual space $\mathfrak{g}^* \oplus \mathfrak{h}^*$ of $\mathfrak{g} \bowtie \mathfrak{h}$ is computed to be

$$\{\mathcal{H},\mathcal{F}\}_{\bowtie}(\mu\oplus\nu) = \underbrace{\pm\left\langle \mu, \left[\frac{\delta\mathcal{H}}{\delta\mu}, \frac{\delta\mathcal{F}}{\delta\mu}\right]\right\rangle \pm \left\langle \nu, \left[\frac{\delta\mathcal{H}}{\delta\nu}, \frac{\delta\mathcal{F}}{\delta\nu}\right]\right\rangle}_{\text{A: direct product}} \tag{4.23}$$

$$\underbrace{\mp\left\langle \mu, \frac{\delta\mathcal{H}}{\delta\nu} \triangleright \frac{\delta\mathcal{F}}{\delta\mu}\right\rangle \pm \left\langle \mu, \frac{\delta\mathcal{F}}{\delta\nu} \triangleright \frac{\delta\mathcal{H}}{\delta\mu}\right\rangle}_{\text{B: via the left action of } \mathfrak{h} \text{ on } \mathfrak{g}} \underbrace{\mp\left\langle \nu, \frac{\delta\mathcal{H}}{\delta\nu} \triangleleft \frac{\delta\mathcal{F}}{\delta\mu}\right\rangle \pm \left\langle \nu, \frac{\delta\mathcal{F}}{\delta\nu} \triangleleft \frac{\delta\mathcal{H}}{\delta\mu}\right\rangle}_{\text{C: via the right action of } \mathfrak{g} \text{ on } \mathfrak{h}}.$$

Notice that, the terms labelled by A are just the sum of individual Poisson brackets on the dual spaces $\mathfrak{g}^*$ and $\mathfrak{h}^*$ of the constitutive Lie subalgebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, while the terms labeled by B come from the left action of $\mathfrak{h}$ on $\mathfrak{g}$. Finally, the terms labelled by C result from the right action of $\mathfrak{g}$ on $\mathfrak{h}$. In the case of one-sided actions, that is the case of the semidirect product theories, B or C drops.

If there is no action then, both B and C drop. In the light of the (plus/minus) *matched* Lie-Poisson bracket (4.23), the *matched* Lie-Poisson equations generated by a Hamiltonian function $\mathcal{H} = \mathcal{H}(\mu, \nu)$ on $\mathfrak{g}^* \oplus \mathfrak{h}^*$ are computed to be

$$\begin{aligned} \dot{\mu} &= \underbrace{\pm ad^*_{\frac{\delta \mathcal{H}}{\delta \mu}}(\mu)}_{\text{Lie-Poisson Eq. on } \mathfrak{g}^*} \ \underbrace{\mp \mu \overset{*}{\triangleleft} \frac{\delta \mathcal{H}}{\delta \nu}}_{\text{action of } \mathfrak{h}} \ \underbrace{\mp \mathfrak{a}^*_{\frac{\delta \mathcal{H}}{\delta \nu}} \nu}_{\text{action of } \mathfrak{g}}, \\ \dot{\nu} &= \underbrace{\pm ad^*_{\frac{\delta \mathcal{H}}{\delta \nu}}(\nu)}_{\text{Lie-Poisson Eq. on } \mathfrak{h}^*} \ \underbrace{\pm \frac{\delta \mathcal{H}}{\delta \mu} \overset{*}{\triangleright} \nu}_{\text{action of } \mathfrak{g}} \ \underbrace{\pm \mathfrak{b}^*_{\frac{\delta \mathcal{H}}{\delta \mu}} \mu}_{\text{action of } \mathfrak{h}}. \end{aligned} \tag{4.24}$$

The first terms on the right hand sides are the individual equations of motions. The rest of the terms are the dual and the cross actions as manifestations of the mutual actions.

Coordinate realizations

Recall the Lie-Poisson bracket in (4.19) and the Lie-Poisson equations (4.20) computed for the unified products. In the double cross sum case, we let the constants that determine the twisted 2-cocycle vanish, that is, $\Phi^{\gamma}_{ba} = 0$, and the structure constants of the Lie algebra $\mathfrak{h}$ to be $\kappa^d_{ba} = D^d_{ba}$. Hence, the (plus/minus) matched Lie-Poisson bracket (4.23) takes the form of

$$\{\mathcal{H}, \mathcal{F}\}_{_{\Phi\bowtie}}(\mu \oplus \nu) = \pm \mu_\alpha C^\alpha_{\beta\gamma} \frac{\partial \mathcal{H}}{\partial \mu_\beta} \frac{\partial \mathcal{F}}{\partial \mu_\gamma} \tag{4.25}$$

$$\pm \nu_a D^a_{bd} \frac{\partial \mathcal{H}}{\partial \nu_b} \frac{\partial \mathcal{F}}{\partial \nu_d} \pm \mu_\alpha L^\alpha_{a\beta} \Big(\frac{\partial \mathcal{H}}{\partial \nu_a} \frac{\partial \mathcal{F}}{\partial \mu_\beta} - \frac{\partial \mathcal{F}}{\partial \nu_a} \frac{\partial \mathcal{H}}{\partial \mu_\beta} \Big) \pm \nu_a R^a_{b\beta} \Big(\frac{\partial \mathcal{H}}{\partial \nu_b} \frac{\partial \mathcal{F}}{\partial \mu_\beta} - \frac{\partial \mathcal{F}}{\partial \nu_b} \frac{\partial \mathcal{H}}{\partial \mu_\beta} \Big).$$

The matched (plus/minus) Lie-Poisson dynamics in (4.24), accordingly, is computed to be

$$\begin{aligned} \dot{\mu}_\beta &= \pm \mu_\rho C^\rho_{\beta\alpha} \frac{\partial \mathcal{H}}{\partial \mu_\alpha} \mp \mu_\alpha L^\alpha_{a\beta} \frac{\partial \mathcal{H}}{\partial \nu_a} \mp \nu_a R^a_{b\beta} \frac{\partial \mathcal{H}}{\partial \nu_b}, \\ \dot{\nu}_d &= \pm \nu_a D^a_{db} \frac{\partial \mathcal{H}}{\partial \nu_b} \pm \nu_a R^a_{d\alpha} \frac{\partial \mathcal{H}}{\partial \mu_\alpha} \pm \mu_\alpha L^\alpha_{d\beta} \frac{\partial \mathcal{H}}{\partial \mu_\beta}. \end{aligned} \tag{4.26}$$

4.3. Lie Poisson dynamics on duals of 2-cocycles

In Subsection 3.3, it is shown that the 2-cocycle extension $\mathfrak{g}\,_\Phi \rtimes \mathfrak{h}$ of a Lie algebra $\mathfrak{h}$ by its representation space $\mathfrak{g}$, as a particular case of the unified product $\mathfrak{g}\,_\Phi \bowtie \mathfrak{h}$. Thus, the Lie-Poisson dynamics on the dual space of a 2-cocycle extension may be derived from the Lie-Poisson dynamics on the dual space of a unified product, which is given in Subsection 4.1. Hence, following Subsection 3.3 we let the Lie bracket on $\mathfrak{g}$ to be trivial. This results with several consequences. The left action $\triangleright$ and the right dual action $\overset{*}{\triangleleft}$ in (4.1) are both trivial, while the coadjoint action on $\mathfrak{g}^*$ becomes identically zero. In addition, in this case, Φ turns out to be a true 2-cocycle, and κ becomes a Lie bracket on $\mathfrak{h}$. Thus, the dual of κ suits the coadjoint action as in (4.21). We apply all these modifications to the Lie-Poisson bracket (4.12) on the dual of a unified product to arrive at the (plus/minus) Lie-Poisson bracket

$$\{\mathcal{H},\mathcal{F}\}_{\Phi\rtimes}(\mu\oplus\nu) = \pm\left\langle \nu,[\frac{\delta\mathcal{H}}{\delta\nu},\frac{\delta\mathcal{F}}{\delta\nu}]\right\rangle \pm \underbrace{\left\langle \mu,\Phi\left(\frac{\delta\mathcal{H}}{\delta\nu},\frac{\delta\mathcal{F}}{\delta\nu}\right)\right\rangle}_{\text{A: 2-cocycle}}, \\ \underbrace{\pm\left\langle \mu,\frac{\delta\mathcal{H}}{\delta\nu}\rhd\frac{\delta\mathcal{F}}{\delta\mu}\right\rangle \mp\left\langle \mu,\frac{\delta\mathcal{F}}{\delta\nu}\rhd\frac{\delta\mathcal{H}}{\delta\mu}\right\rangle}_{\text{B: left action of }\mathfrak{h}\text{ on }\mathfrak{g}} \tag{4.27}$$

on the dual of the 2-cocycle extension $\mathfrak{g}\,{}_\Phi\rtimes\mathfrak{h}$. Here, the first term on the right hand side is the Lie-Poisson bracket on $\mathfrak{h}^*$. The term labelled as A is due to 2-cocycle Φ whereas the terms labelled as B are due to the left action of $\mathfrak{h}$ on $\mathfrak{g}$. For the Lie-Poisson bracket (4.27), the Lie-Poisson equations governed by a Hamiltonian function $\mathcal{H}=\mathcal{H}(\mu,\nu)$ is computed to be

$$\underbrace{\dot{\mu} = \mp\mu\overset{*}{\lhd}\frac{\delta\mathcal{H}}{\delta\nu}}_{\text{action of }\mathfrak{h}}, \qquad \underbrace{\dot{\nu} = \pm ad^*_{\frac{\delta\mathcal{H}}{\delta\nu}}(\nu)}_{\text{Lie-Poisson Eq. on }\mathfrak{h}^*} \underbrace{\pm\Phi^*_{\frac{\delta\mathcal{H}}{\delta\nu}}(\mu)}_{\text{2-cocycle}} \underbrace{\pm\mathfrak{b}^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\mu}_{\text{action of }\mathfrak{h}}. \tag{4.28}$$

A direct observation gives that the Lie-Poisson equation (4.28) is a particular case of the Lie Poisson equation (4.13) where $\overset{*}{\rhd}$ and $\mathfrak{a}^*$ both vanish.

Coordinate realizations

The coordinate expression of a 2-cocycle extension has been mentioned in Subsection 3.3. Therefore, referring to Subsection 4.1, we write the Lie-Poisson bracket (4.27) in coordinates as

$$\{\mathcal{H},\mathcal{F}\}_{\Phi\bowtie}(\mu\oplus\nu) \\ = \pm\nu_a D^a_{bd}\frac{\partial\mathcal{H}}{\partial\nu_b}\frac{\partial\mathcal{F}}{\partial\nu_d} \pm \mu_\alpha\Phi^\alpha_{bk}\frac{\partial\mathcal{H}}{\partial\nu_b}\frac{\partial\mathcal{F}}{\partial\nu_k} \pm \mu_\alpha L^\alpha_{a\beta}\Big(\frac{\partial\mathcal{H}}{\partial\nu_a}\frac{\partial\mathcal{F}}{\partial\mu_\beta} - \frac{\partial\mathcal{F}}{\partial\nu_a}\frac{\partial\mathcal{H}}{\partial\mu_\beta}\Big). \tag{4.29}$$

Notice that, we write the structure constants on $\mathfrak{h}$ as D^a_{bd}. Further, we can write the Lie Poisson equations in (4.28) as

$$\dot{\mu}_\beta = \mp\mu_\alpha L^\alpha_{a\beta}\frac{\partial\mathcal{H}}{\partial\nu_a}, \qquad \dot{\nu}_d = \pm\mu_\alpha\Phi^\alpha_{db}\frac{\partial\mathcal{H}}{\partial\nu_b} \pm \nu_a D^a_{db}\frac{\partial\mathcal{H}}{\partial\nu_b} \pm \mu_\alpha L^\alpha_{d\beta}\frac{\partial\mathcal{H}}{\partial\mu_\beta}. \tag{4.30}$$

5. Illustration: decomposing 3 particles BBGKY hierarchy

In the present section, we shall consider the BBGKY hierarchy in plasma dynamics [39], in order to illustrate the Lie algebraic constructions and the Lie-Poisson structures discussed so far. It is proved in [61] that the BBGKY hierarchy can be recast as a Lie-Poisson equation. The formulations presented therein were for $n>3$. Here, on the other hand, we shall focus on the case $n=3$, which is missing in [61]. Accordingly, we shall first determine the dynamics of the BBGKY hierarchy for $n=3$, and then we shall investigate its Lie-Poisson form. Two decomposition of the BBGKY dynamics will be presented; a double cross sum decomposition, and a genuine unified product decomposition.

5.1. BBGKY dynamics for 3 particles

Assume that a plasma rests in a finite $3D$ manifold Q in $\mathbb{R}^3$. Being a cotangent bundle, $P=T^*Q$ is a symplectic and a Poisson manifold [62, 53]. Define the product symplectic space $P^3 = P \times P \times P$ endowed with the product symplectic and product Poisson structures.

We denote the 3 particle density function by $f_3 = f_3(z_1, z_2, z_3)$ on P^3. The dynamics of 3-particle plasma density function is governed by the Vlasov equation

$$\frac{\partial f_3}{\partial t} = \{H(z_1, z_2, z_3), f_3(z_1, z_2, z_3)\}, \tag{5.1}$$

where H is the total energy of the plasma particles [63, 66, 71]. Here, $\{\bullet, \bullet\}$ denotes the canonical Poisson bracket on P_3 with respect to the variables z_1, z_2, z_3. We refer the reader to [19, 20, 30, 37] for some recent studies on Vlasov motion related with the geometry here. In the present work, we assume that, the particle energy function is in the form

$$\begin{aligned} H &= \sum_i H_1(z_i) + \sum_{i<j} H_2(z_i, z_j) + H_3(z_1, z_2, z_3) \\ &= H_1(z_1) + H_1(z_2) + H_1(z_3) + H_2(z_1, z_2) + H_2(z_1, z_3) \\ &\quad + H_2(z_2, z_3) + H_3(z_1, z_2, z_3). \end{aligned} \tag{5.2}$$

Here, the functions H_2 and H_3 are assumed to be symmetric. We remark that, the dynamical equations in[61] is for $H_3 = 0$, while the presentation here will use a nontrivial H_3.

Dynamics of moments

Now, we determine the moments of the plasma density function $f_3 = f_3(z_1, z_2, z_3)$ on P^3 as

$$\begin{aligned} f_1(z_1) &:= 3 \int f_3(z_1, z_2, z_3) dz_2 dz_3 \\ f_2(z_1, z_2) &:= 6 \int f_3(z_1, z_2, z_3) dz_3. \end{aligned} \tag{5.3}$$

To find the dynamics of the moments f_1 and f_2, we simply take the partial derivatives of (5.3), and then directly substitute the Vlasov equation (5.1) into these expressions.

In order to arrive at the equation governing the moment functions, we record the identity

$$\int \{h(z), f(z)\}_z \, dz = 0, \tag{5.4}$$

which is valid for any two functions, on the Poisson space P. The equation (5.4) is the result of the suppression of the boundary terms. In (5.4), $\{\bullet, \bullet\}_z$ stands for the Poisson bracket on P. Now, we compute the dynamics of f_1 as

$$\begin{aligned} \frac{\partial f_1}{\partial t} &= 3 \int \{H(z_1, z_2, z_3), f_3(z_1, z_2, z_3)\} dz_2 dz_3 \\ &= 3 \int \Big\{ \sum_i H_1(z_i) + \sum_{i<j} H_2(z_i, z_j) + H_3(z_1, z_2, z_3), f_3(z_1, z_2, z_3) \Big\} dz_2 dz_3 \\ &= 3 \int \Big\{ \sum_i H_1(z_1), f_3(z_1, z_2, z_3) \Big\}_{z_1} dz_2 dz_3 \end{aligned}$$

$$
\begin{aligned}
&+3\int\{H_2(z_1,z_2)+H_2(z_1,z_3),f_3(z_1,z_2,z_3)\}_{z_1}dz_2dz_3\\
&+3\int\{H_3(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1}dz_2dz_3\\
=&\Big\{\sum_i H_1(z_1),3\int f_3(z_1,z_2,z_3)dz_2dz_3\Big\}_{z_1}\\
&+\int\Big\{H_2(z_1,z_2),6\int f_3(z_1,z_2,z_3)dz_3\Big\}_{z_1}dz_2\\
&+3\int\{H_3(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1}dz_2dz_3\\
=&\{H_1(z_1),f_1(z_1)\}_{z_1}+\int\{H_2(z_1,z_2),f_2(z_1,z_2)\}_{z_1}dz_2\\
&+3\int\{H_3(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1}dz_2dz_3. \qquad (5.5)
\end{aligned}
$$

In the second equality, we have employed the energy function H in (5.2). In the third equality, we have used the identity (5.4) several times. We have substituted the definitions of the density functions f_1 and f_2 in the last equality. Here, the notation $\{\bullet,\bullet\}_{z_1}$ is the Poisson bracket only for z_1 variable, even if the functions inside the bracket depend on other variables. In a similar fashion, the dynamics of the moment function f_2 may be computed as

$$
\begin{aligned}
\frac{\partial f_2}{\partial t}&=6\int\{H(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\}dz_3\\
&=6\int\Big\{\sum_i H_1(z_i)+\sum_{i<j}H_2(z_i,z_j)+H_3(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\Big\}dz_3\\
&=6\int\{H_1(z_1)+H_1(z_2),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&\quad+6\int\{H_2(z_1,z_2),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&\quad+6\int\{H_2(z_1,z_3)+H_2(z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&\quad+6\int\{H_3(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&=\{H_1(z_1)+H_1(z_2),6\int f_3(z_1,z_2,z_3)dz_3\}_{z_1,z_2}\\
&\quad+\{H_2(z_1,z_2),6\int f_3(z_1,z_2,z_3)dz_3\}_{z_1,z_2}\\
&\quad+6\int\{H_2(z_1,z_3)+H_2(z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&\quad+6\int\{H_3(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&=\{H_1(z_1)+H_1(z_2),f_2(z_1,z_2)\}_{z_1,z_2}+\{H_2(z_1,z_2),f_2(z_1,z_2)\}_{z_1,z_2}\\
&\quad+6\int\{H_2(z_1,z_3)+H_2(z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&\quad+6\int\{H_3(z_1,z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3. \qquad (5.6)
\end{aligned}
$$

Here, $\{\bullet,\bullet\}_{z_1,z_2}$ is the Poisson bracket only for z_1 and z_2 variables even if the functions inside the bracket depend on other variables.

5.2. Lie-Poisson realization of BBGKY hierarchy

Let A_1 denote the space of smooth functions on P. Equipped with the canonical Poisson bracket, A_1 is a Lie algebra. Similarly, we define A_2 and A_3 as Lie algebras of the symmetric functions on P^2 and P^3, respectively. There are, then, hierarchical embeddings which are defined to be

$$\begin{aligned}
&A_1 \longrightarrow A_2, \qquad K_1(z_1)\mapsto K_1^{(2)}(z_1,z_2):=K_1(z_1)+K_1(z_2)\\
&A_1 \longrightarrow A_3, \qquad K_1(z_1)\mapsto K_1^{(3)}(z_1,z_2,z_3):=K_1(z_1)+K_1(z_2)+K_1(z_3)\\
&A_2 \longrightarrow A_3, \qquad K_2(z_1,z_2)\mapsto K_2^{(3)}(z_1,z_2,z_3):=K_2(z_1,z_2)+K_2(z_2,z_1)\\
&\qquad\qquad + K_2(z_1,z_3)+K_2(z_3,z_1)+K_2(z_2,z_3)+K_2(z_3,z_2).
\end{aligned} \tag{5.7}$$

Next, letting

$$\mathcal{A}:=A_3\oplus A_2\oplus A_1, \tag{5.8}$$

we introduce the mapping

$$\alpha:\mathcal{A}\to A_3,\ (K_3,K_2,K_1)\to K_3(z_1,z_2,z_3)+K_2^{(3)}(z_1,z_2,z_3)+K_1^{(3)}(z_1,z_2,z_3) \tag{5.9}$$

from $\mathcal{A}$ to the Lie algebra A_3 of symmetric functions on P^3. Let us note that α in (5.9) turns out to be a Lie algebra homomorphism, provided the domain space $\mathcal{A}$ is equipped with the Lie bracket

$$\begin{aligned}
\big[(K_3,K_2,K_1),(L_3,L_2,L_1)\big]_{\mathcal{A}} &= \Big(\{K_3,L_3\}+\{K_3,L_2^{(3)}\}+\{K_3,L_1^{(3)}\}+\{K_2^{(3)},L_3\}\\
&+\{K_1^{(3)},L_3\}+\{K_2^{(3)},L_2^{(3)}\},\{K_2,L_1^{(2)}\}_{z_1,z_2}+\{K_1^{(2)},L_2\}_{z_1,z_2},\{K_1,L_1\}_{z_1}\Big)
\end{aligned} \tag{5.10}$$

where on the right hand side the notation $\{\bullet,\bullet\}$ without a subscript refers to the Poisson bracket on P_3, while $\{\bullet,\bullet\}_{z_1,z_2}$ denotes the Poisson bracket only for z_1 and z_2 variables, and $\{\bullet,\bullet\}_{z_1}$ is the Poisson bracket only for z_1 variable.

On the other hand, the dual mapping of α in (5.9) is computed to be

$$\begin{aligned}
&\alpha^*:A_3^*\longrightarrow \mathcal{A}^*,\\
&f_3\mapsto\Big(f_3, f_2=\int 6f_3(z_1,z_2,z_3)dz_3, f_1=\int 3f_3(z_1,z_2,z_3)dz_2dz_3\Big),
\end{aligned} \tag{5.11}$$

which happens to be both a momentum and a Poisson map. We note also that the mapping α^* determines the moment functions exhibited in (5.3).

Coadjoint action

Assume that the adjoint action of the $\mathcal{A}$ on itself is the Lie bracket $[\bullet,\bullet]_{\mathcal{A}}$ in (5.10). The coadjoint action of the space $\mathcal{A}$ on its dual $\mathcal{A}^*$ is

$$\begin{aligned}
&\Big\langle ad^*_{(L_3,L_2,L_1)}(f_3,f_2,f_1),(K_3,K_2,K_1)\Big\rangle = -\Big\langle (f_3,f_2,f_1), ad_{(L_3,L_2,L_1)}(K_3,K_2,K_1)\Big\rangle\\
&=\Big\langle (f_3,f_2,f_1),\big[(K_3,K_2,K_1),(L_3,L_2,L_1)\big]_{\mathcal{A}}\Big\rangle\\
&=\Big\langle f_3,\{K_3,L_3\}+\{K_3,L_2^{(3)}\}+\{K_3,L_1^{(3)}\}+\{K_2^{(3)},L_3\}+\{K_1^{(3)},L_3\}+\{K_2^{(3)},L_2^{(3)}\}\Big\rangle\\
&\quad+\big\langle f_2,\{K_2,L_1^{(2)}\}_{z_1,z_2}+\{K_1^{(2)},L_2\}_{z_1,z_2}\big\rangle+\big\langle f_1,\{K_1,L_1\}_{z_1}\big\rangle.
\end{aligned} \tag{5.12}$$

In the second line, we have substituted the Lie algebra bracket (5.10). In the last equality, the first pairing is the one available between A_3^* and A_3 with the symplectic volume $dz_1dz_2dz_3$, whereas the second one is between A_2^* and A_2 with the symplectic volume dz_1dz_2. Finally, the last pairing is the one between A_1^* and A_1 with the symplectic volume dz_1. It is evident that, in order to arrive at the explicit expression of the coadjoint action from (5.12), we need to single out the functions K_3, K_2 and K_1. For this, we first recall the association property

$$\int h(z)\{k(z),l(z)\}_z\ dz = \int k(z)\{l(z),h(z)\}_z\ dz \tag{5.13}$$

of the smooth functions. To see this identity, we simply consider the Leibniz identity

$$\{h(z)k(z),l(z)\}_z = h(z)\{k(z),l(z)\}_z + k(z)\{h(z),l(z)\}_z, \tag{5.14}$$

then, take the integral of this expression. In this case, the left hand side turns out to be zero due to (5.4). The integrals on the right hand side of (5.14) give precisely (5.13) after a reordering. We apply the identity (5.13) to the first pairing on the last equality in (5.12). We thus have

$$\begin{aligned}
&\int f_3(z_1,z_2,z_3)\Big(\{K_3,L_3\}+\{K_3,L_2^{(3)}\}+\{K_3,L_1^{(3)}\}+\{K_2^{(3)},L_3\}+\{K_1^{(3)},L_3\}\\
&\qquad+\{K_2^{(3)},L_2^{(3)}\}\Big)(z_1,z_2,z_3)dz_1dz_2dz_3\\
&=\int\Big(K_3\{L_3,f_3\}(z_1,z_2,z_3)+K_3\{L_2^{(3)},f_3\}(z_1,z_2,z_3)+K_3\{L_1^{(3)},f_3\}(z_1,z_2,z_3)\Big)dz_1dz_2dz_3\\
&\qquad+\int K_2(z_1,z_2)\Big(6\int\{L_3,f_3\}_{z_1,z_2}dz_3\Big)dz_1dz_2+\int K_1(z_1)\Big(3\int\{L_3,f_3\}_{z_1}dz_2dz_3\Big)dz_1\\
&\qquad+2\int K_2(z_1,z_2)\{L_2,f_2\}_{z_1,z_2}dz_1dz_2\\
&\qquad+\int K_2(z_1,z_2)\Big(12\int\{L_2(z_1,z_3)+L_2(z_2,z_3),f_3\}_{z_1,z_2}dz_3\Big)dz_1dz_2,
\end{aligned}\tag{5.15}$$

where we employed the identities (5.4) and (5.13) and the definitions of moments in (5.3). In a similar way, we compute the pairings on the last line of (5.12) as

$$\begin{aligned}
&\int f_2(z_1,z_2)\Big(\{K_2,L_1^{(2)}\}_{z_1,z_2}+\{K_1^{(2)},L_2\}_{z_1,z_2}\Big)(z_1,z_2)dz_1dz_2+\int f_1(z_1)\{K_1,L_1\}_{z_1}(z_1)dz_1\\
&=\int K_2(z_1,z_2)\{L_1^{(2)},f_2\}_{z_1,z_2}(z_1,z_2)dz_1dz_2+\int K_1(z_1)\Big(\int\{L_2(z_1,z_2),f_2(z_1,z_2)\}_{z_1}dz_2\Big)dz_1\\
&\qquad+\int K_1(z_1)\{L_1,f_1\}_{z_1}(z_1)dz_1.
\end{aligned}\tag{5.16}$$

In (5.15) and (5.16), we collect the terms involving K_3, K_2 and K_1 in an order, and then we arrive at the coadjoint flow

$$ad^*_{(L_3,L_2,L_1)}(f_3,f_2,f_1)=(\tilde{f}_3,\tilde{f}_2,\tilde{f}_1), \tag{5.17}$$

where
$$\begin{aligned}
\tilde{f}_3(z_1,z_2,z_3)&=\{L_3+L_2^{(3)}+L_1^{(3)},f_3\}(z_1,z_2,z_3),\\
\tilde{f}_2(z_1,z_2)&=\{L_1^{(2)}+2L_2,f_2\}_{z_1,z_2}(z_1,z_2)\\
&\qquad+12\int\{L_2(z_1,z_3)+L_2(z_2,z_3),f_3(z_1,z_2,z_3)\}_{z_1,z_2}dz_3\\
&\qquad+6\int\{L_3,f_3\}_{z_1,z_2}(z_1,z_2,z_3)dz_3,
\end{aligned}$$

$$\tilde{f}_1(z_1) = \{L_1, f_1\}_{z_1}(z_1) + 2\int \{L_2(z_1,z_2), f_2(z_1,z_2)\}_{z_1} dz_2$$
$$+ 3\int \{L_3(z_1,z_2,z_3), f_3(z_1,z_2,z_3)\}_{z_1} dz_2 dz_3. \tag{5.18}$$

Lie-Poisson equation

Let us now consider the functional

$$\mathcal{H}(f_3, f_2, f_1) = \int H_3(z_1,z_1,z_3) f_3(z_1,z_1,z_3) dz_1 dz_2 dz_3$$
$$+ \int \frac{1}{2} H_2(z_1,z_2) f_2(z_1,z_1) dz_1 dz_2 + \int H_1(z_1) f_2(z_1) dz_1. \tag{5.19}$$

Accordingly, we have

$$\frac{\delta\mathcal{H}}{\delta(f_3,f_2,f_1)} = \Big(\frac{\delta\mathcal{H}}{\delta f_3}, \frac{\delta\mathcal{H}}{\delta f_2}, \frac{\delta\mathcal{H}}{\delta f_1}\Big) = \big(H_3(z_1,z_1,z_3), \frac{1}{2}H_2(z_1,z_2), H_1(z_1)\big) \in \mathcal{A} \tag{5.20}$$

which are the energy function in (5.2) under the isomorphism α in (5.9). If the three tuple $H = (H_3, (1/2)H_2, H_1)$ is substituted in to the coadjoint action, then we arrive at

$$\frac{\partial}{\partial t}(f_3, f_2, f_1) = ad^*_{\delta\mathcal{H}/\delta(f_3,f_2,f_1)}(f_3,f_2,f_1) = ad^*_{(H_3,(1/2)H_2,H_1)}(f_3,f_2,f_1). \tag{5.21}$$

A direct calculation proves that the coadjoint flow (5.21) is precisely the system of equations (5.5), (5.6) and (5.1) governing the dynamics of the moments.

5.3. BBGKY hierarchy as a matched pair

Recall the direct sum $\mathcal{A} = A_3 \oplus A_2 \oplus A_1$ in (5.8). One evident decomposition of $\mathcal{A}$ is given by

$$\mathcal{A} = \mathfrak{g}_{32} \oplus \mathfrak{h}_1, \qquad \mathfrak{g}_{32} = A_3 \oplus A_2 \text{ and } \mathfrak{h}_1 = A_1. \tag{5.22}$$

In the present subsection, we shall examine this realization from the double cross sum point of view.

Decomposition of the Lie algebra

It is a direct calculation to show that the Lie bracket (5.10) is closed if it is restricted to the constitutive subspaces $\mathfrak{g}_{32}$ and $\mathfrak{h}_1$. In other words, both $\mathfrak{g}_{32}$ and $\mathfrak{h}_1$ are Lie subalgebras. So, as a result of Proposition 3.4, $\mathcal{A}$ is a double cross sum of $\mathfrak{g}_{32}$ and $\mathfrak{h}_1$, that is $\mathcal{A} = \mathfrak{g}_{32} \bowtie \mathfrak{h}_1$.

Restricting the bracket (5.10) to the subspaces $\mathfrak{g}_{32}$ and $\mathfrak{h}_1$, the Lie algebra brackets on $\mathfrak{g}_{32}$ and $\mathfrak{h}_1$ may be given by

$$\begin{aligned}
[\bullet,\bullet]_{32} :& \mathfrak{g}_{32} \otimes \mathfrak{g}_{32} \longrightarrow \mathfrak{g}_{32}, \\
& [(K_3,K_2),(L_3,L_2)]_{32} = \Big(\{K_3 + K_2^{(3)}, L_3\} + \{K_3 + K_2^{(3)}, L_2^{(3)}\}, 0\Big), \\
[\bullet,\bullet]_1 :& \mathfrak{h}_1 \otimes \mathfrak{h}_1 \longrightarrow \mathfrak{h}_1, \qquad [K_1, L_1]_1 = \{K_1, L_1\}_{z_1},
\end{aligned} \tag{5.23}$$

respectively. In order to compute mutual actions we recall the identity (3.20). In the present case, we compute

$$\begin{aligned}
\big(K_1 \triangleright (L_3,L_2)\big) \oplus \big(K_1 \triangleleft (L_3,L_2)\big) :&= [(0,0,K_1),(L_3,L_2,0)]_{\mathcal{A}} \\
&= \big(\{K_1^{(3)}, L_3\}, \{K_1^{(2)}, L_2\}_{z_1,z_2}\big) \oplus 0
\end{aligned} \tag{5.24}$$

and conclude that the left action of $\mathfrak{h}_1$ on $\mathfrak{g}_{32}$ and the right action of $\mathfrak{g}_{32}$ on $\mathfrak{h}_1$ are

$$\begin{aligned} \triangleright : \mathfrak{h}_1 \otimes \mathfrak{g}_{32} \longrightarrow \mathfrak{g}_{32}, \qquad & K_1 \triangleright (L_3, L_2) = \big(\{K_1^{(3)}, L_3\}, \{K_1^{(2)}, L_2\}_{z_1,z_2}\big), \\ \triangleleft : \mathfrak{h}_1 \otimes \mathfrak{g}_{32} \longrightarrow \mathfrak{h}_1, \qquad & K_1 \triangleleft (L_3, L_2) = 0, \end{aligned} \tag{5.25}$$

respectively. Notice that, the right action $\triangleleft$ is trivial so that the Lie algebra $\mathcal{A} = \mathfrak{g}_{32} \rtimes \mathfrak{h}_1$ is a semidirect product Lie algebra. The Lie bracket $[\bullet, \bullet]_{\mathcal{A}}$ in (5.10) admits the following decomposition

$$\begin{aligned} &[(K_3, K_2) \oplus K_1, (L_3, L_2) \oplus L_1] \\ &= \big([(K_3, K_2), (L_3, L_2)]_{32} + K_1 \triangleright (L_3, L_2) - L_1 \triangleright (K_3, K_2)\big) \oplus [K_1, L_1]_1. \end{aligned} \tag{5.26}$$

Here, the subalgebras $[\bullet, \bullet]_{32}$ and $[\bullet, \bullet]_1$ are the ones in (5.23), and the left action is in (5.25). This realization is precisely in the matched pair Lie bracket form (3.28) where the left action $\triangleleft$ is trivial. Let us apply this to the Lie-Poisson formulation of the BBGKY dynamics (4.23).

Decomposition of the BBGKY dynamics

The dual spaces of the constitutive Lie subalgebras $\mathfrak{g}_{32}$ and $\mathfrak{h}_1$ are $\mathfrak{g}_{32}^* = A_3^* \oplus A_2^*$ and $\mathfrak{h}_1^* = A_1^*$, respectively. So that, we can write $\mathcal{A}^* = \mathfrak{g}_{32}^* \oplus \mathfrak{h}_1^*$. The coadjoint action of $\mathfrak{g}_{32}$ on $\mathfrak{g}_{32}^*$, and the coadjoint action of $\mathfrak{h}_1$ on $\mathfrak{h}_1^*$ are

$$\begin{aligned} ad_{(L_3,L_2)}(f_3, f_2) = \Big(&\{L_3, f_3\} + 6\int \{L_2(z_1, z_2), f_3(z_1, z_2, z_3)\}_{z_1,z_2} dz_3, \\ &2\{L_2(z_1, z_2), f_2(z_1, z_2)\}_{z_1,z_2} + 6\int \{L_3(z_1, z_2, z_3), f_3(z_1, z_2, z_3)\}_{z_1,z_2} dz_3 \\ &+ 12\int \{L_2(z_1, z_3) + L_2(z_2, z_3), f_3\}_{z_1,z_2} dz_3\Big), \\ ad_{K_1} f_1 = \{&K_1(z_1), f_1(z_1)\}_{z_1}, \end{aligned} \tag{5.27}$$

respectively. Recall the mutual actions of $\mathfrak{g}_{32}$ and $\mathfrak{h}_1$ on each other given in (5.25). The dual of these actions are computed to be

$$\begin{aligned} &\overset{*}{\triangleleft} : \mathfrak{g}_{32}^* \otimes \mathfrak{h}_1 \longrightarrow \mathfrak{g}_{32}^*, \\ &\qquad (f_3, f_2) \overset{*}{\triangleleft} K_1 = \big(3\{f_3(z_1, z_2, z_3), K_1(z_1)\}_{z_1}, 2\{f_2(z_1, z_2), K_1(z_1)\}_{z_1}\big) \\ &\overset{*}{\triangleright} : \mathfrak{g}_{32} \otimes \mathfrak{h}_1^* \longrightarrow \mathfrak{h}_1^*, \quad (L_3, L_2) \overset{*}{\triangleright} f_1 = 0. \end{aligned} \tag{5.28}$$

The mapping $\mathfrak{b}$ in (4.2) and its dual (4.3) are computed to be

$$\mathfrak{b}_{(L_3,L_2)} : \mathfrak{h}_1 \longrightarrow \mathfrak{g}_{32},\ \mathfrak{b}_{(L_3,L_2)}(K_1) := K_1 \triangleright (L_3, L_2) \tag{5.29}$$

$$\mathfrak{b}^*_{(L_3,L_2)} : \mathfrak{g}_{32}^* \longrightarrow \mathfrak{h}_1^*,\ \mathfrak{b}^*_{(L_3,L_2)}(f_3, f_2) = 3\int \{L_3, f_3\}_{z_1} dz_2 dz_3 + 2\int \{L_2, f_2\}_{z_1} dz_2.$$

Since the left action is trivial both the mapping $\mathfrak{a}$ in (4.5) and its dual (4.6) are trivial. It is now straightforward to modify the matched Lie-Poisson equation (4.24) to the present version and determine the coadjoint flow as

$$\begin{aligned} \frac{d(f_3, f_2)}{dt} &= ad^*_{(L_3,L_2)}(f_3, f_2) - (f_3, f_2) \overset{*}{\triangleleft} K_1 \\ \frac{df_1}{dt} &= ad^*_{K_1} f_1 + \mathfrak{b}^*_{(L_3,L_2)}(f_3, f_2). \end{aligned} \tag{5.30}$$

These equations represent precisely the coadjoint flow (5.21) realization of BBGKY dynamics, and they take the classical form if $(L_3, L_2, K_1) = (H_3, (1/2)H_2, H_1)$.

5.4. BBGKY hierarchy as a unified product

Recall once more the direct sum $\mathcal{A} = A_3 \oplus A_2 \oplus A_1$ in (5.8). We have examined a matched pair decomposition (5.22) of this sum. An alternative decomposition of $\mathcal{A}$ can be given by

$$\mathcal{A} = \mathfrak{g}_3 \oplus \mathfrak{h}_{21}, \qquad \mathfrak{g}_3 := A_3, \text{ and } \mathfrak{h}_{21} = A_2 \oplus A_1. \tag{5.31}$$

Decomposition of the Lie algebra

It is straightforward to see that, $\mathfrak{g}_3$ is a subalgebra of $\mathcal{A}$ with induced bracket

$$[\bullet,\bullet]_3 : \mathfrak{g}_3 \otimes \mathfrak{g}_3 \longrightarrow \mathfrak{g}_3, \qquad [K_3, L_3]_3 = \{K_3, L_3\}, \tag{5.32}$$

where $\{\bullet,\bullet\}$ is the Poisson bracket on P^3. On the other hand, the subspace $\mathfrak{h}_{21}$ fails to be so. Indeed, the Lie bracket (5.10) of two generic elements $(0, K_2, K_1)$ and $(0, L_2, L_1)$ in $\mathfrak{h}_2$ is

$$\begin{aligned} &[(0, K_2, K_1), (0, L_2, L_1)]_{\mathcal{A}} \\ &= \underbrace{\{K_2^{(3)}, L_2^{(3)}\} \oplus \big(\{K_2, L_1^{(2)}\}_{z_1,z_2} + \{K_1^{(2)}, L_2\}_{z_1,z_2}, \{K_1, L_1\}_{z_1}\big)}_{\in \mathfrak{g}_3 \oplus \mathfrak{h}_{21}}, \end{aligned} \tag{5.33}$$

where the first term on the right hand side falls into $\mathfrak{g}_3$ whereas the second and the third terms are in $\mathfrak{h}_{21}$. As a result, this decomposition should be analysed from the point of view of the unified products presented in Subsection 3.1. Accordingly, it follows at once from (3.17) that

$$\begin{aligned} &\Phi : \mathfrak{h}_{21} \otimes \mathfrak{h}_{21} \longrightarrow \mathfrak{g}_3, \qquad ((K_2, K_1), (L_2, L_1)) \mapsto \{K_2^{(3)}, L_2^{(3)}\} \\ &\kappa : \mathfrak{h}_{21} \otimes \mathfrak{h}_{21} \longrightarrow \mathfrak{h}_{21}, \\ &\quad ((K_2, K_1), (L_2, L_1)) \mapsto \big(\{K_2, L_1^{(2)}\}_{z_1,z_2} + \{K_1^{(2)}, L_2\}_{z_1,z_2}, \{K_1, L_1\}_{z_1}\big). \end{aligned} \tag{5.34}$$

Now, we are ready to compute mutual *actions* defined in (3.3) and (3.1) between the constitutive spaces $\mathfrak{h}_{21}$ and $\mathfrak{g}_3$. To obtain the fomulas, we employ the identity (3.20) that is

$$\begin{aligned} (K_2, K_1) \triangleright L_3 \oplus (K_2, K_1) \triangleleft L_3 &= [(0, K_2, K_1), (L_3, 0, 0)]_{\mathcal{A}} \\ &= \big(\{K_2^{(3)}, L_3\} + \{K_1^{(3)}, L_3\}\big) \oplus (0, 0) \in \mathfrak{g}_3 \oplus \mathfrak{h}_{21} \end{aligned} \tag{5.35}$$

which, in turn, gives

$$\begin{aligned} &\triangleright : \mathfrak{h}_{21} \otimes \mathfrak{g}_3 \longrightarrow \mathfrak{g}_3, \qquad (K_2, K_1) \triangleright L_3 = \{K_2^{(3)}, L_3\} + \{K_1^{(3)}, L_3\} \\ &\triangleleft : \mathfrak{h}_{21} \otimes \mathfrak{g}_3 \longrightarrow \mathfrak{h}_{21}, \qquad (K_2, K_1) \triangleleft L_3 = (0, 0). \end{aligned} \tag{5.36}$$

Due to Proposition 3.2, the decomposition $\mathfrak{g}_3 \oplus \mathfrak{h}_{21}$ of the Lie algebra $\mathcal{A}$ reads the decomposition of the Lie bracket $\{\bullet,\bullet\}_{\mathcal{A}}$ given in (5.10) into the form (3.4), where the right action $\triangleleft$ is trivial

$$\begin{aligned} [K_3 \oplus (K_2, K_1), L_3 \oplus (L_2, L_1)]_{_\Phi\bowtie} = \big(&\{K_3, L_3\} + (K_2, K_1) \triangleright L_3 - (L_2, L_1) \triangleright K_3 \\ &+ \Phi((K_2, K_1), (L_2, L_1))\big) \oplus \kappa((K_2, K_1), (L_2, L_1)), \end{aligned} \tag{5.37}$$

where the left action is the one given in (5.36) whereas Φ and κ mapping are those available in (5.34).

An alternative decomposition of the Lie algebra

The hierarchy of the moment functions suggests an alternative formulation of Φ and κ in (5.34). This is due to the fact that the term $[K_2^{(3)}, L_2^{(3)}]$ can be written as a sum of some terms in $\mathfrak{h}_{21}$ and some terms in $\mathfrak{g}_3$. Indeed,

$$\begin{aligned} [K_2^{(3)}, L_2^{(3)}] = 2(\{K_2(z_1,z_2), L_2(z_1,z_2)\}_{z_1,z_2})^{(3)}(z_1,z_2,z_3) \\ + 4\{K_2(z_1,z_2), L_2(z_1,z_3) + L_2(z_2,z_3)\}_{z_1,z_2} \\ + 4\{K_2(z_1,z_3), L_2(z_1,z_2) + L_2(z_2,z_3)\}_{z_1,z_3} \\ + 4\{K_2(z_2,z_3), L_2(z_1,z_2) + L_2(z_1,z_3)\}_{z_2,z_3}. \end{aligned} \tag{5.38}$$

Here, as depicted in the display, the first term on the right hand side can be written as the image of the symmetric function $\{K_2(z_1,z_2), L_2(z_1,z_2)\}$ under the embedding $A_2 \mapsto A_3$ given in (5.7). Accordingly, instead of Φ and κ in (5.34), we can propose the following alternatives

$$\begin{aligned} \tilde{\Phi} :& \mathfrak{h}_{21} \otimes \mathfrak{h}_{21} \longrightarrow \mathfrak{g}_3, \\ & ((K_2,K_1),(L_2,L_1)) \mapsto 4\{K_2(z_1,z_2), L_2(z_1,z_3) + L_2(z_2,z_3)\} \\ & \qquad + 4\{K_2(z_1,z_3), L_2(z_1,z_2) + L_2(z_2,z_3)\} \\ & \qquad + 4\{K_2(z_2,z_3), L_2(z_1,z_2) + L_2(z_1,z_3)\} \\ \tilde{\kappa} :& \mathfrak{h}_{21} \otimes \mathfrak{h}_{21} \longrightarrow \mathfrak{h}_{21}, \\ & ((K_2,K_1),(L_2,L_1)) \mapsto \Big(\{K_2(z_1,z_2), L_1(z_1) + L_1(z_2)\}_{z_1,z_2} \\ & \qquad + \{K_1(z_1) + K_1(z_2), L_2(z_1,z_2)\}_{z_1,z_2} \\ & \qquad + 2\{K_2(z_1,z_2), L_2(z_1,z_2)\}_{z_1,z_2}, \{K_1(z_1), L_1(z_1)\}_{z_1}\Big). \end{aligned} \tag{5.39}$$

Evidently, this observation reads an alternative Lie bracket operation on $\mathcal{A}$ as well. We denote this by a tilde notation $[\bullet,\bullet]_{\tilde{\Phi}\bowtie}$ and record as follows

$$\begin{aligned} &[K_3 \oplus (K_2,K_1), L_3 \oplus (L_2,L_1)]_{\tilde{\Phi}\bowtie} = \big(\{K_3,L_3\} + (K_2,K_1) \rhd L_3 \\ &-(L_2,L_1) \rhd K_3 + \tilde{\Phi}((K_2,K_1),(L_2,L_1))\big) \oplus \tilde{\kappa}((K_2,K_1),(L_2,L_1)), \end{aligned} \tag{5.40}$$

where $\{K_3, L_3\}$ is the Poisson bracket on P^3, and $\rhd$ is the left action in (5.36).

Decomposition of the dynamics: $\mathcal{A}^* = \mathfrak{g}_3^* \oplus \mathfrak{h}_{21}^*$

We start with the dualization of the mutual actions in (5.36). Let us first remark that the left action is trivial, so that it induces a trivial dual action. As for the right action, we compute the dual action as

$$\mathfrak{g}_3^* \overset{*}{\lhd} \mathfrak{h}_{21} \longrightarrow \mathfrak{g}_3^*, \qquad f_3 \overset{*}{\lhd} (K_2,K_1) = \{f_3, K_2^{(3)}\} + \{f_3, K_1^{(3)}\}. \tag{5.41}$$

Using the right action in (5.36), and recalling (4.2) and (4.3), we compute the following mapping along with its dual

$$\begin{aligned} &\mathfrak{b}_{L_3} : \mathfrak{h}_{21} \longrightarrow \mathfrak{g}_3, \quad \mathfrak{b}_{L_3}(K_2,K_1) = (K_2,K_1) \rhd L_3 = [K_2^{(3)}, L_3] + [K_1^{(3)}, L_3] \\ &\mathfrak{b}_{L_3}^* : \mathfrak{g}_3^* \longrightarrow \mathfrak{h}_{21}^*, \quad \mathfrak{b}_{L_3}^* f_3 = \Big(6\int \{L_3, f_3\}_{z_1,z_2} dz_3, 3\int \{L_3, f_3\}_{z_1} dz_2 dz_3\Big). \end{aligned} \tag{5.42}$$

Further, according to (4.7) and (4.8), freezing the first entries of $\tilde{\Phi}$ and $\tilde{\kappa}$ in (5.39), we arrive at linear mappings. One of these mappings is $\tilde{\Phi}_{(K_2,K_1)}$ from $\mathfrak{h}_{21}$ to $\mathfrak{g}_3$, and the other is $\tilde{\kappa}_{(K_2,K_1)}$ from $\mathfrak{h}_{21}$ to $\mathfrak{h}_{21}$. Dualizations of these mappings result with

$$
\begin{aligned}
&\tilde{\Phi}^*_{(K_2,K_1)} : \mathfrak{g}_3^* \longrightarrow \mathfrak{h}_{21}^*, \quad \tilde{\Phi}^*_{(K_2,K_1)} f_3 = \Big(24 \int \{K_2(z_1,z_3), f_3(z_1,z_2,z_3)\}_{z_1} dz_3, 0\Big),\\
&\tilde{\kappa}^*_{(K_2,K_1)} : \mathfrak{h}_{21}^* \longrightarrow \mathfrak{h}_{21}^*, \qquad\qquad (5.43)\\
&\tilde{\kappa}^*_{(K_2,K_1)}(f_2,f_1) = \Big(2\{K_1(z_1), f_2(z_1,z_2)\}_{z_1} + 2\{K_2(z_1,z_2), f_2(z_1,z_2)\}_{z_1,z_2},\\
&\qquad\qquad 2\int \{K_2(z_1,z_2), f_2(z_1,z_2)\}_{z_1} dz_2 + \{K_1(z_1), f_1(z_1)\}_{z_1}\Big).
\end{aligned}
$$

Now, recall the decomposed Lie-Poisson equation (4.13). Since, in the present case, the right action is trivial, we take the terms involving $\overset{*}{\triangleleft}$ and $\mathfrak{a}^*$ as zero. After substituting all these into the Lie-Poisson equation, we obtain

$$
\begin{aligned}
\frac{df_3}{dt} &= ad^*_{K_3}(f_3) - f_3 \overset{*}{\triangleleft} (K_2,K_1),\\
\frac{d(f_2,f_1)}{dt} &= \tilde{\kappa}^*_{(K_2,K_1)}(f_2,f_1) + \tilde{\Phi}^*_{(K_2,K_1)} f_3 + \mathfrak{b}^*_{L_3} f_3.
\end{aligned} \tag{5.44}
$$

Here, $ad^*_{K_3}(f_3) = \{K_3, f_3\}$ is the coadjoint action of $\mathfrak{g}_3$ on its dual space $\mathfrak{g}_3^*$. If we take $K_3 = H_3$, $K_2 = (1/2)H_2$ and $K_1 = H_1$ then, the system is exactly the dynamics of the moments in (5.5), (5.6) and (5.1) by decomposing the coadjoint flow (5.21).

6. Coupling of 2-cocycles

In Subsection 3.3, 2-cocycle extensions were exhibited as particular instances of extended structures. In this section, we shall discuss the coupling of two 2-cocycle extensions under mutual actions. This will be achieved by the arguments of Subsection 3.2. Our goal is to explore conditions for a double cross sum of 2-cocycle extensions to be a 2-cocycle extension of a double cross sum. We shall, further, study dynamics on the coupled system to have the Lie-Poisson equations for the collective motion.

6.1. Coupling of 2-cocycle extensions

We start with two Lie algebras, say $\mathfrak{l}$ and $\mathfrak{k}$, and two vector spaces V and W. Assume a V-valued 2-cocycle φ on $\mathfrak{l}$, and a W-valued 2-cocycle ϕ on $\mathfrak{k}$ given by

$$
\varphi : \mathfrak{l} \times \mathfrak{l} \to V, \qquad \phi : \mathfrak{k} \times \mathfrak{k} \to W. \tag{6.1}
$$

Further, we consider a left action of the Lie algebra $\mathfrak{l}$ on the vector space V, and a left action of the Lie algebra $\mathfrak{k}$ on the vector space W, which are denoted by

$$
\begin{aligned}
\rfloor &: \mathfrak{l} \otimes V \to V, \qquad l \otimes v \mapsto l \rfloor v,\\
\lfloor &: \mathfrak{k} \otimes W \to W, \qquad k \otimes w \mapsto k \lfloor w.
\end{aligned} \tag{6.2}
$$

If these actions are compatible with the 2-cocycles in the sense of (3.38), then one arrives at the following 2-cocycle extensions

$$
\mathfrak{g} := V_{\varphi}\rtimes \mathfrak{l}, \qquad \mathfrak{h} := W_{\phi}\rtimes \mathfrak{k}. \tag{6.3}
$$

In the light of the discussions done in Subsection 3.3, proper modifications of (3.40), yield the Lie algebra brackets

$$\begin{aligned} [v\oplus l, v'\oplus l']_{\varphi\rtimes} &= \big(l \downharpoonleft v' - l' \downharpoonleft v + \varphi(l,l')\big) \oplus [l,l'], \\ [w\oplus k, w'\oplus k']_{\phi\rtimes} &= \big(k \downharpoonright w' - k' \downharpoonright w + \phi(k,k')\big) \oplus [k,k'], \end{aligned} \tag{6.4}$$

on $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Here, the bracket $[l,l']$ is the Lie algebra bracket on $\mathfrak{l}$, whereas the bracket $[k,k']$ refers to the Lie bracket on $\mathfrak{k}$.

Match pairs of 2-cocycle extensions

We shall now examine the conditions for $(\mathfrak{g},\mathfrak{h}) = (V_{\varphi}\rtimes \mathfrak{l},\, W_{\phi}\rtimes \mathfrak{k})$ to be a matched pair. To this end, we consider the followings three sets of mappings:

(1) We first consider the mutual actions

$$\begin{aligned} \blacktriangleright &: \mathfrak{k}\otimes\mathfrak{l} \to \mathfrak{l}, \qquad k\otimes l \mapsto k \blacktriangleright l, \\ \blacktriangleleft &: \mathfrak{k}\otimes\mathfrak{l} \to \mathfrak{k}, \qquad k\otimes l \mapsto k \blacktriangleleft l \end{aligned} \tag{6.5}$$

of the Lie algebras $\mathfrak{l}$ and $\mathfrak{k}$ on each other. We assume that these actions satisfy the compatibility conditions in (3.29). Hence, $(\mathfrak{l},\mathfrak{k})$ makes a matched pair of Lie algebras, and determines the double cross sum $\mathfrak{l}\bowtie\mathfrak{k}$.

(2) In order to extend the mutual actions given in (6.5) to the product spaces $\mathfrak{g}$ and $\mathfrak{h}$ in (6.3), we introduce a right action of $\mathfrak{l}$ on W and, a left action $\mathfrak{k}$ on V given by, respectively,

$$\begin{aligned} \curvearrowright &: \mathfrak{k}\otimes V \to V, \qquad k\otimes v \mapsto k \curvearrowright v, \\ \curvearrowleft &: W\otimes\mathfrak{l} \to W, \qquad w\otimes l \mapsto w \curvearrowleft l. \end{aligned} \tag{6.6}$$

(3) In addition, we shall make use of the maps

$$\begin{aligned} \epsilon &: \mathfrak{k}\otimes\mathfrak{l} \to V, \qquad \epsilon(k,l)\in V, \\ \iota &: \mathfrak{k}\otimes\mathfrak{l} \to W, \qquad \iota(k,l)\in W \end{aligned} \tag{6.7}$$

satisfying

$$\begin{aligned} \epsilon([k_1,k_2],l) &= \epsilon(k_1, k_2 \blacktriangleright l) - \epsilon(k_2, k_1\blacktriangleright l), \\ \iota([k_1,k_2],l) &= \iota(k_1, k_2 \blacktriangleright l) - \iota(k_2, k_1\blacktriangleright l) \end{aligned}$$

for any $k_1,k_2\in\mathfrak{k}$ and any $l\in\mathfrak{l}$.

Referring to the mappings (6.5), (6.6) and (6.7), we define mutual actions of 2-cocycle extensions $\mathfrak{g} = V_{\varphi}\rtimes\mathfrak{l}$ and $\mathfrak{h} = W_{\phi}\rtimes\mathfrak{k}$ to be

$$\begin{aligned} \rhd &: (W_{\phi}\rtimes\mathfrak{k})\times(V_{\varphi}\rtimes\mathfrak{l}) \longrightarrow V_{\varphi}\rtimes\mathfrak{l}, \\ &\qquad ((w\oplus k),(v\oplus l)) \mapsto (k\curvearrowright v + \epsilon(k,l))\oplus(k\blacktriangleright l), \\ \lhd &: (W_{\phi}\rtimes\mathfrak{k})\times(V_{\varphi}\rtimes\mathfrak{l}) \longrightarrow W_{\phi}\rtimes\mathfrak{k}, \\ &\qquad ((w\oplus k),(v\oplus l)) \mapsto (w\curvearrowleft l + \iota(k,l))\oplus(k\blacktriangleleft l). \end{aligned} \tag{6.8}$$

It is possible to see that $\rhd$ is a left action whereas $\lhd$ is a right action. In order to construct a matched pair of $\mathfrak{h} = W_{\phi}\rtimes\mathfrak{k}$ and $\mathfrak{g} = V_{\varphi}\rtimes\mathfrak{l}$, one needs to justify the compatibility conditions in (3.29). A direct observation gives that, for the actions (6.8), the compatibility conditions (3.29) consist of 4 equations. Two of them, those for the second terms in the decompositions, involve only the left $\blacktriangleright$ and the right $\blacktriangleleft$ actions in (6.5). These two equations are precisely the matched pair compatibility conditions for $\mathfrak{l}\bowtie\mathfrak{k}$. Since, we assume that $\mathfrak{l}\bowtie\mathfrak{k}$ is a matched pair, these two compatibility conditions are automatically satisfied. So, we are left with the other two compatibility conditions. For any k,k' in $\mathfrak{k}$, l,l' in $\mathfrak{l}$, v,v' in V, and w,w' in W, these equations are computed to be

$$
\begin{aligned}
&k \curvearrowright (l \downharpoonright v' - l' \downharpoonright v + \varphi(l,l')) + \epsilon(k,[l,l'])\\
&\qquad = l \downharpoonright (k \curvearrowright v' + \epsilon(k,l')) - (k \blacktriangleright l') \downharpoonright v + \varphi(l, k \blacktriangleright l') - l' \downharpoonright (k \curvearrowright v + \epsilon(k,l))\\
&\qquad\quad + (k \blacktriangleright l) \downharpoonright v' - \varphi(l', k \blacktriangleright l) + (k \blacktriangleleft l) \curvearrowright v' - (k \blacktriangleleft l') \curvearrowright v\\
&\qquad\quad + \epsilon(k \blacktriangleleft l, l') - \iota(k \blacktriangleleft l', l) - (w \curvearrowleft l' + \iota(k,l')) \curvearrowleft l\\
&(k \downharpoonleft w' - k' \downharpoonleft w + \phi(k,k')) \curvearrowleft l + \iota([k,k'],l)\\
&\qquad = k \downharpoonleft (w' \curvearrowleft l + \iota(k',l)) - (k' \blacktriangleleft l) \downharpoonleft w + \phi(k, k' \blacktriangleleft l)\\
&\qquad\quad - k' \downharpoonleft (w \curvearrowleft l + \iota(k,l)) + (k \blacktriangleleft l) \downharpoonleft w' - \phi(k', k \blacktriangleleft l)\\
&\qquad\quad - w \curvearrowleft (k' \blacktriangleright l) + w' \curvearrowleft (k \blacktriangleright l) + \iota(k', k \blacktriangleright l) - \epsilon(k, k' \blacktriangleright l)\\
&\qquad\quad + k \curvearrowright (k' \curvearrowright v + \epsilon(k',l)),
\end{aligned}
\tag{6.9}
$$

where $\downharpoonright$ and $\downharpoonleft$ are the left actions in (6.2), $\blacktriangleright$ and $\blacktriangleleft$ are actions in (6.5), $\curvearrowright$ and $\curvearrowleft$ are the actions in (6.6), ϵ and ι are the mappings in (6.7). Assuming that these conditions are satisfied, we have the double cross sum Lie algebra

$$
\mathfrak{g}\bowtie\mathfrak{h} = (V_{\varphi}\rtimes \mathfrak{l}) \bowtie (W_{\phi}\rtimes \mathfrak{k}). \tag{6.10}
$$

In view of (3.28) then, the bracket operation on $\mathfrak{g}\bowtie\mathfrak{h}$ takes the form of

$$
\big[\big((v\oplus l)\oplus(w\oplus k)\big),\big((v'\oplus l')\oplus(w'\oplus k')\big)\big]_{\bowtie} = (\bar{v}\oplus\bar{l})\oplus(\bar{w}\oplus\bar{k}), \tag{6.11}
$$

where

$$
\begin{aligned}
\bar{v} &= l \downharpoonright v' - l' \downharpoonright v + k \curvearrowright v' - k' \curvearrowright v + \epsilon(k,l') - \epsilon(k',l) + \varphi(l,l'),\\
\bar{l} &= [l,l'] + k \blacktriangleright l' - k' \blacktriangleright l,\\
\bar{w} &= k \downharpoonleft w' - k' \downharpoonleft w + w \curvearrowleft l' - w' \curvearrowleft l + \iota(k,l') - \iota(k',l) + \phi(k,k'),\\
\bar{k} &= [k,k'] + k \blacktriangleleft l' - k' \blacktriangleleft l.
\end{aligned}
\tag{6.12}
$$

The double cross sum as a 2-cocycle extension

We shall next investigate the conditions for the double cross sum $\mathfrak{g}\bowtie\mathfrak{h}$ of (6.10) itself to be a 2-cocycle extension. To this end, we need to determine a left action, a Lie algebra 2-cocycle on $\mathfrak{g}\bowtie\mathfrak{h}$. Let us determine these one by one.

Left action. Recall the mutual actions in (6.5) and the matched pair algebra $\mathfrak{l}\bowtie\mathfrak{k}$. It is evident that $\mathfrak{l}\bowtie\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}\bowtie\mathfrak{h}$. Define a left action of $\mathfrak{l}\bowtie\mathfrak{k}$ on the product space $V\oplus W$ as follows:

$$
\begin{aligned}
&\triangleright : (\mathfrak{l}\bowtie\mathfrak{k}) \times (V\oplus W) \longrightarrow (V\oplus W),\\
&\quad (l\oplus k)\triangleright(v\oplus w) = \big(l \downharpoonright v + k \curvearrowright v\big) \oplus \big(-w \curvearrowleft l + k \downharpoonleft w\big),
\end{aligned}
\tag{6.13}
$$

where we have employed the left actions $\downharpoonright$ and $\downharpoonleft$ in (6.2), the actions $\curvearrowright$ and $\curvearrowleft$ in (6.6). To be a left action, (6.13) needs to satisfy the first condition in (3.27).

We compute this as

$$\begin{aligned}(k \blacktriangleright l') \downharpoonleft v - (k' \blacktriangleright l) \downharpoonleft v + (k \blacktriangleleft l') \curvearrowright v - (k' \blacktriangleleft l) \curvearrowright v \\ = l \downharpoonleft (k' \curvearrowright v) - l' \downharpoonleft (k \curvearrowright v) k \curvearrowright (l' \downharpoonleft v) - k' \curvearrowright (l \downharpoonleft v)\end{aligned} \tag{6.14}$$

for the first entry in (6.13). Let us note that this is an equation defined on the vector space V. As for the second entry, we have

$$\begin{aligned}- w \curvearrowleft (k \blacktriangleright l') + w \curvearrowleft (k' \blacktriangleright l) + (k \blacktriangleleft l') \downharpoonright w - (k' \blacktriangleleft l) \downharpoonright w \\ = -(k' \downharpoonright w) \curvearrowleft l + (k \downharpoonright w) \curvearrowleft l' - k \downharpoonright (w \curvearrowleft l') + k' \downharpoonright (w \curvearrowleft l),\end{aligned} \tag{6.15}$$

where $\blacktriangleright$ and $\blacktriangleleft$ are the mutual actions in (6.5).

2-cocycle Later, we introduce a $(V \oplus W)$-valued 2-cocycle on $\mathfrak{l} \bowtie \mathfrak{k}$, in terms of the 2-cocycles φ and ϕ given in (6.1), as follows

$$\begin{aligned}&\Theta : (\mathfrak{l} \bowtie \mathfrak{k}) \times (\mathfrak{l} \bowtie \mathfrak{k}) \longrightarrow V \oplus W, \qquad (6.16)\\ &\Theta((l \oplus k), (l' \oplus k')) := \big(\varphi(l, l') + \epsilon(k, l') - \epsilon(k', l)\big) \oplus \big(\phi(k, k') + \iota(k, l') - \iota(k', l)\big),\end{aligned}$$

One needs to check the compatibility conditions in (3.38),

$$\begin{aligned}0 = {} & \varphi(l, k'' \blacktriangleright l' - k' \blacktriangleright l'') + \epsilon(k, [l'', l']) + \epsilon(k, k'' \blacktriangleright l' - k' \blacktriangleright l'') \\ & - \epsilon([k'', k'], l) - \epsilon(k'' \blacktriangleleft l' - k' \blacktriangleleft l'', l) \\ & - l \downharpoonleft (\varphi(l', l'') + \epsilon(k', l'') - \epsilon(k'', l')) - k \curvearrowright (\varphi(l', l'') + \epsilon(k', l'') - \epsilon(k'', l')) \\ & \varphi(l', k \blacktriangleright l'' - k'' \blacktriangleright l) + \epsilon(k', [l, l'']) + \epsilon(k', k \blacktriangleright l'' - k'' \blacktriangleright l) - \epsilon([k, k''], l') \\ & - \epsilon(k \blacktriangleleft l'' - k'' \blacktriangleleft l, l') - l' \downharpoonleft (\varphi(l'', l) + \epsilon(k'', l) - \epsilon(k, l'')) \\ & - k \curvearrowright (\varphi(l'', l) + \epsilon(k'', l) - \epsilon(k, l'')) \\ & \varphi(l'', k' \blacktriangleright l - k \blacktriangleright l') + \epsilon(k'', [l', l]) + \epsilon(k'', k' \blacktriangleright l - k \blacktriangleright l') - \epsilon([k', k], l'') \\ & - \epsilon(k' \blacktriangleleft l - k \blacktriangleleft l', l'') - l'' \downharpoonleft (\varphi(l, l') + \epsilon(k, l') - \epsilon(k', l)) \\ & - k \curvearrowright (\varphi(l, l') + \epsilon(k, l') - \epsilon(k', l)) \\ 0 = {} & \phi(l, k'' \blacktriangleright l' - k' \blacktriangleright l'') + \iota(k, [l'', l']) + \iota(k, k'' \blacktriangleright l' - k' \blacktriangleright l'') - \iota([k'', k'], l) \\ & - \iota(k'' \blacktriangleleft l' - k' \blacktriangleleft l'', l) - l \downharpoonleft (\phi(l', l'') + \iota(k', l'') - \iota(k'', l')) \\ & - k \curvearrowright (\phi(l', l'') + \iota(k', l'') - \iota(k'', l')) \\ & \phi(l', k \blacktriangleright l'' - k'' \blacktriangleright l) + \iota(k', [l, l'']) + \iota(k', k \blacktriangleright l'' - k'' \blacktriangleright l) - \iota([k, k''], l') \\ & - \iota(k \blacktriangleleft l'' - k'' \blacktriangleleft l, l') - l' \downharpoonleft (\phi(l'', l) + \iota(k'', l) - \iota(k, l'')) \\ & - k \curvearrowright (\phi(l'', l) + \iota(k'', l) - \iota(k, l'')) \\ & \varphi(l'', k' \blacktriangleright l - k \blacktriangleright l') + \iota(k'', [l', l]) + \iota(k'', k' \blacktriangleright l - k \blacktriangleright l') - \iota([k', k], l'') \\ & - \iota(k' \blacktriangleleft l - k \blacktriangleleft l', l'') - l'' \downharpoonleft (\phi(l, l') + \iota(k, l') - \iota(k', l)) \\ & - k \curvearrowright (\phi(l, l') + \iota(k, l') - \iota(k', l)).\end{aligned} \tag{6.17}$$

We are ready now to define a 2-cocycle extension of the Lie alegbra $\mathfrak{l} \bowtie \mathfrak{k}$ via $V \oplus W$, which we shall denote by

$$(V \oplus W)_{\Theta} \rtimes (\mathfrak{l} \bowtie \mathfrak{k}). \tag{6.18}$$

To arrive at the Lie bracket on this space, one only needs to employ the explicit definitions of the left action $\triangleright$ in 2-cocycle Θ into the generic formula (3.40) of Lie bracket for 2-cocycle extensions. This results in

$$
\begin{aligned}
&[(v \oplus w) \oplus (l \oplus k), (v' \oplus w') \oplus (l' \oplus k')]_{\Theta \rtimes} \qquad (6.19)\\
&= \big((l \oplus k) \triangleright (v' \oplus w') - (l' \oplus k') \triangleright (v \oplus w) + \Theta((l \oplus k), (l' \oplus k'))\big) \oplus [(l \oplus k), (l' \oplus k')],
\end{aligned}
$$

where $\triangleright$ is the left action in (6.13), and the bracket $[(l \oplus k), (l' \oplus k')]$ is the matched pair Lie bracket on $\mathfrak{l} \bowtie \mathfrak{k}$. It is immediate to see that extended Lie algebra bracket on (6.19) is precisely equal to the matched Lie bracket given in (6.11) and (6.12), up to some reordering. Eventually, we are ready now to collect all the discussions done so far in the following proposition.

Proposition 6.1. *The matched pair Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$, given in* (6.10)*, of 2-cocycle extensions $\mathfrak{g} = V_{\varphi}\rtimes \mathfrak{l}$ and $\mathfrak{h} = W_{\phi}\rtimes \mathfrak{k}$ is a 2-cocycle extension admitting a left action $\triangleright$ in* (6.13) *and a 2-cocycle Θ in* (6.16) *that is*

$$
(V_{\varphi}\rtimes \mathfrak{l}) \bowtie (W_{\phi}\rtimes \mathfrak{k}) \cong (V \oplus W)_{\Theta}\rtimes (\mathfrak{l} \bowtie \mathfrak{k}). \qquad (6.20)
$$

Even though, we have derived all the mappings and conditions up to now explicitly. There is a short but implicit way to arrive at that proposition by employing Proposition 3.4. For this, we first embed Lie subalgebras to the space $(V \oplus W)_{\Theta}\rtimes(\mathfrak{l} \bowtie \mathfrak{k})$ as follows

$$
\begin{aligned}
&\mathfrak{g} \longrightarrow (V \oplus W)_{\Theta}\rtimes(\mathfrak{l} \bowtie \mathfrak{k}), \qquad (v \oplus l) \mapsto (v \oplus 0) \oplus (l \oplus 0)\\
&\mathfrak{h} \longrightarrow (V \oplus W)_{\Theta}\rtimes(\mathfrak{l} \bowtie \mathfrak{k}), \qquad (w \oplus k) \mapsto (0 \oplus w) \oplus (0 \oplus k).
\end{aligned} \qquad (6.21)
$$

Then Proposition 3.4 implies that the total space admits a matched pair decomposition.

A particular case

For future reference, we now examine a particular case of Proposition 6.1. First, we choose the left actions $\downharpoonleft$ and $\downharpoonright$ in (6.2) to be trivial. So, the Lie brackets on the 2-cocycle extensions in (6.4) turn out to be

$$
\begin{aligned}
[v \oplus l, v' \oplus l']_{\varphi\rtimes} &= \varphi(l, l') \oplus [l, l'],\\
[w \oplus k, w' \oplus k']_{\phi\rtimes} &= \phi(k, k') \oplus [k, k'].
\end{aligned} \qquad (6.22)
$$

In addition, consider that the mutual actions of $\mathfrak{k}$ and $\mathfrak{l}$ in (6.5) and the mappings in (6.6) are all zero. Hence, we have the mutual actions of $\mathfrak{h}$ and $\mathfrak{g}$ in (6.8) are

$$
\begin{aligned}
\rhd &: ((w \oplus k), (v \oplus l)) \mapsto (\epsilon(k, l) \oplus 0),\\
\lhd &: ((w \oplus k), (v \oplus l)) \mapsto (\iota(k, l) \oplus 0).
\end{aligned} \qquad (6.23)
$$

A direct calculation gives us that, in the present setting, $\rhd$ is a left action if and only if $\epsilon(k, [l, l']) = 0$, and $\lhd$ is a right action if and only if $\iota([k, k'], l) = 0$. Eventually, we claim that, the assumptions in this case reduce the matched Lie algebra bracket in (6.11) and (6.12) as

$$
\begin{aligned}
&\big[\big((v \oplus l) \oplus (w \oplus k)\big), \big((v' \oplus l') \oplus (w' \oplus k')\big)\big]_{\bowtie} \qquad (6.24)\\
&= \Big(\big(\epsilon(k, l') - \epsilon(k', l) + \varphi(l, l')\big) \oplus 0\Big) \oplus \Big(\big(\iota(k, l') - \iota(k', l) + \phi(k, k')\big) \oplus 0\Big)
\end{aligned}
$$

where ϵ and ι are as in (6.7).

In order to implement Proposition 6.1, we exploit a proper left action and a 2-cocycle operator. Here, according to (6.13), we take the left action as trivial whereas the 2-cocycle Θ is precisely equal to the one in (6.16).

6.2. Lie Poisson dynamics on the duals of the double cross sums of 2-cocycle extensions

We have presented the double cross sum of 2-cocycle extensions in the previous subsection. To arrive at Hamiltonian dynamics on the dual picture, we make use the Lie-Poisson formalism on 2-cocycle extensions, that is, the theory in Subsection 4.3. In that subsection, there exist both the Lie-Poisson bracket (4.27) and the Lie-Poisson equations (4.28) for the case of 2-cocycle extensions.

Lie-Poisson brackets

Let us consider the following notation on the dual spaces

$$\mu = \mu_{V^*} \oplus \mu_{W^*} \in V^* \oplus W^*, \qquad \nu = \nu_{\mathfrak{l}^*} \oplus \nu_{\mathfrak{k}^*} \in \mathfrak{l}^* \oplus \mathfrak{k}^*. \tag{6.25}$$

Substitute the 2-cocycle Θ in (6.16) and the left action $\vartriangleright$ in (6.13) in the Lie-Poisson bracket (4.27). Therefore, for this situation, (plus/minus) Lie-Poisson bracket is

$$\begin{aligned} \{\mathcal{H},\mathcal{F}\}_{_\Theta\rtimes}(\mu\oplus\nu) = &\pm \Big\langle \nu_{\mathfrak{l}^*}\oplus\nu_{\mathfrak{k}^*}, \big[\big(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\big),\big(\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\big)\big]\Big\rangle \\ &\pm \Big\langle \mu_{V^*}\oplus\mu_{W^*}, \Theta\Big(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}},\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\Big)\Big\rangle \\ &\pm \Big\langle \mu_{V^*}\oplus\mu_{W^*}, \big(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\big)\vartriangleright\big(\frac{\delta\mathcal{F}}{\delta\mu_{V^*}}\oplus\frac{\delta\mathcal{F}}{\delta\mu_{W^*}}\big)\Big\rangle \\ &\mp \Big\langle \mu_{V^*}\oplus\mu_{W^*}, \big(\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\big)\vartriangleright\big(\frac{\delta\mathcal{H}}{\delta\mu_{V^*}}\oplus\frac{\delta\mathcal{H}}{\delta\mu_{W^*}}\big)\Big\rangle, \end{aligned} \tag{6.26}$$

where the bracket on the first line is the matched pair Lie bracket on $\mathfrak{l}\bowtie\mathfrak{k}$. Here, the first pairing is the one between $\mathfrak{l}^*\times\mathfrak{k}^*$ and $\mathfrak{l}\bowtie\mathfrak{k}$, the others are the pairing between $V^*\oplus W^*$ and $V\oplus W$. On the other hand, needless to say that we assume all the vector spaces to be reflexive. If the explicit expressions for the Lie bracket on $\mathfrak{l}\bowtie\mathfrak{k}$, the left action $\vartriangleright$ in (6.13), and the 2-cocycle Θ in (6.16) are substituted into the bracket (6.26), one arrives at

$$\begin{aligned} \{\mathcal{H},\mathcal{F}\}_{_\Theta\rtimes}(\mu\oplus\nu) = &\pm\big\langle \nu_{\mathfrak{l}^*}, \big[\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}},\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\big] + \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\blacktriangleright\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}} - \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\blacktriangleright\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\big\rangle \\ &\pm\big\langle \nu_{\mathfrak{k}^*}, \big[\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}},\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\big] + \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\blacktriangleleft\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}} - \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\blacktriangleleft\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\big\rangle \\ &\pm\big\langle \mu_{V^*}, \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\rfloor\frac{\delta\mathcal{F}}{\delta\mu_{V^*}} + \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\curvearrowright\frac{\delta\mathcal{F}}{\delta\mu_{V^*}} + \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\rfloor\frac{\delta\mathcal{H}}{\delta\mu_{V^*}} + \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\curvearrowright\frac{\delta\mathcal{H}}{\delta\mu_{V^*}}\big\rangle \\ &\pm\big\langle \mu_{V^*}, \epsilon\big(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}},\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\big) - \epsilon\big(\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}},\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\big) + \varphi\big(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}},\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\big) \\ &\mp\big\langle \mu_{W^*}, -\frac{\delta\mathcal{F}}{\delta\mu_{W^*}}\curvearrowleft\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}} + \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\lfloor\frac{\delta\mathcal{F}}{\delta\mu_{W^*}} - \frac{\delta\mathcal{H}}{\delta\mu_{W^*}}\curvearrowleft\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}} + \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\lfloor\frac{\delta\mathcal{H}}{\delta\mu_{W^*}}\big\rangle. \\ &\pm\big\langle \mu_{W^*}, \iota\big(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}},\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}\big) - \iota\big(\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}},\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\big) + \phi\big(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}},\frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}\big)\big\rangle. \end{aligned} \tag{6.27}$$

Here, $\blacktriangleright$ and $\blacktriangleleft$ are the actions in (6.5), $\curvearrowleft$ and $\curvearrowright$ are in (6.6) whereas $\downharpoonleft$ and $\downharpoonleft$ are those in (6.2). In view of Proposition 6.1, one can write the bracket (6.26) as a matched pair Lie-Poisson bracket. To this end, by reordering the elements in (6.25), we set

$$\tilde{\mu} = \mu_{V^*} \oplus \nu_{\mathfrak{l}^*} \in \mathfrak{g} = V_{\varphi} \rtimes \mathfrak{l}, \qquad \tilde{\nu} = \mu_{W^*} \oplus \nu_{\mathfrak{k}^*} \in \mathfrak{h} = W_{\phi} \rtimes \mathfrak{k}. \tag{6.28}$$

Thus, the matched pair Lie-Poisson bracket in (4.23) takes the form

$$\begin{aligned}
&\{\mathcal{H},\mathcal{F}\}_{\bowtie}(\tilde{\mu}\oplus\tilde{\nu}) = \pm\{\mathcal{H},\mathcal{F}\}_{\varphi\rtimes}(\tilde{\mu}) \pm \{\mathcal{H},\mathcal{F}\}_{\phi\rtimes}(\tilde{\nu}) \\
&\mp\Big\langle \mu_{V^*}\oplus\nu_{\mathfrak{l}^*}, (\frac{\delta\mathcal{H}}{\delta\mu_{W^*}} \oplus \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}) \rhd (\frac{\delta\mathcal{F}}{\delta\mu_{V^*}} \oplus \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}) - (\frac{\delta\mathcal{F}}{\delta\mu_{W^*}} \oplus \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}) \rhd (\frac{\delta\mathcal{H}}{\delta\mu_{V^*}} \oplus \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}})\Big\rangle \\
&\mp\Big\langle \mu_{W^*}\oplus\nu_{\mathfrak{k}^*}, (\frac{\delta\mathcal{H}}{\delta\mu_{W^*}} \oplus \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}) \lhd (\frac{\delta\mathcal{F}}{\delta\mu_{V^*}} \oplus \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{l}^*}}) - (\frac{\delta\mathcal{F}}{\delta\mu_{W^*}} \oplus \frac{\delta\mathcal{F}}{\delta\nu_{\mathfrak{k}^*}}) \lhd (\frac{\delta\mathcal{H}}{\delta\mu_{V^*}} \oplus \frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}})\Big\rangle,
\end{aligned} \tag{6.29}$$

where the actions $\lhd$ and $\rhd$ are in (6.8). Notice that, the Lie-Poisson brackets on the right hand side of the first line are the individual Lie-Poisson brackets on the dual spaces $\mathfrak{g}^*$ and $\mathfrak{h}^*$, respectively. Those terms available in the second line of (6.29) are the manifestations of the left action of $\mathfrak{h}$ on $\mathfrak{g}$. The third line, on the other hand, is due to the right action of $\mathfrak{g}$ on $\mathfrak{h}$. A direct computation gives that the Lie-Poisson bracket (6.29) is equal to the Lie-Poisson bracket in (6.27).

Dual actions:

First, define the dual actions of $\curvearrowright$ and $\curvearrowleft$ in (6.6) as

$$\begin{aligned}
&\overset{*}{\curvearrowleft} : V^* \to V^*, \qquad \langle \mu_{V^*} \overset{*}{\curvearrowleft} k, v\rangle = \langle \mu_{V^*}, k \curvearrowright v\rangle, \\
&\overset{*}{\curvearrowright} : W^* \to W^*, \qquad \langle l \overset{*}{\curvearrowright} \mu_{W^*}, w\rangle = \langle \mu_{W^*}, w \curvearrowleft l\rangle,
\end{aligned} \tag{6.30}$$

respectively. It follows at once that $\overset{*}{\curvearrowleft}$ is a right action whereas $\overset{*}{\curvearrowright}$ is a left action. Let us, furthermore, introduce the dual (right) actions of $\downharpoonleft$ and $\downharpoonright$ in (6.2) as

$$\begin{aligned}
&\overset{*}{\downharpoonleft} k : W^* \to W^*, \qquad \langle \mu_{W^*} \overset{*}{\downharpoonleft} k, w\rangle = \langle \mu_{W^*}, k \downharpoonright w\rangle, \\
&\overset{*}{\downharpoonright} l : V^* \to V^*, \qquad \langle \mu_{V} \overset{*}{\downharpoonright} l, v\rangle = \langle \mu_{V^*}, l \downharpoonleft v\rangle,
\end{aligned} \tag{6.31}$$

respectively. Next, we determine the dual actions of $\blacktriangleleft$ and $\blacktriangleright$ in (6.5) to be

$$\begin{aligned}
&\overset{*}{\blacktriangleleft} : \mathfrak{k}\otimes\mathfrak{l}^* \longrightarrow \mathfrak{l}^*, \qquad \langle k \overset{*}{\blacktriangleleft} \nu_{\mathfrak{l}^*}, l\rangle = \langle \nu_{\mathfrak{l}^*}, k \blacktriangleright l\rangle, \\
&\overset{*}{\blacktriangleright} : \mathfrak{l}\otimes\mathfrak{k}^* \longrightarrow \mathfrak{k}^*, \qquad \langle l \overset{*}{\blacktriangleright} \nu_{\mathfrak{k}^*}, k\rangle = \langle \nu_{\mathfrak{k}^*}, k \blacktriangleleft l\rangle,
\end{aligned} \tag{6.32}$$

respectively. We define the dual mappings of the 2-cocycles φ and ϕ as

$$\begin{aligned}
&\varphi_l^* : V^* \longrightarrow \mathfrak{l}^*, \qquad \langle \varphi_l^*\mu_{V^*}, l'\rangle = -\langle \mu_{V^*}, \varphi_l l'\rangle, \\
&\phi_k^* : W^* \longrightarrow \mathfrak{k}^*, \qquad \langle \phi_k^*\mu_{W^*}, k'\rangle = -\langle \mu_{W^*}, \phi_k k'\rangle.
\end{aligned} \tag{6.33}$$

Lastly, the duals of ϵ and ι in (6.7) appear as

$$\begin{aligned}
&\epsilon_k^* : V^* \longrightarrow \mathfrak{l}^*, \qquad \langle \epsilon_k^*\mu_{V^*}, l\rangle = -\langle \mu_{V^*}, \epsilon_k l\rangle, \\
&\iota_k^* : W^* \longrightarrow \mathfrak{l}^*, \qquad \langle \iota_k^*\mu_{W^*}, l\rangle = -\langle \mu_{W^*}, \iota_k l\rangle.
\end{aligned} \tag{6.34}$$

Lie-Poisson equations

According to the equations (4.28), it suffices to define dual mappings of the action $\triangleright$ (6.13) and Θ (6.16) to formulate the Lie-Poisson dynamics. For $\triangleright$, by definition, we compute

$$\begin{aligned}\langle(\mu_{V^*}\oplus\mu_{W^*})\overset{*}{\triangleleft}(l\oplus k),(v\oplus w)\rangle &= \langle\mu_{V^*}, l\downharpoonright v + k\curvearrowright v\rangle + \langle\mu_{W^*}, -w\curvearrowleft l + k\downharpoonleft w\rangle,\\ &= \langle\mu_{V^*}, l\downharpoonright v\rangle + \langle\mu_{V^*}, k\curvearrowright v\rangle + \langle\mu_{W^*}, -w\curvearrowleft l\rangle + \langle\mu_{W^*}, k\downharpoonleft w\rangle. \qquad (6.35)\end{aligned}$$

Then, we have

$$(\mu_{V^*}\oplus\mu_{W^*})\overset{*}{\triangleleft}(l\oplus k) = \big(\mu_{V^*}\overset{*}{\curvearrowleft} k + \mu_{V^*}\overset{*}{\downharpoonleft} l\big)\oplus\big(l\overset{*}{\curvearrowright}\mu_{W^*} + \mu_{W^*}\overset{*}{\downharpoonright} k\big). \qquad (6.36)$$

For Θ, we compute

$$\Theta^*_{(l\oplus k)}(\mu_{V^*}\oplus\mu_{W^*}) = (\varphi^*_l\mu_{V^*} + \epsilon^*_k\mu_{V^*} - \iota^*_k\mu_{W^*})\oplus(\phi^*_k\mu_{W^*} + \iota^*_k\mu_{W^*} - \epsilon^*_k\mu_{V^*}), \qquad (6.37)$$

where φ^*_l, ϕ^*_k are defined as (6.33), and ϵ^*_k, ι^*_k are given by (6.34). There are two more dual mappings we need to get for the left side of the equation (4.28). These are $\mathfrak{b}^*_{(l\oplus k)}(\mu_{V^*}\oplus\mu_{W^*})$ and the coadjoint action of $(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}})$ on the dual element $(\nu_{\mathfrak{l}^*}\oplus\nu_{\mathfrak{k}^*})$. Being a matched pair, we can employ the equation (4.22) for the case of $\mathfrak{l}\bowtie\mathfrak{k}$. Accordingly, we arrive at

$$\begin{aligned}&ad^*_{(\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\oplus\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}})}(\nu_{\mathfrak{l}^*}\oplus\nu_{\mathfrak{k}^*}) \qquad (6.38)\\ &= \Big(ad^*_{\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}}\nu_{\mathfrak{l}^*} + \nu_{\mathfrak{l}^*}\overset{*}{\blacktriangleleft}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}} + \nu_{\mathfrak{k}^*}\overset{*}{\blacktriangleright}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\Big)\oplus\Big(ad^*_{\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}}\nu_{\mathfrak{k}^*} - \nu_{\mathfrak{l}^*}\overset{*}{\blacktriangleleft}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}} - \nu_{\mathfrak{k}^*}\overset{*}{\blacktriangleright}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\Big),\end{aligned}$$

where $\overset{*}{\blacktriangleleft}$ and $\overset{*}{\blacktriangleright}$ denote dual actions in the equation (6.32). Finally, using equation (4.3), we arrive at

$$\mathfrak{b}^*_{(l,k)}\mu_{V^*} = (\mu_{V^*}\overset{*}{\downharpoonleft} l + \mu_{V^*}\overset{*}{\curvearrowleft} k)\oplus(\mu_{W^*}\overset{*}{\downharpoonright} k - \mu_{W^*}\overset{*}{\curvearrowright} l). \qquad (6.39)$$

Hence, according to the equations (6.36), (6.37), (6.38), (6.39) the (plus/minus) Lie-Poisson equations (governed by the Hamiltonian function $\mathcal{H}=\mathcal{H}((\mu_{V^*},\mu_{W^*}),(\nu_{\mathfrak{l}^*},\nu_{\mathfrak{k}^*}))$ are computed as

$$\begin{aligned}\dot{\mu}_{V^*} &= \mu_{V^*}\overset{*}{\downharpoonleft}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}} + \mu_{V^*}\overset{*}{\curvearrowleft}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}},\\ \dot{\mu}_{W^*} &= -\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\overset{*}{\curvearrowright}\mu_{W^*} + \mu_{W^*}\overset{*}{\downharpoonright}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}, \qquad (6.40)\\ \dot{\nu}_{\mathfrak{l}^*} &= \varphi^*_l\mu_{V^*} + \epsilon^*_k\mu_{V^*} - \iota^*_k\mu_{W^*} + ad^*_{\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}}\nu_{\mathfrak{l}^*} + \nu_{\mathfrak{l}^*}\overset{*}{\blacktriangleleft}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}} + \nu_{\mathfrak{k}^*}\overset{*}{\blacktriangleright}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}\\ &\quad + \mu_{V^*}\overset{*}{\downharpoonleft} l + \mu_{V^*}\overset{*}{\curvearrowleft} k,\\ \dot{\nu}_{\mathfrak{k}^*} &= \phi^*_k\mu_{W^*} + \iota^*_k\mu_{W^*} - \epsilon^*_k\mu_{V^*} + ad^*_{\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{k}^*}}}\nu_{\mathfrak{k}^*} - \nu_{\mathfrak{l}^*}\overset{*}{\blacktriangleleft}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}}\\ &\quad - \nu_{\mathfrak{k}^*}\overset{*}{\blacktriangleright}\frac{\delta\mathcal{H}}{\delta\nu_{\mathfrak{l}^*}} + \mu_{W^*}\overset{*}{\downharpoonright} k - \mu_{W^*}\overset{*}{\curvearrowright} l.\end{aligned}$$

Particular case

Now, we examine how Lie-Poisson equations look like for the particular case we gave in (6.1). Let us recall briefly that we took the left actions $\downharpoonright$ and $\downharpoonleft$ in (6.2), the mutual actions of $\mathfrak{k}$ and $\mathfrak{l}$ in (6.5) and the mappings in (6.6) to be trivial.

Since the calculations of these choices are made in the previous section, it is possible to see the effects directly for Lie-Poisson equations (6.40). Accordingly, $\dot{\mu}_{V^*}$ and $\dot{\mu}_{W^*}$ both vanish, and

$$\begin{aligned} \dot{\nu}_{\mathfrak{l}^*} &= \varphi_l^* \mu_{V^*} + \epsilon_k^* \mu_{V^*} - \iota_k^* \mu_{W^*} + ad^*_{\frac{\delta \mathcal{H}}{\delta \nu_{\mathfrak{l}^*}}} \nu_{\mathfrak{l}^*} \\ \dot{\nu}_{\mathfrak{k}^*} &= \phi_k^* \mu_{W^*} + \iota_k^* \mu_{W^*} - \epsilon_k^* \mu_{V^*} + ad^*_{\frac{\delta \mathcal{H}}{\delta \nu_{\mathfrak{k}^*}}} \nu_{\mathfrak{k}^*}. \end{aligned} \tag{6.41}$$

7. Couplings of dissipative systems

In the present section, we consider two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ under mutual actions satisfying the compatibility conditions in (3.29). Hence, we have a well-defined double cross sum Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$ equipped with the Lie bracket (3.28). As explained in Subsection 4.2, on the dual space $\mathfrak{g}^* \oplus \mathfrak{h}^*$ there exists the Lie-Poisson bracket $\{\bullet, \bullet\}_{\bowtie}$ displayed in (4.23). This Poisson structure lets us to arrive at the matched Lie-Poisson equations (4.24) governing the collective motion of the individual Lie-Poisson dynamics on $\mathfrak{g}^*$ and $\mathfrak{h}^*$. Following the discussions on the coupling of Rayleigh dissipation in the next subsection, we shall examine the coupling problem of the symmetric brackets.

7.1. Rayleigh type dissipation

In (2.14), Rayleigh type dissipation was introduced by means of the coadjoint action, and a linear operator. In the present subsection, we shall provide a way to couple two Lie-Poisson dynamics admitting Rayleigh type dissipative terms. To this end, we first determine the dynamics of the constitutive systems. Let us now assume that, on the dual space $\mathfrak{g}^*$ the Rayleigh type dissipation is provided by a linear operator $\Upsilon^{\mathfrak{g}} : \mathfrak{g}^* \mapsto \mathfrak{g}$, that is,

$$\dot{\mu} \mp ad^*_{\frac{\delta \mathcal{H}}{\delta \mu}} \mu = \mp ad^*_{\Upsilon^{\mathfrak{g}}(\mu)} \mu. \tag{7.1}$$

Similarly, on $\mathfrak{h}^*$, we let the Rayleigh type dissipation is given by a linear operator $\Upsilon^{\mathfrak{h}} : \mathfrak{h}^* \mapsto \mathfrak{h}$. Namely,

$$\dot{\nu} \mp ad^*_{\frac{\delta \mathcal{H}}{\delta \nu}} \nu = \mp ad^*_{\Upsilon^{\mathfrak{h}}(\nu)} \nu. \tag{7.2}$$

To couple the dynamics in (7.1) and (7.2), we introduce a linear operator from the dual space $\mathfrak{g}^* \oplus \mathfrak{h}^*$ to the double cross sum Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$ given by

$$\mathfrak{g}^* \oplus \mathfrak{h}^* \longrightarrow \mathfrak{g} \oplus \mathfrak{h}, \qquad (\mu \oplus \nu) \mapsto (\Upsilon^{\mathfrak{g}}(\mu) \oplus \Upsilon^{\mathfrak{h}}(\nu)), \tag{7.3}$$

where $\Upsilon^{\mathfrak{g}}$ and $\Upsilon^{\mathfrak{h}}$ are the linear mappings, in (7.1) and (7.2), generating the dissipation for the individual systems. The dissipative term generated by the mapping Λ is computed to be

$$\begin{aligned} &\mp ad^*_{\Upsilon^{\mathfrak{g}}(\mu) \oplus \Upsilon^{\mathfrak{h}}(\nu)} (\mu \oplus \nu) \\ &= \big(\mp ad^*_{\Upsilon^{\mathfrak{g}}(\mu)} \mu \pm \mu \overset{*}{\triangleleft} \Upsilon^{\mathfrak{h}}(\nu) \pm \mathfrak{a}^*_{\Upsilon^{\mathfrak{h}}(\nu)} \nu \big) \oplus \big(\mp ad^*_{\Upsilon^{\mathfrak{h}}(\nu)} \nu \mp \Upsilon^{\mathfrak{g}}(\mu) \overset{*}{\triangleright} \nu \mp \mathfrak{b}^*_{\Upsilon^{\mathfrak{g}}(\mu)} \mu \big), \end{aligned} \tag{7.4}$$

where the dual actions $\overset{*}{\triangleleft}$ and $\overset{*}{\triangleright}$ are those given in (4.1) and (4.4), respectively. Notice that, the cross actions $\mathfrak{a}^*$ and $\mathfrak{b}^*$ are the ones in (4.6) and (4.3), respectively. The dissipation, then, was obtained as above. Observe that, while coupling the dissipative terms in (7.4), we respect the mutual actions. So that, the collective dissipative term manifests the mutual actions. It reduces to the direct sum of the dissipative terms of the individual motions if the actions are trivial.

Obeying the general construction in (2.14), we merge the dissipative terms in (7.4) with the matched Lie-Poisson equations (4.24). This leads to the coupled system

$$\begin{aligned} &\dot{\mu} \mp ad^{*}_{\delta\mathcal{H}/\delta\mu}\mu \pm \frac{\delta\mathcal{H}}{\delta\mu} \overset{*}{\triangleleft} \nu \pm \mathfrak{a}^{*}_{\delta\mathcal{H}/\delta\nu}\nu = \mp ad^{*}_{\Upsilon^{\mathfrak{g}}(\mu)}\mu \pm \mu \overset{*}{\triangleleft} \Upsilon^{\mathfrak{h}}(\nu) \pm \mathfrak{a}^{*}_{\Upsilon^{\mathfrak{h}}(\nu)}\nu \\ &\dot{\nu} \mp ad^{*}_{\delta\mathcal{H}/\delta\nu}\nu \mp \frac{\delta\mathcal{H}}{\delta\nu} \overset{*}{\triangleright} \mp \mathfrak{b}^{*}_{\delta\mathcal{H}/\delta\mu}\mu = \mp ad^{*}_{\Upsilon^{\mathfrak{h}}(\nu)}\nu \mp \Upsilon^{\mathfrak{g}}(\mu) \overset{*}{\triangleright} \nu \mp \mathfrak{b}^{*}_{\Upsilon^{\mathfrak{g}}(\mu)}\mu. \end{aligned} \tag{7.5}$$

It is evident that, this formulation respects the mutual actions. By taking one these actions, one arrives at the semidirect product theory for the Lie-Poisson system with Rayleigh type dissipation. If both actions are trivial, it is immediate to see that the system (7.5) turns out to be a simple collection of the individual motions in (7.1) and (7.2).

7.2. Matched double bracket

Recall that, in (2.21), we have presented the double bracket in terms of the structure constants of the Lie algebra. Therefore, to have a symmetric bracket on the matched Lie-Poisson geometry, we first recall the structure constants of the double cross sum Lie algebra given in (3.36). Then referring to the coordinate realization of the matched Lie-Poisson bracket in (4.19), we compute the associated Poisson bivector Λ as

$$\begin{aligned} &\Lambda_{\alpha\beta} = \pm C^{\gamma}_{\beta\alpha}\mu_{\gamma}, && \Lambda_{\alpha b} = \mp R^{d}_{b\alpha}\nu_{d} \mp L^{\gamma}_{b\alpha}\mu_{\gamma}, \\ &\Lambda_{a\beta} = \pm R^{d}_{\beta a}\nu_{d} \pm L^{\gamma}_{\beta a}\mu_{\gamma}, && \Lambda_{ab} = \pm D^{d}_{ab}\nu_{d}, \end{aligned} \tag{7.6}$$

where $C^{\gamma}_{\beta\alpha}$'s are structure constants on $\mathfrak{g}$, D^{d}_{ab}'s are structure constants on $\mathfrak{h}$. Here, $R^{d}_{b\alpha}$'s and $L^{\gamma}_{b\alpha}$'s are constants defining the right and the left actions according to the exhibitions in (3.22), respectively. In accordance with this coordinate realizations and in view of the definition (2.21), matched Double bracket dissipation $(\mathcal{F},\mathcal{S})^{(mD)}$ for two functions $\mathcal{F}$ and $\mathcal{S}$ defined on $\mathfrak{g}^{*}\oplus\mathfrak{h}^{*}$ is

$$\begin{aligned} (\mathcal{F},\mathcal{S})^{(mD)}(\mu,\nu) &= \Big[\sum_{b}\Lambda_{\alpha b}\Lambda_{\beta b} + \sum_{\gamma}\Lambda_{\alpha\gamma}\Lambda_{\beta\gamma}\Big]\frac{\partial\mathcal{F}}{\partial\mu_{\alpha}}\frac{\partial\mathcal{S}}{\partial\mu_{\beta}} \\ &+ \Big[\sum_{b}\Lambda_{ab}\Lambda_{\alpha b} + \sum_{\gamma}\Lambda_{a\gamma}\Lambda_{\alpha\gamma}\Big]\frac{\partial\mathcal{F}}{\partial\nu_{a}}\frac{\partial\mathcal{S}}{\partial\mu_{\alpha}} + \Big[\sum_{a}\Lambda_{ba}\Lambda_{ca} + \sum_{\beta}\Lambda_{b\beta}\Lambda_{c\beta}\Big]\frac{\partial\mathcal{F}}{\partial\nu_{b}}\frac{\partial\mathcal{S}}{\partial\nu_{c}} \\ &+ \Big[\sum_{\beta}\Lambda_{\alpha\beta}\Lambda_{b\beta} + \sum_{a}\Lambda_{\alpha a}\Lambda_{ba}\Big]\frac{\partial\mathcal{F}}{\partial\mu_{\alpha}}\frac{\partial\mathcal{S}}{\partial\nu_{b}}. \end{aligned} \tag{7.7}$$

Referring to this bracket, the dissipative dynamics (2.19) for $a=1$, generated by a functional $\mathcal{S}$, is computed to be

$$\begin{aligned} \dot{\mu}_{\beta} &= (\sum_{b}\Lambda_{\beta b}\Lambda_{\alpha b} + \sum_{\gamma}\Lambda_{\beta\gamma}\Lambda_{\alpha\gamma})\frac{\partial\mathcal{S}}{\partial\mu_{\alpha}} + (\sum_{\alpha}\Lambda_{\beta\alpha}\Lambda_{a\alpha} + \sum_{b}\Lambda_{\beta b}\Lambda_{ab})\frac{\partial\mathcal{S}}{\partial\nu_{a}} \\ \dot{\nu}_{d} &= (\sum_{b}\Lambda_{db}\Lambda_{\alpha b} + \sum_{\gamma}\Lambda_{d\gamma}\Lambda_{\alpha\gamma})\frac{\partial\mathcal{S}}{\partial\mu_{\alpha}} + (\sum_{a}\Lambda_{da}\Lambda_{na} + \sum_{\alpha}\Lambda_{d\alpha}\Lambda_{n\alpha})\frac{\partial\mathcal{S}}{\partial\nu_{n}}. \end{aligned} \tag{7.8}$$

In order to arrive at the explicit expression of the symmetric bracket (7.7), and the dissipative dynamics in (7.8) in terms of the local characterizations of left and the right actions and the structure constants of the constitutive subalgebras, one needs to substitute the calculations (7.6) into (7.7) and (7.8).

Now, we add the matched Lie-Poisson bracket $\{\bullet,\bullet\}_{\bowtie}$, given in (4.25), and the matched double bracket $(\bullet,\bullet)^{(mD)}$ in (7.7). This reads the matched metriplectic bracket. The matched metriplectic, dynamics generated by a Hamiltonian function $\mathcal{H}$ and an entropy type function $\mathcal{S}$, is computed to be

$$\dot{\mathbf{z}} = [|\mathbf{z},\mathcal{H}|]_{\bowtie,D} = \{\mathbf{z},\mathcal{H}\}_{\bowtie} + a(z,\mathcal{S})^{(mD)}. \tag{7.9}$$

In order to arrive at the explicit expression of the equations of motion in (7.9), it is enough to add the reversible matched pair dynamics in (4.26) and the irreversible matched pair dynamics in (7.8). Taking one of the actions to be trivial, one arrives at the semidirect product metriplectic bracket and the semidirect product dynamical equation. If, both of the actions are trivial, then the coupling turns out to be a simple addition.

7.3. Matched Cartan-Killing bracket

Once more, we recall the structure constants (3.36) of the matched pair Lie algebra. Referring to (2.25), we first compute the matched Cartan metric $\bar{\mathcal{G}}_{a\beta}$ and $\bar{\mathcal{G}}_{\alpha b}$ through

$$\begin{aligned}
&\bar{\mathcal{G}}_{\alpha b} = -R^d_{a\alpha}D^a_{bd} - L^\beta_{a\alpha}R^a_{b\beta} + C^\gamma_{\alpha\beta}L^\beta_{b\gamma}, \qquad &&\bar{\mathcal{G}}_{ab} = L^\beta_{a\alpha}L^\alpha_{b\beta} + D^k_{ad}D^d_{bk}\\
&\bar{\mathcal{G}}_{a\beta} = L^\gamma_{a\epsilon}C^\epsilon_{\beta\gamma} - D^d_{ab}R^b_{d\beta} - R^d_{a\gamma}L^\gamma_{d\beta}, \qquad &&\bar{\mathcal{G}}_{\alpha\beta} = R^b_{a\alpha}R^a_{b\beta} + C^\epsilon_{\alpha\gamma}C^\gamma_{\beta\epsilon}
\end{aligned} \tag{7.10}$$

on the matched pair Lie algebra $\mathfrak{g}\oplus\mathfrak{h}$. We write an element of $\mathfrak{g}^*\oplus\mathfrak{h}^*$ as $(\mu,\nu) = \mu_\alpha\bar{\mathbf{e}}^\alpha + \nu_a\bar{\mathbf{e}}^a$. We compute the matched pair Cartan-Killing bracket, defined in (2.26), as

$$\begin{aligned}
(\mathcal{F},\mathcal{H})^{(mCK)} &= \frac{\partial\mathcal{F}}{\partial\mu_\alpha}\bar{\mathcal{G}}_{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\mu_\beta} + \frac{\partial\mathcal{F}}{\partial\mu_\alpha}\bar{\mathcal{G}}_{\alpha b}\frac{\partial\mathcal{H}}{\partial\nu_b} + \frac{\partial\mathcal{F}}{\partial\nu_a}\bar{\mathcal{G}}_{a\beta}\frac{\partial\mathcal{H}}{\partial\mu_\beta} + \frac{\partial\mathcal{F}}{\partial\nu_a}\bar{\mathcal{G}}_{ab}\frac{\partial\mathcal{H}}{\partial\nu_b} \\
&= (R^b_{\alpha a}R^a_{\beta b} + C^\epsilon_{\alpha\gamma}C^\gamma_{\beta\epsilon})\frac{\partial\mathcal{F}}{\partial\mu_\alpha}\frac{\partial\mathcal{H}}{\partial\mu_\beta} + (-R^d_{a\alpha}D^a_{bd} - L^\beta_{a\alpha}R^a_{b\beta} + C^\gamma_{\alpha\beta}L^\beta_{\gamma b})\frac{\partial\mathcal{F}}{\partial\mu_\alpha}\frac{\partial\mathcal{H}}{\partial\nu_b}\\
&\quad + (-R^d_{\gamma a}L^\gamma_{d\beta} + L^\gamma_{\alpha a}C^\alpha_{\beta\gamma} - D^d_{ab}R^b_{d\beta})\frac{\partial\mathcal{F}}{\partial\nu_a}\frac{\partial\mathcal{H}}{\partial\mu_\beta} + (L^\beta_{\alpha a}L^\alpha_{\beta b} + D^k_{ad}D^d_{bk})\frac{\partial\mathcal{F}}{\partial\nu_a}\frac{\partial\mathcal{H}}{\partial\nu_b}.
\end{aligned} \tag{7.11}$$

According to formulation (7.11) the equation of motion for a functional $\mathcal{S}$ is

$$\dot{\mu}_\beta = \bar{\mathcal{G}}_{\beta a}\frac{\partial S}{\partial\nu_a} + \bar{\mathcal{G}}_{\beta\alpha}\frac{\partial S}{\partial\mu_\alpha}, \qquad \dot{\nu}_d = \bar{\mathcal{G}}_{d\beta}\frac{\partial S}{\partial\mu_\beta} + \bar{\mathcal{G}}_{da}\frac{\partial S}{\partial\nu_a}. \tag{7.12}$$

We substitute the explicit representations of the metric (7.10) into the system (7.12) and arrive at

$$\begin{aligned}
\dot{\mu}_\beta &= (-R^d_{a\beta}D^a_{bd} - L^\alpha_{a\beta}R^a_{b\alpha} + C^\gamma_{\beta\alpha}L^\alpha_{\gamma b})\frac{\partial S}{\partial\nu_b} + (R^b_{\alpha a}R^a_{\beta b} + C^\epsilon_{\alpha\gamma}C^\gamma_{\beta\epsilon})\frac{\partial S}{\partial\mu_\beta},\\
\dot{\nu}_d &= (-R^a_{\gamma d}L^\gamma_{a\beta} + L^\gamma_{\alpha d}C^\alpha_{\beta\gamma} - D^a_{db}R^b_{a\beta})\frac{\partial S}{\partial\mu_\beta} + (L^\beta_{\alpha a}L^\alpha_{\beta d} + D^k_{ab}D^b_{dk})\frac{\partial S}{\partial\nu_a}.
\end{aligned} \tag{7.13}$$

7.4. Matched Casimir dissipation bracket

Recall the Casimir dissipation bracket given in (2.29). In order to carry this discussion to coupled systems on $\mathfrak{g}^*\times\mathfrak{h}^*$, as it may be deduced from that equation, we first need to determine a real valued bilinear operator on the double cross sum Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$. We then employ the pairing equipped with a Casimir function(al). Let us first determine the dissipations individually on $\mathfrak{g}^*$ and $\mathfrak{h}^*$, and then we couple them.

Consider symmetric bilinear operators on $\mathfrak{g}$ ve $\mathfrak{h}$, denoted by ψ and ϑ, respectively. Assume that $\mathcal{C}$ is a Casimir function $\mathfrak{g}^*$ and $\mathcal{D}$ is a Casimir function on $\mathfrak{h}^*$. Then, the Casimir dissipation brackets are

$$\begin{aligned} (\mathcal{F},\mathcal{H})^{(CD)}_{\mathfrak{g}^*}(\mu) &= -\psi\left(\left[\frac{\delta \mathcal{F}}{\delta \mu},\frac{\delta \mathcal{H}}{\delta \mu}\right]_{\mathfrak{g}},\left[\frac{\delta \mathcal{C}}{\delta \mu},\frac{\delta \mathcal{H}}{\delta \mu}\right]_{\mathfrak{g}}\right)\\ (\mathcal{F},\mathcal{H})^{(CD)}_{\mathfrak{h}^*}(\nu) &= -\vartheta\left(\left[\frac{\delta \mathcal{F}}{\delta \nu},\frac{\delta \mathcal{H}}{\delta \nu}\right]_{\mathfrak{h}},\left[\frac{\delta \mathcal{D}}{\delta \nu},\frac{\delta \mathcal{H}}{\delta \nu}\right]_{\mathfrak{h}}\right). \end{aligned} \tag{7.14}$$

Define a real valued symmetric bilinear operator on $\mathfrak{g}\oplus\mathfrak{h}$, using ψ and ϑ, as

$$(\psi,\vartheta):(\mathfrak{g}\bowtie\mathfrak{h})\times(\mathfrak{g}\bowtie\mathfrak{h})\longrightarrow\mathbb{R},\qquad (\xi\oplus\eta,\xi'\oplus\eta')\mapsto\psi(\xi,\xi')+\vartheta(\eta,\eta'). \tag{7.15}$$

In terms of the Casimir functions $\mathcal{C}$ and $\mathcal{D}$ on $\mathfrak{g}^*$ and $\mathfrak{h}^*$, respectively, we define a Casimir function $(\mathcal{C},\mathcal{D})$ on $\mathfrak{g}^*\times\mathfrak{h}^*$, for example, as follows

$$(\mathcal{C},\mathcal{D})(\mu,\nu)=\mathcal{C}(\mu)+\mathcal{D}(\nu). \tag{7.16}$$

So that, matched Casimir dissipation bracket is defined to be

$$\begin{aligned} &(\mathcal{F},\mathcal{H})^{(mCD)}(\mu\oplus\nu) \\ &= -(\psi,\vartheta)\left(\left[\left(\frac{\delta \mathcal{F}}{\delta \mu}\oplus\frac{\delta \mathcal{F}}{\delta \nu}\right),\left(\frac{\delta \mathcal{H}}{\delta \mu}\oplus\frac{\delta \mathcal{H}}{\delta \nu}\right)\right]_{\bowtie},\left[\left(\frac{\delta \mathcal{C}}{\delta \mu}\oplus\frac{\delta \mathcal{D}}{\delta \nu}\right),\left(\frac{\delta \mathcal{H}}{\delta \mu}\oplus\frac{\delta \mathcal{H}}{\delta \nu}\right)\right]_{\bowtie}\right) \end{aligned} \tag{7.17}$$

where $[\bullet,\bullet]_{\bowtie}$ is the matched Lie algebra bracket in (3.28). Referring to this bracket, the dissipative dynamics (2.19) for $a=1$, generated by a functional $\mathcal{H}$, is a system of equations. The dynamics on $\mathfrak{g}^*$ is

$$\begin{aligned} \dot{\mu} = &-ad^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\left[\frac{\delta \mathcal{C}}{\delta \mu},\frac{\delta \mathcal{H}}{\delta \mu}\right]^\flat - ad^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\left(\frac{\delta \mathcal{D}}{\delta \nu}\rhd\frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat + ad^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\left(\frac{\delta \mathcal{H}}{\delta \nu}\rhd\frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat \\ &-\left[\frac{\delta \mathcal{C}}{\delta \mu},\frac{\delta \mathcal{H}}{\delta \mu}\right]^\flat \overset{*}{\lhd}\frac{\delta \mathcal{H}}{\delta \nu} - \left[\frac{\delta \mathcal{D}}{\delta \nu}\rhd\frac{\delta \mathcal{H}}{\delta \mu}\right]^\flat \overset{*}{\lhd}\frac{\delta \mathcal{H}}{\delta \nu} + \left[\frac{\delta \mathcal{H}}{\delta \nu}\rhd\frac{\delta \mathcal{C}}{\delta \mu}\right]^\flat \overset{*}{\lhd}\frac{\delta \mathcal{H}}{\delta \nu} \\ &-\mathfrak{a}^*_{\frac{\delta\mathcal{H}}{\delta\nu}}\left[\frac{\delta \mathcal{D}}{\delta \nu},\frac{\delta \mathcal{H}}{\delta \nu}\right]^\flat - \mathfrak{a}^*_{\frac{\delta\mathcal{H}}{\delta\nu}}\left(\frac{\delta \mathcal{D}}{\delta \nu}\lhd\frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat - \mathfrak{a}^*_{\frac{\delta\mathcal{H}}{\delta\nu}}\left(\frac{\delta \mathcal{H}}{\delta \nu}\lhd\frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat \end{aligned} \tag{7.18}$$

whereas the dynamics on $\mathfrak{h}^*$ is

$$\begin{aligned} \dot{\nu} = &-ad^*_{\frac{\delta\mathcal{H}}{\delta\nu}}\left[\frac{\delta \mathcal{D}}{\delta \nu},\frac{\delta \mathcal{H}}{\delta \nu}\right]^\flat - ad^*_{\frac{\delta\mathcal{H}}{\delta\nu}}\left(\frac{\delta \mathcal{D}}{\delta \nu}\lhd\frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat + ad^*_{\frac{\delta\mathcal{H}}{\delta\nu}}\left(\frac{\delta \mathcal{H}}{\delta \nu}\lhd\frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat \\ &-\frac{\delta \mathcal{H}}{\delta \mu}\overset{*}{\rhd}\left[\frac{\delta \mathcal{D}}{\delta \nu},\frac{\delta \mathcal{H}}{\delta \nu}\right]^\flat - \frac{\delta \mathcal{H}}{\delta \mu}\overset{*}{\rhd}\left[\frac{\delta \mathcal{D}}{\delta \nu}\lhd\frac{\delta \mathcal{H}}{\delta \mu}\right]^\flat + \frac{\delta \mathcal{H}}{\delta \mu}\overset{*}{\rhd}\left[\frac{\delta \mathcal{H}}{\delta \nu}\lhd\frac{\delta \mathcal{C}}{\delta \mu}\right]^\flat \\ &-\mathfrak{b}^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\left[\frac{\delta \mathcal{C}}{\delta \mu},\frac{\delta \mathcal{H}}{\delta \mu}\right]^\flat - \mathfrak{b}^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\left(\frac{\delta \mathcal{D}}{\delta \nu}\rhd\frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat + \mathfrak{b}^*_{\frac{\delta\mathcal{H}}{\delta\mu}}\left(\frac{\delta \mathcal{H}}{\delta \nu}\rhd\frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat. \end{aligned} \tag{7.19}$$

Here, the right action $\triangleright$ and the left action $\triangleleft$ are those in (3.26), whereas $\overset{*}{\triangleright}$ and $\overset{*}{\triangleleft}$ are the dual actions in (4.1) and (4.4), respectively. We note that the dual operators $\mathfrak{a}^*$ is in (4.6) , and $\mathfrak{b}^*$ is in (4.3). Here, superscript $\flat$ denotes the dualization obtained through for the symmetric operators ψ and ϑ given by

$$\langle \xi^\flat, \xi' \rangle = \psi(\xi, \xi'), \qquad \langle \eta^\flat, \eta' \rangle = \vartheta(\eta, \eta'). \tag{7.20}$$

In order to prevent to notation inflation, we denote these two mappings by the same notation as we did while denoting the Lie algebra brackets on $\mathfrak{g}$ and $\mathfrak{h}$.

We can couple the matched irreversible motion, that is matched Casimir dissipation motion, in (7.18) and (7.19) with the matched reversible motion, that is the matched Lie-Poisson dynamics in (4.24). This results with the matched metriplectic system involving Casimir dissipation terms, which is simply achieved by adding the right hand sides of the systems obeying the order. This collective motion, can be determined by a single matched metriplectic bracket

$$[|\mathcal{F}, \mathcal{H}|]_{\bowtie, CD} = \{\mathcal{F}, \mathcal{H}\}_{\bowtie} + a(\mathcal{F}, \mathcal{H})^{(mCD)},$$

where $\{\bullet, \bullet\}_{\bowtie}$ is the matched Lie-Poisson bracket in (4.23) and $(\bullet, \bullet)^{(mCD)}$ is the matched Casimir dissipation bracket in (7.17). In this case, the dynamics governed by a Hamiltonian function(al) $\mathcal{H}$, is implicitly written by

$$(\dot{\mu} \oplus \dot{\nu}) = [|(\mu \oplus \nu), \mathcal{H}|]_{\bowtie, CD}.$$

7.5. Matched Hamilton dissipation bracket

We first recall the Hamilton dissipation bracket given in (2.34), and the pure irreversible motion in (2.36). In this subsection, we shall couple (match) two Hamilton dissipation brackets in the form (2.34), and two pure irreversible motions in (2.36). Accordingly, obeying the notation presented in the previous subsection we introduce the Hamilton dissipations brackets on the constitutive spaces $\mathfrak{g}^*$ and $\mathfrak{h}^*$, for two bilinear operators ψ and ϑ, as

$$\begin{aligned} (\mathcal{F}, \mathcal{H})^{HD}_{\mathfrak{g}}(\mu) &= -\psi\left(\left[\frac{\delta \mathcal{F}}{\delta \mu}, \frac{\delta \mathcal{C}}{\delta \mu}\right]_{\mathfrak{g}}, \left[\frac{\delta \mathcal{H}}{\delta \mu}, \frac{\delta \mathcal{C}}{\delta \mu}\right]_{\mathfrak{g}}\right) \\ (\mathcal{F}, \mathcal{H})^{HD}_{\mathfrak{h}}(\nu) &= -\vartheta\left(\left[\frac{\delta \mathcal{F}}{\delta \nu}, \frac{\delta \mathcal{D}}{\delta \nu}\right]_{\mathfrak{h}}, \left[\frac{\delta \mathcal{H}}{\delta \nu}, \frac{\delta \mathcal{D}}{\delta \nu}\right]_{\mathfrak{h}}\right), \end{aligned} \tag{7.21}$$

where $\mathcal{C}$ and $\mathcal{D}$ are Casimir functions on $\mathfrak{g}^*$ and $\mathfrak{h}^*$, respectively. In order to match these symmetric brackets, we recall the real valued bilinear map (7.15) defined on the double cross sum Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$. Then, we introduce matched Hamilton dissipation bracket

$$\begin{aligned} &(\mathcal{F}, \mathcal{H})^{(mHD)}(\mu \oplus \nu) \\ &= -(\psi, \vartheta)\left(\left[\left(\frac{\delta \mathcal{F}}{\delta \mu} \oplus \frac{\delta \mathcal{F}}{\delta \nu}\right), \left(\frac{\delta \mathcal{C}}{\delta \mu} \oplus \frac{\delta \mathcal{D}}{\delta \nu}\right)\right], \left[\left(\frac{\delta \mathcal{H}}{\delta \mu} \oplus \frac{\delta \mathcal{H}}{\delta \nu}\right), \left(\frac{\delta \mathcal{C}}{\delta \mu} \oplus \frac{\delta \mathcal{D}}{\delta \nu}\right)\right]\right) \end{aligned} \tag{7.22}$$

where the brackets inside the pairing are the matched Lie bracket in (3.28).

Irreversible dynamics on $\mathfrak{g}^* \times \mathfrak{h}^*$ can hence be obtained as

$$
\begin{aligned}
\dot{\mu} = & -ad^*_{\frac{\delta \mathcal{C}}{\delta \mu}} \left[\frac{\delta \mathcal{H}}{\delta \mu}, \frac{\delta \mathcal{C}}{\delta \mu}\right]^\flat - ad^*_{\frac{\delta \mathcal{C}}{\delta \mu}} \left(\frac{\delta \mathcal{H}}{\delta \nu} \rhd \frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat + ad^*_{\frac{\delta \mathcal{C}}{\delta \mu}} \left(\frac{\delta \mathcal{D}}{\delta \nu} \rhd \frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat \\
& - [\frac{\delta \mathcal{H}}{\delta \mu}, \frac{\delta \mathcal{C}}{\delta \mu}]^\flat \overset{*}{\lhd} \frac{\delta \mathcal{D}}{\delta \nu} - [\frac{\delta \mathcal{H}}{\delta \nu} \rhd \frac{\delta \mathcal{C}}{\delta \mu}]^\flat \overset{*}{\lhd} \frac{\delta \mathcal{D}}{\delta \nu} + [\frac{\delta \mathcal{D}}{\delta \nu} \rhd \frac{\delta \mathcal{H}}{\delta \mu}]^\flat \overset{*}{\lhd} \frac{\delta \mathcal{D}}{\delta \nu} \\
& - \mathfrak{a}^*_{\frac{\delta \mathcal{D}}{\delta \nu}} [\frac{\delta \mathcal{H}}{\delta \nu}, \frac{\delta \mathcal{D}}{\delta \nu}]^\flat - \mathfrak{a}^*_{\frac{\delta \mathcal{D}}{\delta \nu}} \left(\frac{\delta \mathcal{H}}{\delta \nu} \lhd \frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat - \mathfrak{a}^*_{\frac{\delta \mathcal{D}}{\delta \nu}} \left(\frac{\delta \mathcal{D}}{\delta \nu} \lhd \frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat \\
\dot{\nu} = & -ad^*_{\frac{\delta \mathcal{D}}{\delta \nu}} [\frac{\delta \mathcal{H}}{\delta \nu}, \frac{\delta \mathcal{D}}{\delta \nu}]^\flat - ad^*_{\frac{\delta \mathcal{D}}{\delta \nu}} \left(\frac{\delta \mathcal{H}}{\delta \nu} \lhd \frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat + ad^*_{\frac{\delta \mathcal{D}}{\delta \nu}} \left(\frac{\delta \mathcal{D}}{\delta \nu} \lhd \frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat \\
& - \frac{\delta \mathcal{C}}{\delta \mu} \overset{*}{\rhd} [\frac{\delta \mathcal{H}}{\delta \nu}, \frac{\delta \mathcal{D}}{\delta \nu}]^\flat - \frac{\delta \mathcal{C}}{\delta \mu} \overset{*}{\rhd} [\frac{\delta \mathcal{H}}{\delta \nu} \lhd \frac{\delta \mathcal{C}}{\delta \mu}]^\flat + \frac{\delta \mathcal{C}}{\delta \mu} \overset{*}{\rhd} [\frac{\delta \mathcal{D}}{\delta \nu} \lhd \frac{\delta \mathcal{H}}{\delta \mu}]^\flat \\
& - \mathfrak{b}^*_{\frac{\delta \mathcal{C}}{\delta \mu}} [\frac{\delta \mathcal{H}}{\delta \mu}, \frac{\delta \mathcal{C}}{\delta \mu}]^\flat - \mathfrak{b}^*_{\frac{\delta \mathcal{C}}{\delta \mu}} \left(\frac{\delta \mathcal{H}}{\delta \nu} \rhd \frac{\delta \mathcal{C}}{\delta \mu}\right)^\flat + \mathfrak{b}^*_{\frac{\delta \mathcal{C}}{\delta \mu}} \left(\frac{\delta \mathcal{D}}{\delta \nu} \rhd \frac{\delta \mathcal{H}}{\delta \mu}\right)^\flat . \qquad (7.23)
\end{aligned}
$$

Here, the right action $\rhd$ and the left action $\lhd$ are those in (3.26), whereas $\overset{*}{\rhd}$ and $\overset{*}{\lhd}$ are the dual actions in (4.1) and (4.4), respectively. Also, $\mathfrak{a}$ and $\mathfrak{a}^*$ were defined as in (4.5) and (4.6), while $\mathfrak{b}$ and $\mathfrak{b}^*$ were defined as in (4.2) and (4.3).

8. Illustration: Heisenberg algebras in mutual actions

8.1. Heisenberg algebra and Lie-Poisson dynamics

We start with a 3 dimensional Heisenberg algebra which we denote by $\mathfrak{g}$, see [58]. Assuming a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathfrak{g}$, the Lie bracket may be given by

$$
[\mathbf{e}_1, \mathbf{e}_3] = 0, \qquad [\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \qquad [\mathbf{e}_2, \mathbf{e}_3] = 0. \qquad (8.1)
$$

Heisenberg algebra may be expressed as a 2-cocycle extension Lie algebra. Hence, we can examine it through the discussions done in Subsection 3.3. To see this, referring to the basis of the algebra $\mathfrak{g}$, we define two linear spaces $V = \langle \mathbf{e}_3 \rangle$ and $\mathfrak{l} = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. Accordingly, we introduce a V-valued skew-symmetric bilinear mapping on $\mathfrak{l}$ as

$$
\varphi : \mathfrak{l} \times \mathfrak{l} \longrightarrow V, \qquad \varphi(\mathbf{e}_1, \mathbf{e}_1) = 0, \quad \varphi(\mathbf{e}_2, \mathbf{e}_2) = 0, \quad \varphi(\mathbf{e}_1, \mathbf{e}_2) = \mathbf{e}_3. \qquad (8.2)
$$

It is straightforward to verify that φ is a 2-cocycle. Taking the left action of $\mathfrak{l}$ on V (see the first action in the list (6.2)) to be trivial, we observe at once that (8.1) is indeed in the form of (3.40).

Coadjoint flow

Let now a basis for the dual space $\mathfrak{g}^*$ be $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$. Let also $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathfrak{g}$, and $\mu = (\mu_1, \mu_2, \mu_3)$ in $\mathfrak{g}^*$. Then, the coadjoint action of a Lie algebra element ξ in $\mathfrak{g}$ to a dual element μ in $\mathfrak{g}^*$ is computed to be

$$
ad^* : \mathfrak{g} \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \qquad ad^*_\xi \mu = (\mu_3 \xi^2, -\mu_3 \xi^1, 0). \qquad (8.3)
$$

Referring to this calculation, we write the Lie-Poisson dynamics (2.13) generated by a Hamiltonian function $\mathcal{H}$ as

$$
\dot{\mu}_1 = \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2}, \qquad \dot{\mu}_2 = -\mu_3 \frac{\partial \mathcal{H}}{\partial \mu_1} \qquad \dot{\mu}_3 = 0. \qquad (8.4)
$$

Here, the latter gives that μ_3 is a constant. In equation (8.4), if we choose $\mu_1 = q$, $\mu_2 = p$, and $\mu_3 = 1$, then we arrive at the Hamilton's equations in its very classical form

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}. \tag{8.5}$$

Therefore we claim that, in the present geometry, the Hamiltonian dynamics can be realized as a coadjoint flow.

Double bracket dissipation

For the present case, the symmetric double bracket in (2.21) is computed to be

$$(\mathcal{F}, \mathcal{S})^{(D)}(\mu) = \mu_3^2\Big(\frac{\partial \mathcal{S}}{\partial \mu_1}\frac{\partial \mathcal{F}}{\partial \mu_1} + \frac{\partial \mathcal{S}}{\partial \mu_2}\frac{\partial \mathcal{F}}{\partial \mu_2}\Big). \tag{8.6}$$

Therefore, for a function $\mathcal{S}$, the irreversible dynamics due to the symmetric bracket is computed to be

$$\dot{\mu}_1 = \mu_3^2\frac{\partial \mathcal{S}}{\partial \mu_1}, \qquad \dot{\mu}_2 = \mu_3^2\frac{\partial \mathcal{S}}{\partial \mu_2}, \qquad \dot{\mu}_3 = 0. \tag{8.7}$$

Then the metriplectic equations of motion (2.24) are computed to be

$$\dot{\mu}_1 = \mu_3\frac{\partial \mathcal{H}}{\partial \mu_2} + \mu_3^2\frac{\partial \mathcal{S}}{\partial \mu_1}, \qquad \dot{\mu}_2 = -\mu_3\frac{\partial \mathcal{H}}{\partial \mu_1} + \mu_3^2\frac{\partial \mathcal{S}}{\partial \mu_2}, \qquad \dot{\mu}_3 = 0. \tag{8.8}$$

If we choose $\mu_1 = q$, $\mu_2 = p$, and $\mu_3 = 1$, then the metriplectic dynamics (8.8) turns out to be

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} + \frac{\partial \mathcal{S}}{\delta q}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} + \frac{\partial \mathcal{S}}{\partial p}. \tag{8.9}$$

Two interesting particular instances of the present dynamics are the following.

(1) Let us take the Hamiltonian function $\mathcal{H} = p^2 + V(q)$ to be the total energy of the system, and $S = S(q)$. Then, the system (8.9) reduces to

$$\ddot{q} - S_{qq}\dot{q} - V_q = 0. \tag{8.10}$$

We cite [70] for a more elegant geometrization of the second order ODE (8.10) in terms of the GENERIC framework.

(2) As another naive application of the dissipative system (8.9), we consider a general Hamiltonian function H, along with $S = ap^2/2$ for a scalar a. Then, a fairly straightforward calculation gives (8.9) in the form of

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -ap + \frac{\partial \mathcal{H}}{\partial q}. \tag{8.11}$$

This is the conformal Hamiltonian dynamics as described in [68]. To see the geometry behind this dynamics consider first the vector field

$$X = (\partial \mathcal{H}/\partial p)\partial_q + (-ap + \partial \mathcal{H}/\partial q)\partial_p$$

generating (8.11) and then define the symplectic two-form $\Omega = dq \wedge dp$. A Hamiltonian vector field preserves the symplectic two-forms, but the vector field X satisfies

$$\mathfrak{L}_X\Omega = d\big(d\mathcal{H} - apdq\big) = adq \wedge dp = a\Omega, \tag{8.12}$$

where $\mathfrak{L}$ denotes the Lie derivative. In other words, X preserves the symplectic two-form up to a conformal factor.

8.2. Coupling of two Heisenberg algebras and matched Lie-Poisson dynamics

Consider now two copies of $3D$ Heisenberg algebras $\mathfrak{g}$ and $\mathfrak{h}$, with bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ respectively. Accordingly, given any ξ and ξ' in $\mathfrak{g}$, and any η and η' in $\mathfrak{h}$, we let

$$\begin{aligned} \xi &= \xi^1\mathbf{e}_1 + \xi^2\mathbf{e}_2 + \xi^3\mathbf{e}_3, & \xi' &= \xi^{1'}\mathbf{e}_1 + \xi^{2'}\mathbf{e}_2 + \xi^{3'}\mathbf{e}_3, \\ \eta &= \eta^1\mathbf{f}_1 + \eta^2\mathbf{f}_2 + \eta^3\mathbf{f}_3, & \eta' &= \eta^{1'}\mathbf{f}_1 + \eta^{2'}\mathbf{f}_2 + \eta^{3'}\mathbf{f}_3. \end{aligned} \tag{8.13}$$

As a result, the Lie brackets on $\mathfrak{g}$ and $\mathfrak{h}$ can be exhibited in the form

$$[\xi, \xi'] = (\xi^1\xi^{2'} - \xi^{1'}\xi^2)\mathbf{e}_3, \qquad [\eta, \eta'] = (\eta^1\eta^{2'} - \eta^{1'}\eta^2)\mathbf{f}_3 \tag{8.14}$$

respectively. Following the notation in (3.3), and referring to the coordinate realizations (8.13), we introduce a right action of $\mathfrak{g}$ on $\mathfrak{h}$, and a left action $\mathfrak{h}$ on $\mathfrak{g}$ as [58]

$$\begin{aligned} \triangleright &: \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}, & \eta \triangleright \xi &= -\eta^1\xi^2\mathbf{e}_3, \\ \triangleleft &: \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}, & \eta \triangleleft \xi &= -\xi^1\eta^2\mathbf{f}_3. \end{aligned} \tag{8.15}$$

This means that the only non-zero action constants are $R_{12}^3 = -1$, $L_{12}^3 = -1$. It then takes a straightforward computation to observe that these actions indeed satisfy the compatibility conditions (3.29) for a matched pair of Lie algebras. Accordingly, for the present coupling, the Lie algebra bracket on $\mathfrak{g} \bowtie \mathfrak{h}$ takes the form

$$\begin{aligned} &[(\xi, \eta), (\xi', \eta')]_{\bowtie} \\ &= (\xi^1\xi^{2'} - \xi^2\xi^{1'} - \eta^1\xi^{2'} + \eta^{1'}\xi^2)\mathbf{e}_3 \oplus (\eta^1\eta^{2'} - \eta^2\eta^{1'} - \eta^2\xi^{1'} + \eta^{2'}\xi^1)\mathbf{f}_3. \end{aligned} \tag{8.16}$$

As coupling of two cocycle extensions

Considering the bases of the Heisenberg algebras $\mathfrak{g}$ and $\mathfrak{h}$, their 2-cocycle extension realizations appear as

$$V_\varphi \rtimes \mathfrak{l} := \langle \mathbf{e_3} \rangle_\varphi \rtimes \langle \mathbf{e_1}, \mathbf{e_2} \rangle, \qquad W_\phi \rtimes \mathfrak{k} := \langle \mathbf{f_3} \rangle_\phi \rtimes \langle \mathbf{f_1}, \mathbf{f_2} \rangle, \tag{8.17}$$

respectively, where

$$\begin{aligned} &\varphi : \mathfrak{l} \times \mathfrak{l} \longrightarrow V, & &\varphi(\mathbf{e}_1, \mathbf{e}_1) = 0, & &\varphi(\mathbf{e}_2, \mathbf{e}_2) = 0, & &\varphi(\mathbf{e}_1, \mathbf{e}_2) = \mathbf{e}_3, \\ &\phi : \mathfrak{k} \times \mathfrak{k} \longrightarrow W, & &\phi(\mathbf{f}_1, \mathbf{f}_1) = 0, & &\phi(\mathbf{f}_2, \mathbf{f}_2) = 0, & &\phi(\mathbf{f}_1, \mathbf{f}_2) = \mathbf{f}_3, \end{aligned} \tag{8.18}$$

and the left actions in (6.2) are zero. We now exercise Proposition 6.1 for two copies of the Heisenberg algebra to form a matched pair. That is, we show that $\mathfrak{g} \bowtie \mathfrak{h}$ is a 2-cocycle extension by itself. To this end, following the notation in Subsection 6.1, we shall take both **(1)** the mutual actions $\blacktriangleleft$ and $\blacktriangleright$ of $\mathfrak{l}$ and $\mathfrak{k}$ on each other exhibited in (6.5), and **(2)** the cross actions $\curvearrowleft$ and $\curvearrowright$ in (6.6) to be trivial, whereas **(3)** the mappings ϵ and ι given in (6.7) as

$$\epsilon(\mathbf{f_1}, \mathbf{e_2}) = -\mathbf{e_3} \qquad \iota(\mathbf{f_2}, \mathbf{e_1}) = -\mathbf{f_3}, \tag{8.19}$$

while they vanish on the rest of the basis elements. Keeping these settings in mind, it is now straightforward to observe that both the left action $\triangleright$ and the right action $\triangleleft$ in (8.15) can be recast in the form of (6.8). Equivalently, the Lie bracket (8.16) on the sum of the two copies of Heisenberg algebras fits into (6.11).

On the other hand, we shall now show that the double cross sum $\mathfrak{g} \bowtie \mathfrak{h}$ is a 2-cocycle extension of $\mathfrak{l} \bowtie \mathfrak{k} = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \bowtie \langle \mathbf{f}_1, \mathbf{f}_2 \rangle$ over $V \oplus W := \langle \mathbf{e}_3 \rangle \oplus \langle \mathbf{f}_3 \rangle$. It follows at once, by the above settings, that the left action $\triangleright$ of the Lie algebra $\mathfrak{l} \bowtie \mathfrak{k}$ onto $V \oplus W$ is trivial. The 2-cocycle

$$\begin{aligned} \Theta &: (\mathfrak{l} \bowtie \mathfrak{k}) \times (\mathfrak{l} \bowtie \mathfrak{k}) \longrightarrow V \oplus W \\ &: \big(\langle \mathbf{e}_1, \mathbf{e}_2 \rangle \bowtie \langle \mathbf{f}_1, \mathbf{f}_2 \rangle\big) \times \big(\langle \mathbf{e}_1, \mathbf{e}_2 \rangle \bowtie \langle \mathbf{f}_1, \mathbf{f}_2 \rangle\big) \longrightarrow \langle \mathbf{e}_3 \rangle \oplus \langle \mathbf{f}_3 \rangle, \end{aligned} \tag{8.20}$$

then, following (6.16), can be exploited as

$$\begin{aligned} &\Theta(\xi^1 \mathbf{e}_1 + \xi^2 \mathbf{e}_2 + \eta^1 \mathbf{f}_1 + \eta^2 \mathbf{f}_2, \xi^{1'} \mathbf{e}_1 + \xi^{2'} \mathbf{e}_2 + \eta^{1'} \mathbf{f}_1 + \eta^{2'} \mathbf{f}_2) \\ &= (\xi^1 \xi^{2'} - \xi^{1'} \xi^2 - \eta^1 \xi^{2'} + \eta^{1'} \xi^2) \mathbf{e}_3 \oplus (\eta^1 \eta^{2'} - \eta^2 \eta^{1'} - \eta^2 \xi^{1'} + \eta^{2'} \xi^1) \mathbf{f}_3. \end{aligned} \tag{8.21}$$

Finally, it follows from Proposition 6.1 that

$$\mathfrak{g} \bowtie \mathfrak{h} = (V_{\varphi} \rtimes \mathfrak{l}) \bowtie (W_{\phi} \rtimes \mathfrak{k}) \cong (V \oplus W)_{\Theta} \rtimes (\mathfrak{l} \bowtie \mathfrak{k}).$$

Matched Lie-Poisson equations

Let us first fix the notation for the dual elements as

$$\mu = \mu_1 \mathbf{e}^1 + \mu_2 \mathbf{e}^2 + \mu_3 \mathbf{e}^3 \in \mathfrak{g}^*, \qquad \nu = \nu_1 \mathbf{f}^1 + \nu_2 \mathbf{f}^2 + \nu_3 \mathbf{f}^3 \in \mathfrak{h}^*. \tag{8.22}$$

The dual actions $\overset{*}{\triangleleft}$ in (4.4) and $\overset{*}{\triangleright}$ in (4.1), the cross actions $\mathfrak{a}^*$ in (4.6) and $\mathfrak{b}^*$ in (4.3) are computed to be

$$\mu \overset{*}{\triangleleft} \eta = -\mu_3 \eta^1 \mathbf{e}^2, \;\; \xi \overset{*}{\triangleright} \nu = -\nu_3 \xi^1 \mathbf{f}^2, \;\; \mathfrak{b}^*_{\xi} \mu = -\mu_3 \xi^2 \mathbf{f}^1, \;\; \mathfrak{a}^*_{\eta} \nu = -\nu_3 \eta^2 \mathbf{e}^1. \tag{8.23}$$

Then the matched Lie-Poisson equations (4.24) generated by a Hamiltonian function $\mathcal{H} = \mathcal{H}(\mu, \nu)$ on $\mathfrak{g}^* \oplus \mathfrak{h}^*$ is computed to be

$$\begin{aligned} \dot{\mu}_1 &= -\nu_3 \frac{\partial \mathcal{H}}{\partial \nu_2} + \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2}, & \dot{\mu}_2 &= \mu_3 \frac{\partial \mathcal{H}}{\partial \nu_1} - \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_1}, & \dot{\mu}_3 &= 0 \\ \dot{\nu}_1 &= -\mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2} + \nu_3 \frac{\partial \mathcal{H}}{\partial \nu_2}, & \dot{\nu}_2 &= -\nu_3 \frac{\partial \mathcal{H}}{\partial \nu_1} + \nu_3 \frac{\partial \mathcal{H}}{\partial \mu_1}, & \dot{\nu}_3 &= 0. \end{aligned} \tag{8.24}$$

Let us study some particular instances on this system of equations. In equation (8.24), if we choose $\mu_1 = q$, $\mu_2 = p$, $\nu_1 = u$, $\nu_2 = w$, and $\mu_3 = \nu_3 = 1$, then we arrive at the Hamilton's equations in a coupled form

$$\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} - \frac{\partial \mathcal{H}}{\partial w}, & \dot{p} &= \frac{\partial \mathcal{H}}{\partial u} - \frac{\partial \mathcal{H}}{\partial q}, \\ \dot{u} &= -\frac{\partial \mathcal{H}}{\partial p} + \frac{\partial \mathcal{H}}{\partial w}, & \dot{w} &= -\frac{\partial \mathcal{H}}{\partial u} + \frac{\partial \mathcal{H}}{\partial q}. \end{aligned} \tag{8.25}$$

Rayleigh type dissipation

For the present case, addition of a Rayleigh type dissipation term as

$$\begin{aligned} &\mathfrak{g}^* \oplus \mathfrak{h}^* \longrightarrow \mathfrak{g} \bowtie \mathfrak{h}, \\ &\mu \oplus \nu \;\mapsto\; (\Upsilon^{\mathfrak{g}^1} \mathbf{e}_1 + \Upsilon^{\mathfrak{g}^2} \mathbf{e}_2 + \Upsilon^{\mathfrak{g}^3} \mathbf{e}_3, \Upsilon^{\mathfrak{h}^1} \mathbf{f}_1 + \Upsilon^{\mathfrak{h}^2} \mathbf{f}_2 + \Upsilon^{\mathfrak{h}^3} \mathbf{f}_3) \end{aligned} \tag{8.26}$$

to the Lie-Poisson dynamics on $\mathfrak{g}^* \oplus \mathfrak{h}^*$ yields, in view of (7.5), the matched Lie-Poisson equations in the form of

$$\dot{\mu}_1 - \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2} + \nu_3 \frac{\partial \mathcal{H}}{\partial \nu_2} = \nu_3 \Upsilon^{\mathfrak{h}^2} - \mu_3 \Upsilon^{\mathfrak{g}^2}, \quad \dot{\mu}_2 - \mu_3 \frac{\partial \mathcal{H}}{\partial \nu_1} + \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_1} = \mu_3 \Upsilon^{\mathfrak{g}^1} - \mu_3 \Upsilon^{\mathfrak{h}^1},$$
$$\dot{\nu}_1 + \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2} - \nu_3 \frac{\partial \mathcal{H}}{\partial \nu_2} = \mu_3 \Upsilon^{\mathfrak{g}^2} - \nu_3 \Upsilon^{\mathfrak{h}^2}, \quad \dot{\nu}_2 + \nu_3 \frac{\partial \mathcal{H}}{\partial \nu_1} - \nu_3 \frac{\partial \mathcal{H}}{\partial \mu_1} = \nu_3 \Upsilon^{\mathfrak{h}^1} - \nu_3 \Upsilon^{\mathfrak{g}^1},$$
$$\dot{\mu}_3 = 0, \qquad \dot{\nu}_3 = 0. \tag{8.27}$$

Double bracket dissipation

For the present discussion, the matched double bracket (7.7) takes the particular form

$$\begin{aligned}(\mathcal{F}, \mathcal{H})^{(mD)}(\mu \oplus \nu) = \mu_3^2 \frac{\partial \mathcal{F}}{\partial \mu_1} \frac{\partial \mathcal{H}}{\partial \mu_1} &+ (2\mu_3^2 + 2\nu_3\mu_3 + \nu_3^2) \frac{\partial \mathcal{F}}{\partial \mu_2} \frac{\partial \mathcal{H}}{\partial \mu_2} \\ &- (\mu_3 + \nu_3)^2 \frac{\partial \mathcal{F}}{\partial \nu_2} \frac{\partial \mathcal{H}}{\partial \mu_2} + (2\nu_3^2 + 2\mu_3\nu_3 + \mu_3^2) \frac{\partial \mathcal{F}}{\partial \nu_2} \frac{\partial \mathcal{H}}{\partial \nu_2} \\ &- (\mu_3 + \nu_3)^2 \frac{\partial \mathcal{F}}{\partial \mu_2} \frac{\partial \mathcal{H}}{\partial \nu_2} + \nu_3^2 \frac{\partial \mathcal{F}}{\partial \nu_1} \frac{\partial \mathcal{H}}{\partial \nu_1}.\end{aligned} \tag{8.28}$$

Then the irreversible dynamics (7.8) is computed to be

$$\begin{aligned}\dot{\mu}_1 &= \frac{\partial \mathcal{S}}{\partial \mu_1} \mu_3^2, & \dot{\mu}_2 &= \frac{\partial \mathcal{S}}{\partial \mu_2} \mu_3^2, & \dot{\mu}_3 &= 0 \\ \dot{\nu}_1 &= \frac{\partial \mathcal{S}}{\partial \nu_1} \nu_3^2, & \dot{\nu}_2 &= \frac{\partial \mathcal{S}}{\partial \nu_2} \nu_3^2, & \dot{\nu}_3 &= 0.\end{aligned} \tag{8.29}$$

Let us collect the reversible Lie-Poisson dynamics in (8.24), and the irreversible dynamics generated by $\mathcal{S}$ under the realm of the symmetric (double) bracket in (8.28). Then the metriplectic equations of motion are computed to be

$$\begin{aligned}\dot{\mu}_1 &= \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2} - \nu_3 \frac{\partial \mathcal{H}}{\partial \nu_2} + \frac{\partial \mathcal{S}}{\partial \mu_1} \mu_3^2 & \dot{\mu}_2 &= \mu_3 \frac{\partial \mathcal{H}}{\partial \nu_1} - \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_1} + \frac{\partial \mathcal{S}}{\partial \mu_2} \mu_3^2, \\ \dot{\nu}_1 &= -\mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2} - \nu_3 \frac{\partial \mathcal{H}}{\partial \nu_2} + \frac{\partial \mathcal{S}}{\partial \nu_1} \nu_3^2, & \dot{\nu}_2 &= \nu_3 \frac{\partial \mathcal{H}}{\partial \nu_1} + \nu_3 \frac{\partial \mathcal{H}}{\partial \mu_1} + \frac{\partial \mathcal{S}}{\partial \nu_2} \nu_3^2.\end{aligned} \tag{8.30}$$

Accordingly, setting $\mu_1 = q$, $\mu_2 = p$, $\nu_1 = u$, $\nu_2 = w$, and $\mu_3 = \nu_3 = 1$, we arrive at

$$\begin{aligned}\dot{q} &= \frac{\partial \mathcal{H}}{\partial p} - \frac{\partial \mathcal{H}}{\partial w} + \frac{\partial \mathcal{S}}{\partial q}, & \dot{p} &= \frac{\partial \mathcal{H}}{\partial u} - \frac{\partial \mathcal{H}}{\partial q} + \frac{\partial \mathcal{S}}{\partial p} \\ \dot{u} &= -\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial \mathcal{H}}{\partial w} + \frac{\partial \mathcal{S}}{\partial u}, & \dot{w} &= \frac{\partial \mathcal{H}}{\partial u} + \frac{\partial \mathcal{H}}{\partial q} + \frac{\partial \mathcal{S}}{\partial w}.\end{aligned} \tag{8.31}$$

In particular, taking the Hamiltonian function $\mathcal{H} = 1/2(p^2 + w^2) + V(q, u)$ to be the total energy of the system, from (8.31) we obtain

$$\begin{aligned}\ddot{q} - \mathcal{S}_{qq}\dot{q} + 2V_q + \mathcal{S}_w - \mathcal{S}_p = 0, \\ \ddot{u} - \mathcal{S}_{uu}\dot{u} + 2V_u + \mathcal{S}_w + \mathcal{S}_p = 0.\end{aligned} \tag{8.32}$$

9. Illustration: Rigid bodies

We consider two identical three dimensional Euclidean spaces denoted by $\mathfrak{g} = \mathbb{R}^3$ and $\mathfrak{h} = \mathbb{R}^3_{\mathbf{k}}$, which are Lie algebras equipped with the Lie brackets given by

$$[\xi, \xi']_{\mathbb{R}^3} = \xi \times \xi', \qquad [\eta, \eta']_{\mathbb{R}^3_{\mathbf{k}}} = \mathbf{k} \times (\eta \times \eta'), \tag{9.1}$$

respectively, [57]. Here, $\times$ stands for the cross product on $\mathbb{R}^3$, and $\mathbf{k}$ is the unit vector $(0, 0, 1)$ available in the standard basis on $\mathbb{R}^3$. We note that the subscript $\mathbb{R}^3_{\mathbf{k}}$ is just a reminder that the bracket is not the classical cross product on $\mathbb{R}^3$, instead, it's the second bracket in (9.1). We shall follow the notations in Subsection 3.2. The coadjoint action of $\mathfrak{g}$ on it dual $\mathfrak{g}^* \simeq \mathbb{R}^3$, and the coadjoint action of $\mathfrak{h}$ on it dual $\mathfrak{h}^* \simeq \mathbb{R}^3$ are computed to be

$$ad^*_\xi \mu = \mu \times \xi, \qquad ad^*_\eta \nu = (\nu \cdot \eta)\mathbf{k} - \nu(\mathbf{k} \cdot \eta). \tag{9.2}$$

Let, now, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denotes a basis for $\mathbb{R}^3$, and let $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be a basis for $\mathbb{R}^3_{\mathbf{k}}$, where

$$\mathbf{e}_1 = \mathbf{f}_1 = (1, 0, 0)\,, \quad \mathbf{e}_2 = \mathbf{f}_2 = (0, 1, 0)\,, \quad \mathbf{e}_3 = \mathbf{f}_3 = \mathbf{k} = (0, 0, 1)\,. \tag{9.3}$$

Accordingly, the non-zero structure constants in (3.34) turn out to be

$$C^3_{12} = C^2_{31} = C^1_{23} = 1, \qquad D^1_{13} = D^2_{23} = 1. \tag{9.4}$$

Mutual actions

The left action of $\mathbb{R}^3_{\mathbf{k}}$ on $\mathbb{R}^3$, and the right action $\mathbb{R}^3$ on $\mathbb{R}^3_{\mathbf{k}}$ are defined through

$$\begin{aligned} \triangleleft : \mathbb{R}^3_{\mathbf{k}} \times \mathbb{R}^3 \to \mathbb{R}^3_{\mathbf{k}}, \qquad & \eta \triangleleft \xi := \eta \times \xi \\ \triangleright : \mathbb{R}^3_{\mathbf{k}} \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad & \eta \triangleright \xi := \eta \times (\xi \times \mathbf{k}). \end{aligned} \tag{9.5}$$

It is straightforward to prove that these actions satisfy the matched pair compatibility conditions in (3.29). Referring to the notations fixed in (3.22), the non-zero constants determining the actions (9.5) are computed to be

$$\begin{aligned} &L^3_{11} = L^3_{22} = -1, && L^2_{32} = L^1_{31} = 1, \\ &R^3_{12} = R^2_{31} = R^1_{23} = -1, && R^2_{13} = R^3_{21} = R^1_{32} = 1. \end{aligned} \tag{9.6}$$

As a result, the bracket on $\mathbb{R}^3 \bowtie \mathbb{R}^3_{\mathbf{k}}$ takes the form of

$$\begin{aligned} &[\xi \oplus \eta, \xi' \oplus \eta']_{\bowtie} \\ &= \big(\xi \times \xi' + \eta \times (\xi' \times \mathbf{k}) - \eta' \times (\xi \times \mathbf{k})\big) \oplus \big(\mathbf{k} \times (\eta \times \eta') + \eta \times \xi' - \eta' \times \xi\big). \end{aligned} \tag{9.7}$$

Matched Lie-Poisson dynamics

The Lie-Poisson dynamics on the dual space $\mathfrak{g}^* \times \mathfrak{h}^* \simeq \mathbb{R}^3 \times \mathbb{R}^3$ can be obtained via Proposition 4.2. To have this, first we compute the duals of the actions (9.5), in a respective order, as

$$\mu \overset{*}{\triangleleft} \eta = \mu(\eta \cdot \mathbf{k}) - (\mu \cdot \mathbf{k})\eta, \qquad \xi \overset{*}{\triangleright} \nu = \xi \times \nu \tag{9.8}$$

whereas the mappings (4.6) and (4.3) are

$$\mathfrak{a}^*_\eta \nu = \nu \times \eta, \qquad \mathfrak{b}^*_\xi \mu = (\mu \cdot \xi)\mathbf{k} - (\mu \cdot \mathbf{k})\xi. \tag{9.9}$$

Notice that, the Lie-Poisson formulation on $\mathfrak{g}^*$ corresponds to the rigid body dynamics in $3D$, [40, 62]. So we can consider the matched pair dynamics in this setting as the coupling of two bodies in $3D$. Since the dynamics for the rigid bodies are given by the Lie-Poisson equations of negative sign, we consider the opposite Lie-Poisson bracket and the negative Lie-Poisson equations.

Given any $\mu = (\mu_1, \mu_2, \mu_3)$ and $\nu = (\nu_1, \nu_2, \nu_3)$, we recall that the explicit realization of the matched Lie-Poisson bivector, defined for the matched Lie-Poisson bracket (4.29), has already been given in (7.6). In the following table, we present the coefficients of the matched Lie-Poisson bivector for the present case.

Λ	$\Lambda_{\alpha\beta}$	$\Lambda_{\alpha b}$	$\Lambda_{a\beta}$	Λ_{ab}
Λ_{11}	0	μ_3	$-\mu_3$	0
Λ_{12}	$-\mu_3$	$-\nu_3$	ν_3	0
Λ_{13}	μ_2	$\nu_2-\mu_1$	$-\nu_2+\mu_1$	ν_1
Λ_{21}	μ_3	ν_3	$-\nu_3$	0
Λ_{22}	0	μ_3	$-\mu_3$	0
Λ_{23}	μ_1	$-\nu_1-\mu_2$	$\nu_1+\mu_2$	ν_2
Λ_{31}	$-\mu_2$	$-\nu_2$	ν_2	$-\nu_1$
Λ_{32}	$-\mu_1$	ν_1	$-\nu_1$	$-\nu_2$
Λ_{33}	0	0	0	0

We remark that the first column determined the Lie-Poisson bivector on $\mathfrak{g}^*$, whereas the last one is the Lie-Poisson bivector on $\mathfrak{h}^*$. The second and third columns are manifestations of the mutual actions (9.5). Collecting all these results, we compute the matched Lie-Poisson equations (4.24) generated by a Hamiltonian function $\mathcal{H}$ on the dual space as

$$
\begin{aligned}
\dot{\mu} &= \frac{\partial \mathcal{H}}{\partial \mu} \times \mu + (\frac{\partial \mathcal{H}}{\partial \nu} \cdot \mathbf{k})\mu - (\mu \cdot \mathbf{k})\frac{\partial \mathcal{H}}{\partial \nu} - \nu \times \frac{\partial \mathcal{H}}{\partial \nu}, \\
\dot{\nu} &= (\mathbf{k} \cdot \frac{\partial \mathcal{H}}{\partial \nu})\nu - (\nu \cdot \frac{\partial \mathcal{H}}{\partial \nu})\mathbf{k} + \frac{\partial \mathcal{H}}{\partial \mu} \times \nu + (\mu \cdot \mathbf{k})\frac{\partial \mathcal{H}}{\partial \mu} - (\mu \cdot \frac{\partial \mathcal{H}}{\partial \mu})\mathbf{k}.
\end{aligned}
\tag{9.10}
$$

In Section 7, couplings of various ways of dissipations are listed. We shall now examine these couplings for the concrete example given in the previous subsection.

Rayleigh dissipation

In this $3D$ framework, and for a linear operator

$$
(\mathbb{R}^3)^* \oplus (\mathbb{R}^3_{\mathbf{k}})^* \longrightarrow \mathbb{R}^3 \bowtie \mathbb{R}^3_{\mathbf{k}}, \qquad \mu \oplus \nu \mapsto (\Upsilon^{\mathfrak{g}}(\mu) \oplus \Upsilon^{\mathfrak{h}}(\nu)), \tag{9.11}
$$

the matched Lie-Poisson systems with Rayleigh type dissipations, that is the system (7.5), takes the particular form

$$
\begin{aligned}
&\dot{\mu} - \frac{\partial \mathcal{H}}{\partial \mu} \times \mu - \mu(\frac{\partial \mathcal{H}}{\partial \nu} \cdot \mathbf{k}) + (\mu \cdot \mathbf{k})\frac{\partial \mathcal{H}}{\partial \nu} + \nu \times \frac{\partial \mathcal{H}}{\partial \nu} \\
&\quad = \mu \times \Upsilon^{\mathfrak{g}}(\mu) - \mu(\Upsilon^{\mathfrak{h}}(\nu) \cdot \mathbf{k}) + (\mu \cdot \mathbf{k})\Upsilon^{\mathfrak{h}}(\nu) + \nu \times \Upsilon^{\mathfrak{h}}(\nu),
\end{aligned}
$$

$$\dot{\nu} - \nu(\mathbf{k}\cdot\frac{\partial\mathcal{H}}{\partial\nu}) + (\nu\cdot\frac{\partial\mathcal{H}}{\partial\nu})\mathbf{k} - \frac{\partial\mathcal{H}}{\partial\mu}\times\nu - (\mu\cdot\mathbf{k})\frac{\partial\mathcal{H}}{\partial\mu} + (\mu\cdot\frac{\partial\mathcal{H}}{\partial\mu})\mathbf{k} \tag{9.12}$$
$$= (\nu\cdot\Upsilon^{\mathfrak{h}}(\nu))\mathbf{k} - \nu(\mathbf{k}\cdot\Upsilon^{\mathfrak{h}}(\nu)) - \Upsilon^{\mathfrak{h}}(\nu)\times\nu + (\mu\cdot\Upsilon^{\mathfrak{g}}(\mu))\mathbf{k} - (\mu\cdot\mathbf{k})\Upsilon^{\mathfrak{g}}(\mu).$$

Cartan-Killing dissipation

The matched Cartan-Killing metric (7.10), on the other hand, is determined as

$$[\mathcal{G}_{ij}] = \begin{bmatrix} -4 & 0 & 0 & 0 & -5 & 0 \\ 0 & -4 & 0 & 5 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 5 & 0 & -2 & 0 & 0 \\ -5 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9.13}$$

Then, the irreversible dynamics generated by a function $\mathcal{S}$, presented in (7.12), is computed to be

$$\begin{aligned} &\dot{\mu}_1 = -4\frac{\partial\mathcal{S}}{\partial\mu_1} + 5\frac{\partial\mathcal{S}}{\partial\mu_2}, \qquad \dot{\mu}_2 = -4\frac{\partial\mathcal{S}}{\partial\mu_2} - 5\frac{\partial\mathcal{S}}{\partial\mu_1}, \qquad \dot{\mu}_3 = -4\frac{\partial\mathcal{S}}{\partial\mu_3}, \\ &\dot{\nu}_1 = -5\frac{\partial\mathcal{S}}{\partial\nu_2} - 2\frac{\partial\mathcal{S}}{\partial\nu_1}, \qquad \dot{\nu}_2 = 5\frac{\partial\mathcal{S}}{\partial\nu_1} - 2\frac{\partial\mathcal{S}}{\partial\nu_2}, \qquad \dot{\nu}_3 = \frac{\partial\mathcal{S}}{\partial\nu_3}. \end{aligned} \tag{9.14}$$

Acknowledgments. This paper is a part of the project “Matched pairs of Lagrangian and Hamiltonian Systems” supported by TÜBİTAK (the Scientific and Technological Research Council of Turkey) with the project number 117F426, the support of which is acknowledged by the authors.

The first named author (OE) is grateful to Prof. Miroslav Grmela, Prof. Partha Guha, Prof. Michal Pavelka, and Prof. Petr Vágner for enlightening discussions on GENERIC. We also thank Begüm Ateşli for the discussions on the Lie algebra extensions considered here.

Finally, we appreciate the anonymous referee’s comments on the history and the terminology regarding the metriplectic systems, and acknowledge hereby their immense positive impact to the exposition.

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Received January 13, 2021
and in final form December 2, 2021